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HAUSDORFF DIMENSION OF QUASI-CIRCLES

by RUFUS BOWEN (1)

Let G be the group of all linear fractional transformations taking the unit disk U onto itself. One calls a discrete subgroup $\Gamma \subset G$ a *surface group* if U/Γ is a compact surface without branch points. This paper concerns the relation between two such groups Γ_1 and Γ_2 yielding the same topological surface. This is a classical and well-developed problem [6]; what is novel here is the application of the Gibbs measures of statistical mechanics and dynamical systems.

The groups Γ_1 and Γ_2 as above are isomorphic since each is isomorphic to the fundamental group of the surface. Furthermore, for any isomorphism $\alpha : \Gamma_1 \to \Gamma_2$ there is an interesting homeomorphism $h : S^1 \to S^1$ of the circle $S^1 = \{z \in \mathbf{C} : |z| = 1\}$ so that:

$$h(\psi z) = \alpha(\psi) h(z) \quad \text{for} \quad \psi \in \Gamma_1, \ z \in S^1.$$

This homeomorphism h, which is unique, is called the *boundary correspondence* (Fenchel and Nielsen, see [11], [14], or [22]). Let G^* be the group of linear fractional transformations taking S^1 onto itself. (G has index 2 in G^*). We will give a new proof of the following result of Mostow [13]:

Theorem 1. — The boundary homeomorphism h is a linear fractional transformation if it is absolutely continuous.

Now recall the quasi-Fuchsian group $\Lambda = \Lambda(\Gamma_1, \Gamma_2, \alpha)$ [6] associated to the pair of Fuchsian groups Γ_1, Γ_2, and a given abstract isomorphism $\alpha : \Gamma_1 \to \Gamma_2$. This is a discrete group Λ of linear fractional transformations of the extended complex plane $(\hat{\mathbf{C}} = S^2)$ which simultaneously uniformizes the surfaces U/Γ_i in the following sense:

(i) There is a Jordan curve γ in S^2 (called a *quasi-circle*) with $\psi(\gamma) = \gamma$ for all elements ψ in Λ, and each orbit of Λ is dense in γ.

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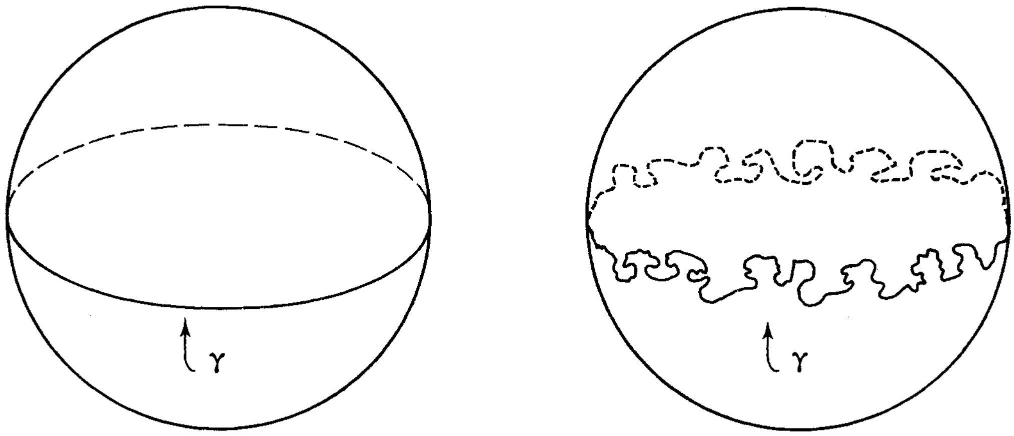
(ii) There are analytic diffeomorphisms $e_i : U \to D_i$ where D_1, D_2 are connected components of $S^2 \setminus \gamma$ so that $\alpha_i(\psi) = e_i^{-1} \circ \psi \circ e_i$ belongs to Γ_i for all $\psi \in \Lambda$ and $\alpha_i : \Lambda \to \Gamma_i$ is an isomorphism, and

(iii) $\alpha_2 \circ \alpha_1^{-1}$ is the given isomorphism $\alpha : \Gamma_1 \to \Gamma_2$.

The curve γ is generally not smooth nor even rectifiable [5, p. 263].



γ is a round circle (Theorem 1) γ is a genuine quasi-circle (Theorem 2)

FIG. 1

Theorem 2. — Suppose Γ_1 and Γ_2 are not conjugate as Fuchsian groups, namely via $\beta \in G^*$. *Then the Hausdorff dimension a of γ is greater than* 1. *Furthermore* $0 < \nu_a(\gamma) < \infty$, *where ν_a is a-dimensional Hausdorff measure, and $\nu_a | \gamma$ is ergodic under Λ.*

The paper starts with a variant of the Nielsen development. This associates symbol sequences (generalized " decimal expansion ") to points in S^1 in a way determined by Γ_i. The reader familiar with dynamical systems will recognize this as a " Markov partition for Γ_i ". *Via* this construction functions and measures are transferred from the circle (where they have geometric meaning) to the Cantor set of symbol sequences (where they can be analyzed). The paper ends with results on Schottky groups, where the symbolic sequences are quite transparent.

The author thanks Dennis Sullivan for introducing him to Kleinian groups and Hedlund for his paper [9] which motivated the present one.

1. Nielsen Development.

Let $\Gamma \subset G^*$ be a surface group. A piecewise smooth map $f : S^1 \to S^1$ is called *Markov* for Γ if one can partition S^1 into segments $I_1, \ldots, I_m$ so that:

(i) $f | I_k = f_k | I_k$ some $f_k \in \Gamma$, and

(ii) for each k, $f(I_k)$ is the union of various I_j's.

Condition (ii) can be rephrased as follows. Letting W be the set of endpoints of the I_k's, consider each such point to be really two points, depending on which I_k it is associated with. Then condition (ii) is equivalent to $f(W) \subset W$. The idea of a Markov map is to replace the action of the group Γ on S^1 by the single map f.

For each $g \geqslant 2$ let Φ_g be a surface group of genus g whose fundamental domain R in U is a regular $4g$-sided noneuclidean polygon ([9], [12], [18, p. 89]). The Nielsen development is a certain Markov map for Φ_g ([9], [14, pp. 211-217]). We shall construct variants which are more suitable for our purposes.

Each angle of R is $\alpha = \dfrac{\pi}{2g}$ and each vertex of R belongs to $4g$ distinct translates $\varphi(R)$, $\varphi \in \Phi_g$. The net $\mathfrak{N}$ is defined to be the collection of all sides and vertices of all translates $\varphi(R)$, $\varphi \in \Phi_g$. This net has the following crucial property:

(*) *the entire noneuclidean geodesic passing through any edge in $\mathfrak{N}$ is contained in $\mathfrak{N}$.*

Let V be the set of vertices in $\mathfrak{N}$ which are adjacent in $\mathfrak{N}$ to vertices of R but are not themselves vertices of R. The set V is contained in the set of vertices of the noneuclidean polygon $\widetilde{R}$ which is the union of R plus all the translates $\varphi(R)$ which touch R. The polygon $\widetilde{R}$ is convex since each interior angle equals $2\alpha < \pi$. For each vertex p of $\mathfrak{N}$ let W_p be the set of $4g$ points on S^1 which are the points at infinity of the $2g$ noneuclidean geodesics in $\mathfrak{N}$ passing through p. Letting $W = \bigcup_{p \in V} W_p$, property (*) implies that:

$$W_q \subset W \quad \text{for } q \text{ a vertex of R.}$$

Recall now a set of generators for Φ_g ([9], [12], [18]). Divide the sides of R into g groups of 4 consecutive sides; label the j-th group a_k, b_k, a_k^{-1}, b_k^{-1}. Call a_k and a_k^{-1} (and b_k and b_k^{-1}) corresponding sides. For each side s of R there is an element $\varphi_s \in \Phi_g$ so that:

$$\varphi_s(s) = R \cap \varphi_s(R) = \text{side corresponding to } s.$$

The set $\{\varphi_s\}$ generates Φ_g. Let J_s be the smaller segment of S^1 whose endpoints are the points at infinity of the noneuclidean geodesic through s.

Lemma 1. — $\varphi_s(J_s \cap W) \subset W$.

Proof. — The geodesics passing through nonconsecutive sides of R do not intersect. One way of checking this well-known fact is by an area argument. If two such geodesics intersected (perhaps at ∞), then, they together with m sides $(m \geqslant 1)$ of R would form a polygon containing R and with noneuclidean area less than $m\pi \leqslant (4g-3)\pi$. This polygon contains at least $2g$ translates of R, each having area $(4g-4)\pi$. So $2g(4g-4) \leqslant 4g-3$, which is impossible since $g \geqslant 2$.

We claim that $J_s \cap W \subset \bigcup_{p \in T_s} W_p$ where T_s is the set of vertices in $\mathfrak{N}$ adjacent to an endpoint of s. Now $\varphi_s(T_s) \subset V \cup \{\text{vertices of R}\}$ because $\varphi_s(s)$ is a side of R. Hence the above claim would yield:

$$\varphi_s(J_s \cap W) \subset \bigcup_{\varphi_s(T_s)} W_p \subset W.$$

Suppose $u \in J_s \cap W$ but $u \notin \bigcup_{p \in T_s} W_p$. Let γ be a geodesic in $\mathfrak{N}$ from some $q \in V \backslash T_s$ to u. Let q_1, q_2 be the vertices of $\widetilde{R}$ on the geodesic α through s; let r_1, r_2 be the vertices of $\widetilde{R}$ adjacent to q_1, q_2 respectively and exterior to the domain bounded by α and J_s. Let β_i be the geodesic containing a side of $\widetilde{R}$, passing through r_i but not q_i. Then β_i and α do not intersect as they pass through nonconsecutive sides of an image $\varphi(R)$.

Since the endpoints of J_s are in $W_{q_1} \cup W_{q_2}$, and $q_i \in T_s$, one has $u \in \operatorname{int} J_s$. Therefore the geodesic γ must cut the geodesic α; this intersection $\widetilde{q}$ is a vertex of the net. If q lay between q_1 and q_2, then $\widetilde{q}$ would be an endpoint of s and $u \in \bigcup_{p \in T_s} W_p$. Suppose $\widetilde{q}$ lies between q_1 and S^1 on α.

Consider the region Q in the unit disk exterior to the half disks bounded by the geodesics α, β_1 and β_2. Because β_1, β_2 are sides of the convex polygon $\widetilde{R}$, the point $q \in \widetilde{R}$ above lies in $\overline{Q}$. Because $\overline{Q}$ is convex the geodesic $\widetilde{\gamma} \subset \gamma$ from q to $\widetilde{q}$ lies in $\overline{Q}$. $\widetilde{\gamma}$ must now intersect $\widehat{r_1 q_1}$; this intersection is a vertex of the next, hence r_1 or q_1. Either case gives a contradiction. ∎

Let v be a vertex of R, belonging to the sides s and s' of R. We will construct a segment $J(v) \subset \operatorname{int}(J_s \cap J_{s'})$. Let $p, p' \in V$ be the vertices of the net $\mathfrak{N}$ adjacent to v in $\mathfrak{N}$ and lying on the geodesics α and α' through s and s'. Choose vertices q, q' of R so that q, p, v, p', q' are consecutive vertices of a translate $\varphi(R)$, $\varphi \in \Phi_g$. The geodesics β, β' containing $\widehat{pq}$, $\widehat{p'q'}$ do not intersect; they intersect S^1 at points $w(v)$, $w'(v) \in \operatorname{int}(J_s \cap J_{s'})$. Let $J(v)$ be the interval $[w(v), w'(v)] \subset S^1$. The endpoints $w(v)$, $w'(v)$ of $J(v)$ are in W; arguments analogous to the proof of Lemma 1 show that $W \cap \operatorname{int} J(v) = \emptyset$.

Theorem 0. — A surface group $\Gamma \subset G$ *has transitive Markov maps on* S^1.

Proof. — For some $g \geq 2$ there is an isomorphism $A : \Gamma \to \Phi_g$ and a boundary correspondence $H : S^1 \to S^1$ so that $H(\psi z) = A(\psi) H(z)$ for $\psi \in \Gamma$, $z \in S^1$. It is enough to produce Markov maps for Φ_g and pull them back by H.

The set $W \subset S^1$ above partitions S^1 into closed segments $I_1, \ldots, I_m$. By Lemma 1 we get a Markov f simply by requiring:

$$f | I_k = \varphi_s | I_k \quad \text{where} \quad J_s \supset I_k.$$

Since some I_k's belong to more than one J_s, there are a number of ways of doing this. Letting v, v' be the vertices of side s we can arrange that:

(i) the set $\widetilde{J}_s$ where $f = \varphi_s$ is a closed segment with

$$\operatorname{int} J_s \supset \widetilde{J}_s \supset J_s \backslash (J(v) \cup J(v')),$$

and:

(ii) either $J(v) \subset \widetilde{J}_s$ or

$$\operatorname{int}(J(v) \cap \widetilde{J}_s) = \emptyset \text{ as desired.}$$

This is possible since the interval $J(v)$ is an I_k and $J(v) \subset \text{int}(J_s \cap J_{s'})$ for v a common vertex of s and s'. The flexibility stated in (ii) will be important later. From now on f will denote a Markov map for Φ_g chosen as in (8).

Let us write $k \to j$ if $f(I_k) \supset I_j$. Transitivity means that for any k, j one can find $x_0, x_1, \dots, x_n$ so that:

$$k = x_0 \to x_1 \to x_2 \to \dots \to x_{n-1} \to x_n = j.$$

This is the same as saying that $f^n(I_k) \supset I_j$. First we claim that some iterate $f^n(I_k)$ contains $J(v)$ for some vertex v of R. Otherwise f is continuous and one-to-one on $f^n(I_k)$, and $f^{n+1}(I_k)$ is an interval longer than $f^n(I_k)$ by a factor $\inf |f'| > 1$ everywhere. This cannot continue indefinitely.

Given a vertex v of R and a translate $S = \varphi(R)$ containing v we will construct a segment $J(v, S) \subset S^1$ with endpoints in W. Let q_1, q_2, v, q_3, q_4 be consecutive vertices of S. Continue the directed geodesic rays $\widehat{q_2 q_1}$ and $\widehat{q_3 q_4}$ until they intersect S^1 at q_0 and q_5. Let $J(v, S)$ be the segment $\widehat{q_0 q_5}$, chosen so that the interior of $q_0 q_1 q_2 v q_3 q_4 q_5$ is (noneuclidean) convex. Notice that $J(v) = J(v, S)$ for the translate $S = \varphi(R)$ opposite R at v. The intervals $J(v, S)$ are unions of various I_k's. We say intervals $J(v, R)$ are of *type 0* and $J(v, S)$ has *type n* $(n \geq 1)$ when there are $n-1$ translates $\varphi(R)$ between S and R angularly at v. In particular $J(v)$ has type $2g$, the maximum type.

When $J(v, S)$ has type $n \geq 1$, notice that $f(J(v, S)) = J(v', S')$ has type $n-1$. We now see that some iterate $f^m(I_k)$ of any I_k contains a $J(v, R)$. Since

$$J(v, R) \cup f J(v, R) = S^1,$$

f is transitive. ∎

Let $\varepsilon > 0$ be smaller than the minimum distance along S^1 between points in W. For each k let $D_k \subset \mathbf{C}$ be the closed disk containing I_k whose boundary circle ∂D_k is perpendicular to S^1, with $S^1 \cap \partial D_k$ the two points on S^1 at distance $\varepsilon/2$ from the endpoints of I_k.

Lemma 2. — If $\lambda = \inf\{ |f_k'(z)| : z \in D_k,\ 1 \leq k \leq m\}$, *then* $\lambda > 1$, *and* $D_j \subset \text{int} f_k(D_k)$ *for* $k \to j$.

Proof. — $f_k = \varphi_s$ for some $J_s \supset I_k$. Because R is a regular $4g$-gon centered at the origin, one sees that $|\varphi_s'(z)| = 1$ on the circle containing s and $|\varphi_s'(z)| > 1$ inside this circle. In particular $\lambda > 1$. Since $f_k(I_k) \supset I_j$ and $f_k | D_k$ expands distances, $D_j \cap S^1 \subset \text{int}(f_k(D_k \cap S^1))$ as subsets of S^1. As $f_k(D_k)$ is a disk with boundary perpendicular to S^1 and $f_k(D_k) \cap S^1 = f_k(D_k \cap S^1)$, it follows that $D_j \subset \text{int} f_k(D_k)$. ∎

We will now review the standard construction of symbolic dynamics. A finite sequence $(x_0, x_1, \dots, x_n) \in \{1, \dots, m\}^{n+1}$ or an infinite sequence

$$x = \{x_i\}_{i=0}^{\infty} \in \prod_{i=0}^{\infty} \{1, \dots, m\}$$

is *admissible* if $x_i \to x_{i+1}$ for every i. The set of all admissible infinite sequences is denoted Σ_f; it is a closed subset of $\prod_{i=0}^{\infty}\{1, \ldots, m\}$ and is homeomorphic to the Cantor set. The shift map $\sigma : \Sigma_f \to \Sigma_f$ is defined by $\sigma\{x_i\}_{i=0}^{\infty} = \{x_{i+1}\}_{i=0}^{\infty}$.

For $x_0, x_1, \ldots, x_n$ admissible, the map $F_{x_0,\ldots,x_n} = f_{x_0}^{-1} \circ \ldots \circ f_{x_{n-2}}^{-1} \circ f_{x_{n-1}}^{-1}$ maps D_{x_n} onto a disk $D(x_0, \ldots, x_n) = F_{x_0,\ldots,x_n}(D_{x_n}) \subset \operatorname{int} D_{x_0}$ with diameter $\leqslant \lambda^{-n}$. By induction one sees that $D(x_0, \ldots, x_n) \cap S^1$ is an interval intersecting I_{x_0}. Clearly:

$$D(x_0, \ldots, x_n) \supset D(x_0, \ldots, x_{n+1})$$

and so, for $\underline{x} \in \Sigma_f$, $\pi(\underline{x}) = \bigcap_{n=0}^{\infty} D(x_0, \ldots, x_n)$ is a single point of I_{x_0}. The map $\pi : \Sigma_f \to S^1$ is continuous since $\pi\{\underline{y} : y_i = x_i, i = 0, \ldots, n\} \subset D(x_0, \ldots, x_n)$ has diameter $\leqslant \lambda^{-n}$. It is onto because $\pi\Sigma_f$ is compact and contains every z not in the countable set $\bigcup_{n=0}^{\infty} f^{-n}W$. (For such a z one has $z = \pi(\underline{x})$ where $\underline{x} \in \Sigma_f$ is defined by $x_k =$ the j with $f^k z \in I_j$.)

We now return to the group Γ. The boundary correspondence $H : S^1 \to S^1$ can be extended to a homeomorphism of $S^2 = \mathbf{C} \cup \{\infty\}$ so that $H(\psi z) = \alpha(\psi)H(z)$ for $z \in S^2$, $\psi \in \Gamma$ (H is not unique on all of S^2 but it is unique on S^1 [11]). Let $f_\Gamma = H^{-1} f H$, a Markov map for Γ.

Lemma 3. — There is an N *so that, if* $\mu = \inf\{|(f_\Gamma^N)'(z)| : z \in S^1\}$, *then* $\mu > 1$.

Proof. — Define:

$$D_{\Gamma,k} = H^{-1}D_k, \quad f_{\Gamma,k} = H^{-1}f_k H, \quad I_{\Gamma,k} = H^{-1}I_k,$$
$$F_{\Gamma,x_0,\ldots,x_n} = H^{-1}F_{x_0,\ldots,x_n}H, \text{ etc.}$$

Let γ_k be a smooth Jordan curve interior to $D_{\Gamma,k}$ and surrounding $I_{\Gamma,k}$. For $y \in I_{x_n}$ and $F = F_{\Gamma,x_0,\ldots,x_n}$, Cauchy's formula gives:

$$|F'(y)| = \frac{1}{2\pi}\left|\int_{\gamma_{x_n}} \frac{F(\zeta)\,d\zeta}{(\zeta - y)^2}\right| \leqslant \text{constant}.\operatorname{diam} F_{\Gamma,x_0,\ldots,x_n}(D_{\Gamma,x_n}).$$

Since $F_{\Lambda,x_0,\ldots,x_n}(D_{\Gamma,x_n}) = H^{-1}D(x_0, \ldots, x_n)$ and $\operatorname{diam} D(x_0, \ldots, x_n) \leqslant \lambda^{-n}$, $|F'(y)| \to 0$ uniformly as $n \to \infty$. In particular, for some N,

$$\sup\{|F'_{\Gamma,x_0,\ldots,x_N}(y)| : y \in I_{x_N}, x_0, \ldots, x_N \text{ admissible}\} < 1.$$

But $|(f_\Gamma^N)'(z)| = |F'_{\Gamma,x_0,\ldots,x_N}(y)|^{-1}$ for some $x_0, \ldots, x_N$, where $y = f_\Gamma^N(z)$. ■

Define the continuous surjection $\pi_\Gamma : \Sigma_f \to S^1$ by:

$$\pi_\Gamma(\underline{x}) = H^{-1}\pi(\underline{x}) = \bigcap_{n=0}^{\infty} D_\Gamma(x_0, \ldots, x_n).$$

Notice that:

$$f_{\Gamma,x_0}\pi_\Gamma(\underline{x}) = \bigcap_{n=0}^{\infty} f_{\Gamma,x_0} D_\Gamma(x_0, \ldots, x_n)$$
$$= \bigcap_{n=0}^{\infty} D_\Gamma(x_1, \ldots, x_n) = \pi_\Gamma(\sigma\underline{x}).$$

2. Measures.

Define the real-valued function φ_Γ on Σ_f by:

$$\varphi_\Gamma(\underline{x}) = -\ln |f'_{\Gamma,x_0}(\pi_\Gamma \underline{x})|.$$

Lemma 4. — There are $\alpha \in (0, 1)$ *and* $c > 0$ *so that* $|\varphi_\Gamma(\underline{x}) - \varphi_\Gamma(\underline{y})| \leqslant c\alpha^n$ *when* $x_i = y_i$, *for* $i = 0, 1, \ldots, n$.

Proof. — The set $E = \bigcap_{j=0}^{n} f_\Gamma^{-j} I_{x_j} = \pi_\Gamma\{\underline{y} : y_i = x_i \text{ for } i = 0, 1, \ldots, n\}$ is a closed segment; $f_\Gamma^n = f_{\Gamma,x_{n-1}} \circ \ldots \circ f_{\Gamma,x_1} \circ f_{\Gamma,x_0}$ is continuous on E and $f_\Gamma^n E = I_{x_0}$. By Lemma 3:

$$|(f_\Gamma^n)'(u)| \geqslant \text{const.}\, \mu^{n/N} \quad \text{with} \quad \mu > 1.$$

Hence E has length at most const.α^n where $\alpha = \mu^{-1/N} \in (0, 1)$. The lemma follows because $\pi_\Gamma \underline{x}, \pi_\Gamma \underline{y} \in E$ and $-\ln |f'_\Gamma|$ is Lipschitz on each I_k. ■

Lemma 5. — There is a constant $d > 1$ *so that the segment* $E_n(\underline{x}) = \bigcap_{j=0}^{n} f_\Gamma^{-j} I_{x_j}$, *for* $\underline{x} \in \Sigma_f$, *has length in the interval* $\exp\left(\sum_{j=0}^{n-1} \varphi_\Gamma(\sigma^j \underline{x})\right)[d^{-1}, d]$.

Proof. — Each $y \in E = E_n(\underline{x})$ is $\pi_\Gamma(\underline{y})$ for some $\underline{y} \in \Sigma_f$ with $y_i = x_i$, for $i = 0, 1, \ldots, n$.

Then
$$|(f_\Gamma^n)'(y)| = \prod_{j=0}^{n-1} |f'_{\Gamma,x_j}(f^j y)| = \prod_{j=0}^{n-1} |f'_{\Gamma,x_j}(\pi_\Gamma(\sigma^j \underline{y}))|$$
$$= \exp\left(-\sum_{j=0}^{n-1} \varphi_\Gamma(\sigma^j \underline{y})\right).$$

Since $\sigma^j \underline{x}$, $\sigma^j \underline{y}$ agree in places $0, 1, \ldots, n-j$,

$$|\varphi_\Gamma(\sigma^j \underline{x}) - \varphi_\Gamma(\sigma^j \underline{y})| \leqslant c\alpha^{n-j}$$

and
$$\left|\sum_{j=0}^{n-1} \varphi_\Gamma(\sigma^j \underline{x}) - \sum_{j=0}^{n-1} \varphi_\Gamma(\sigma^j \underline{y})\right| \leqslant \sum_{j=0}^{n-1} c\alpha^{n-j} \leqslant \frac{c}{1-\alpha}.$$

Hence $|(f_\Gamma^n)'(y)|$ differs from $\exp\left(-\sum_{j=0}^{n-1} \varphi_\Gamma(\sigma^j \underline{x})\right)$ by at most a bounded factor; the lemma follows because $\ell(I_{x_0}) = \ell(f_\Gamma^n . E) = \int_E |(f_\Gamma^n)'(y)|\, dy$. ■

The theory of Gibbs states ([7], [16], [19]) constructs a Borel probability measure μ_Γ on Σ_f so that:

(i) μ_Γ is invariant and ergodic under σ, and
(ii) there are constants b and P so that:

$$\mu_\Gamma\{\underline{y} : y_i = x_i \text{ for } i = 0, \ldots, n\}$$

is in the interval:

$$\exp\left(-Pn + \sum_{j=0}^{n-1} \varphi_\Gamma(\sigma^j \underline{x})\right)[b^{-1}, b] \quad \text{for all} \quad x \in \Sigma_f,\ n \geqslant 0.$$

The map $\pi_\Gamma : \Sigma_f \to S^1$ is one-to-one except on the countable set $\bigcup_{n=0}^{\infty} f^{-n}W$ (where it is two-to-one). Since μ_Γ is nonatomic, $\tilde{\mu}_\Gamma = \pi_\Gamma^* \mu_\Gamma$ is a Borel probability measure on S^1 and π_Γ gives an isomorphism of measure spaces $(\Sigma_f, \mu_\Gamma) \simeq (S^1, \tilde{\mu}_\Gamma)$. By Lemma 5, normalized Lebesgue measure $d\ell$ on S^1 and $\tilde{\mu}_\Gamma$ differ by a multiplicative factor in $\exp(-Pn)[(bd)^{-1}, bd]$ on sets $E_n(\underline{x})$. Calculating the total measure of S^1 one has $1 \in \exp(-Pn)[(bd)^{-1}, bd]$; letting $n \to \infty$, $P = 0$. It now follows that $\tilde{\mu}_\Gamma$ is equivalent to $d\ell$.

Theorem 1. — *Suppose Γ_1 and Γ_2 are isomorphic surface groups. If the boundary correspondence corresponding to this isomorphism is absolutely continuous, then the isomorphism is conjugation by an element of* G^*.

Proof. — Choose consistent isomorphisms of Γ_1 and Γ_2 with Φ_g. Let $H : S^1 \to S^1$ be the boundary correspondence for $\Gamma_1 \simeq \Gamma_2$. As H is absolutely continuous and $\tilde{\mu}_{\Gamma_1} \sim d\ell$, $H_{\tilde{\mu}_{\Gamma_1}} \succ \tilde{\mu}_{\Gamma_2}$; hence $\mu_{\Gamma_1} \succ \mu_{\Gamma_2}$ on Σ_f. Since these measures are invariant and ergodic under σ, $\mu_{\Gamma_1} = \mu_{\Gamma_2}$. So $H_{\tilde{\mu}_{\Gamma_1}} = \tilde{\mu}_{\Gamma_2}$ and H^{-1} is absolutely continuous.

Since H and H^{-1} are absolutely continuous, $w(\underline{x}) = -\ln |H'(\pi_{\Gamma_1}(\underline{x}))|$ is defined for $(\mu_{\Gamma_1} = \mu_{\Gamma_2})$-almost all $\underline{x} \in \Sigma_f$. Differentiating the equation $f_{\Gamma_1} = H^{-1} f_{\Gamma_2} H$ gives $\varphi_{\Gamma_1}(\underline{x}) - \varphi_{\Gamma_2}(\underline{x}) = w(\underline{x}) - w(\sigma\underline{x})$ for almost all $\underline{x}$. Now the theory of Gibbs measures ([10], [19], [7, p. 40]) produces a continuous $u : \Sigma_f \to R$ so that $\varphi_{\Gamma_1}(\underline{x}) - \varphi_{\Gamma_2}(\underline{x}) = u(\underline{x}) - u(\sigma\underline{x})$ for every $\underline{x}$. It follows that:

$$u(\underline{x}) - w(\underline{x}) = u(\sigma\underline{x}) - w(\sigma\underline{x})$$

for almost all $\underline{x}$. Since μ_{Γ_1} is ergodic, $u(\underline{x}) - w(\underline{x})$ is constant almost everywhere. Subtracting this constant from $u(\underline{x})$ we may assume $w(\underline{x}) = u(\underline{x})$ almost everywhere [1].

Lemma 6. — *If* $\pi(\underline{x}) = \pi(\underline{y})$ *and* $x_0 = y_0$, *then* $u(\underline{x}) = u(\underline{y})$.

Proof. — Find a sequence of indices $\{a_k\}_{k=1}^{\infty}$ so that $a_1 \to x_0 = y_0$ and $a_{k+1} \to a_k$. If $\underline{x}^{(n)} = .a_n a_{n-1} \dots a_1 \underline{x}$ and $\underline{y}^{(n)} = .a_n a_{n-1} \dots a_1 \underline{y}$, then $\sigma\underline{x}^{(n+1)} = \underline{x}^{(n)}$ yields:

$$u(\underline{x}^{(n)}) = u(\underline{x}^{(n+1)}) + \varphi_{\Gamma_2}(\underline{x}^{(n+1)}) - \varphi_{\Gamma_1}(\underline{x}^{(n+1)})$$

and
$$u(\underline{x}) = u(\underline{x}^{(N)}) + \sum_{n=1}^{N} (\varphi_{\Gamma_2}(\underline{x}^{(n)}) - \varphi_{\Gamma_1}(\underline{x}^{(n)})).$$

Now $\pi(\underline{x}^{(n)}) = \pi(\underline{y}^{(n)})$ gives $\varphi_{\Gamma_i}(\underline{x}^{(n)}) = \varphi_{\Gamma_i}(\underline{y}^{(n)})$ and $u(\underline{x}) - u(\underline{y}) = u(\underline{x}^{(N)}) - u(\underline{y}^{(N)})$. This last expression tends to 0 as $N \to \infty$ since u is continuous. ■

Lemma 7. — *Each* $H : \tilde{J}_{\Gamma_1, s} \to \tilde{J}_{\Gamma_2, s}$ *is* C^1.

Proof. — Let $\tilde{J}_{\Gamma_1, s} = I_{\Gamma_1, j} \cup I_{\Gamma_1, j+1} \cup \dots \cup I_{\Gamma_1, r}$. First we show that $\tilde{u}(\pi_{\Gamma_1}\underline{x}) = u(\underline{x})$ defines a continuous function $\tilde{u}$ on $\tilde{J}_{\Gamma_1, s}$. Lemma 6 proves this for each $\tilde{u} | I_{\Gamma_1, k}$. Suppose

[1] At this point one has that the derivative of H is a.e. bounded. It then follows that corresponding elements have the same eigenvalues at their fixed points. It is then classical that the two Fuchsian groups are conjugate (editor).

that p is a common endpoint of $I_{\Gamma_1,i}$ and $I_{\Gamma_1,i+1}$; let $p=\pi_{\Gamma_1}(\underline{x})=\pi_{\Gamma_2}(\underline{y})$ where $x_0=i$, $y_0=i+1$. Now $f_{\Gamma_1}=\varphi_{\Gamma_1,s}$ on $\widetilde{J}_{\Gamma_1,s}$ has image $\varphi_{\Gamma_1,s}(\widetilde{J}_{\Gamma_1,s})\supset\widetilde{J}_{\Gamma_1,s}$. Hence one can find sequences $\{a_k\}_{k=1}^{\infty}$, $\{b_k\}_{k=1}^{\infty}$ so that $\underline{x}^{(n)}=.a_n a_{n-1}\dots a_1\underline{x}$, $\underline{y}^{(n)}=.b_n b_{n-1}\dots b_1\underline{y}$ are admissible and furthermore $a_k, b_k\in[j,r]$. Then $\pi(\underline{x}^{(n)})=\pi(\underline{y}^{(n)})$, $\varphi_{\Gamma_i}(\underline{x}^{(n)})=\varphi_{\Gamma_i}(\underline{y}^{(n)})$ and the proof of Lemma 6 gives $u(\underline{x})=u(\underline{y})$. So $\widetilde{u}$ is continuous on $\widetilde{J}_{\Gamma_1,s}$.

Because $H'(x)=e^{-\widetilde{u}}(x)$ for almost all $x\in\widetilde{J}_{\Gamma_1,s}$, fixing $p\in\widetilde{J}_{\Gamma_1,s}$ one has:

$$\begin{aligned} H(x)&=H(p)+\int_p^x H'(t)\,dt\\ &=H(p)+\int_p^x e^{-u(t)}\,dt. \end{aligned}$$

By the fundamental theorem of calculus $H'(x)=e^{-\widetilde{u}(x)}$. ■

Lemma 8. — For each s, $H|\widetilde{J}_{\Gamma_1,s}=\beta_s|\widetilde{J}_{\Gamma_1,s}$ *for some* $\beta_s\in G^*$.

Proof. — Let p_i be the fixed points of $\varphi_{\Gamma_i,s}|J_{\Gamma_i,s}$. Since H gives a local C^1 conjugacy between these points, $\tau=\varphi'_{\Gamma_1,s}(p_1)=\varphi'_{\Gamma_2,s}(p_2)$. These points are sources and $\tau>1$ by Lemma 3. Changing coordinates by linear fractional transformations each $\varphi^{-1}_{\Gamma_i,s}|\widetilde{J}_{\Gamma_i,s}$ is conjugate to $x\mapsto\tau^{-1}x$ on a real interval L_i containing 0 (p_i is sent to 0). In these coordinates $H:\widetilde{J}_{\Gamma_1,s}\to\widetilde{J}_{\Gamma_2,s}$ is transformed into $h:L_1\to L_2$ such that $\tau^{-1}h(x)=h(\tau^{-1}x)$ and $h(0)=0$. Writing $h(x)=\tau h(\tau^{-1}x)$ and iterating one gets:

$$h(x)=\tau^n h(\tau^{-n}x)=x\left(\frac{h(\tau^{-n}x)-h(0)}{\tau^{-n}x}\right)\to h'(0)x.$$

So $h(x)$ is linear. Unravelling coordinates H equals a linear fractional transformation $\beta_s\in G^*$ on $\widetilde{J}_{\Gamma_1,s}$. ■

Finally we prove Theorem 1. For two consecutive sides s, s' of R each of $H|\widetilde{J}_{\Gamma_1,s}$ and $H|\widetilde{J}_{\Gamma_1,s'}$ are in G^*. Now there is an interval $I_{\Gamma_1,j}\subset J_{\Gamma_1,s}\cap J_{\Gamma_1,s'}$ which can be in either $\widetilde{J}_{\Gamma_1,s}$ or $\widetilde{J}_{\Gamma_1,s'}$ depending on the choice of f. Since $H|I_{\Gamma_1,g}$ does not depend on f, it follows that $H|\widetilde{J}_{\Gamma_1,s}$ and $H|\widetilde{J}_{\Gamma_2,s'}$ are restrictions on the same element of G^*. Continuing around the circle one has $H=\beta|S^1$ for some $\beta\in G^*$.

3. Quasi-Fuchsian Groups.

This section proves Theorem 2. Let $\Lambda=\Lambda(\Gamma_1,\Gamma_2,\alpha)$ as in the introduction. It is classical that e_i extends to a homeomorphism $e_i:\overline{U}\to\overline{D}_i=D_i\cup\gamma$ (e.g. [20], p. 121). It is well-known that if γ is rectifiable, then $e_i:S^1\to\gamma$ is absolutely continuous (parametrizing γ by arc length) and so is $e_i^{-1}:\gamma\to S^1$ (*e.g.* [21, p. 293]). Then the boundary correspondence $e_2^{-1}\circ e_1:S^1\to S^1$ would be absolutely continuous. Under the hypotheses of Theorem 2 therefore, Theorem 1 tells us that γ is not rectifiable.

Since $\Lambda|\gamma$ is topologically conjugate to $\Gamma_1|S^1$ and $\Gamma_2|S^1$, there is a natural Markov map f_Λ of γ and projection $\pi_\Lambda : \Sigma_f \to \gamma$. Since $\Lambda | D_i \cup \gamma$ is topologically conjugate to $\Gamma_i | \overline{U}_i$, one finds closed neighborhoods $D_{\Lambda,k}$ of the topological intervals $I_{\Lambda,k}$ by

$$D_{\Lambda,k} = e_1(D_{\Gamma_1,k} \cap \overline{U}) \cup e_2(D_{\Gamma_2,k} \cap \overline{U}).$$

Then $\operatorname{int} f_{\Lambda,k}(D_{\Lambda,k}) \supset D_{\Lambda,j}$ and the proof of lemma 3 shows that the function

$$F_{\Lambda,x_0,\ldots,x_n} = f_{\Lambda,x_0}^{-1} \circ \ldots \circ f_{\Lambda,x_{n-1}}^{-1}$$

has derivative

$$|F'_{\Lambda,x_0,\ldots,x_n}(z)| \leqslant \text{const}.\alpha^n \quad \text{for} \quad z \in D_{\Lambda,x_n}$$

where $\alpha \in (0, 1)$. (Here one uses topological disks $E_{\Lambda,k}$ slightly larger than $D_{\Lambda,k}$ having the same properties and integrates around $\gamma_k \subset E_k$ surrounding $D_{\Lambda,k}$.) As in the proof of Lemma 5 we find a constant independent of n so that:

$$|F'_{\Lambda,x_0,\ldots,x_n}(z_1)| \leqslant \text{const}.\,|F'_{\Lambda,x_0,\ldots,x_n}(z_2)| \quad \text{for} \quad z_1, z_2 \in D_{\Lambda,x_n}.$$

(Choose the $E_{\Lambda,k}$'s with smooth boundary so that any $z_1, z_2 \in D_{\Lambda,x_n}$ can be joined by an arc in D_{Λ,x_n} of bounded length.)

Lemma 9. — There are positive constants c_1 and c_2 so that the following is true:

$$B_{c_1 r}(\pi_\Lambda \underline{x}) \subset D_\Lambda(x_0, \ldots, x_n) = F_{\Lambda,x_0,\ldots,x_n}(D_{\Lambda,x_n}) \subset B_{c_2 r}(\pi_\Lambda x)$$

where $r = F'_{\Lambda,x_0,\ldots,x_n}(\pi(\sigma^n \underline{x}))$.

Proof. — Now $D_{\Lambda,x_n} \supset B_{d_1}(\pi_\Lambda(\sigma^n \underline{x}))$ for some $d_1 > 0$ independent of n and $\underline{x} \in \Sigma_f$, since $\sigma^n \underline{x} \in I_{x_n}$. By Koebe's one-quarter theorem take $c_1 = d_1/4$. Assuming D_{Λ,x_n} has a smooth boundary (perturb if necessary) there is a $d_2 > 0$ so that every z_1, z_2 in the same D_{Λ,x_n} can be joined by a smooth curve of length d_2. Since $|F'(z)|$ varies by at most a bounded factor on D_{Λ,x_n} one has:

$$\operatorname{diam} F(D_{\Lambda,x_n}) \leqslant 2d_2.\text{const}.\,|F'(\pi_\Lambda(\sigma^n \underline{x}))|. \quad \blacksquare$$

Let $\varphi(\underline{x}) = -\ln |f'_{\Lambda,x_0}(\pi_\Lambda \underline{x})|$ for $\underline{x} \in \Sigma_f$. As before $|\varphi(\underline{x}) - \varphi(\underline{y})| \leqslant c\alpha^n$ when $x_i = y_i$ for all $i = 0, 1, \ldots, n$. The inequality $|F'(z)| \leqslant \text{const}.\alpha^n$ implies that for some $N > 0$

$$S_N \varphi(\underline{x}) = \sum_{k=0}^{N-1} \varphi(\sigma^k \underline{x}) \leqslant -\varepsilon < 0.$$

There is a unique $a > 0$ so that the topological pressure $P(a\varphi) = 0$. To see this consider the variational formula (see [7]):

$$P(a\varphi) = \sup_\mu \left(h_\mu(\sigma) + \int a\varphi \, d\mu \right)$$
$$= \sup_\mu \left(h_\mu(\sigma) + \frac{a}{N} \int S_N \varphi \, d\mu \right)$$

where μ varies over all σ-invariant measures on Σ_f. When $a=0$, one has

$$P(a\varphi)=P(0)>0$$

and when a is sufficiently large $P(a\varphi)<0$ (since $S_N \leqslant -\varepsilon$). The formula shows that $P(a\varphi)$ strictly decreases as a increases; since $P(a\varphi)$ is continuous in a, there is a unique a with $P(a\varphi)=0$.

Lemma 10. — The Hausdorff dimension of γ is a. The a-dimensional Hausdorff measure ν_a on γ is finite and equivalent to $\pi_\Lambda^* \mu_{a\varphi}$.

Proof. — Here $\mu_{a\varphi}$ is the Gibbs measure for $a\varphi$. It is invariant and ergodic under σ and there is a constant $u>1$ so that:

$$(\pi_\Lambda^* \mu_{a\varphi})(E_n(\underline{x})) \in \exp\left(\sum_{k=0}^{n-1} a\varphi(\sigma^k x)\right)[u^{-1}, u]$$

for any $\underline{x}\in\Sigma_f$ and $E_n(\underline{x})=\bigcap_{k=0}^{n} f^{-k} I_{\Lambda, x_k}=\pi_\Lambda\{y\in\Sigma_f : y_i=x_i \text{ for } i=0, 1, \ldots, n\}$. Here we use that $P=P(a\varphi)=0$. The family $\mathscr{D}_n=\{D_\Lambda(x_0, \ldots, x_n)\}$ covers γ, each member has diameter $\leqslant$ const. α^n, and

$$\begin{aligned}\sum_{(x_0, \ldots, x_n)} (\operatorname{diam} D_\Lambda(x_0, \ldots, x_n))^a &\leqslant \text{const}. \sum_{(x_0, \ldots, x_n)} |F'_{\Lambda, x_0, \ldots, x_n}(\pi_\Lambda \underline{x})|^a \\ &\leqslant \text{const}. \sum_{(x_0, \ldots, x_n)} \exp\left(\sum_{k=0}^{n-1} a\varphi(\sigma^k \underline{x})\right) \\ &\leqslant \text{const}. \sum_{(x_0, \ldots, x_n)} (\pi_\Lambda^* \mu_{a\varphi})(E_n(x_0, \ldots, x_n)) \\ &\leqslant \text{const}. 1.\end{aligned}$$

It follows that the Hausdorff dimension of γ is at most a; restricting ourselves to $(x_0, \ldots, x_n)$ beginning with a given sequence $(y_0, \ldots, y_m)$ the above shows that

$$\nu_a(E_m(y_0, \ldots, y_m)) \leqslant \text{const}. (\pi_\Lambda^* \mu_{a\varphi})(E_m(y_0, \ldots, y_m)).$$

We will now find an ε_0 such that the following is true: if $\{U_j\}_{j=1}^\infty$ is an open cover of $E_m(y_0, \ldots, y_m)$, then $\sum_{j=1}^{\infty} (\operatorname{diam} U_j)^a \geqslant \varepsilon_0 (\pi_\Lambda^* \mu_{a\varphi})(E_m(y_0, \ldots, y_m))$. Since every $(\pi_\Lambda^* \mu_{a\varphi})(E_m(\underline{y}))>0$, this will prove the lemma.

Suppose the inequality is false (ε_0 as yet undetermined). For each U_j pick $z_j=\pi_\Lambda \underline{x}^{(j)} \in U_j \cap E_m(y_0, \ldots, y_m)$. Then $U_j \subset B_{2t_j}(z_j)$ where $t_j=\operatorname{diam} U_j$. Since $\sum_{k=0}^{N-1} \varphi(\sigma^k \underline{x})=-\ln|(f^N)'(\pi_\Lambda \underline{x})| \leqslant \text{const}+N\log\alpha$ where $\alpha\in(0, 1)$, there is an n_j so that:

$$\ln\frac{2t_j}{c_1} \in \left(\sum_{k=0}^{n_j} \varphi(\sigma^k \underline{x}^{(j)}), \sum_{k=0}^{n_j-1} \varphi(\sigma^k \underline{x}^{(j)})\right).$$

From Lemma 9 one has $U_j \subset D(x_0^{(j)}, \ldots, x_{n_j}^{(j)})$. Also

$$t_j \geqslant \left(\frac{c_1}{2} e^{-\|\varphi\|_\infty}\right) \exp \sum_{k=0}^{n_j-1} \varphi(\sigma^k \underline{x}^{(j)})$$

and so

$$t_j^a \geqslant \left(\frac{c_1}{2} e^{-\|\varphi\|_\infty}\right)^a u^{-1} (\pi_\Lambda^* \mu_{a\varphi}) (E(x_0^{(j)}, \ldots, x_{n_j}^{(j)})).$$

Our hypothesis that the desired inequality failed implies:

$$(\pi_\Lambda^* \mu_{a\varphi}) E(y_0, \ldots, y_m) \geqslant \widetilde{\varepsilon}_0 \sum_j (\pi_\Lambda^* \mu_{a\varphi}) (E(x_0^{(j)}, \ldots, x_{n_j}^{(j)}))$$

where $\widetilde{\varepsilon}_0 = \left(\frac{c_1}{2} e^{-\|\varphi\|_\infty}\right)^a u^{-1} \varepsilon_0^{-1}$.

The disks $D_{\Lambda,k}$ were chosen so as to intersect two of the intervals $I_{\Lambda,j}$ other than $I_{\Lambda,k}$, one containing each endpoint of $I_{\Lambda,k}$ (true for Φ_g and then pulled back). By induction one sees that $D_\Lambda(x_0, \ldots, x_n)$, which contains $E(x_0, \ldots, x_n)$, intersects at most two other $E(w_0, \ldots, w_n)$'s (1). As each of these contains a point of $E(x_0, \ldots, x_n)$, the various estimates on measures gives us:

$$(\pi_\Lambda^* \mu_{a\varphi}) (D_\Lambda(x_0, \ldots, x_n) \cap \gamma) \leqslant \text{const.} (\pi_\Lambda^* \mu_{a\varphi}) (E(x_0, \ldots, x_n)).$$

Since the $D_\Lambda(x_0^{(j)}, \ldots, x_{n_j}^{(j)})$'s cover $E(y_0, \ldots, y_m)$, we now have:

$$(\pi_\Lambda^* \mu_{a\varphi}) E(y_0, \ldots, y_m) > \frac{\text{const}}{\varepsilon_0} \cdot (\pi_\Lambda^* \mu_{a\varphi}) (E(y_0, \ldots, y_m)).$$

For small ε_0 this is a contradiction. ∎

The Hausdorff measure $\nu_a(\gamma)$ is finite and positive by the above inequalities. Now $a \geqslant 1$ since γ is topologically a circle; $a \neq 1$ because then $\nu_a(\gamma) < +\infty$ would imply γ is rectifiable. Hence $a > 1$, $\nu_a | \gamma$ is ergodic for f_Λ because it is equivalent to $\mu_{a\varphi}$ for σ and $\nu_a | \gamma$ is ergodic under Λ because f_Λ is locally in Λ. This finishes Theorem 2.

4. Schottky Groups.

Let $C_1, \ldots, C_{2p}$ be $2p$ Jordan curves in $\mathbf{C} \cup \infty$ $(p \geqslant 2)$ whose interior domains are pairwise disjoint. Suppose for each k such that $1 \leqslant k \leqslant p$ we are given a linear fractional transformation g_k mapping the domain interior to C_k onto that exterior to C_{2p+1-k}. Then the g_k's freely generate a Kleinian group Γ called a *Schottky group* [5].

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(1) One can ignore this point if he omits the countable set $\bigcup_n f^{-n} W$ and computes the Hausdorff measure of the rest using the estimates valid on the open intervals (editor).

Let $D_{\Gamma,k} = C_k \cup$ (domain interior to C_k), $f_k = g_k$ for $1 \leqslant k \leqslant p$, $f_k = g_{p+1-k}^{-1}$ for $p < k \leqslant 2p$, and $i \to j$ unless $i + j = 2p + 1$. Then $D_{\Gamma,j} \subset \operatorname{int} f_i(D_{\Gamma,i})$ when $i \to j$ and the set Σ of admissible sequences $x \in \prod_{i=0}^{\infty} \{1, \ldots, 2p\}$ consists of those with $x_i + x_{i+1} \neq 2p + 1$, for all i. These sequences correspond naturally to the unending irreducible words in the g_k's and g_k^{-1}'s. By a quasi-conformal change of variables the Schottky group Γ can be conjugated into another one Γ^* where the C_k^*'s are isometric circles for the g_k^*'s ([4], [8]). Here $|f'_{\Gamma^*,k}|$ is larger than and bounded away from 1 on each $D_{\Gamma^*,k}$. Defining f_{Γ^*} on $\bigcup_k D_{\Gamma^*,k}$ by $f_{\Gamma^*} | D_{\Gamma^*,k} = f_{\Gamma^*,k}$, one can see there is a unique point $\pi_{\Gamma^*}(\underline{x}) \in \bigcap_{j=0} f_{\Gamma^*}^{-j}(D_{\Gamma^* x_j})$ for each $\underline{x} \in \Sigma$. The set $L(\Gamma^*) = \{\pi_{\Gamma^*}(\underline{x}) : \underline{x} \in \Sigma\}$ is just the *limit set* of Γ^* (see [5]) and $\pi_{\Gamma^*} : \Sigma \to L(\Gamma^*)$ is a homeomorphism of Cantor sets.

For the original Γ we now assume the C_k's came from the C_k^*'s by the quasi-conformal change of variables. Defining $\pi_\Gamma : \Sigma \to L(\Gamma)$ by $\pi_\Gamma(\underline{x}) = \bigcap_{j=0}^{\infty} f_\Gamma^{-1}(D_{\Gamma,x_j})$, the proof of Lemma 3 in section 1 goes through to show $|(f_\Gamma^N)'(z)| \geqslant \mu > 1$ on the domain of definition of f_Γ^N, for some N. Letting $\varphi(\underline{x}) = -\ln |f'_{\Gamma,x_0}(\pi_\Gamma \underline{x})|$ it follows that for some $c > 0$, $\alpha \in (0, 1)$, one has $|\varphi(\underline{x}) - \varphi(\underline{y})| \leqslant c\alpha^n$ when $x_i = y_i$, for $i = 0, 1, \ldots, n$. The analysis of section 3 carries over. Things only become simpler because π_Γ is a homeomorphism instead of a surjection which is 2-to-1 over some points. The proof of Theorem 2 gives us

Theorem 3. — Let Γ *be a Schottky group with limit set* $L(\Gamma)$ *as above. The Hausdorff dimension a of* Λ *is positive. Furthermore* $0 < \nu_a(L(\Gamma)) < \infty$ *where* ν_a *is the Hausdorff a-dimensional measure, and* $\nu_a | L(\Gamma)$ *is ergodic under* Γ (1).

This theorem and the next one contain a number of earlier known results, namely those in [1], [2], [3], [15] concerning Fuchsian groups without cusps. Since $L(\Gamma)$ has zero 2-dimensional measure, one has $a < 2$ also.

Theorem 4. — Let Γ *be a Schottky group and a the Hausdorff dimension of* $L(\Gamma)$. *For* $z \notin L(\Gamma)$ *the absolute Poincaré series* $\sum_{g \in \Gamma} |g'(z)|^s$ *converges iff* $s > a$ (1).

Proof. — A fundamental domain R for Γ consists of the region exterior to all the C_k's. We omit from the series any term with $|g'(z)| = \infty$ (at most one). Alternatively, we use a metric on S^2 to compute derivatives. Recall that $a > 0$ was the unique positive number such that the pressure $P(a\varphi) = 0$. It is enough to check the statement for $z \in \overline{R}$. Under the correspondence:

$$g_k \Leftrightarrow k, \qquad g_k^{-1} \Leftrightarrow p + 1 - k \qquad (1 \leqslant k \leqslant p),$$

reduced words in the $\{g_k, g_k^{-1}\}$ correspond to finite admissible strings $x_0 x_1 \ldots x_n$. Now:

$$f_{\Gamma,x_n} \circ f_{\Gamma,x_{n-1}} \circ \ldots \circ f_{\Gamma,x_0} D_\Gamma(x_0, \ldots, x_n) \supset R$$

(1) For "Fuchsian Schottky groups" this is contained in Theorem (4.1) and (7.1) of [15] (editor).

where $D_\Gamma(x_0, \ldots, x_n) = f_{\Gamma,x_0}^{-1} \circ \ldots \circ f_{\Gamma,x_{n-1}}^{-1}(D_{\Gamma,x_n})$ is a disk. The derivative of

$$f_{x_n,\ldots,x_0} = f_{\Gamma,x_n} \circ \ldots \circ f_{\Gamma,x_0}$$

varies by at most a bounded factor over D_{Γ,x_n}, independent of n (using metric on S^2, as in section 3). It follows that $|(f_{x_n,\ldots,x_0}^{-1})'(z)|^a$, $z \in \overline{R}$ differs by a bounded factor from

$$\mu_{a\varphi}\{\underline{y} \in \Sigma : y_i = x_i,\ i = 0, \ldots, n\} = \mu(x_0, \ldots, x_n).$$

As $x_0, \ldots, x_n$ runs over all admissible strings, $f_{x_n,\ldots,x_0}$ runs over $\Gamma \backslash \{e\}$. So $\sum_{g \in \Gamma} |g'(z)|^s$ converges or diverges as $\Sigma \mu(x_0, \ldots, x_n)^{s/a}$.

For $s \leqslant a$ this diverges, since the sum for each fixed n is $\geqslant 1$. Because:

$$\sum_{k=0}^{N-1} \varphi(\sigma^k \underline{x}) \leqslant \varepsilon_0 < 0,$$

it follows that $\mu(x_0, \ldots, x_n) \leqslant c\beta^n$ for some $c > 0$, $\beta \in (0, 1)$. If $s > a$, then

$$\frac{\mu(x_0, \ldots, x_n)^{s/a}}{\mu(x_0, \ldots, x_n)} \leqslant \mu(x_0, \ldots, x_n)^{\left(\frac{s}{a}-1\right)} \leqslant \tilde{c}\tilde{\beta}^n, \qquad \tilde{\beta} \in (0, 1)$$

and $\sum_{n \text{ fixed}} \mu(x_0, \ldots, x_n)^{s/a} \leqslant \tilde{c}\tilde{\beta}^n$. ■

Remarks:

a) Let T be a finite union of pairwise disjoint rectifiable arcs with endpoints on the boundaries of the $D_{\Gamma,k}$'s and not intersecting the interior of the $D_{\Gamma,k}$'s. Choose T so that two endpoints are on each $\partial D_{\Gamma,k}$, $S^2 \backslash (T \cup \bigcup_k D_{\Gamma,k})$ has two connected components (each simply connected), and f_k takes the two endpoints of T on $\partial D_{\Gamma,k}$ onto the two on $\partial D_{\Gamma,p+1-k}$. Then $\gamma = L(\Gamma) \cup \bigcup_{g \in \Gamma} g(T)$ is a quasi-circle for Γ. The proof above shows that γ is rectifiable iff the Hausdorff dimension a of $L(\Gamma)$ is less than 1. Furthermore, if Γ had a rectifiable quasi-circle γ, then the above constructs γ from $T = \gamma \backslash (\bigcup_k \operatorname{int} D_{\Gamma,k})$.

b) Finally, we mention a zeta function. For $g \in \Gamma$, $g \neq e$, let $\rho(g) = g'(p)$ where p is the fixed source of g. Then $\rho(g)$ is constant on conjugacy classes. Call a conjugacy class *primitive* if g in the class cannot be written as h^n with $h \in \Gamma$, $n > 1$. The function:

$$\zeta_\Gamma(s) = \prod_{\substack{\text{primitive} \\ \text{classes}}} (1 - \rho(g)^s)$$

converges for $\operatorname{Re} s > a$ and continues to a meromorphic function on all of **C** (the values of $\rho(g)^s$ being chosen in a proper consistent fashion). This is seen by the method of Ruelle [17]. The zeta function above is of course an analogue of Selberg's zeta function for surface groups. Ruelle's paper generalizes the meromorphy of Selberg's function. Presumably, the Nielsen development and Ruelle's method gives meromorphy of $\zeta_\Lambda(s)$ when Λ is a quasi-Fuchsian group corresponding to surface groups, as in section 3.

Problems (1) (added November 1978 by the editor)

a) (Generalizing Ahlfors measure problem). For a (finitely generated) Kleinian group is it true that either the limit set is all of S^2 or has Hausdorff dimension strictly less than 2?

b) If $\bar{T}$ denotes the closure of Teichmüller space in Kleinian groups and Σ denotes the Cantor set of symbols above, is there a continuous parametrization of the limit set, $\bar{T}\times\Sigma\to S^2$, so that image $(t, \Sigma)=$ limit set Γ_t?

Is the Hausdorff dimension of $\Lambda(\Gamma_t)$ continuous in t ranging over $\bar{T}$?

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(1) These were the last two of over 150 items in a private notebook of problems and questions found in Bowen's papers. (Editor).