HAUSDORFF DIMENSION OF SINGULAR VECTORS

YITWAH CHEUNG AND NICOLAS CHEVALLIER

ABSTRACT. We prove that the set of singular vectors in $\mathbb{R}^d, d \geq 2$, has Hausdorff dimension $\frac{d^2}{d+1}$ and that the Hausdorff dimension of the set of ε-Dirichlet improvable vectors in $\mathbb{R}^d$ is roughly $\frac{d^2}{d+1}$ plus a power of ε between $\frac{d}{2}$ and d. As a corollary, the set of divergent trajectories of the flow by $\operatorname{diag}(e^t, \ldots, e^t, e^{-dt})$ acting on $\mathrm{SL}_{d+1}(\mathbb{R})/\mathrm{SL}_{d+1}(\mathbb{Z})$ has Hausdorff codimension $\frac{d}{d+1}$. These results extend the work of the first author in [6].

1. Introduction

Singular vectors were introduced by A. Khintchine in the twenties (see [16]). Recall that $\theta = (\theta_1, ..., \theta_d) \in \mathbb{R}^d$ is *singular* if for every $\varepsilon > 0$, there exists T_0 such that for all $T > T_0$ the system of inequalities

$$(1.1) \qquad \max_{1 \leq i \leq d} |q\theta_i - p_i| < \frac{\varepsilon}{T^{1/d}} \quad \text{and} \quad 0 < q < T$$

admits an integer solution $(p, q) \in \mathbb{Z}^d \times \mathbb{Z}$. In dimension one, only rational numbers are singular. The existence of singular vectors that do not lie in a rational subspace was proved by Khintchine for all dimensions $d \geq 2$. Thus, singular vectors exhibit phenomena that cannot occur in dimension one. For instance, when $\theta \in \mathbb{R}^d$ is singular, the sequence $0, \theta, 2\theta, \ldots, n\theta, \ldots$ fills the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ in such a way that there exists a point y whose distance in the torus to the set $\{0, \theta, \ldots, n\theta\}$, times $n^{1/d}$, goes to infinity as n tends to infinity (see [4], Chapter V). While it is easy to see that the set $\mathrm{Sing}(d)$ of all singular vectors in $\mathbb{R}^d$ has zero Lebesgue measure, it is hard to compute its Hausdorff dimension. After a few partial results by Baker and Rynne (see [2], [3] and [24]), it has been proved in [6] by the first author that the Hausdorff dimension of $\mathrm{Sing}(2)$ is $\frac{4}{3}$.

In this paper, we extend this result to higher dimensions.

Theorem 1.1. *For $d \geq 2$ the Hausdorff dimension of* $\mathrm{Sing}(d)$ *is* $\frac{d^2}{d+1}$.

Date: September 21, 2015.

1991 *Mathematics Subject Classification.* 11J13, 11K55, 37A17.

Key words and phrases. singular vectors; self-similar coverings; multi-dimensional continued fractions; simultaneous Diophantine approximation; best approximations; divergent trajectories.

The first author is partially supported by NSF grant DMS 0956209.

More generally, a real $d \times c$ matrix θ is a *singular system of d linear forms in c variables* if for every $\varepsilon > 0$, there exists T_0 such that for all $T > T_0$ the system of inequalities

$$\|\theta q - p\| < \frac{\varepsilon}{T^{c/d}} \quad \text{and} \quad 0 < \|q\| < T \tag{1.2}$$

admits an integer solution $(q, p) \in \mathbb{Z}^c \times \mathbb{Z}^d$. The terminology reflects the fact that the set of singular matrices has Lebesgue measure zero, as proved by Khintchine in [16]. There is a well-known dynamical interpretation of what it means for a $d \times c$ matrix θ to be singular in terms of the flow on the space $G/\Gamma = \mathrm{SL}_{d+c}(\mathbb{R})/\,\mathrm{SL}_{d+c}(\mathbb{Z})$ induced by the action of $g_t = \mathrm{diag}(e^{ct}, \ldots, e^{ct}, e^{-dt}, \ldots, e^{-dt})$: *$\theta$ is singular if and only if the forward g_t-orbit of the coset $\begin{pmatrix} \mathbb{1}_d & \theta \\ 0 & \mathbb{1}_c \end{pmatrix}\Gamma$ is divergent, meaning that it eventually leaves every compact subset of G/Γ.* Thus, Khintchine's measure zero result can be derived from the ergodicity of the g_t-action.

The study of singular systems can thus be viewed as a special case of the problem of understanding divergent trajectories of a one-parameter diagonal action where there are exactly one positive and one negative Lyapunov exponent. This viewpoint was initiated by S.G. Dani in [10], where Khintchine's results are generalized to this broader setup, and further developed by Barak Weiss ([29] and [30]) and by Kleinbock and Weiss [17].

In this broader context, we have the following corollary of Theorem 1.1.

Corollary 1.2. *The Hausdorff dimension of the union of all divergent g_t-orbits in G/Γ is $\dim G - \frac{d}{d+1}$ when $c = 1$ and $d \geq 2$.*

Proof. The trajectory $\{g_t h_\theta \Gamma\}_{t \geq 0}$ is divergent if and only if $\{g_t p h_\theta \Gamma\}_{t \geq 0}$ is divergent for every p in the subgroup $P = \{p : g_t p g_{-t}$ stays bounded in G/Γ as $t \to \infty\}$, a.k.a. the normalizer of the *stable horospherical subgroup* of g_1. Since P is a manifold, the Hausdorff codimension of the union of divergent g_t-orbits in G/Γ agrees with that of $\mathrm{Sing}(d)$ in $\mathbb{R}^d$. □

Recently, the dimension upper bound in Corollary 1.2 was generalized by S. Kadyrov, D. Kleinbock, E. Lindenstrauss and G. Margulis [14] to $\dim G - \frac{cd}{c+d}$; this upper bound is actually obtained for a larger set of g_t-orbits that "escape on average" and they conjecture it to be sharp. Corollary 1.2 verifies sharpness in the case $\min(c, d) = 1, c + d \geq 3$; sharpness is also known in the case $c = d = 1$ by a result of Kadyrov and A. Pohl [15]. In terms of approximation theory, a real $d \times c$ matrix is "singular on average" if *for any $\varepsilon > 0$,*

$$\lim_{N \to \infty} \frac{1}{N} \operatorname{card} \left\{ \ell \in \{1, \ldots, N\} : \exists (q, p) \in \mathbb{Z}^c \times \mathbb{Z}^d \text{ s.t. } (1.2) \text{ holds for } T = 2^\ell \right\} = 1. \tag{1.3}$$

The conjectured value for the Hausdorff dimension of the set of such matrices is $cd - \frac{cd}{c+d}$. It is also conjectured in [14] that the Hausdorff dimension of the set of real $d \times c$ matrices that are singular on average coincides with that of singular matrices unless $c = d = 1$.

We note that L. Yang [31] has computed the Hausdorff dimension of divergent trajectories in the special case of a reducible flow on a product of rank one spaces with a unique positive as well as a unique negative Lyapunov exponent.

Let us also mention some related results on Hausdorff dimension of divergent trajectories of the Teichmüller geodesic flow. In the paper [21], H. Masur showed that every nonergodic measured foliation on a closed Riemann surface determines a Teichmüller geodesic ray with a divergent image in Riemann space and further that the set of such *divergent directions* in any Teichmüller disk has Hausdorff dimension at most $\frac{1}{2}$. Consequently, the union of divergent Teichmüller geodesic rays has Hausdorff dimension at most $\dim C - \frac{1}{2}$ for any connected component C of any stratum of abelian differentials.[1] On the other hand, the set of holomorphic differentials with nonergodic vertical foliation was shown by Masur and J. Smillie [22] to have Hausdorff dimension strictly greater than $\dim C - 1$, apart from several low dimensional exceptional cases covered by the Weyl theorem. Recently, it was shown by J. Athreya and J. Chaika in [1] that Masur's upper bound is sharp in the stratum of genus two abelian differentials with a double zero. Finally, we mention for certain Teichmüller disks it can happen that the set of nonergodic directions has Hausdorff dimension strictly less than that of the set of divergent directions [7].

To state our second main result, let ε be a fixed positive real number. Recall that a vector x in $\mathbb{R}^d$ is *ε-Dirichlet improvable* if the system of inequalities (1.1) admits a solution for T large enough. H. Davenport and W. Schmidt have proved that the set $\mathrm{DI}_\varepsilon(d)$ of ε-Dirichlet improvable vectors has measure zero for all $\varepsilon < 1$ ([11] and [12]). This result has been generalized in several directions (see for instance [18]) but less is known about the Hausdorff dimension of $\mathrm{DI}_\varepsilon(d)$. The intersection of the sets $\mathrm{DI}_\varepsilon(d)$, $\varepsilon > 0$, is the set of singular vectors, so it is natural to expect that the Hausdorf dimension of $\mathrm{DI}_\varepsilon(d)$ decreases to the Hausdorff dimension of $\mathrm{Sing}(d)$ as ε goes to zero. When $d = 2$, the bounds on $\mathrm{Hdim}\,\mathrm{DI}_\varepsilon(d)$ obtained in [5] imply this continuity at $\varepsilon = 0$ of the Hausdorff dimension.

Our second result extends these bounds to higher dimensions.

Theorem 1.3. *Let $d \geq 2$ be an integer and let t be any positive real number $> d$. There is a constant C such that for ε small enough,*

$$\frac{d^2}{d+1} + \varepsilon^t \leq \mathrm{Hdim}\,\mathrm{DI}_\varepsilon(d) \leq \frac{d^2}{d+1} + C\varepsilon^{d/2}. \tag{1.4}$$

Actually, we give a slightly better lower bound; see Corollary 6.12. In the two dimensional case, this result already improves the lower bound of [6] but there is a further improvement giving a precise rate of convergence when ε goes to zero.

Theorem 1.4. *For all real numbers $t > 1$, we have*

$$\frac{4}{3} + \varepsilon^t \leq \mathrm{Hdim}\,\mathrm{DI}_\varepsilon(2)$$

for ε small enough. Consequently, when $d = 2$,

$$\lim_{\varepsilon \to 0^+} \frac{\log(\mathrm{Hdim}\,\mathrm{DI}_\varepsilon(d) - \frac{d^2}{d+1})}{\log \varepsilon} = 1. \tag{1.5}$$

[1] The argument here uses a result of Y. Minsky and B. Weiss, see Theorem 1.6 of [1].

Our proof of the above Theorem does not work when the dimension is ≥ 3. In particular, our analysis stops short of establishing the existence of the limit in (1.5) for $d \geq 3$. It seems that our approach could potentially be carried further to establish existence of the above limit, which we conjecture to lie in the interval $[d-1, d[$.

At last we have to say that results about singular linear forms can be deduced using a transference Theorem ([4], Chapter V, corollary section II), but a direct proof is not clear.

Overview of the paper. The proofs rely on three main tools. The first is Geometry of Numbers, especially Minkowski's theorems and reduced bases. The second is the notion of self-similar covering introduced in [5] and [6]. The third is the notion of best Diophantine approximation. Section 2 is devoted to Geometry of Numbers and counting results. Section 3 is devoted to Hausdorff dimension bounds using self-similar covers and Section 4 is devoted to best Diophantine approximations. The proofs of the above Theorems actually begin in Section 5. Proving the upper bounds is the easiest parts of the proofs. While the upper bounds are proved in Section 5, the lower bounds need Sections 6, 7 and 8.

Sketch of the proofs. The guideline for the proofs relies on two simple results. For each primitive vector $x = (p, q)$ in $\mathbb{Z}^d \times \mathbb{Z}_{>0}$ let $\lambda_1(x)$ denote the length of the shortest vector in the lattice $\Lambda_x = \mathbb{Z}^d + \mathbb{Z}\frac{p}{q}$. The first result is: θ in $\mathbb{R}^d$ is singular if and only if

$$\lim_{n\to\infty} \lambda_1(x_n)\, |x_n|^{1/d} = 0$$

where $x_n = (p_n, q_n)$ is the sequence of best approximation vectors of θ and $|x_n| = |q_n|$ (see the beginning of Section 4.1 and Corollary 4.4). A similar result holds for Dirichlet improvable vectors as well. Roughly, a vector θ is in $\mathrm{DI}_\varepsilon(d)$ if and only if $\lambda_1(x_n)\, |x_n|^{1/d} \leq \varepsilon$ for n large enough (see again Corollary 4.4). The second result is a multidimensional extension of Legendre's Theorem about convergents of ordinary continued fraction expansions: if $x = (p, q)$ is a best approximation vector of θ, then $\theta \in B(\frac{p}{q}, \frac{2\lambda_1(x)}{|x|})$ and conversely, if $\theta \in B(\frac{p}{q}, \frac{\lambda_1(x)}{2|x|})$, then x is a best approximation vector of θ (see Lemma 4.2). Then we use the strategy that works for computing the Hausdorff dimension of Cantor sets defined by a nested tree of intervals, in which the children of an interval are defined as the immediate successors with respect to the partial order induced by containment of intervals. The diameter of one interval raised to the power s has to be compared with the sum over all the children intervals of the diameters raised to the power s.

For the upper bound, consider for each primitive vector $x = (p, q)$ in $\mathbb{Z}^d \times \mathbb{Z}_{>0}$ with $\lambda_1(x)\, |x|^{1/d} \leq \varepsilon$, the set $\sigma_\varepsilon(x)$ of children of x. The first idea is to take $\sigma_\varepsilon(x)$ the set of all possible primitive vectors y in $\mathbb{Z}^d \times \mathbb{Z}_{>0}$ with $\lambda_1(y)\, |y|^{1/d} \leq \varepsilon$ such that x and y are two consecutive best approximation vectors of some θ in $\mathbb{R}^d$. If for all x,

$$\sum_{y=(u,v)\in\sigma_\varepsilon(x)} \left(\operatorname{diam} B\left(\frac{u}{v}, \frac{2\lambda_1(y)}{|y|}\right)\right)^s \leq \left(\operatorname{diam} B\left(\frac{p}{q}, \frac{2\lambda_1(x)}{|x|}\right)\right)^s$$

then the Hausdorff dimension of $\mathrm{DI}_\varepsilon(d)$ is $\leq s$. We make precise this statement by using self similar covering introduced by the first author (see [6] and Theorem 3.2). However the above inequality does not hold and we modify the definition of the set $\sigma_\varepsilon(x)$ with an "acceleration" by considering only a subsequence of the sequence of best approximations. The best approximations y that follow x are skipped while the hyperplane that best fits with the lattice Λ_x also fits with the lattice Λ_y, see Definition 5.1. We note that this is not a straightforward extension of the first acceleration in [6]. Another point is that it is better to use a larger radius than $\frac{2\lambda_1(x)}{|x|}$, for it avoids the second acceleration used in [6] (see comments in Section 3 about the choice of the radius). With these ingredients the proof the upper bound follows readily; see Section 5.2.

The lower bound is trickier. The idea is to find a Cantor set included in $\mathrm{DI}_\varepsilon(d)$. This Cantor set has an "inhomogeneous" tree structure. For each $x = (p, q)$, we want to define a finite set $\sigma_\varepsilon(x)$ such that for all $y = (u, v)$ in $\sigma_\varepsilon(x)$, the balls $B(y) = B(\frac{u}{v}, \frac{\lambda_1(y)}{2|y|})$ are included in the ball $B(x) = B(\frac{p}{q}, \frac{\lambda_1(x)}{2|x|})$ and are well-separated; we also want that x and y are consecutive best approximation vectors of all θ in $B(y)$. This last condition is useful, for it allows to control $\lambda_1(y)$. Once this is done, the inequality

$$\sum_{y=(u,v)\in\sigma_\varepsilon(x)} \left(\operatorname{diam} B\left(\frac{u}{v}, \frac{\lambda_1(y)}{2\,|y|}\right)\right)^s \geq \left(\operatorname{diam} B\left(\frac{p}{q}, \frac{\lambda_1(x)}{2\,|x|}\right)\right)^s$$

implies that the Hausdorff dimension of $\mathrm{DI}_\varepsilon(I)$ is $\geq s$ (see Theorem 3.4).

There are at least two difficulties in this program. The first is to find conditions ensuring that x and y are consecutive best approximation vectors (see Definition 6.3 and Lemma 6.4). The second is to be sure that the set $\sigma_\varepsilon(x)$ is big enough (see Proposition 6.11); this is done in Section 8. The issues of nestedness and spacing of balls are relatively straightforward. The proofs of Theorems 1.1 and 1.3 end in Section 6.4.

Theorem 1.4 requires one more step, which is modifying the definition of $\sigma_\varepsilon(x)$ to make it even bigger; this is the goal of Section 7. This new definition of $\sigma_\varepsilon(x)$ follows more closely the definition used in the upper bounds, leading to an improvement of Theorem 1.3. However, when $d \geq 3$ some information is lost (see Lemmas 7.1 and 7.2) and the improvement works only in the two-dimensional case.

Acknowledgments. The authors would like to thank the referee(s) for a careful reading of the manuscript and their many suggestions for improving the readability of the paper; in particular, the inclusion of the flowchart in the Appendix as an aid for reading the proofs was called for given the complexity of the main argument.

2. Counting lattice points

Notation. In a metric space $d(A, B) = \inf\{d(x, y) : (x, y) \in A \times B\}$ denotes the "distance" between the two subsets A and B. We shall use the standard notations $\ll, \gg, \asymp$ in number theory:

$$f(a, b, ...) \ll g(a, b, ...)$$

means that

$$f(a,b,...) \leq C g(a,b,...)$$

where C is a constant that does not depend on the parameters $a, b, ...$ etc. The symbol $\gg$ is used for the reverse inequality and $\asymp$ when both inequalities hold. Unless we specify what the constant C depends on, in all the statements of the paper, C depends either only on the dimension, or only on the norm and the dimension.

We shall also use the nonstandard notation $x \asymp_2 y$ which means $\frac{1}{2}y \leq x \leq 2y$.

For a lattice L in $\mathbb{R}^n$ and $1 \leq k \leq n$, we denote by

$$\lambda_k(L) = \inf\{r \geq 0 : B(0,r) \text{ contains at least } k \text{ linearly independent vectors of } L\}$$

the k^{th} minimum of L, which depends on a choice of a norm $\|\cdot\|$ on $\mathbb{R}^n$ that we assume to be fixed. The successive minima of a lattice L in $\mathbb{R}^n$ are the real numbers $\lambda_1(L), \ldots, \lambda_n(L)$. The rescaled version of the k-th minimum is denoted

$$\widehat{\lambda}_k(L) = \lambda_k(L)\,\mathrm{vol}(L)^{-\frac{1}{n}} = \lambda_k(\tilde{L})$$

where $\tilde{L}$ is the lattice homothetic to L with covolume 1.

2.1. **Successive minima and reduced basis.** Assume that $\mathbb{R}^n$ is equipped with a norm. By the (second) Minkowski theorem, the successive minima of a lattice L in $\mathbb{R}^n$ satisfy (see [4])

$$\text{(M)} \qquad \frac{2^n}{n!}\,\mathrm{vol}(L) \leq \lambda_1(L)\cdots\lambda_n(L)\,\mathrm{vol}(B(0,1)) \leq 2^n\,\mathrm{vol}(L).$$

where $\mathrm{vol}(L)$ is the covolume of the lattice L. We shall use this fact many times and simply write

$$\lambda_1(L)\cdots\lambda_n(L) \asymp \mathrm{vol}(L)$$

where the constants involved in $\asymp$ depend only on the dimension and on the volume of the unit ball.

We shall need reduced bases in a lattice. Minkowski reduced bases are well-suited to our needs. Recall that a basis $v_1, ..., v_n$ of a lattice L is Minkowski reduced if v_1 is a shortest vector of L and if for all $i > 1$, the vector v_i is a vector with minimal norm among the vectors v such that $v_1, ..., v_{i-1}, v$ can be extended into a basis of L. Since a shortest vector can be extended into a basis, Minkowski reduced basis always exist. Other kinds of reduced bases could work as well. We shall use the following properties of a Minkowski reduced basis $v_1, \ldots, v_n$. For all i,

$$\text{(M')} \qquad \lambda_i(L) \leq \|v_i\| \leq 2^i \lambda_i(L),$$

A proof of this inequality can be found in [28]. Actually, Minkowski reduced bases are associated to Euclidean norms and the above inequality holds only in these cases. When dealing with a non Euclidean norm, we consider a reduced basis associated to the standard Euclidean norm and by the norm equivalence, one has $\|v_i\| \asymp \lambda_i$ where the constants involved depend not only on the dimension but also on the norm.

An important property of a reduced basis is that the angles between a vector of the basis and the subspace spanned by other vectors cannot be too small. More precisely, there is positive constant c_n depending only on the dimension such that

$$\text{(M")} \qquad \|v_i^*\| \geq c_n \lambda_i(L)$$

where v_i^* is the orthogonal projection of v_i on the line orthogonal to the subspace spanned by the other basis vectors. Once again, the above inequality holds for Euclidean norms and for other norms, one has

$$\|v_i^*\| \gg \lambda_i(L)$$

where the implicit constants depend on the norm. To prove (M"), note that by (M') one has

$$\lambda_1(L) \cdots \lambda_{i-1}(L) \, \|v_i^*\| \, \lambda_{i+1}(L) \cdots \lambda_n(L) \gg |\det(v_1, \ldots, v_{i-1}, v_i^*, v_{i+1}, \ldots, v_n)| = \mathrm{vol}(L)$$

and then use the Minkowski theorem.

For sake of simplicity, the reader can suppose that all the results are stated for the standard Euclidean norm. However, it is clear that the results of Section 2 hold with any norm with constants possibly depending on norms. The results in Section 4 hold for any norm.

2.2. **Codimension one sublattice of minimal covolume.** Fix a norm $\|\cdot\|$ on $\mathbb{R}^n$. For a subset $X \subset \mathbb{R}^n$, we denote the radius of the largest ball disjoint from X by

$$\begin{aligned} e(X) &= \sup\{r > 0 : \exists p \text{ s.t. } B(p,r) \cap X = \varnothing\} = \sup\{d(p,X) : p \in \mathbb{R}^d\} \\ &= \inf\{r > 0 : \forall x \in \mathbb{R}^n \ \exists p \in X \text{ s.t. } d(p,y) < r\}. \end{aligned}$$

Lemma 2.1. *The successive minima satisfy $\lambda_n(L) \leq 2e(L) \leq \lambda_1(L) + \cdots + \lambda_n(L)$.*

Proof. There is a parallelepiped $\mathcal{P}$ with edge lengths given by the successive minima of the lattice L, the translates of which cover $\mathbb{R}^n$, i.e. $\mathcal{P} + L = \mathbb{R}^n$. Therefore, $2e(L) \leq \lambda_1(L) + \cdots + \lambda_n(L)$. Furthermore, the intersection of L with the (open) ball $B(0, \lambda_n(L))$ is contained in an $(n-1)$-dimensional subspace F. Let v be a vector such that $v + F$ is a supporting hyperplane of the ball $B(0, \lambda_n(L))$. The open ball $B(\frac{1}{2}v, \frac{\lambda_n(L)}{2})$ is contained in $B(0, \lambda_n(L))$ and does not meet F; therefore, $e(L) \geq \frac{\lambda_n(L)}{2}$. □

Lemma 2.2. *Let L be a lattice in $\mathbb{R}^n$, $L' \subset L$ a sublattice of codimension one with minimal covolume, and H' the real span of L'. Then*

$$2e(L + H') = \inf\{\|v\| : v \in (L + H') \setminus H'\} \asymp \lambda_n(L).$$

Proof. Since $L + H'$ is a regularly spaced net of hyperplanes, we have $2e(L+H') = \inf\{\|v\| : v \in (L + H') \setminus H'\}$. Let L'' be a codimension one sublattice of L spanned by linearly independent vectors $v_1, \ldots, v_{n-1} \in L$ such that $\|v_i\| = \lambda_i(L)$. Making use of the Minkowski theorem, we obtain

$$\begin{aligned} \lambda_1(L) \cdots \lambda_{n-1}(L) &= \|v_1\| \cdots \|v_{n-1}\| \\ &\geq \lambda_1(L'') \cdots \lambda_{n-1}(L'') \\ &\gg \mathrm{vol}(L'') \geq \mathrm{vol}(L'). \end{aligned}$$

Since L' is a subset of L, $\lambda_i(L') \geq \lambda_i(L)$ for $i = 1, ..., n-1$. Again with the Minkowski theorem, we obtain $\mathrm{vol}(L') \gg \lambda_1(L) \cdots \lambda_{n-1}(L)$. Combining this with the above and using the Minkowski theorem once more, we get

$$\frac{\mathrm{vol}(L)}{\mathrm{vol}(L')} \asymp \lambda_n(L).$$

Since L' is a primitive sublattice of L, there exists a vector w in L such that $L = L' + \mathbb{Z}w$. We have $\mathrm{vol}(L) \asymp \mathrm{vol}(L')\|w^\perp\|$ where $w^\perp$ is the orthogonal projection onto the line orthogonal to H'. Hence $\|w^\perp\| \asymp \lambda_n(L)$. Now a vector $v = kw + v'$ with k in $\mathbb{R}$ and v' in H' has a norm $\gg |k|\|w^\perp\|$. Therefore,

$$\inf\{\|v\| : v \in (L + H') \setminus H'\} \gg \|w^\perp\| \asymp \lambda_n(L).$$

For the reverse inequality, just observe that $w^\perp \in (L + H') \setminus H'$. □

2.3. **Coarse asymptotics.** We shall need several counting estimates in lattices. All these estimates are coarse asymptotics that are well known to specialists. The proofs of these estimates, which are dispersed in many papers, are included for the convenience of the reader.

Lemma 2.3. *Let L be a lattice in $\mathbb{R}^n$ and let $x \in \mathbb{R}^n$. Then for all $r > 0$,*

$$\mathrm{card}\,(L \cap B(x,r)) \ll 1 + \sum_{i=1}^{n} \frac{r^i}{\lambda_1(L) \ldots \lambda_i(L)}.$$

In particular, for any positive constant c and any $r \geq c\lambda_n(L)$,

$$\mathrm{card}\,(L \cap B(x,r)) \ll \frac{r^n}{\min(1, c^n)\,\mathrm{vol}(L)}.$$

Proof. It is enough to prove that

$$\mathrm{card}\,(L \cap B(x,r)) \ll 1 + \sum_{i=1}^{n} \frac{r^i}{\|e_1\| \ldots \|e_i\|}$$

where $e_1, \ldots, e_n$ is a reduced basis of L. We prove by induction on n that the above inequality holds for any ball $B(x,r)$. If $n = 1$ the result is clear. Suppose $n > 1$ and denote $\Gamma = \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_{n-1}$. By induction hypothesis for any $x, y \in \mathbb{R}^d$,

$$\mathrm{card}\,((y + \Gamma) \cap B(x,r)) \ll 1 + \sum_{i=1}^{n-1} \frac{r^i}{\|e_1\| \ldots \|e_i\|}.$$

Moreover since the basis is reduced, $\|\sum_{i=1}^n a_i e_i\| \leq r$ implies $|a_n| \leq c_n \frac{r}{\|e_n\|}$ where c_n is a constant depending only on the norm. It follows that

$$L \cap B(x,r) \subset \bigcup_{a \in \mathbb{Z},\, |a - x_n| \leq c_n \frac{r}{\|e_n\|}} (ae_n + \Gamma) \cap B(x,r)$$

where $x = \sum_{i=1}^n x_i e_i$, thus

$$\begin{aligned}
\operatorname{card}(L \cap B(x,r)) &\ll \sum_{a\in\mathbb{Z},\, |a-x_n|\le c_n \frac{r}{\|e_n\|}} \left(1 + \sum_{i=1}^{n-1} \frac{r^i}{\|e_1\|\dots\|e_i\|}\right) \\
&\ll \left(1+\frac{r}{\|e_n\|}\right)\left(1 + \sum_{i=1}^{n-1} \frac{r^i}{\|e_1\|\dots\|e_i\|}\right) \\
&\ll 1 + \sum_{i=1}^{n} \frac{r^i}{\|e_1\|\dots\|e_i\|}.
\end{aligned}$$

□

Lemma 2.4. *Let L be a lattice in $\mathbb{R}^n$, t a real number $> n$, and R a positive real number. Then*

$$\sum_{x\in L, \|x\|\ge R} \frac{1}{\|x\|^t} \ll \frac{2^{nt}}{t-n} \sum_{k=1}^{n} \frac{1}{\lambda_1(L)\dots\lambda_k(L) R^{t-k}}.$$

In particular, for any positive constant c and any $R \ge c\lambda_n(L)$,

$$\sum_{x\in L, \|x\|\ge R} \frac{1}{\|x\|^t} \ll \frac{2^{nt}}{(t-n)\min(c^n,1)\operatorname{vol}(L) R^{t-n}}.$$

Proof. Let $e_1, \dots, e_n$ be a reduced basis of L. Since $\|e_i\| \asymp \lambda_i(L)$, the lemma will follow from the inequality

$$S := \sum_{x\in L,\ \|x\|\ge R} \frac{1}{\|x\|^t} \ll \frac{2^{nt}}{t-n} \sum_{k=1}^{n} \frac{1}{\|e_1\|\dots\|e_k\| R^{t-k}}.$$

We will use the elementary inequality: if $a > 0$ and $s > 1$ then

$$\sum_{m\ge a} \frac{1}{m^s} \le \frac{2^s}{s-1}\frac{1}{a^{s-1}}.$$

We shall drop the factors of 2^s in the proof.

Let $x = \sum_{i=1}^n x_i e_i$ be in L. Consider for each nonempty subset I of $\{1, \dots, n\}$ and each bijection $\sigma : \{1, \dots, |I| = \operatorname{card} I\} \to I$, the subsets of L

$$L_I = \left\{ \sum_{i\in I^C} x_i e_i \in L : \frac{R}{n} \ge |x_i|\|e_i\| \text{ for } i \in I^C \right\},$$

$$L'_{I,\sigma} = \left\{ \sum_{i\le |I|} x_{\sigma(i)} e_{\sigma(i)} \in L : \frac{R}{n} \le |x_{\sigma(1)}|\|e_{\sigma(1)}\| \le |x_{\sigma(2)}|\|e_{\sigma(2)}\| \le \dots \le |x_{\sigma(|I|)}|\|e_{\sigma(|I|)}\| \right\},$$

and the sum

$$S_{I,\sigma} = \sum_{x \in L_I \oplus L'_{I,\sigma}} \frac{1}{\|x\|^t}.$$

Since $\|\sum x_i e_i\| \le \sum |x_i|\|e_i\|$, we have $\|\sum x_i e_i\| \ge R$ implies $\max |x_i|\|e_i\| \ge \frac{R}{n}$. Therefore, $L \setminus B(0,R) \subset \bigcup_{I,\sigma}(L_I \oplus L'_{I,\sigma})$, so it is enough to bound each sum $S_{I,\sigma}$ from above. Let V_i be the subspace spanned by the e_j except e_i. Since the basis is reduced, $\|x\| \gg |x_i|\|e_i\|$ for all i. It follows that for $x \in L_I \oplus L'_{I,\sigma}$, $\|x\| \gg |x_{\sigma(|I|)}|\|e_{\sigma(|I|)}\|$, thus

$$\begin{aligned}
S_{I,\sigma} &\ll \sum_{x \in L_I \oplus L'(I,\sigma)} \frac{1}{|x_{\sigma(|I|)}|^t \|e_{\sigma(|I|)}\|^t} \\
&\le \prod_{i \in I^C} \left(1 + \frac{2R}{n\|e_i\|}\right) \sum_{x \in L'_{I,\sigma}} \frac{1}{|x_{\sigma(|I|)}|^t \|e_{\sigma(|I|)}\|^t} \\
&\ll \prod_{i \in I^C} \left(1 + \frac{2R}{n\|e_i\|}\right) \sum_{x \in L''} \frac{1}{t-1} \frac{1}{\left(\frac{|x_{\sigma(|I|-1)}|\|e_{\sigma(|I|-1)}\|}{\|e_{\sigma(|I|)}\|}\right)^{t-1} \|e_{\sigma(|I|)}\|^t}, \\
&= \prod_{i \in I^C} \left(1 + \frac{2R}{n\|e_i\|}\right) \sum_{x \in L''} \frac{1}{t-1} \frac{1}{\left(|x_{\sigma(|I|-1)}|\|e_{\sigma(|I|-1)}\|\right)^{t-1} \|e_{\sigma(|I|)}\|},
\end{aligned}$$

where

$$L'' = \left\{ \sum_{i<|I|} x_{\sigma(i)} e_{\sigma(i)} \in L : \frac{R}{n} \le |x_{\sigma(1)}|\|e_{\sigma(1)}\| \le |x_{\sigma(2)}|\|e_{\sigma(2)}\| \le \dots \le |x_{\sigma(|I|-1)}|\|e_{\sigma(|I|-1)}\| \right\}.$$

Continuing inductively, we have

$$\begin{aligned}
S_{I,\sigma} &\ll \frac{1}{t-|I|} \prod_{i \in I^C} \left(1 + \frac{2R}{n\|e_i\|}\right) \frac{1}{\left(\frac{R}{n}\right)^{t-|I|} \|e_{\sigma(1)}\| \dots \|e_{\sigma(|I|)}\|}, \\
&\ll \frac{1}{t-|I|} \prod_{i \in I^C} \left(1 + \frac{R}{\|e_i\|}\right) \frac{1}{R^{t-|I|} \|e_{\sigma(1)}\| \dots \|e_{\sigma(|I|)}\|}, \\
&= \frac{1}{t-|I|} \sum_{J \supset I} \frac{1}{R^{t-|J|} \prod_{j \in J} \|e_j\|}.
\end{aligned}$$

□

The above two Lemmas will be used in the proofs of Hausdorff dimension upper bounds while the next Lemma will be used in the proof of Hausdorff dimension lower bounds.

Lemma 2.5. *Let L be a lattice in $\mathbb{R}^n$ and let $x \in \mathbb{R}^n$. There exists a constant C_n depending only on the dimension and on the norm such that for all $r > C_n \lambda_n(L)$, $\mathrm{card}(L \cap B(x,r)) \gg \frac{r^n}{\mathrm{vol}(L)}$. Moreover, when $x = 0$, $\mathrm{card}(L \cap B(x,r)) \gg \frac{r^n}{\mathrm{vol}(L)}$ for all nonnegative r.*

Proof. Let $e_1, \dots, e_n$ be a basis for L, and $\mathcal{P} = \{\sum_{i=1}^n a_i e_i : a_i \in [0,1[\}$ the fundamental parallelepiped associated with the basis. Each parallelepiped $y + \mathcal{P}$ contains exactly one point of L. The number of parallelepipeds of the form $y + \mathcal{P}, y \in L$ that are contained in $B(x,r)$ is at least the number parallelepipeds intersecting $B(x,r)$ minus the number of parallelepipeds intersecting the boundary of $B(x,r)$. It follows that

$$\operatorname{card}(L \cap B(x,r)) \geq \frac{\operatorname{vol}(B(x,r))}{\operatorname{vol}(L)} - \frac{\operatorname{vol}(B(x, r + \operatorname{diam}\mathcal{P})) - \operatorname{vol}(B(x, r - \operatorname{diam}\mathcal{P}))}{\operatorname{vol}(L)}.$$

If the basis is reduced, then $\operatorname{diam}\mathcal{P} \ll \lambda_n(L)$ and the first inequality of the lemma follows.

For $i \leq n$, consider the sublattice $L_i = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_i$ where the v_i's are linearly independent vectors with $\|v_i\| = \lambda_i(L)$. The first inequality gives

$$\operatorname{card}(L \cap B(0,r)) \geq \operatorname{card}(L_i \cap B(0,r)) \gg \frac{r^i}{\operatorname{vol}(L_i)}$$

for all $r \geq C_n \lambda_i(L)$ (we can assume that the sequence $(C_n)_{n\geq 1}$ is nondecreasing). Now, by the second Minkowski theorem, if $C_n\lambda_i(L) \leq r \leq C_n\lambda_{i+1}(L)$ then

$$\frac{r^i}{\operatorname{vol}(L_i)} = \frac{r^i \lambda_{i+1}(L)\cdots\lambda_n(L)}{\operatorname{vol}(L_i)\lambda_{i+1}(L)\cdots\lambda_n(L)} \gg \frac{r^n}{\operatorname{vol}(L)},$$

and if $r < C_n\lambda_1(L)$ then $L \cap B(0,r)$ contains at least one point ($x = 0$) while $\frac{r^n}{\operatorname{vol}(L)} \ll 1$. ☐

Remark 2.6. In Lemma 2.5, the condition $r > C_n\lambda_n(L)$ is necessary. Indeed, by Lemma 2.2, $e(L) \gg \lambda_n(L)$ which implies that there are balls with radius $\asymp \lambda_n(L)$ that contain no points of L.

Remark 2.7. It is possible to make the constant C_n depend only on the dimension because any norm is equivalent to a Euclidean norm with constants depending only on the dimension (see [27]).

The estimates of this section will be enough for the upper bound calculations. For the lower bound calculations, we need to consider sums that are over restricted subsets (introduced in Section 6) of the lattice, a counting problem that we shall deal with in Section 8.

3. Self-similar coverings

3.1. **Upper estimates.**

Definition 3.1. Let Y be a metric space. A *self-similar structure* on Y is a triple (J, σ, B) where J is countable, σ is a subset of J^2 and B is a map from J into the set of bounded subsets of Y. A *σ-admissible sequence* is a sequence (x_n) in J such that

(i) for all integers n, $(x_n, x_{n+1}) \in \sigma$.

Let X be a subset of Y. A *self-similar covering* of X is a self-similar structure (J, σ, B) such that for all θ in X, there exists a σ-admissible sequence $(x_n)_{n\in\mathbb{N}}$ in J such that

(ii) $\lim_{n\to\infty} \operatorname{diam} B(x_n) = 0$,

(iii) $\bigcap_{n\in\mathbb{N}} B(x_n) = \{\theta\}$.

The set covered by a self-similar structure (J, σ, B) is the set all θ in Y with the above property.

Notation. $\sigma(x)$ denotes the set of y in J such that $(x, y) \in \sigma$.

There is an easy procedure for constructing self-similar coverings. For simplicity, let us assume that a countable dense subset J of Y is given together with a function $r : J \to]0, \infty[$ going to zero at infinity, where $x \to \infty$ in J means leaving each finite subset of J. Suppose we have a map $\eta : X \to J^{\mathbb{N}}$ that associates to each $\theta \in X$ a sequence (x_n) in J tending to infinity with $d(x_n, \theta) \leq r(x_n)$ for all n. For each x in J, let $B_\eta(x)$ be the set of all θ in X whose associated sequence contains x. Define σ_η by the criterion that $y \in \sigma_\eta(x)$ if there is a sequence associated to some θ in X that contains both x and y, with y following immediately after x. Then (J, σ_η, B_η) is a self-similar covering of X.

Theorem 3.2. ([6]) *Let Y be a metric space, let X be a subset of Y that admits a self-similar covering (J, σ, B) and let s be a positive real number. If for all x in J,*

$$\sum_{y\in\sigma(x)} \operatorname{diam} B(y)^s \leq \operatorname{diam} B(x)^s, \tag{3.1}$$

then $\operatorname{Hdim} X \leq s$.

Remark. This Theorem has been proved in the case $X = Y$ in [5]. In [6], it has already been noted that in the more general case of a subset X of Y, the arguments of the proof given in [5] apply with essentially no changes.

Definition 3.3. Let $s(J, \sigma, B)$ be the best possible upper bound provided by Theorem 3.2 applied to a fixed self-similar covering; in other words, $s(J, \sigma, B) = \sup_{x\in J} s(x)$ where $s(x)$ is the infimum over all positive real numbers satisfying (3.1).

It is possible to have $s(J, \sigma, B) = \infty$, which is a situation we want to avoid. Note that if the self-similar coverings (J, σ_1, B) and (J, σ_2, B) are such that $\sigma_1(x) \subset \sigma_2(x)$ for all x in J, then $s(J, \sigma_1, B) \leq s(J, \sigma_2, B)$.

The problem of computing $s(J, \sigma_\eta, B_\eta)$ reduces to the problem of enumerating the set $\sigma_\eta(x)$ for which an exact description is most likely impossible. Also, the need to use estimates on $\operatorname{diam} B_\eta(x)$ (as opposed to an exact formula) can further complicate calculations. To overcome these difficulties, observe that if $\sigma \subset J^2$ and a map B from J to the set of bounded subsets of Y are such that for all $x \in J$, $\sigma(x) \supset \sigma_\eta(x)$ and $B(x) \supset B_\eta(x)$, then (J, σ, B) is also a self-similar covering of X. The choice of σ and B may considerably simplify the evaluation of $s(J, \sigma, B)$ (for example, by choosing $B(x)$ a metric ball of some specified radius) and one may reasonably expect $s(J, \sigma, B)$ to be close to $s(J, \sigma_\eta, B_\eta)$ if σ is a sufficiently good approximation to σ_η.[2]

Our initial approach is to let η be the encoding by the sequence of best approximation vectors. There are two natural choices for $B(x)$, one given by Lemma 4.2 below and one

[2]It is not always optimal to choose B approximating B_η as efficiently as possible. See Remark 5.3.

given by Dirichlet theorem; but regardless of the choice, the ratio $\frac{\operatorname{diam} B(y)}{\operatorname{diam} B(x)}$ may be close to one as y ranges over $\sigma_\eta(x)$, which leads to ineffective upper bounds. This obstacle can easily be overcome by using an *acceleration* of η instead, i.e. an encoding $\eta' : X \to J^{\mathbb{N}}$ such that for each $x \in J$, $\eta'(x)$ is a subsequence of $\eta(x)$. A potential drawback of using the acceleration is that $\sigma_{\eta'}$ may be more difficult to approximate than σ_η.

In Section 5.1, we specify the particular approximation to $\sigma_{\eta'}$ that we will use. We do not specify η' explicitly, although it can easily be inferred from the proof of Lemma 5.2.

3.2. **Lower estimates.**

Theorem 3.4. *Let Y be a metric space and let X be a subset of Y. Suppose that there is a self-similar structure (J, σ, B) that covers a subset of X (see Definition 3.1) and constants $c, s \geq 0$ and $\rho \in]0,1[$ such that*

1. *for all $x \in J$, $\sigma(x)$ is finite, $B(x)$ is a nonempty compact subset of Y, and for all $y \in \sigma(x)$ we have $B(y) \subset B(x)$,*
2. *for each admissible sequence (x_n), $\lim_{n\to\infty} \operatorname{diam} B(x_n) = 0$,*
3. *for each $x \in J$, and each $y \in \sigma(x)$, there are at most c points z in $\sigma(x) \setminus \{y\}$ such that*
$$d(B(y), B(z)) \leq \rho \operatorname{diam} B(x),$$
4. *for every $x \in J$,*
$$\sum_{y \in \sigma(x)} \operatorname{diam} B(y)^s \geq (c+1) \operatorname{diam} B(x)^s,$$
5. *for all $x \in J$, and each $y \in \sigma(x)$, $\operatorname{diam} B(y) < \operatorname{diam} B(x)$.*

Then X contains a subset of positive s-dimensional Hausdorff measure.

To obtain the dimension lower bound for singular vectors, we shall require the following *weighted* analog of Theorem 3.4. We only prove this analog, as Theorem 3.4 is a special case.

Definition 3.5. By a *strictly nested* self-similar structure we mean one that satisfies the conditions 1, 2 and 5 of Theorem 3.4.

Theorem 3.6. *Let Y be a metric space and let X be a subset of Y. Suppose that there is a self-similar structure (J, σ, B) that covers a subset of X, a subset $J_0 \subset J$ that contains a tail of any σ-admissible sequence, a function $\rho : J \to]0,1[$ and two constants $c, s \geq 0$ such that*

1. *for all $x \in J_0$, $\sigma(x) \subset J_0$ and is finite, $B(x)$ is a nonempty compact subset of Y, and for all $y \in \sigma(x)$ we have $B(y) \subset B(x)$,*
2. *for each admissible sequence (x_n), $\lim_{n\to\infty} \operatorname{diam} B(x_n) = 0$,*
3. *for each $x \in J_0$, and each $y \in \sigma(x)$, there are at most c points z in $\sigma(x) \setminus \{y\}$ such that*
$$d(B(y), B(z)) \leq \rho(x) \operatorname{diam} B(x),$$

4. *for every $x \in J_0$,*

$$\sum_{y \in \sigma(x)} (\rho(y) \operatorname{diam} B(y))^s \geq (c+1)(\rho(x) \operatorname{diam} B(x))^s,$$

5. *for all $x \in J_0$, and each $y \in \sigma(x)$, $\rho(y) \operatorname{diam} B(y) < \rho(x) \operatorname{diam} B(x)$.*

Then X contains a subset of positive s-dimensional Hausdorff measure.

Proof. By Theorem 3.3 of [5], it is enough to construct for each x in J_0, a subset $\sigma'(x)$ of $\sigma(x)$ such that

- $\sigma'(x)$ contains at least two points,
- for any pair of distinct y, z in $\sigma'(x)$, $d(B(y), B(z)) > \rho(x) \operatorname{diam} B(x)$,
- $\sum_{y \in \sigma'(x)} (\rho(y) \operatorname{diam} B(y))^s \geq (\rho(x) \operatorname{diam} B(x))^s$.

Choose y_0 in $\sigma_0(x) = \sigma(x)$ such that $\rho(y_0) \operatorname{diam} B(y_0)$ is maximal. Remove from $\sigma(x)$ the subset $\tau(y_0)$ consisting of y in $\sigma(x)$ such that $d(B(y_0), B(y)) \leq \rho(x) \operatorname{diam} B(x)$. This leaves a new set $\sigma_1(x) = \sigma(x) \setminus \tau(y_0)$. Note that y_0 is in $\tau(y_0)$. Next,

- Choose y_1 in $\sigma_1(x)$ such that $\rho(y_1) \operatorname{diam} B(y_1)$ is maximal.
- Let $\sigma_2(x) = \sigma_1(x) \setminus \tau(y_1)$ where $\tau(y_1)$ is the subset consisting of y in $\sigma_1(x)$ such that $d(B(y_1), B(y)) \leq \rho(x) \operatorname{diam} B(x)$.

We continue this process until we arrive at the set $\sigma_n(x)$ being empty.

Since $\sigma(x)$ is a finite set, we get a finite sequence $y_0, \ldots, y_{n-1}$ such that for $i = 0, \ldots, n-1$,

- $\rho(y_i) \operatorname{diam} B(y_i) \geq \rho(y) \operatorname{diam} B(y)$ for all y in $\sigma_i(x)$,
- $d(B(y_i), B(y)) > \rho(x) \operatorname{diam} B(x)$ for all y in $\sigma_{i+1}(x)$,
- $\sigma(x)$ is the disjoint union of the $\tau(y_i)$, $i = 0, ..., n-1$.

It follows that

- $\rho(y_i) \operatorname{diam} B(y_i) \geq \rho(y) \operatorname{diam} B(y)$ for all y in $\tau(y_i)$,
- $d(B(y_i), B(y_j)) > \rho(x) \operatorname{diam} B(x)$ for all $0 \leq i \neq j < n$.

Set $\sigma'(x) = \{y_0, \ldots, y_{n-1}\}$. We have

$$\begin{aligned}
\sum_{y \in \sigma(x)} (\rho(y) \operatorname{diam} B(y))^s &= \sum_{i=0}^{n-1} \sum_{y \in \tau(y_i)} (\rho(y) \operatorname{diam} B(y))^s \\
&\leq \sum_{i=0}^{n-1} (\operatorname{card} \tau(y_i)) \times (\rho(y_i) \operatorname{diam} B(y_i))^s \\
&\leq \sum_{i=0}^{n-1} (c+1)(\rho(y_i) \operatorname{diam} B(y_i))^s \\
&= (c+1) \sum_{y \in \sigma'(x)} (\rho(y) \operatorname{diam} B(y))^s,
\end{aligned}$$

therefore

$$\sum_{y\in\sigma'(x)} (\rho(y)\operatorname{diam} B(y))^s \geq (\rho(x)\operatorname{diam} B(x))^s.$$

Since by condition 5, $\rho(y)\operatorname{diam} B(y) < \rho(x)\operatorname{diam} B(x)$, the above inequality ensures that $\sigma'(x)$ contains at least two points. $\square$

Remark 3.7. Theorem 3.6 is invoked only in the proof of Theorem 1.1. We shall use a function ρ decreasing to zero with a polynomial decay while the diameters of the sets $B(x)$ decrease to zero exponentially fast along the admissible sequences.

4. Farey lattices

Let the set of primitive vectors in $\mathbb{Z}^{d+1}$ corresponding to rationals in $\mathbb{Q}^d$ in their "lowest terms representation" be denoted by

$$Q = \{(p_1,\dots,p_d,q) \in \mathbb{Z}^{d+1} : \gcd(p_1,\dots,p_d,q) = 1, q > 0\}.$$

Given $x = (p,q) \in Q$, where $p \in \mathbb{Z}^d$, we use the notation

$$|x| = q \quad \text{and} \quad \widehat{x} = \frac{p}{q}.$$

For x in Q, we define the *Farey lattice*

$$\Lambda_x := \mathbb{Z}^d + \mathbb{Z}\widehat{x} = \pi_x(\mathbb{Z}^{d+1})$$

where $\pi_x : \mathbb{R}^{d+1} \to \mathbb{R}^d$ is the "projection along lines parallel to x" given by the formula $\pi_x(m,n) = m - n\widehat{x}$ for $(m,n) \in \mathbb{R}^d \times \mathbb{R}$. (The terminology suggests the idea of the lattice Λ_x as an algebraic embodiment of the multi-dimensional Farey sequence of order $|x|$, which is embedded in Λ_x via the map $m/n \mapsto \pi_x(m,n)$.) Observe that $\operatorname{vol}\Lambda_x = |x|^{-1}$.

Given a norm on $\mathbb{R}^d$, we denote successive minima of Λ_x by $\lambda_i(x)$ and the *normalized* successive minima by

$$\widehat{\lambda}_i(x) := \widehat{\lambda}_i(\Lambda_x) = |x|^{1/d}\lambda_i(x) \quad \text{for} \quad i = 1,\dots,d.$$

4.1. **Inequalities of best approximation.** We gather some well-known lemmas about best simultaneous approximations and, for the convenience of the reader, reproduce their proofs in our notation. For example, the proof of Lemma 4.1 below is taken from [8], while Lemma 4.2 and Lemma 4.3(iii) appear as Theorems 2.11 and 2.15 in [6], respectively.

Recall that the sequence $(q_n)_{n\geq 0}$ of *best simultaneous approximation denominators* of $\theta \in \mathbb{R}^d$ with respect to the norm $\|\cdot\|$ is defined by the recurrence relation

$$q_0 = 1, \quad q_{n+1} = \min\{q \in \mathbb{N} : q > q_n, \operatorname{dist}(q\theta,\mathbb{Z}^d) < \operatorname{dist}(q_n\theta,\mathbb{Z}^d)\}.$$

By definition, the sequence $(q_n)_n$ is strictly increasing, while the sequence $(r_n)_n$ where $r_n = \operatorname{dist}(q_n\theta,\mathbb{Z}^d)$, is strictly decreasing. These sequences are infinite if and only if $\theta \in \mathbb{R}^d \setminus \mathbb{Q}^d$. For each $n \geq 0$, we let $x_n = (p_n,q_n) \in \mathbb{Z}^d \times \mathbb{Z}_{>0}$ where p_n is chosen so that $\|q_n\theta - p_n\| = r_n$. It is customary to refer to $(x_n)_{n\geq 0}$ as *the sequence of best simultaneous approximation vectors*, even though the choice of p_n need not be unique.[3] See [8], [19], [20], [23] for more about best approximations. In what follows we shall often write best approximate instead of best simultaneous approximation vector.

Lemma 4.1. *If $x_n = (p_n, q_n)$ and $x_{n+1} = (p_{n+1}, q_{n+1}) \in \mathbb{Z}^d \times \mathbb{Z}_{>0}$ are two consecutive best approximation vectors of $\theta \in \mathbb{R}^d$, then*

(i) $q_{n+1}\|q_n\theta - p_n\|^d \ll 1$, *and*
(ii) $\frac{1}{2}\lambda_1(x_{n+1}) < \|q_n\theta - p_n\| < 2\lambda_1(x_{n+1})$

where for the second inequality in (ii) we additionally require that $\lambda_1(x_{n+1}) < \lambda_1(\mathbb{Z}^d)$.

Proof. By Minkowski's theorem, $\widehat{\lambda}_1(x_{n+1}) \ll 1$ so that (i) is a consequence of the second inequality in (ii). For the first inequality in (ii), note that $\pi_{x_{n+1}}(x_n)$ is a nonzero element of $\Lambda_{x_{n+1}}$ whose length is $\|p_n - q_n\widehat{x}_{n+1}\| = |x_n| d(\widehat{x}_n, \widehat{x}_{n+1})$. Therefore,

$$\begin{aligned}
\lambda_1(x_{n+1}) &\leq |x_n| d(\widehat{x}_n, \widehat{x}_{n+1}) \\
&\leq |x_n|\|\widehat{x}_n - \theta\| + |x_n|\|\widehat{x}_{n+1} - \theta\| \\
&= \|q_n\theta - p_n\| + \frac{|x_n|}{|x_{n+1}|}\|q_{n+1}\theta - p_{n+1}\| \\
&< 2\|q_n\theta - p_n\|.
\end{aligned}$$

For the second inequality in (ii), since $\lambda_1(x_{n+1}) < \lambda_1(\mathbb{Z}^d)$, the first minimum of the lattice $\Lambda_{x_{n+1}}$ cannot be reached at an integer vector and $\lambda_1(x_{n+1}) = \operatorname{dist}(q\widehat{x}_{n+1}, \mathbb{Z}^d)$ for some integer $q \in [1, q_{n+1} - 1]$. Since $\operatorname{dist}(q\widehat{x}_{n+1}, \mathbb{Z}^d) = \operatorname{dist}((q_{n+1} - q)\widehat{x}_{n+1}, \mathbb{Z}^d)$, we may assume $q \leq \frac{q_{n+1}}{2}$ by replacing q with $q_{n+1} - q$, if necessary. Therefore,

$$\begin{aligned}
\lambda_1(x_{n+1}) &= \operatorname{dist}(q\widehat{x}_{n+1}, \mathbb{Z}^d) = \operatorname{dist}(q\theta + q(\widehat{x}_{n+1} - \theta), \mathbb{Z}^d) \\
&\geq \operatorname{dist}(q\theta, \mathbb{Z}^d) - \operatorname{dist}(q\theta + q(\widehat{x}_{n+1} - \theta), q\theta) \\
&\geq \|q_n\theta - p_n\| - \frac{q}{q_{n+1}}\|q_{n+1}\theta - p_{n+1}\| \\
&> \frac{1}{2}\|q_n\theta - p_n\|.
\end{aligned}$$

□

We note that the condition needed for the second inequality of Lemma 4.1(ii) holds as soon as

$$|x_{n+1}| > \left(\frac{\mu_d}{\lambda_1(\mathbb{Z}^d)}\right)^d$$

where μ_d is the supremum of $\lambda_1(L)$ over all d-dimensional lattices $L \subset \mathbb{R}^d$ of covolume 1.

[3] It is unique as soon as q_n is large enough, e.g. if $q_n > (4\mu_d/\lambda_1(\mathbb{Z}^d))^d$. See [19] or Remark 2.13 of [6].

Lemma 4.2 (Thm. 2.11 of [6])**.** *For $x \in Q$, let $\Delta(x) = \{\theta : \widehat{x}$ is a best approximate of $\theta\}$. If $|x| > \left(\frac{\mu_d}{\lambda_1(\mathbb{Z}^d)}\right)^d$, then*[4]

$$\bar{B}\left(\widehat{x}, \frac{\lambda_1(x)}{2|x|}\right) \subset \Delta(x) \subset B\left(\widehat{x}, \frac{2\lambda_1(x)}{|x|}\right)$$

where $\bar{B}$ denotes the closed ball.

Proof. Let θ be in $\Delta(x)$ and let $x_n = (p_n, q_n)$ be the sequence of best approximation vectors of θ. Then $x = x_n$ for some n. By Lemma 4.1, $\|q_n\theta - p_n\| \leq \|q_{n-1}\theta - p_{n-1}\| < 2\lambda_1(x)$, hence

$$\left\|\theta - \frac{p_n}{q_n}\right\| < \frac{2\lambda_1(x)}{q_n}$$

giving the second inclusion. For the first inclusion, suppose that $\theta \in \bar{B}(\widehat{x}, \frac{\lambda_1(x)}{2|x|})$. For all positive integers $q < |x|$ and all $p \in \mathbb{Z}^d$ we have

$$\|q\theta - p\| + q\|\theta - \widehat{x}\| \geq \|q\widehat{x} - p\| \geq \lambda_1(x)$$

so that

$$\|q\theta - p\| \geq \lambda_1(x) - q\frac{\lambda_1(x)}{2|x|} > \frac{\lambda_1(x)}{2}$$

whereas

$$\|\,|x|\theta - |x|\widehat{x}\| = |x|\,\|\theta - \widehat{x}\| \leq \frac{\lambda_1(x)}{2}.$$

Therefore, by definition, x is a best approximation vector of θ. $\square$

We remark that (iii) of the next Lemma is Theorem 2.15 of [6]. Recall that the notation $x \asymp_2 y$ means $\frac{1}{2}y \leq x \leq 2y$.

Lemma 4.3. *Let $x_n = (p_n, q_n), n \geq 0$ be the sequence of best approximation vectors of $\theta \in \mathbb{R}^d$. Then*

(i) $\|\widehat{x}_n - \widehat{x}_{n+1}\| < \frac{4\lambda_1(x_{n+1})}{|x_n|}$.

(ii) *For all $k \geq 0$,* $\|\widehat{x}_n - \widehat{x}_{n+k}\| < \frac{4\lambda_1(x_n)}{|x_n|}$.

(iii) *For all $y = (p, q) \in \mathbb{Z}^{d+1}$ with $0 < q < |x_n|$,* $\|p - q\theta\| \asymp_2 \|p - q\widehat{x}_n\|$.

Proof. (i) By Lemma 4.1, $\|q_n\theta - p_n\| < 2\lambda_1(x_{n+1})$. Therefore,

$$\begin{aligned}\|q_n\widehat{x}_{n+1} - p_n\| &\leq \|q_n\theta - p_n\| + \|q_n(\widehat{x}_{n+1} - \theta)\| \\ &= \|q_n\theta - p_n\| + \frac{q_n}{q_{n+1}}\|p_{n+1} - q_{n+1}\theta\| \\ &< 2\|q_n\theta - p_n\| < 4\lambda_1(x_{n+1}).\end{aligned}$$

[4] The first inclusion generalizes *Legendre's Theorem: p/q is a convergent of θ if $|\theta - p/q| < 1/2q^2$.*

(ii)

$$\begin{aligned}\|\widehat{x}_n - \widehat{x}_{n+k}\| &\le \|\widehat{x}_n - \theta\| + \|\widehat{x}_{n+k} - \theta\| \\ &= \frac{\|q_n\theta - p_n\|}{q_n} + \frac{\|p_{n+k} - q_{n+k}\theta\|}{q_{n+k}} \\ &\le \frac{2\|q_n\theta - p_n\|}{q_n} < \frac{2\|q_{n-1}\theta - p_{n-1}\|}{q_n} < \frac{4\lambda_1(x_n)}{q_n}.\end{aligned}$$

(iii) Let $(p', q') = x_n - y$. Since

$$q_n\|\widehat{x}_n - \theta\| = \|p_n - q_n\theta\| < \|p' - q'\theta\| \le \|p' - q'\widehat{x}_n\| + q'\|\widehat{x}_n - \theta\|,$$

we have

$$\left(1 - \frac{q'}{q_n}\right)\|p_n - q_n\theta\| < \|p' - q'\widehat{x}_n\| = \|p - q\widehat{x}_n\|,$$

thus

$$\|p_n - q_n\theta\| < \frac{q_n}{q}\|p - q\widehat{x}_n\|.$$

Using once again the triangle inequality, we get

$$\begin{aligned}\|p - q\theta\| &\le \|p - q\widehat{x}_n\| + q\|\widehat{x}_n - \theta\| \\ &< 2\|p - q\widehat{x}_n\|.\end{aligned}$$

The proof of the opposite inequality is straightforward:

$$\begin{aligned}\|p - q\widehat{x}_n\| &\le \|p - q\theta\| + q\|\theta - \widehat{x}_n\| \\ &= \|p - q\theta\| + \frac{q}{q_n}\|q_n\theta - p_n\| \\ &\le \|p - q\theta\| + \frac{q}{q_n}\|q\theta - p\| \\ &< 2\|p - q\theta\|.\end{aligned}$$

□

For $\varepsilon > 0$ and a given norm $\|\cdot\|$ on $\mathbb{R}^d$, we let $\mathrm{DI}_\varepsilon(d)$ denote the set of $\theta \in \mathbb{R}^d$ such that the system of inequalities (compare (1.1))

(1.1')
$$\|q\theta - p\| < \frac{\varepsilon}{T^{1/d}} \quad \text{and} \quad 0 < q < T$$

admits a solution $(p, q) \in \mathbb{Z}^d \times \mathbb{Z}_{>0}$ for T large enough. Observe that this set depends on the norm but that thanks to norm equivalence, the statements in Theorems 1.3 and 1.4 do not depend on the norm.

Observe that the condition $\theta \in \mathrm{DI}_\varepsilon(d)$ can be rewritten in terms of the sequence of best approximates as:

$$q_n^{1/d}\|q_{n-1}\theta - p_{n-1}\| < \varepsilon \quad \text{for all sufficiently large } n. \tag{4.1}$$

Indeed, (4.1) implies that if T is large enough so that $q_{n-1} < T \le q_n$ for some sufficiently large n, then (p_{n-1}, q_{n-1}) solves (1.1'). On the other hand, given any solution (p,q) to (1.1) with $T = q_n$, we deduce $\|q_{n-1}\theta - p_{n-1}\| \le \|q\theta - p\| < \varepsilon/q_n^{1/d}$.

The following *near* characterization of ε-Dirichlet improvable vectors has the advantage that there is no explicit dependence on θ.[5]

Corollary 4.4. *Let (x_n) be the sequence of best approximation vectors of some $\theta \in \mathbb{R}^d$. Then $\theta \in \mathrm{DI}_{\varepsilon/2}(d)$ implies $\widehat{\lambda}_1(x_n) < \varepsilon$ for all large enough n, which in turn implies $\theta \in \mathrm{DI}_{2\varepsilon}(d)$. In particular, $\theta \in \mathrm{Sing}(d)$ if and only if $\widehat{\lambda}_1(x_n) \to 0$ as $n \to \infty$.*

Proof. By Lemma 4.1(ii), for n large enough $\widehat{\lambda}_1(x_n) \asymp_2 q_n^{1/d}\|q_{n-1}\theta - p_{n-1}\|$. □

5. Proofs of Theorems: upper bounds

We want to use self-similar covers and Theorem 3.2 to give upper bounds on Hausdorff dimension. The main difficulty is to define the map σ of the self-similar structure (J, σ, B). We shall use the encoding via best approximates together with an acceleration as suggested in Section 3.1.

5.1. **Acceleration.** For each x in Q we fix once and for all a codimension one sublattice of Λ_x of minimal volume and call it Λ'_x. Let $H_x = \pi_x^{-1} H'_x$ where H'_x is the real span of Λ'_x. Thus,

$$\Lambda'_x = \Lambda_x \cap H'_x.$$

Definition 5.1. Let ε be a positive real number and let x be an element in Q.

(1) $D(x)$ is the set of $y \in Q$ such that $|x| \le |y|, y \in H_x$, and $\|\widehat{x} - \widehat{y}\| \le \frac{4\lambda_1(x)}{|x|}$.
(2) Let y be in $D(x)$. $E(x,y,\varepsilon)$ is the set of $z \in Q$ such that $|y| < |z|$, $z \notin H_x$, and $\|\widehat{y} - \widehat{z}\| < \frac{\varepsilon}{|y||z|^{1/d}}$.
(3) $\sigma_\varepsilon(x) = \sigma(x,\varepsilon) = \bigcup_{y \in D(x)} E(x,y,\varepsilon)$.

Let $Q_\varepsilon = \{x \in Q : \widehat{\lambda}_1(x) < \varepsilon\}$, $B(x) = B(\widehat{x}, \frac{2\mu_d}{|x|^{1+1/d}})$ where μ_d is the supremum of $\lambda_1(L)$ over all d-dimensional lattices $L \subset \mathbb{R}^d$ of covolume 1, and let

$$\widetilde{\mathrm{DI}}_\varepsilon(d) = \{\theta \in \mathbb{R}^d : \text{the best approximates } x_n \text{ of } \theta \text{ are in } Q_\varepsilon \text{ for } n \text{ large enough}\}$$

and

$$\widetilde{\mathrm{DI}}^*_\varepsilon(d) = \{\theta \in \widetilde{\mathrm{DI}}_\varepsilon(d) : 1, \theta_1, \ldots, \theta_d \text{ are linearly independent over } \mathbb{Q}\}.$$

From Corollary 4.4, it is clear that

$$\widetilde{\mathrm{DI}}_{\varepsilon/2}(d) \subset \mathrm{DI}_\varepsilon(d) \subset \widetilde{\mathrm{DI}}_{2\varepsilon}(d).$$

[5] The characterization given here differs slightly from the one in [6].

So $\mathrm{Sing}(d) = \cap_{\varepsilon>0}\widetilde{\mathrm{DI}}_\varepsilon(d)$ and we can deal with $\widetilde{\mathrm{DI}}_\varepsilon(d)$ instead of $\mathrm{DI}_\varepsilon(d)$.

Lemma 5.2. *$(Q_\varepsilon, \sigma_\varepsilon, B)$ is a self-similar covering of $\widetilde{\mathrm{DI}}^*_{\varepsilon/4}(d)$.*

Proof. For any best approximate x of θ, Lemma 4.2 implies $\Delta(x) \subset B(x, \frac{2\lambda_1(x)}{|x|}) \subset B(x)$ so that $\theta \in B(x)$. We need to verify that for any $\theta \in \widetilde{\mathrm{DI}}^*_{\varepsilon/4}(d)$, the sequence of best approximates (x_n) contains a σ_ε-admissible subsequence. By definition, there exists an n_0 such that $x_n \in Q_{\varepsilon/4}$ for all $n > n_0$. Given n_i, let n_{i+1} be the smallest integer $m > n_i$ such that $x_m \notin H_{x_{n_i}}$. Since $1, \theta_1, \ldots, \theta_d$ are linearly independent over $\mathbb{Q}$, n_i is defined for all $i \in \mathbb{Z}_{\geq 0}$. Now suppose $x = x_{n_i}$ for some i. Let $y = x_{n_{i+1}-1}$ and $z = x_{n_{i+1}}$. Then $|x| \leq |y| < |z|$, $y \in H_x$, $z \notin H_x$, and by Lemma 4.3, $\|\widehat{x} - \widehat{y}\| < \frac{4\lambda_1(x)}{|x|}$ and $\|\widehat{y} - \widehat{z}\| < \frac{4\lambda_1(z)}{|y|} = \frac{4\widehat{\lambda}_1(z)}{|y||z|^{1/d}} < \frac{\varepsilon}{|y||z|^{1/d}}$. □

Remark 5.3. It could seem that $B'(x) = B(\widehat{x}, \frac{2\lambda_1(x)}{|x|})$ is a better choice for the map B than $B(x) = B(\widehat{x}, \frac{2\mu_d}{|x|^{1+1/d}})$ because the balls $B'(x)$ are included in the balls $B(x)$. However, the exponent $s(Q_\varepsilon, \sigma_\varepsilon, B')$ is more difficult to estimate and might be greater than $s(Q_\varepsilon, \sigma_\varepsilon, B)$. Indeed, $\widehat{\lambda}_1(x)/\varepsilon$ small does not imply $\widehat{\lambda}_1(z)/\varepsilon$ small for z in $\sigma_\varepsilon(x)$. The only constraint on z is $\widehat{\lambda}_1(z) < \varepsilon$. Therefore, for some x in Q_ε, the optimal value of the exponent s may increase when using B' instead of B.

5.2. **Geometry associated with the subspace H_x.** We prove some simple facts that will be used both in the proof of the upper bounds and in the proof the lower bounds.

For $x \in Q$, we shall simply write e_x for $e(\Lambda_x)$ (see Section 2.2).

Lemma 5.4. *For all $y \in D(x)$, $e_y \leq e_x + 4\lambda_1(x)$.*

Proof. Assume $r := e_y - 4\lambda_1(x) > 0$ for otherwise the result is clear. Let p be a point such that $B(p, e_y) \cap \Lambda_y = \varnothing$. It is enough to prove that $B(p, r) \cap \Lambda_x = \varnothing$. Suppose not. Then there exists $(a, b) \in \mathbb{Z} \times \mathbb{Z}^d$ with $0 \leq a < |x|$ such that $d(p, a\widehat{x} + b) < r$. But then $d(a\widehat{x} + b, a\widehat{y} + b) = a\|\widehat{x} - \widehat{y}\| < 4\lambda_1(x)$ so that $a\widehat{y} + b \in B(p, e_y) \cap \Lambda_y$. □

Lemma 5.5. *Let x and y be in Q. Then $y \in H_x$ if and only if $\widehat{y} \in \widehat{x} + H'_x$.*

Proof. Note that $y \in H_x$ if and only if $\pi_x(y) \in \Lambda'_x$ and that $\widehat{y} - \widehat{x} = |y|^{-1}\pi_x(y)$. □

Lemma 5.6. *Let $y \in D(x)$ and $\alpha \in \Lambda_y \setminus H'_x$. Then $\lambda_d(x) \asymp \lambda_d(y) \ll \|\alpha\|$.*

Proof. Since $\widehat{y} - \widehat{x} \in H'_x$, we have $\Lambda_y \subset \Lambda_x + \mathbb{R}(\widehat{y} - \widehat{x}) \subset \Lambda_x + H'_x$. Hence, by Lemma 2.2, $\|\alpha\| \gg e(\Lambda_x + H'_x) \asymp \lambda_d(x)$. Now Lemma 5.4 implies $e_y \ll e_x \ll \lambda_d(x)$, whereas $e(\Lambda_y) \geq e(\Lambda_x + H'_x) \gg \lambda_d(x)$. And since $e_y \asymp \lambda_d(y)$, by Lemma 2.1, it follows that $\lambda_d(x) \asymp \lambda_d(y)$. □

5.3. **Upper bound calculation and results.** Thanks to Theorem 3.2 and Lemma 5.2, the proofs of the upper estimates on Hausdorff dimension reduce to upper bounds on the sum

$$\sum_{z \in \sigma_\varepsilon(x)} \frac{(\operatorname{diam} B(z))^s}{(\operatorname{diam} B(x))^s}.$$

We split this sum in two sums, $\sum_{y\in D(x)}$ and $\sum_{z\in E(x,y,\varepsilon)}$. The first Lemma below deals with the first sum while the second Lemma deals with second sum. Ultimately, the second sum is further split due to a regrouping of its terms according to their projection $\alpha = \pi_y(z)$.

Lemma 5.7. *For $t > d$ we have*

$$\sum_{y\in D(x)} \left(\frac{|x|}{|y|}\right)^t \ll \frac{1}{t-d}.$$

Proof. For each $k > 0$, let $D_k := \{y \in D(x) : k|x| \le |y| < (k+1)|x|\}$. Since

$$\sum_{y\in D(x)} \left(\frac{|x|}{|y|}\right)^t = \sum_{k=1}^{\infty}\sum_{y\in D_k} \left(\frac{|x|}{|y|}\right)^t \le \sum_{k=1}^{\infty} \frac{\operatorname{card} D_k}{k^t}$$

it is enough to estimate $\operatorname{card} D_k$. For all $y = (p, q) \in D_k$,

$$\|\pi_x(y)\| = \|p - q\widehat{x}\| = q\|\widehat{y} - \widehat{x}\| \le q\frac{4\lambda_1(x)}{|x|} \le 8k\lambda_1(x).$$

Since H_x is a hyperplane which contains x, its projection is a subspace of $\mathbb{R}^d$ of dimension $d-1$. Since x is a primitive element of $\mathbb{Z}^{d+1}$, the projection π_x induces a bijection from $\{(u, v) \in \mathbb{Z}^{d+1} : k|x| \le v < (k+1)|x|\}$ onto Λ_x. It follows that π_x induces a bijection from $\{(u, v) \in H_x \cap \mathbb{Z}^{d+1} : k|x| \le v < (k+1)|x|\}$ onto $\Lambda'_x = \Lambda_x \cap \pi_x(H_x)$. So we have to count the number of elements of the lattice Λ'_x inside the ball $B(0, 8k\lambda_1(x))$.

Since $\operatorname{rank} \Lambda'_x = d - 1$, making use of Lemma 2.3, we find that

$$\begin{aligned}\operatorname{card} D_k &\ll 1 + \sum_{i=1}^{d-1} \frac{(k\lambda_1(x))^i}{\lambda_1(\Lambda'_x)\dots\lambda_i(\Lambda'_x)}\\ &\ll 1 + \sum_{i=1}^{d-1} \frac{(k\lambda_1(x))^i}{\lambda_1(x)\dots\lambda_i(x)} \ll k^{d-1}\end{aligned}$$

because $\Lambda'_x \subset \Lambda_x$ implies $\lambda_i(\Lambda'_x) \ge \lambda_i(x)$ for $i = 1, \dots, d-1$. □

Lemma 5.8. *For $t > d$ we have*

$$\sum_{z\in E(x,y,\varepsilon)} \left(\frac{|y|}{|z|}\right)^t \ll \frac{\varepsilon^t}{t-d}.$$

Proof. Note that $E(x, y, \varepsilon)$ is the set of $z \in Q$ such that $\alpha := \pi_y(z) \in \Lambda_y \setminus H'_x$, and $\|\alpha\||y| < \varepsilon|z|^{1-1/d}$. Hence, we have a double sum of the form $\sum_\alpha \sum_z$ where α ranges over the set $\Lambda_y \setminus H'_x$ and z ranges over the subset of $\pi_y^{-1}\alpha$ satisfying

$$\frac{|z|}{|y|} > \frac{1}{|y|}\left(\frac{\|\alpha\||y|}{\varepsilon}\right)^{d/(d-1)} = \left(\frac{\|\alpha\||y|^{1/d}}{\varepsilon}\right)^{d/(d-1)}$$

Since $z \in \pi_y^{-1}\alpha$ is uniquely determined by the integer part of $|z|/|y|$, we have

$$\sum_{z\in E(x,y,\varepsilon)} \left(\frac{|y|}{|z|}\right)^t \asymp \sum_{\alpha\in\Lambda_y\setminus H'_x} \sum_{k>k_\alpha} \frac{1}{k^t} \ll \sum_{\alpha\in\Lambda_y\setminus H'_x} \left(\frac{\varepsilon}{\|\alpha\||y|^{1/d}}\right)^{t'}$$

where $k_\alpha = \left(\frac{\|\alpha\||y|^{1/d}}{\varepsilon}\right)^{\frac{d}{d-1}}$ and $t' = \frac{t-1}{d-1}d > d$. Since $\alpha \notin H'_x$, Lemma 5.6 implies

$$\|\alpha\| \gg \lambda_d(y) \asymp \lambda_d(x) \gg |x|^{-1/d}.$$

Applying Lemma 2.4 with the lattice Λ_y, the exponent t', and $R = \inf\{\|\alpha\| : \alpha \in \Lambda_y \setminus H'_x\}$, we obtain

$$\sum_{z\in E(x,y,\varepsilon)} \left(\frac{|y|}{|z|}\right)^t \ll \frac{\varepsilon^{t'}\lambda_d(y)^{(d-t')}}{(t'-d)\operatorname{vol}(\Lambda_y)|y|^{t'/d}} \leq \frac{\varepsilon^t}{(t-d)}\left(\frac{|x|}{|y|}\right)^{(t'-d)/d} \leq \frac{\varepsilon^t}{(t-d)}.$$

□

Corollary 5.9. $\operatorname{Hdim}\mathrm{DI}_\varepsilon(d) \leq \frac{d^2}{d+1} + O(\varepsilon^{d/2})$.

Proof. Applying the preceding two lemmas with $t = (1+1/d)s$ we have for some $C > 0$

$$\sum_{z\in\sigma(x,\varepsilon)} \left(\frac{|x|}{|z|}\right)^{(1+1/d)s} \leq \sum_{y\in D(x)} \left(\frac{|x|}{|y|}\right)^t \sum_{z\in E(x,y,\varepsilon)} \left(\frac{|y|}{|z|}\right)^t \leq \frac{C\varepsilon^t}{(t-d)^2} \tag{5.1}$$

which is ≤ 1 provided $t > d + \sqrt{C}\varepsilon^{t/2}$, which holds for any $s > \frac{d^2}{d+1} + \frac{d+1}{d}\sqrt{C}\varepsilon^{d/2}$. Hence, by Lemma 5.2 and Theorem 3.2, $\operatorname{Hdim}\widetilde{\mathrm{DI}}^*_{\varepsilon/4}(d) \leq \frac{d^2}{d+1} + O(\varepsilon^{d/2})$ and, since this upper bound exceeds $d-1$, the same holds for $\operatorname{Hdim}\widetilde{\mathrm{DI}}_{\varepsilon/4}(d)$, as well as $\operatorname{Hdim}\mathrm{DI}_\varepsilon(d)$, by Corollary 4.4. □

Corollary 5.10. $\operatorname{Hdim}\mathrm{Sing}(d) \leq \frac{d^2}{d+1}$.

6. Geometry of quotient lattices

Given $y \in Q$, we wish to define a set of children of y whose elements z are best approximation vectors that follow y directly. Actually, we focus on one property of consecutive best approximation vectors: if y and z are consecutive best approximation vectors then $\lambda_1(z)$ is given by y up to a multiplicative factor 4 (see Lemma 4.3 (i)). The process for finding such z is one of the main ideas in the proof of the lower bound and is described below. Lemma 6.1 gives a sufficient condition ensuring that y and z are "consecutive".

For the remainder of the paper, we assume the Euclidean norm on $\mathbb{R}^d$. Given $y \in Q$ and a primitive element α in Λ_y, we let

$$\Lambda_{\alpha^\perp} = \pi_\alpha^\perp(\Lambda_y)$$

where $\pi_\alpha^\perp$ is the orthogonal projection of $\mathbb{R}^d$ onto $\alpha^\perp$, the subspace of vectors of $\mathbb{R}^d$ orthogonal to α.

For any $z \in Q$ such that $\pi_y(z) = \alpha$, the $(d-1)$-volume of $\Lambda_{\alpha^\perp}$ satisfies

$$\mathrm{vol}(\Lambda_{\alpha^\perp}) = \frac{\mathrm{vol}(\Lambda_y)}{\|\alpha\|} = \frac{1}{\|\alpha\||y|} = \frac{1}{|y \wedge z|}.$$

Here, the quantity $|y \wedge z|$ is the 2-volume of the orthogonal projection of $y \wedge z \in \Lambda^2\mathbb{R}^{d+1}$ onto the subspace spanned by $e_1 \wedge e_{d+1}, \ldots, e_d \wedge e_{d+1}$. Equivalently, (see §2 of [5])

$$|y \wedge z| = |y||z|d(\widehat{y}, \widehat{z}).$$

Denote the first successive minimum of $\Lambda_{\alpha^\perp}$ by $\lambda_1(\alpha)$ and its normalized version by

$$\widehat{\lambda}_1(\alpha) := \mathrm{vol}(\Lambda_{\alpha^\perp})^{-1/(d-1)}\lambda_1(\alpha) = |y \wedge z|^{1/(d-1)}\lambda_1(\alpha).$$

Note that for any $z \in Q$, we have $\pi_z(y) \in \Lambda_z$ and $\|\pi_z(y)\| = \frac{|y\wedge z|}{|z|}$. Thus, $\lambda_1(z) \leq \frac{|y\wedge z|}{|z|}$ provided y is in Q and $y \neq z$. Many of the estimates we need rely on a lower bound for the quantity $\lambda_1(z)$, which is the main point of the next lemma.

Lemma 6.1. *Let $y \in Q$ and let α be a primitive element of Λ_y. Suppose z is an element in Q such that $\pi_y(z) = \alpha$. Then $\frac{|y\wedge z|}{|z|} \leq \lambda_1(\alpha)$ implies $\lambda_1(z) = \frac{|y\wedge z|}{|z|}$.*

Proof. Note that since α is a primitive element of Λ_y, $y \wedge z$ is a primitive element of the lattice $\bigwedge^2 \mathbb{Z}^{d+1}$ (i.e., $y \wedge z$ is not of the form ku with an integer $k > 1$ and $u \in \bigwedge^2 \mathbb{Z}^{d+1}$), which in turn implies that $\beta = \pi_z(y)$ is a primitive element in Λ_z. Let $\Lambda_{\beta^\perp} = \pi_\beta^\perp(\Lambda_z)$ be the orthogonal projection onto $\beta^\perp$. Note that α and β are proportional to $\widehat{z} - \widehat{y}$ so that $\pi_\alpha^\perp(k - l\widehat{y}) = \pi_\beta^\perp(k - l\widehat{y}) = \pi_\beta^\perp(k - l\widehat{z})$ for all $(k, l) \in \mathbb{Z}^{d+1}$. Hence, $\Lambda_{\alpha^\perp} = \Lambda_{\beta^\perp}$.

Now suppose $\frac{|y\wedge z|}{|z|} \leq \lambda_1(\alpha)$ and consider a nonzero $\gamma \in \Lambda_z$. If γ is not a real multiple of β, then $\|\gamma\| \geq \lambda_1(\Lambda_{\beta^\perp}) = \lambda_1(\alpha) \geq \frac{|y\wedge z|}{|z|} = \|\beta\|$. On the other hand, if $\gamma = t\beta$ for some $t \in \mathbb{R}$ then since β is primitive, $0 \neq t \in \mathbb{Z}$ so that $\|\gamma\| = |t|\|\beta\| \geq \|\beta\|$. In either case $\|\gamma\| \geq \|\beta\|$. Therefore, $\lambda_1(z) = \|\beta\| = \frac{|y\wedge z|}{|z|}$. □

Definition 6.2. Given a positive real number ε and y in Q, let

$$\Lambda_y(\varepsilon) := \{\alpha \in \Lambda_y : \alpha \textit{ is primitive and } \widehat{\lambda}_1(\alpha) > \varepsilon\}.$$

The sets $\Lambda_y(\varepsilon)$ will be involved in the definition of the self-similar structure we construct for the lower bound argument. Thanks to Lemma 6.1, we can construct $z \in Q$ such that the first minimum of Λ_z is given by y (see Lemma 6.4). In Section 8 we study the distribution of $\Lambda_y(\varepsilon)$ inside Λ_y.

6.1. **Self-similar structure for Dirichlet improvable set.** Recall the subspace H'_x defined as the real span of Λ'_x, which is a distinguished codimension one sublattice of Λ_x of minimal volume. By a *primitive* coset of H'_x we shall mean a coset H' such that $\Lambda_x \cap H' \neq \varnothing$ and $\Lambda_x \setminus H'_x \subset \bigcup_{n\in\mathbb{Z}} nH'$. There are exactly two possibilities for H' and we fix a choice once and for all. The unique element of H' that is perpendicular to H'_x will be denoted by $\alpha_x^\perp$. Note that, from the proof of Lemma 2.2, we have

$$\lambda_d(x) \asymp \|\alpha_x^\perp\| = \frac{\mathrm{vol}(\Lambda_x)}{\mathrm{vol}(\Lambda'_x)}.$$

The coset $n\alpha_x^\perp + H'_x$ will be denoted by $H'_x(n)$. Since, by Lemma 5.5, $\Lambda_y + H'_x = \Lambda_x + H'_x$ for any $y \in H_x \cap Q$, we have for all $y \in D(x)$

$$\Lambda_y \setminus H'_x \subset \bigcup_{n\in\mathbb{Z},\, n\neq 0} H'_x(n).$$

Let $\mathcal{C}(x)$ be the set of vectors in $\mathbb{R}^d$ whose angle with $\alpha_x^\perp$ is at most $\tan^{-1} A_d$, where the constant A_d, which depends only on the dimension, will be determined later in the proof of Lemma 8.6. Let

$$\mathcal{C}'_n(x) = \mathcal{C}(x) \cap H'_x(n) \quad \text{ and } \quad \mathcal{C}_N(x) = \bigcup_{n=1}^{N} \mathcal{C}'_n(x).$$

Definition 6.3. Let N be a positive integer, ε be a positive real number, x in Q and y in $D(x)$. $F(x, y, \varepsilon)$ is the set of $z \in Q$ such that $\pi_y(z) \in \mathcal{C}(x) \cap \Lambda_y(\varepsilon)$ and $|z| \in \frac{|y\wedge z|^{d/(d-1)}}{\varepsilon^{d/(d-1)}}]1, 2[$. The subset of $F(x, y, \varepsilon)$ consisting of $z \in Q$ such that $\pi_y(z) \in \mathcal{C}'_n(x)$ for some $n \in \{1, \ldots, N\}$ will be denoted by $F_N(x, y, \varepsilon)$. Equivalently,

$$F_N(x, y, \varepsilon) = \bigcup_{\alpha\in\Lambda_y(\varepsilon)\cap\mathcal{C}_N(x)} \zeta(y, \alpha, \varepsilon)$$

where $\zeta(y, \alpha, \varepsilon) = \left\{ z \in Q : \pi_y(z) = \alpha, |z| \in \frac{|y\wedge z|^{d/(d-1)}}{\varepsilon^{d/(d-1)}}]1, 2[\right\}$.

The sets $\sigma(x) = F_N(x, x, \varepsilon)$ provide the main example of a self-similar structure for the lower bound argument, enough for the proofs of Theorems 1.1 and 1.3. Only the proof of Theorem 1.4 will require $y \neq x$. It is worth noting that in the proofs of Theorems 1.1 and 1.3, the truncated cone $\mathcal{C}_N(x)$ could have been replaced by a ball of radius $N\lambda_d(x)$ from which H'_x is removed. We shall use the inequality $\|\alpha\| \gg \lambda_d(y)$ many times; it holds for all $\alpha \in \Lambda_y \setminus H_x$, by Lemma 5.6. The cone $\mathcal{C}_N(x)$ is only used in the proof of Theorem 1.4. (See the proof of Lemma 7.8.) It is also important to notice that the results of Section 8 do not depend on the results of the previous sections but Section 8 is used in Sections 6 and 7. So we first choose the constant A_d in order that the conclusion of Lemma 8.6 holds, then ε small enough is chosen and finally N large enough is chosen.

We need to verify that the set covered by this self-similar structure is contained in the Dirichlet set (see Proposition 6.6 below). For that, we first need a couple of lemmas.

Lemma 6.4. *Let* $0 < \varepsilon \leq 1$ *and* $z \in F(x, y, \varepsilon)$. *Then* $\lambda_1(z) = \frac{|y\wedge z|}{|z|}$ *and* $\frac{\varepsilon}{2} < \widehat{\lambda}_1(z) < \varepsilon$. *Moreover* $|z| > |y|$ *when* ε *is small enough.*

Proof. Let $\alpha = \pi_y(z)$. Since $|z| \geq \frac{|y\wedge z|^{d/(d-1)}}{\varepsilon^{d/(d-1)}}$ and $\widehat{\lambda}_1(\alpha) = |y \wedge z|^{1/(d-1)}\lambda_1(\alpha) > \varepsilon$,

$$\frac{|y \wedge z|}{|z|} \leq \frac{\varepsilon^{d/(d-1)}}{|y \wedge z|^{1/(d-1)}} < \varepsilon^{1/(d-1)}\lambda_1(\alpha). \tag{6.1}$$

Since α is primitive and $\varepsilon \leq 1$, it follows from Lemma 6.1 that $\lambda_1(z) = \frac{|y\wedge z|}{|z|}$. Therefore, $\widehat{\lambda}_1(z) = |z|^{1/d}\lambda_1(z) = \frac{|y\wedge z|}{|z|^{1-1/d}} \in \left]\frac{\varepsilon}{2^{1-1/d}}, \varepsilon\right[\subset \left]\frac{\varepsilon}{2}, \varepsilon\right[$.

By Lemma 5.6, $\|\alpha\| \gg \lambda_d(y)$. Hence

$$\frac{|y|}{|z|} \leq \frac{|y|\varepsilon^{\frac{d}{d-1}}}{(|y|\|\alpha\|)^{\frac{d}{d-1}}} \ll \frac{\varepsilon^{\frac{d}{d-1}}}{(|y|^{\frac{1}{d}}\lambda_d(y))^{\frac{d}{d-1}}} = \frac{\varepsilon^{\frac{d}{d-1}}}{\widehat{\lambda}_d(y)^{\frac{d}{d-1}}} \ll \varepsilon^{\frac{d}{d-1}}, \tag{6.2}$$

and therefore $|z| > |y|$ when ε is small enough. □

6.2. **Nestedness and containment.** The next Lemma will be used in the proofs of Propositions 6.6 and 6.13, and Lemma 7.4. It involves a constant C_d depending on the dimension. In the proof of Propositions 6.6 and 6.13, it is enough to use the Lemma with $C_d = 1$ while in the proof of Lemma 7.4, C_d must be the constant involved in Lemma 7.2.

Lemma 6.5. *Let ε be a positive real number and let $x \in Q$ and $y \in D(x)$ satisfy $\widehat{\lambda}_1(y) \leq C_d\varepsilon$. Then for all $z \in F(x, y, \varepsilon)$, the ball $B(\widehat{z}, \frac{2\lambda_1(z)}{|z|})$ is contained in the ball $B(\widehat{y}, \frac{\lambda_1(y)}{2|y|})$ when ε is small enough.*

Proof. Let $\alpha = \pi_y(z)$.

Step 1. $\|\alpha\| > \lambda_1(y)$ when ε is small enough.
By Minkowski's theorem, $1 \ll \widehat{\lambda}_1(y)\widehat{\lambda}_d(y)^{d-1}$ and since $\widehat{\lambda}_1(y) \ll \varepsilon$, we have

$$\widehat{\lambda}_d(y) \gg \frac{1}{\widehat{\lambda}_1(y)^{\frac{1}{d-1}}} > \varepsilon^{-\frac{1}{d-1}} \gg \varepsilon^{-\frac{d}{d-1}}\widehat{\lambda}_1(y).$$

Since $\alpha \notin H'_x$, it follows from Lemma 5.6 that $\|\alpha\| \gg \lambda_d(y) \gg \varepsilon^{-\frac{d}{d-1}}\lambda_1(y)$ so that $\|\alpha\| > \lambda_1(y)$ when ε is small enough.

Step 2. $\lambda_1(\alpha) \leq \lambda_1(y)$.
Let $\beta \in \Lambda_y$ be such that $\|\beta\| = \lambda_1(y)$. Then β is a primitive element whose length is strictly smaller than that of α, by *Step 1*. Hence, it cannot be an integer multiple of α. Thus, $\pi_\alpha(\beta)$ is a nonzero vector in $\Lambda_{\alpha^\perp}$, whose length is bounded above by $\lambda_1(y)$.

Step 3. Estimate $d(\widehat{y}, \widehat{z})$ in terms of $\frac{\lambda_1(y)}{|y|}$.
From (6.1) in the proof of Lemma 6.4 together with *Step 2*, we have

$$d(\widehat{y}, \widehat{z}) = \frac{|y \wedge z|}{|y||z|} < \varepsilon^{1/(d-1)}\frac{\lambda_1(\alpha)}{|y|} \leq \varepsilon^{1/(d-1)}\frac{\lambda_1(y)}{|y|}.$$

Step 4. Estimate $\frac{\lambda_1(z)}{|z|}$ *in terms of* $\frac{\lambda_1(y)}{|y|}$.
Applying Lemma 6.4 together with *Step 3*, we have

$$\frac{\lambda_1(z)}{|z|} = \frac{|y \wedge z|}{|z|^2} = \frac{|y|}{|z|} d(\widehat{y}, \widehat{z}) < \varepsilon^{1/(d-1)} \frac{\lambda_1(y)}{|z|}.$$

Since $|z| \geq \frac{|y\wedge z|^{d/(d-1)}}{\varepsilon^{d/(d-1)}}$, we get

$$\frac{\lambda_1(z)}{|z|} < \frac{\varepsilon^{\frac{d+1}{d-1}} \lambda_1(y)}{|y \wedge z|^{d/(d-1)}}.$$

Since $\alpha \notin H'_x$, we have $|y \wedge z| = \|\alpha\||y| \gg \lambda_d(y)|y| = \widehat{\lambda}_d(y)|y|^{1-1/d} \gg |y|^{(d-1)/d}$ so that

$$\frac{\lambda_1(z)}{|z|} \ll \varepsilon^{\frac{d+1}{d-1}} \frac{\lambda_1(y)}{|y|}.$$

Step 5. $B(\widehat{z}, \frac{2\lambda_1(z)}{|z|}) \subset B(\widehat{y}, \frac{\lambda_1(y)}{2|y|})$ *when* ε *is small enough.*
This is a consequence of the preceding two steps by the triangle inequality. □

Proposition 6.6. *Let* $\sigma_{\varepsilon,N}(x) = F_N(x, x, \varepsilon)$, $B(x) = B(\widehat{x}, \frac{\lambda_1(x)}{2|x|})$ *and* $Q_{\varepsilon,N} = \bigcup_{x\in Q} \sigma_{\varepsilon,N}(x)$. *Then* $(Q_{\varepsilon,N}, \sigma_{\varepsilon,N}, B)$ *is a strictly nested self-similar structure (see Definitions 3.1 and 3.5) covering a subset of* $\mathrm{DI}_{2\varepsilon}(d)$ *provided* ε *is small enough.*

Proof. We need to verify that for each $\sigma_{\varepsilon,N}$-admissible sequence (x_k),

(i) for all k, $B(x_{k+1}) \subset B(x_k)$ and $\operatorname{diam} B(x_{k+1}) < \operatorname{diam} B(x_k)$,
(ii) $\lim_{k\to\infty} \operatorname{diam} B(x_k) = 0$ and $\bigcap B(x_k)$ is a point contained in $\mathrm{DI}_{2\varepsilon}(d)$.

Note that Lemma 6.5 implies (i) when ε is small enough. Moreover, by Lemma 6.4, $|x_k| \to \infty$, hence $\lim_{k\to\infty} \operatorname{diam} B(x_k) = 0$ and $\bigcap_k B(x_k)$ is a single point θ. Since $\theta \in B(x_k)$ which is included in $\Delta(x_k)$ by Lemma 4.3, each element x_k of the $\sigma_{\varepsilon,N}$-admissible sequence is a best approximate to θ. It remains to show that $\theta \in \mathrm{DI}_{2\varepsilon}(d)$.

Let $|x_k| \leq q < |x_{k+1}|$. By Lemma 4.3 (iii), we have

$$\begin{aligned}\min_{0<l\leq q} d(l\theta, \mathbb{Z}^d) &\leq \|q_k\theta - p_k\| \\ &< 2\|q_k\widehat{x}_{k+1} - p_k\|.\end{aligned}$$

Observe that the latter inequality only uses the fact that x_{k+1} is a best approximation vector. On the other hand, by definition of $F_N(x_k, x_k, \varepsilon)$, $|x_{k+1}| > (|x_k \wedge x_{k+1}|/\varepsilon)^{d/(d-1)}$, so that

$$\|q_k\widehat{x}_{k+1} - p_k\| = \frac{|x_k \wedge x_{k+1}|}{|x_{k+1}|} < \varepsilon|x_{k+1}|^{-1/d}.$$

Therefore,

$$\min_{0<l\leq q} d(l\theta, \mathbb{Z}^d) < 2\varepsilon q^{-1/d}.$$

□

Remark 6.7. We note that Proposition 6.6 remains valid if we make $B(x)$ closed balls so that $B(x)$ is compact as required by the assumptions of Theorems 3.4 and 3.6.

6.3. **Spacing and local finiteness.** In order to estimate the distance between two balls $B(z, \frac{\lambda_1(z)}{2|z|})$ and $B(z', \frac{\lambda_1(z')}{2|z'|})$ where z and z' are in Q, we compare the distance between the points $\widehat{z}$ and $\widehat{z'}$ with the diameters of the balls. So we introduce the ratio

$$R(z,z') = \frac{d(\widehat{z}, \widehat{z'})}{\frac{\lambda_1(z)}{|z|} + \frac{\lambda_1(z')}{|z'|}}.$$

Lemma 6.8. *Let $0 < \varepsilon \leq 1$ and $y \in D(x)$. For z and z' in $F_N(x, y, \varepsilon)$, let B and B' denote the balls centered at $\widehat{z}$ and $\widehat{z'}$ with radii $r = \frac{\lambda_1(z)}{2|z|}$ and $r' = \frac{\lambda_1(z')}{2|z'|}$, respectively. Then for all z and z' in $F_N(x, y, \varepsilon)$, $R(z, z') > 1$ implies*

$$d(B, B') \gg \left(\frac{1}{N}\right)^{\frac{d+1}{d-1}} \frac{\varepsilon^{\frac{2d}{d-1}}}{\widehat{\lambda}_d(y)^{\frac{d}{d-1}}} \frac{\lambda_1(y)}{|y|}.$$

Proof. Let z, z' be in $F(x, y, \varepsilon)$ and $\alpha = \pi_y(z)$. By definition of $R(z, z')$,

$$d(B, B') = d(\widehat{z}, \widehat{z'}) - r - r' = (2R(z, z') - 1)\,(r + r').$$

Hence, $R(z, z') > 1$ implies

$$d(B, B') > r + r' \gg \frac{\lambda_1(z)}{|z|}$$

and it is enough to bound the ratio $(\frac{\lambda_1(z)}{|z|})/(\frac{\lambda_1(y)}{|y|})$ from below.

Applying Lemma 6.4 together with $|z| \leq 2\frac{|y \wedge z|^{d/(d-1)}}{\varepsilon^{d/(d-1)}}$ we have

$$\frac{\lambda_1(z)}{|z|} = \frac{|y \wedge z|}{|z|^2} \geq \frac{\varepsilon^{\frac{2d}{d-1}}}{4|y \wedge z|^{\frac{d+1}{d-1}}} = \frac{\varepsilon^{\frac{2d}{d-1}}}{4\|\alpha\|^{\frac{d+1}{d-1}}|y|^{\frac{d+1}{d-1}}}$$

so that

$$\begin{aligned}
\frac{\lambda_1(z)}{|z|}\frac{|y|}{\lambda_1(y)} &\gg \frac{\varepsilon^{\frac{2d}{d-1}}}{\|\alpha\|^{\frac{d+1}{d-1}}} \frac{1}{|y|^{\frac{d+1}{d-1}}} \times \frac{|y|}{\widehat{\lambda}_1(y)|y|^{-\frac{1}{d}}} \\
&= \frac{\varepsilon^{\frac{2d}{d-1}}}{\|\alpha\|^{\frac{d+1}{d-1}}} \frac{1}{|y|^{\frac{d+1}{d(d-1)}}} \frac{1}{\widehat{\lambda}_1(y)} \\
&= \left(\frac{\lambda_d(y)}{\|\alpha\|}\right)^{\frac{d+1}{d-1}} \frac{\varepsilon^{\frac{2d}{d-1}}}{\widehat{\lambda}_1(y)} \frac{1}{(\lambda_d(y)|y|^{\frac{1}{d}})^{\frac{d+1}{d-1}}}.
\end{aligned}$$

By Minkowski's theorem, $\widehat{\lambda}_1(y)^{d-1}\widehat{\lambda}_d(y) \ll 1$, hence

$$\frac{\lambda_1(z)}{|z|}\frac{|y|}{\lambda_1(y)} \gg \left(\frac{\lambda_d(y)}{\|\alpha\|}\right)^{\frac{d+1}{d-1}} \frac{\varepsilon^{\frac{2d}{d-1}}}{\widehat{\lambda}_d(y)^{\frac{d}{d-1}}}.$$

Since the angle α makes with the normal to H'_x is $\ll 1$, we have $\|\alpha\| \ll N\lambda_d(y)$ so that

$$\frac{\lambda_1(z)}{|z|}\frac{|y|}{\lambda_1(y)} \gg \left(\frac{1}{N}\right)^{\frac{d+1}{d-1}} \frac{\varepsilon^{\frac{2d}{d-1}}}{\widehat{\lambda}_d(y)^{\frac{d}{d-1}}}.$$

□

Lemma 6.9. *Let ε be a positive real number and $y \in D(x)$. For all $z \in F(x,y,\varepsilon)$, the number of $z' \in F(x,y,\varepsilon)$ with $R(z,z') \leq 1$ is $\ll 1$ when ε is small enough.*

Proof. Let $\alpha = \pi_y(z)$ and $\alpha' = \pi_y(z')$ be the projections of z and z' in $F(x,y,\varepsilon)$. We first show that if $\frac{\|\alpha'\|}{\|\alpha\|} \geq 4^{d-1}$ then $R(z,z') > 1$. Observe that it implies that $R(z,z') > 1$ when $\frac{\|\alpha\|}{\|\alpha'\|} \geq 4^{d-1}$. Since $|z| \in \frac{|y\wedge z|^{d/(d-1)}}{\varepsilon^{d/(d-1)}}[1,2]$,

$$\|\widehat{y} - \widehat{z}\| = \frac{\|\alpha\|}{|z|} \geq \frac{\|\alpha\|\varepsilon^{\frac{d}{d-1}}}{2|y\wedge z|^{\frac{d}{d-1}}} = \frac{\varepsilon^{\frac{d}{d-1}}}{2|y|^{\frac{d}{d-1}}\|\alpha\|^{\frac{1}{d-1}}},$$

and similarly,

$$\left\|\widehat{y} - \widehat{z'}\right\| = \frac{\|\alpha'\|}{|z'|} \leq \frac{\|\alpha'\|\varepsilon^{\frac{d}{d-1}}}{|y\wedge z'|^{\frac{d}{d-1}}} = \frac{\varepsilon^{\frac{d}{d-1}}}{|y|^{\frac{d}{d-1}}\|\alpha'\|^{\frac{1}{d-1}}}, \tag{6.3}$$

It follows that

$$\begin{aligned}\left\|\widehat{z} - \widehat{z'}\right\| &\geq \|\widehat{y} - \widehat{z}\| - \left\|\widehat{y} - \widehat{z'}\right\| \\ &\geq \frac{\varepsilon^{\frac{d}{d-1}}}{2|y|^{\frac{d}{d-1}}\|\alpha\|^{\frac{1}{d-1}}} - \frac{\varepsilon^{\frac{d}{d-1}}}{|y|^{\frac{d}{d-1}}\|\alpha'\|^{\frac{1}{d-1}}} \geq \frac{\varepsilon^{\frac{d}{d-1}}}{4|y|^{\frac{d}{d-1}}\|\alpha\|^{\frac{1}{d-1}}}.\end{aligned}$$

Since $\lambda_1(z) \leq \frac{|y\wedge z|}{|z|}$, the denominator of $R(z,z')$ is bounded above by

$$|y|\left(\frac{\|\alpha\|}{|z|^2} + \frac{\|\alpha'\|}{|z'|^2}\right)$$

which can be bounded above as before: dividing (6.3) by $|z'|$, we get the upper bound

$$\frac{\varepsilon^{\frac{2d}{d-1}}}{|y|^{\frac{d+1}{d-1}}}\left(\frac{1}{\|\alpha\|^{\frac{d+1}{d-1}}} + \frac{1}{\|\alpha'\|^{\frac{d+1}{d-1}}}\right) \leq \frac{\varepsilon^{\frac{2d}{d-1}}}{|y|^{\frac{d+1}{d-1}}} \times \frac{2}{\|\alpha\|^{\frac{d+1}{d-1}}}.$$

It follows that

$$R(z,z') \geq \frac{|y|^{\frac{1}{d-1}}\|\alpha\|^{\frac{d}{d-1}}}{8\varepsilon^{\frac{d}{d-1}}}.$$

Since $z \in F(x,y,\varepsilon)$, Lemma 5.6 implies $\|\alpha\| \gg \lambda_d(y)$, so that, by Minkowski's theorem, $|y|^{\frac{1}{d-1}}\|\alpha\|^{\frac{d}{d-1}} \gg 1$. Therefore $R(z,z') > 1$ for ε small enough.

Suppose now that $\frac{1}{4^{d-1}}\|\alpha\| \leq \|\alpha'\| \leq 4^{d-1}\|\alpha\|$. Again, since $|z| \in \frac{|y\wedge z|^{d/(d-1)}}{\varepsilon^{d/(d-1)}}[1,2]$, and $|z'| \in \frac{|y\wedge z'|^{d/(d-1)}}{\varepsilon^{d/(d-1)}}[1,2]$ we have

$$\frac{|z'|}{|z|} \asymp \left(\frac{|y\wedge z'|}{|y\wedge z|}\right)^{\frac{d}{d-1}} = \left(\frac{\|\alpha'\|}{\|\alpha\|}\right)^{\frac{d}{d-1}} \asymp 1.$$

Furthermore, by lemma 6.4, $\lambda_1(z) = \frac{|y\wedge z|}{|z|} = \frac{|y|\|\alpha\|}{|z|}$ and $\lambda_1(z') = \frac{|y|\|\alpha'\|}{|z'|}$, hence

$$\frac{\lambda_1(z')}{|z'|} \asymp \frac{\lambda_1(z)}{|z|}.$$

Assume now that z is fixed. We wanted to count the number of $z' = (u',v') \in F(x,y,\varepsilon)$ such that $R(z,z') \leq 1$. With the above inequality $R(z,z') \leq 1$ implies $\|\widehat{z'} - \widehat{z}\| \ll \frac{\lambda_1(z)}{|z|}$. It follows that $\|u' - v'\widehat{z}\| \ll |z'|\frac{\lambda_1(z)}{|z|} \ll \lambda_1(z)$. On the one hand, by Lemma 2.3 the number of points of the lattice Λ_z in a ball $B(0, r\lambda_1(z))$ is $\ll 1 + r^d$. On the other hand, since z is primitive, if z' and z'' are in the lattice $\mathbb{Z}^{d+1}$ and if $\pi_z(z') = \pi_z(z'')$ then $z' - z'' \in \mathbb{Z}z$; together with the inequality $|z'| \asymp |z|$ this implies that there are at most $O(1)$ $z's$ for a given value of $z' \wedge z$. It follows that the number of z' in $F(x,y,\varepsilon)$ with $R(z,z') \leq 1$ is $\ll 1$. □

6.4. **Lower bound calculations.** The next proposition shows that to get lower bounds on Hausdorff dimension, it is enough to control the distribution of Farey lattices with bounded distortion.

Proposition 6.10. *Let ε be a positive real number and let y be in $D(x)$. If ε is small enough, then for all $N \in \mathbb{Z}_{>0}$ and all real numbers s and t in $[0, 2d]$,*[6]

$$\sum_{z\in F_N(x,y,\varepsilon)} \frac{\widehat{\lambda}_1(z)^s}{|z|^t} \gg S_1(N,t)\frac{\varepsilon^{s+(t-1)\frac{d}{d-1}}}{\widehat{\lambda}_d(y)^{(t-d)\frac{d}{d-1}}|y|^t}$$

where

$$S_1(N,t) = \sum_{n=1}^{N} \frac{1}{n^{1+(t-d)\frac{d}{d-1}}}\frac{\operatorname{card}(\mathcal{C}'_n(x)\cap\Lambda_y(\varepsilon))}{\operatorname{card}(\mathcal{C}'_n(x)\cap\Lambda_y)}.$$

Proof. Since $||\alpha|| \gg \lambda_d(y)$,

$$\frac{1}{|y|}\left(\frac{||\alpha|||y|}{\varepsilon}\right)^{\frac{d}{d-1}} \geq \left(\frac{\widehat{\lambda}_d(y)}{\varepsilon}\right)^{\frac{d}{d-1}}$$

which is > 1 when ε is small enough. This implies both $\operatorname{card}\zeta(y,\alpha,\varepsilon) \geq 1$ and

$$\operatorname{card}\zeta(y,\alpha,\varepsilon) \asymp \left(\frac{||\alpha|||y|^{1/d}}{\varepsilon}\right)^{\frac{d}{d-1}}.$$

[6]The constant implicit in $\gg$ depends on the choice of this bounded interval.

By Lemma 6.4, $\widehat{\lambda}_1(z) \asymp \varepsilon$. Also, for any $\alpha \in \Lambda_y \cap \mathcal{C}'_n(x)$, $\|\alpha\| \asymp n\lambda_d(x) \asymp n\lambda_d(y)$. Thus, splitting the sum according to $\alpha = \pi_y(z) \in \mathcal{C}'_n(x)$, we get

$$\sum_{z \in F_N(x,y,\varepsilon)} \frac{\widehat{\lambda}_1(z)^s}{|z|^t} \asymp \sum_{\alpha \in \Lambda_y(\varepsilon) \cap \mathcal{C}_N(x)} \sum_{z \in \zeta(y,\alpha,\varepsilon)} \frac{\varepsilon^s}{|y|^t} \left(\frac{\varepsilon}{\|\alpha\| |y|^{1/d}} \right)^{t\frac{d}{d-1}}$$
$$\asymp \sum_{n=1}^{N} \sum_{\alpha \in \Lambda_y(\varepsilon) \cap \mathcal{C}'_n(x)} \frac{\varepsilon^s}{|y|^t} \left(\frac{\varepsilon}{\|\alpha\| |y|^{1/d}} \right)^{(t-1)\frac{d}{d-1}}$$
$$\asymp \sum_{n=1}^{N} \frac{\varepsilon^{s+(t-1)\frac{d}{d-1}}}{n^{(t-1)\frac{d}{d-1}} |y|^t} \left(\frac{\operatorname{card}(\Lambda_y(\varepsilon) \cap \mathcal{C}'_n(x))}{\widehat{\lambda}_d(y)^{(t-1)\frac{d}{d-1}}} \right).$$

By Lemma 2.5,

$$\operatorname{card}(\Lambda_y \cap \mathcal{C}'_n(x)) \gg \frac{(n\lambda_d(y))^{d-1}}{\operatorname{vol}(\Lambda_y \cap H'_x)} = \frac{n^{d-1}\lambda_d(y)^d}{\operatorname{vol}(\Lambda_y \cap H'_x)\lambda_d(y)}$$
$$\asymp n^{d-1}\lambda_d(y)^d |y| = n^{d-1}\widehat{\lambda}_d(y)^d,$$

thus

$$\sum_{z \in F_N(x,y,\varepsilon)} \frac{\widehat{\lambda}_1(z)^s}{|z|^t} \gg \sum_{n=1}^{N} \frac{\varepsilon^{s+(t-1)\frac{d}{d-1}}}{n^{1+(t-1)\frac{d}{d-1}-d}|y|^t} \left(\frac{1}{\widehat{\lambda}_d(y)^{(t-1)\frac{d}{d-1}-d}} \right) \times \frac{\operatorname{card}(\Lambda_y(\varepsilon) \cap \mathcal{C}'_n(x))}{\operatorname{card}(\Lambda_y \cap \mathcal{C}'_n(x))}$$
$$= \frac{S_1(N,t)\varepsilon^{s+(t-1)\frac{d}{d-1}}}{\widehat{\lambda}_d(y)^{(t-d)\frac{d}{d-1}}|y|^t}.$$

□

The next Proposition shows that the distribution of Farey lattices with bounded distortion can be controlled.

Proposition 6.11. *Let ε be a positive real number and let y be in $D(x)$. If ε is small enough, then for all $N \in \mathbb{Z}_{>0}$, and $t > d$,*

$$S_1(N,t) \gg \sum_{n=1}^{N} \frac{1}{n^{1+(t-d)\frac{d}{d-1}}}. \tag{6.4}$$

Observe that the right hand side of (6.4) is $\gg \frac{1}{t-d}$ when $t > d$ provided N is large enough, e.g. $N^{t-d} \geq 2$.

Proposition 6.11 will be proved Section 8. For the rest of this section, we carry out the lower bound calculations assuming (6.4).

Corollary 6.12. *Let ε be a positive real number and assume $d \geq 2$. For any function $f :]0,1] \to]0,\infty[$ with $\lim_{u\to 0} f(u) = 0$, $\operatorname{Hdim} \mathrm{DI}_\varepsilon(d) \geq \frac{d^2}{d+1} + f(\varepsilon)\varepsilon^d$ for ε small enough.*

Proof. Let $(Q_{\varepsilon,N}, \sigma_{\varepsilon,N}, B)$ be the self-similar structure as in Proposition 6.6. Let $x \in Q_{\varepsilon,N}$ and $z, z' \in F_N(x, x, \varepsilon)$ with $z \neq z'$. By Lemma 6.4, $\widehat{\lambda}_1(z) \asymp \varepsilon$, so that, by definition of $Q_{\varepsilon,N}$, $\widehat{\lambda}_1(x) \asymp \varepsilon$. By Minkowski's theorem, $\widehat{\lambda}_1(x)^{d-1}\widehat{\lambda}_d(x) \ll 1$ so that $\widehat{\lambda}_d(x)^{-1} \gg \widehat{\lambda}_1(x)^{d-1} \gg \varepsilon^{d-1}$. By Lemma 6.8, $R(z, z') > 1$ implies $d(B(z), B(z')) > \rho \operatorname{diam} B(x)$ for a constant

$$\rho \gg \left(\frac{1}{N}\right)^{\frac{d+1}{d-1}} \varepsilon^{\frac{2d}{d-1}+d}.$$

By Lemma 6.9, there is a constant $c > 0$ independent of N and ε, such that for each $z \in F_N(x, x, \varepsilon)$ there are at most c points $z' \in F_N(x, x, \varepsilon) \setminus \{z\}$ such that $d(B(z), B(z')) \leq \rho \operatorname{diam} B(x)$.

Now Proposition 6.10 together with (6.4) implies that, with an N depending only on $t-d$,

$$\sum_{z \in F_N(x,x,\varepsilon)} \frac{\widehat{\lambda}_1(z)^s}{|z|^t} \gg \frac{\varepsilon^{s+(t-1)\frac{d}{d-1}} \widehat{\lambda}_1(x)^{d(t-d)}}{(t-d)|x|^t} \gg \frac{\varepsilon^{s+(t-1)\frac{d}{d-1}+d(t-d)}}{(t-d)|x|^t} = \frac{\varepsilon^{s+d+\frac{d^2(t-d)}{d-1}}}{(t-d)|x|^t}$$

so that, setting $t = (1 + 1/d)s$, we have

$$\sum_{z \in F_N(x,x,\varepsilon)} \frac{\operatorname{diam} B(z)^s}{\operatorname{diam} B(x)^s} \gg \frac{\varepsilon^{d+\frac{d^2(t-d)}{d-1}}}{t-d}.$$

If $t-d = f(\varepsilon)\varepsilon^d$ then the right side in the above tends to infinity when ε goes to zero, so that the sum on the left side is $\geq (c+1)$ when ε is small enough. By Theorem 3.4, $\operatorname{Hdim} \mathrm{DI}_{2\varepsilon}(d) \geq \frac{d^2}{d+1} + f(\varepsilon)\varepsilon^d$, from which we deduce the desired statement for $\operatorname{Hdim} \mathrm{DI}_\varepsilon(d)$. □

Proposition 6.13. *For $d \geq 2$, $\operatorname{Hdim} \operatorname{Sing}(d) \geq \frac{d^2}{d+1}$.*
Moreover $\operatorname{Sing}(d)$ has positive $\frac{d^2}{d+1}$-dimensional Hausdorff measure.

Proof. Let $(\varepsilon_i)_{i\geq 0}$ be a decreasing sequence of positive real numbers and $(N_i)_{i\geq 0}$ an increasing sequence of positive integers such that $\varepsilon_{i+1} \asymp \varepsilon_i$, $N_{i+1} \asymp N_i$,

$$\lim_{i\to\infty} \varepsilon_i = 0 \quad \text{and} \quad \lim_{i\to\infty} \varepsilon_i^d \log N_i = \infty.$$

For example, $N_i = i + 10$ and $\varepsilon_i = (\log\log N_i)^{\frac{-1}{d}}$.

Let $x_0 \in F_{N_0}(x_{-1}, x_{-1}, 2\varepsilon_0)$ for some arbitrary x_{-1} in Q. Define $Q_i, i \geq 0$ recursively by

$$Q_0 = \{x_0\}, \qquad Q_{i+1} = \bigcup_{x \in Q_i} F_{N_i}(x, x, \varepsilon_i).$$

Let $Q' = \bigcup_{i\geq 0} Q_i$ be a formal[7] disjoint union and for each $x \in Q'$, let $\sigma'(x) = F_{N_i}(x, x, \varepsilon_i)$.[8] Let $B(x) = B(\widehat{x}, \frac{\lambda_1(x)}{2|x|})$ as before. The same argument as in the proof of Proposition 6.6 shows that (Q', σ', B) is a strictly nested self-similar structure covering a subset of $\operatorname{Sing}(d)$.

[7]For example, $Q' = \bigcup_{i\geq 0} Q_i \times \{i\}$ and $\sigma'(x) = F_{N_i}(\tilde{x}, \tilde{x}, \varepsilon_i) \times \{i\}$ where $x = (\tilde{x}, i)$.

[8]We represent the first coordinate of an element in Q' by the same name (e.g. in the case of $x = (\tilde{x}, i)$, we drop the tilde) and let the context decide whether an element of Q' or one of Q is intended.

By Lemma 6.8, we can fix a constant $c_\rho > 0$ depending only on the dimension such that the function $\rho : Q' \to]0,1[$ given by ($x \in Q'$)

$$\rho(x) = c_\rho \left(\frac{1}{N_i}\right)^{\frac{d+1}{d-1}} \varepsilon_i^{\frac{2d}{d-1}+d}$$

has the property that for any $z, z' \in F_{N_i}(x, x, \varepsilon_i)$ such that $R(z, z') > 1$, we have

$$d(B(z), B(z')) > \rho(x) \operatorname{diam} B(x).$$

By Lemma 6.9, there is a constant $c > 0$ such that for each $z \in F_{N_i}(x, x, \varepsilon_i)$ there are at most c points $z' \in F_{N_i}(x, x, \varepsilon_i) \setminus \{z\}$ such that $d(B(z), B(z')) \leq \rho(x) \operatorname{diam} B(x)$. Moreover, since $\rho(x)$ decreases with i, Lemma 6.5 implies both $B(z) \subset B(x)$ and

$$\rho(z) \operatorname{diam} B(z) < \rho(x) \operatorname{diam} B(x)$$

for all $z \in \sigma'(x)$ when ε_i is small enough.

Proposition 6.11 implies $S_1(N, d) \gg \log N$ so that Proposition 6.10 with $t = (1+1/d)s = d$ now implies, as in the proof of Corollary 6.12

$$\sum_{z \in F_{N_i}(x,x,\varepsilon_i)} \frac{\widehat{\lambda}_1(z)^s}{|z|^t} \gg \frac{\varepsilon_i^{s+d}}{|x|^t} \log N_i$$

so that

$$\sum_{z \in F_{N_i}(x,x,\varepsilon_i)} \frac{(\rho(z) \operatorname{diam} B(z))^s}{(\rho(x) \operatorname{diam} B(x))^s} \gg \varepsilon_i^d \log N_i.$$

For i large enough, the sum exceeds $c+1$, so that by Theorem 3.6, $\operatorname{Hdim} \operatorname{Sing}(d) \geq \frac{d^2}{d+1}$. ☐

7. Disjoint spheres

We wish to enhance the lower bound of Theorem 1.3 in the two dimensional case. The two next Lemmas are important ingredients of this improvement. Observe that these Lemmas are optimal when $d = 2$. This is the reason we shall assume that $d = 2$ after their proofs.

Lemma 7.1. *Let x and y be in Q and $\varepsilon > 0$. Suppose that $\widehat{\lambda}_{d-1}(x) \leq \varepsilon$, $\|\widehat{x} - \widehat{y}\| \leq \frac{\lambda_1(x)}{10|x|}$, $|x| \leq |y|$, and $y \in H_x$. Then for all $q \in \{|x|, |x|+1, \ldots, |y|\}$,*

$$d(\{\widehat{y}, \ldots, q\widehat{y}\}, \mathbb{Z}^{d+1}) \ll \varepsilon q^{-1/d}$$

when ε is small enough.

Proof. Since $\|\widehat{x} - \widehat{y}\| \leq \frac{\lambda_1(x)}{10|x|}$, by Lemma 4.2, x is a best approximation vector of the rational vector $\theta = y$. Let $x_j = x, \ldots, x_k = y$ be all the intermediate best approximation vectors of y. Consider the net $\mathcal{H}$ of parallel hyperplanes $\Lambda_x + H'_x$ in $\mathbb{R}^d$. Since $y \in H_x$, $\Lambda_y \subset \mathcal{H}$.

Let us show that the lattices Λ_{x_i}, $i = j+1, ..., k-1$ are also included in $\mathcal{H}$. Let $1 \leq q < |x_i|$. Writing $x_l = (p_l, q_l)$, we have

$$\|q\widehat{x}_i - q\widehat{y}\| = \frac{q}{q_i}\|q_i\widehat{y} - p_i\| \leq \|q_j\widehat{y} - p_j\| \leq 2\lambda_1(x)$$

by Lemma 4.1. Making use of the Minkowski theorem, we see that the assumption $\widehat{\lambda}_{d-1}(x) \leq \varepsilon$ implies $\widehat{\lambda}_d(x) \gg \frac{\widehat{\lambda}_{d-1}(x)}{\varepsilon^d}$ and therefore the same inequality holds without the hat. Now by Lemma 2.2, $e(\mathcal{H}) \asymp \lambda_d(x)$, hence $e(\mathcal{H}) > 10\lambda_1(x)$ when ε is small enough. With the above inequality, it follows that the entire lattice Λ_{x_i} is within $e(\mathcal{H})/5$-neighborhood of $\mathcal{H}$. Looking at the picture in the quotient $\mathbb{R}^d/H'_x$, we see that $\Lambda_{x_i} \subset \mathcal{H}$.

Let i be in $\{j, ..., k\}$. We have $\mathrm{vol}(\Lambda_{x_i} \cap H'_x) \times e(\mathcal{H}) = \mathrm{vol}(\Lambda_{x_i}) = \frac{1}{|x_i|}$, and the first $d-1$ minima of the lattice $\Lambda_{x_i} \cap H'_x$ are larger than those of Λ_{x_i}. By Minkowski theorem, it follows that

$$\lambda_1(x_i) \cdots \lambda_{d-1}(x_i) \ll \frac{1}{e(\mathcal{H})|x_i|},$$
$$\lambda_1(x_j) \cdots \lambda_{d-1}(x_j) \asymp \frac{1}{e(\mathcal{H})|x_j|}.$$

Since $|x_i| \geq |x_j|$, we obtain

$$\frac{\widehat{\lambda}_1(x_i)}{\widehat{\lambda}_1(x_j)} \times \cdots \times \frac{\widehat{\lambda}_{d-1}(x_i)}{\widehat{\lambda}_{d-1}(x_j)} \ll \frac{|x_j|^{1/d}}{|x_i|^{1/d}} \leq 1.$$

Therefore, one of the $d-1$ ratios is $\ll 1$. This implies that for some l,

$$\widehat{\lambda}_1(x_i) \leq \widehat{\lambda}_l(x_i) \ll \widehat{\lambda}_l(x_j) \leq \widehat{\lambda}_{d-1}(x_j) \leq \varepsilon.$$

Let q be an integer between $|x_{i-1}|$ and $|x_i|$. By Lemma 4.1,

$$\begin{aligned} d(\{\widehat{y}, ..., q\widehat{y}\}, \mathbb{Z}^d) &\leq \|q_{i-1}\widehat{y} - p_{i-1}\| \leq 2\lambda_1(x_i) \\ &= 2\widehat{\lambda}_1(x_i)|x_i|^{-1/d} \\ &\ll \varepsilon q^{-1/d}. \end{aligned}$$

□

Lemma 7.2. *For all x in Q and y in $D(x)$ (see Definition 5.1 for $D(x)$)*

$$\widehat{\lambda}_1(y) \ll \widehat{\lambda}_1(x)^{\frac{1}{(d-1)^2}}.$$

Proof. By Lemma 5.6,

$$\widehat{\lambda}_d(y) \asymp \frac{|y|^{\frac{1}{d}}}{|x|^{\frac{1}{d}}}\widehat{\lambda}_d(x)$$

and by Minkowski's theorem we have both

$$\widehat{\lambda}_1(y)^{d-1}\widehat{\lambda}_d(y) \ll 1 \text{ and } \widehat{\lambda}_d(x)^{d-1}\widehat{\lambda}_1(x) \gg 1;$$

therefore,

$$\widehat{\lambda}_1(y) \ll \widehat{\lambda}_1(x)^{\frac{1}{(d-1)^2}} \left(\frac{|x|}{|y|}\right)^{\frac{1}{d(d-1)}} .$$

□

We assume until the end of this section that $d = 2$.

Definition 7.3. Let ε be a positive real number and x in Q.

(i) $D'(x)$ is the set of $y \in Q$ such that $|x| \leq |y|$, $y \in H_x$, $\|\widehat{x} - \widehat{y}\| < \frac{\lambda_1(x)}{10|x|}$.

(ii) The subset of $D'(x)$ consisting of $y \in Q$ such that $|x| \leq |y| < N|x|$ will be denoted by $D'_N(x)$. Set

$$\sigma'_{\varepsilon,N}(x) = \bigcup_{y \in D'_N(x)} F_N(x, y, \varepsilon)$$

($F_N(x, y, \varepsilon)$ is defined Section 6.1), $Q'_{\varepsilon,N} = \bigcup_{x \in Q} \sigma'_{\varepsilon,N}(x)$ and $B(x) = B(\widehat{x}, \frac{\lambda_1(x)}{2|x|})$.

Lemma 7.4. *For ε small enough, $(Q'_{\varepsilon,N}, \sigma'_{\varepsilon,N}, B)$ is a strictly nested self-similar structure covering of a subset of $\mathrm{DI}_{c_0\varepsilon}(2)$ for some absolute constant $c_0 > 0$.*

Proof. We need to verify the conditions (i) and (ii) in the proof of Proposition 6.6 for each $\sigma'_{\varepsilon,N}$-admissible sequence (x_k). Let y_k be the element in $D'_N(x_k)$ such that $x_{k+1} \in F_N(x_k, y_k, \varepsilon)$. First observe that by Lemma 6.4, for all $x \in Q'_{\varepsilon,N}$, $\widehat{\lambda}_1(x) \leq \varepsilon$, hence by Lemma 7.2, $\widehat{\lambda}_1(y_k) \ll \widehat{\lambda}_1(x_k) \leq \varepsilon$. Next, by Lemma 6.5, $B(x_{k+1}) \subset B(y_k)$ so that to prove (i) it is enough to verify that $B(y_k) \subset B(x_k)$. We can assume $y_k \neq x_k$. Since

$$\frac{\lambda_1(x_k)}{|y_k|} \leq \frac{\|\pi_{x_k}(y_k)\|}{|y_k|} = d(\widehat{x}_k, \widehat{y}_k) < \frac{\lambda_1(x_k)}{10|x_k|}$$

we have $|y_k| > 10|x_k|$. Similarly,

$$\frac{\lambda_1(y_k)}{|x_k|} \leq \frac{\|\pi_{y_k}(x_k)\|}{|x_k|} = d(\widehat{x}_k, \widehat{y}_k) < \frac{\lambda_1(x_k)}{10|x_k|}$$

so that $\lambda_1(y_k) < \frac{\lambda_1(x_k)}{10}$. Therefore,

$$\frac{\lambda_1(y_k)}{|y_k|} < \frac{\lambda_1(x_k)}{100|x_k|}.$$

Now, given $z \in B(y_k)$ we have

$$\begin{aligned}\|\widehat{x}_k - \widehat{z}\| &\leq \|\widehat{x}_k - \widehat{y}_k\| + \|\widehat{y}_k - \widehat{z}\| \\ &< \frac{\lambda_1(x_k)}{10|x_k|} + \frac{\lambda_1(y_k)}{2|y_k|} < \frac{\lambda_1(x_k)}{2|x_k|}\end{aligned}$$

so that $z \in B(x_k)$.

For (ii), let θ be in $\bigcap_k B(x_k)$ and consider $|x_k| \le q < |x_{k+1}|$. In the case $|y_k| \le q < |x_{k+1}|$ we have

$$\min_{1\le l\le q} d(l\theta, \mathbb{Z}^d) \le \|q_k'\theta - p_k'\| < 2\|q_k'\widehat{x}_{k+1} - p_k'\|$$

by Lemma 4.3(iii) where $y_k = (p_k', q_k')$. On the other hand, by Lemma 6.4,

$$\|q_k'\widehat{x}_{k+1} - p_k'\| = \frac{|y_k \wedge x_{k+1}|}{|x_{k+1}|} < \varepsilon |x_{k+1}|^{-1/d}$$

so that

$$\min_{1\le l\le q} d(l\theta, \mathbb{Z}^d) < 2\varepsilon q^{-1/d}.$$

In the case $|x_k| \le q < |y_k|$ we have

$$\min_{1\le l\le q} d(l\theta, \mathbb{Z}^d) \le 2 \min_{1\le l\le q} d(l\widehat{y}_k, \mathbb{Z}^d)$$

by Lemma 4.3(iii) and because $\theta \in B(x_{k+1}) \subset B(y_k)$. Since $\widehat{\lambda}_1(x_k) \le \varepsilon$, we can use Lemma 7.1 to conclude that

$$\min_{1\le l\le q} d(l\theta, \mathbb{Z}^d) \ll \varepsilon q^{-1/d}.$$

□

Definition 7.5. Let $x, y \in Q$ with $y \in H_x$. The sphere of diameter $\frac{\lambda_1(y)}{|y|}$ that is tangent to $\widehat{x} + H_x'$ at $\widehat{y}$, with inward normal $\alpha_x^\perp$ will be denoted $S(x, y)$.

Lemma 7.6. *Let $x \in Q$, $y \ne y' \in H_x \cap Q$. Then $S(x, y)$ and $S(x, y')$ have disjoint interiors.*

Proof. The condition for $S(x, y)$ and $S(x, y')$ to have disjoint interiors is

$$\left\|\widehat{y} - \widehat{y'}\right\|^2 + \left(\frac{\lambda_1(y)}{2|y|} - \frac{\lambda_1(y')}{2|y'|}\right)^2 \ge \left(\frac{\lambda_1(y)}{2|y|} + \frac{\lambda_1(y')}{2|y'|}\right)^2$$

or equivalently,

$$\|\widehat{y} - \widehat{y'}\|^2 \ge \frac{\lambda_1(y)\lambda_1(y')}{|y||y'|}.$$

Since

$$|y||y'|\|\widehat{y} - \widehat{y'}\|^2 = \frac{|y \wedge y'|}{|y|}\frac{|y \wedge y'|}{|y'|} = \|\pi_y(y')\|\|\pi_{y'}(y)\| \ge \lambda_1(y)\lambda_1(y')$$

the lemma follows. □

Lemma 7.7. *Let χ be a chord of a circle S with diameter one that makes an angle ϕ with each of the radial segments joining its endpoints to the center. Let $\rho(\delta)$ be the radius of the largest disk contained in S centered at a point on the chord a distance δ from the endpoints. Then*

$$\rho(\delta) \ge \frac{\delta}{2}\cos\phi.$$

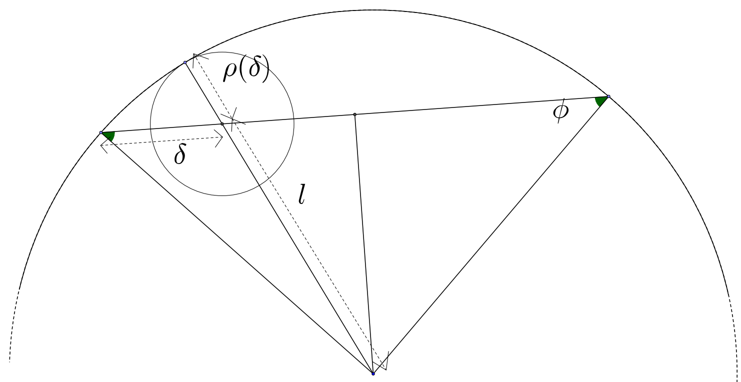


FIGURE 1. Proof of Lemma 7.7.

Proof. We can assume that the radius of the circle is 1 without loss of generality. Let ℓ be the distance of the point from the center of the circle. By Pythagoras' theorem,

$$\ell^2 = (\cos\phi - \delta)^2 + \sin^2\phi = 1 - \delta(2\cos\phi - \delta).$$

Thus,

$$\rho(\delta) = 1 - \ell \geq 1 - \left(1 - \frac{1}{2}\delta(2\cos\phi - \delta)\right)$$

and since $\delta \leq \cos\phi$,

$$\rho(\delta) \geq \frac{\delta}{2}\cos\phi.$$

□

Lemma 7.8. *Let $y \in D(x)$ for some $x, y \in Q$ and let $z \in F(x, y, \varepsilon)$. Then the ball $B(\widehat{z}, \frac{2\lambda_1(z)}{|z|})$ is contained in the interior of the sphere $S(x, y)$ when ε is small enough.*

Proof. Let $\rho(z)$ be the radius of the largest ball centered at $\widehat{z}$ that is contained in $S(x, y)$. Since the point $\widehat{z}$ is in the cone $\widehat{y} + \mathcal{C}(x)$, the angle between $\widehat{z} - \widehat{y}$ and $\alpha_x^{\perp}$ is $\tan^{-1} A_d$. By Lemma 7.7, considering the chord going through $\widehat{y}$ and $\widehat{z}$, we see that

$$d(\widehat{y}, \widehat{z}) \leq \frac{1}{\sqrt{1 + A_d^2}} \frac{\lambda_1(y)}{4|y|} \quad \Longrightarrow \quad \rho(z) \geq \frac{d(\widehat{y}, \widehat{z})}{2\sqrt{1 + A_d^2}}.$$

Now, by Lemma 6.4,

$$\frac{\lambda_1(z)}{|z|} = \frac{|y \wedge z|}{|z|^2} = \frac{|y|}{|z|} d(\widehat{y}, \widehat{z})$$

and by inequality (6.2), $\frac{|y|}{|z|} \ll \varepsilon^{\frac{d}{d-1}}$, hence the ball $B(\widehat{z}, \frac{2|y| d(\widehat{y}, \widehat{z})}{|z|})$ is contained in the interior of the sphere $S(x, y)$ when ε is small enough. □

Corollary 7.9. *Let $y, y' \in D(x)$ for some $x, y, y' \in Q$ and let $z \in F(x, y, \varepsilon)$ and $z' \in F(x, y', \varepsilon)$. If $y \neq y'$ then $R(z, z') > 1$ and $d(B(z), B(z')) > \frac{\lambda_1(z)}{|z|}$ when ε small enough.*

Proof. The balls $B(\widehat{z}, \frac{2\lambda_1(z)}{|z|})$ and $B(\widehat{z}', \frac{2\lambda_1(z')}{|z'|})$ have same centers as $B(z)$ and $B(z')$ with quadruple radii, and by Lemmas 7.6 and 7.8, they do not meet. □

Lemma 7.10. *Let ε be small enough, N a positive integer and $x \in Q$. Suppose that $\widehat{\lambda}_1(x) \in [\frac{1}{2}\varepsilon, \varepsilon]$. There are constants $c > 0$ and $\rho > 0$ such that for all $z \in \sigma'_{\varepsilon,N}(x)$, there are at most c elements $z' \in \sigma'_{\varepsilon,N}(x) \setminus \{z'\}$ such that*

$$d(B(z), B(z')) < \rho \frac{\varepsilon^6}{N^6} \frac{\lambda_1(x)}{|x|}. \tag{7.1}$$

Proof. Step 1. For any $y \in D'_N(x)$,

$$\frac{\lambda_1(y)}{|y|} \gg \frac{1}{N^2} \frac{\lambda_1(x)}{|x|}.$$

Since by Lemma 5.6, $\lambda_2(y) \asymp \lambda_2(x)$, Minkowski's theorem implies

$$|y|\lambda_1(y) \asymp |x|\lambda_1(x).$$

Therefore, we have

$$\frac{\lambda_1(y)}{|y|} \asymp \frac{|x|\lambda_1(x)}{|y|^2} \geq \frac{1}{N^2} \frac{\lambda_1(x)}{|x|}.$$

Step 2. $R(z, z') > 1$ implies (7.1) does not hold if ρ is small enough.
In the case z, z' belongs to the same set $F(x, y, \varepsilon)$ and $R(z, z') > 1$, making use of Lemma 6.8 we obtain

$$d\left(B(z), B(z')\right) \gg \left(\frac{1}{N}\right)^3 \frac{\varepsilon^4}{\widehat{\lambda}_2(y)^2} \frac{\lambda_1(y)}{|y|}$$

Noticing that $\widehat{\lambda}_2(y) \ll \widehat{\lambda}_2(x)(\frac{|y|}{|x|})^{\frac{1}{2}} \leq \widehat{\lambda}_2(x)N^{\frac{1}{2}}$ and that $\widehat{\lambda}_2(x) \asymp \widehat{\lambda}_1(x)^{-1} \asymp \varepsilon^{-1}$, together with *Step 1*, we see that

$$d\left(B(z), B(z')\right) \gg \frac{\varepsilon^6}{N^6} \frac{\lambda_1(x)}{|x|}.$$

In the case z, z' belongs to two different sets $F(x, y, \varepsilon)$ and $F(x, y', \varepsilon)$, Corollary 7.9 and the inequality $\frac{\lambda_1(z)}{|z|} \gg \frac{\varepsilon^4}{N^3\widehat{\lambda}_2(y)^2} \frac{\lambda_1(y)}{|y|}$ from the proof of Lemma 6.8, imply

$$d(B(z), B(z')) > \frac{\lambda_1(z)}{|z|} \gg \frac{\varepsilon^6}{N^6} \frac{\lambda_1(x)}{|x|}.$$

Step 3. For all $z \in \sigma'_{\varepsilon,N}(x)$, the number of $z' \in \sigma'_{\varepsilon,N}(x)$ such that (7.1) holds is $\ll 1$.
This is a consequence of Lemma 6.9, Corollary 7.9 and *Step 2.* □

Lemma 7.11. *Let $L = \mathbb{Z}u \oplus \mathbb{Z}v$ be a two dimensional lattice, let n be a positive integer and let $I_1, ..., I_n$ be n intervals in $\mathbb{Z}$ of same length $l \geq n$. Then the number of primitive points of the lattice L in the region*

$$\mathcal{R} = \{au + bv : a \in \{1, ..., n\}, b \in I_a\}$$

is $\gg ln$.

Proof. Denote $\phi(n)$ the Euler function, i.e. $\phi(n) =$ the number of positive integers $\leq n$ prime with n, and $\Phi(n) = \sum_{k \leq n} \phi(k)$. It is well known that $\Phi(n) \gg n^2$ (see [13]). Since each interval of length k contains at least $\phi(k)$ integers prime with k, the number of primitive elements in $\mathcal{R}$ is at least

$$\begin{aligned}\sum_{k=1}^{n} \phi(k) \left\lfloor \frac{l}{k} \right\rfloor &\asymp l \sum_{k=1}^{n} \frac{\phi(k)}{k} \\ &\geq l \sum_{k=1}^{n} \Phi(k) \left(\frac{1}{k} - \frac{1}{k+1} \right) \gg ln\end{aligned}$$

where $\lfloor x \rfloor$ stands for the integer part of the real number x. ☐

Proposition 7.12. *For $x \in Q$ such that $\widehat{\lambda}_1(x) \asymp \varepsilon$,*

$$\sum_{z \in \sigma'_{\varepsilon,N}(x)} \frac{\widehat{\lambda}_1(z)^s}{|z|^t} \gg S_2(N,t) \frac{\varepsilon^{s+2(t-1)+\delta}}{|x|^t}$$

where $\delta = 2(t-2)$ and

$$S_2(N,t) = S_1(N,t) \times \sum_{k=1}^{\log_2 N} 2^{(2-t-\delta/2)k}.$$

Proof. Set $\nu(\varepsilon, N, s, t) = S_1(N,t) \varepsilon^{s+(t-1)\frac{2}{2-1}}$ and

$$D'(l,k) = \{y \in D'(x) : l|x| \leq |y| \leq k|x|\}.$$

From the proof of Lemma 5.7 we know that π_x induces a bijection from

$$\left\{(u,v) \in H_x \cap \mathbb{Z}^{d+1} : k|x| \leq v < (k+1)|x|\right\}$$

onto $\Lambda'_x = \Lambda_x \cap \pi_x(H_x)$. Making use of Lemma 7.11, we obtain

$$\operatorname{card} \left\{ y \in Q : (k+1)|x| \leq |y| \leq 2k|x|,\ \|\pi_x(y)\| \leq \frac{k}{10}\lambda_1(x) \right\} \gg k^2,$$

and therefore

$$\operatorname{card} D'(k+1, 2k) \gg k^2.$$

Observe that for each $z \in \sigma'_{\varepsilon,N}(x)$, Lemma 6.4 implies $\lambda_1(z) = ||\pi_z(y)||$ so that $\pi_z(y)$ is a shortest vector of Λ_z. And since $|y| < |z|$ (by Lemma 6.4 again) for each of the $O(1)$ possibilities for $\pi_z(y)$ there are at most one $y \in D'(x)$ such that $z \in F_N(x, y, \varepsilon)$. Therefore, by Proposition 6.10,

$$\sum_{z \in \sigma'_{\varepsilon,N}(x)} \frac{\widehat{\lambda}_1(z)^s}{|z|^t} \gg \sum_{y \in D'_N(x)} \sum_{z \in F_N(x,y,\varepsilon)} \frac{\widehat{\lambda}_1(z)^s}{|z|^t} \gg \sum_{y \in D'_N(x)} \frac{\nu(\varepsilon, N, s, t)}{\lambda_2(y)^\delta |y|^{t+\delta/2}}$$

and by Lemma 5.6, $\lambda_2(y)^\delta \asymp \lambda_2(x)^\delta$, thus, grouping the y in the sets $D'(2^k+1, 2^{k+1})$, $0 \le k \le \log_2 N$, we obtain

$$\sum_{z\in\sigma'_{\varepsilon,N}(x)} \frac{\widehat{\lambda}_1(z)^s}{|z|^t} \gg \sum_{k=1}^{\log_2 N} \frac{\nu(\varepsilon,N,s,t)}{\lambda_2(x)^\delta} \frac{2^{2k}}{(2^k|x|)^{t+\delta/2}}$$
$$= \sum_{k=1}^{\log_2 N} \frac{\nu(\varepsilon,N,s,t)}{\widehat{\lambda}_2(x)^\delta} \frac{2^{(2-t-\delta/2)k}}{|x|^t}.$$

The Proposition follows making use of the lower bound $\widehat{\lambda}_2(x) \asymp \widehat{\lambda}_1(x)^{-1}$. □

Corollary 7.13. *Let ε be a positive real number. For any function $f :]0,1] \to]0,\infty[$ with $\lim_{u\to 0} f(u) = 0$, $\operatorname{Hdim}\operatorname{DI}_\varepsilon(2) \ge \frac{4}{3} + f(\varepsilon)\varepsilon$ for ε is small enough.*

Proof. By Lemma 7.4, $(Q'_{\varepsilon,N}, \sigma'_{\varepsilon,N}, B)$ is a strictly self-similar structure that covers a subset of $\operatorname{DI}_{c_0\varepsilon}(2)$ where c_0 is an absolute constant. Let $x \in Q'_{\varepsilon,N}$. By Lemma 7.10, there are constants $c > 0$ and $\rho > 0$ such that for all $z \in \sigma'_{\varepsilon,N}(x)$, there are at most c elements $z' \in \sigma'_{\varepsilon,N}(x) \setminus \{z'\}$ such that

$$d(B(z), B(z')) < \rho \frac{\varepsilon^6}{N^6} \frac{\lambda_1(x)}{|x|}. \tag{7.2}$$

Now Proposition 7.12 together with (6.4) implies

$$\sum_{z\in\sigma'_{\varepsilon,N}(x)} \frac{\widehat{\lambda}_1(z)^s}{|z|^t} \gg S_2(N,t) \frac{\varepsilon^{s+2(t-1)+\delta}}{|x|^t}$$
$$\gg \frac{\varepsilon^{s+2(t-1)+\delta}}{|x|^t} \times \frac{1}{t-2} \times \sum_{k=1}^{\log_2 N} 2^{(2-t-\delta/2)k}$$
$$\gg \frac{\varepsilon^{s+2(t-1)+\delta}}{|x|^t} \times \frac{1}{(t-2)(t+\delta/2-2)}$$

provide that $N^{(t-2)} \ge 2$. So that, setting $t = (1+1/d)s$, we have

$$\sum_{z\in\sigma'_{\varepsilon,N}(x)} \frac{\operatorname{diam} B(z)^s}{\operatorname{diam} B(x)^s} \gg \frac{\varepsilon^{2(t-1)+\delta}}{(t-2)^2}.$$

If $t-2 = f(\varepsilon)\varepsilon$ then the right side in the above tends to infinity when ε goes to zero, so that the sum on the left side is $\ge (c+1)$ when ε is small enough. By Lemma 7.4 and Theorem 3.4, we obtain $\operatorname{Hdim}\operatorname{DI}_{c_0\varepsilon}(2) \ge \frac{4}{3} + f(\varepsilon)\varepsilon$, from which we deduce the desired statement for $\operatorname{Hdim}\operatorname{DI}_\varepsilon(2)$. □

8. Bounded distortion, proof of Proposition 6.11

Notation. For a positive integer n, denote $\phi(n)$ the number of positive integers $j \leq n$ prime with n, i.e. the Euler function, and $D_1(n)$ the sum of the divisors of n: $D_1(n) = \sum_{k|n} k$.

Proposition 8.1. *There is a constant C depending only on the dimension such that for all positive integers n, all x in Q, and all y in $D(x)$,*

$$\frac{\mathrm{card}\,(\mathcal{C}'_n(x) \cap \Lambda_y(\varepsilon))}{\mathrm{card}\,(\mathcal{C}'_n(x) \cap \Lambda_y)} \gg \frac{\phi(n)}{n}\left(1 - C\varepsilon^{d-1}\frac{D_1(n)}{n}\right).$$

When $d = 2$, the proof of the above Proposition is rather simple because the lattice $\Lambda_{\alpha^\perp}$ is one dimensional. It implies $\widehat{\lambda}_1(\alpha) = 1$ and therefore α is in $\Lambda_y(\varepsilon)$ provided that α is primitive and $\varepsilon < 1$. In that case the proof reduces to a lower bound on the number of primitive points in $\mathcal{C}'_n(x)$, which is easy. We give the proof for all $d \geq 2$; it needs a few lemmas.

Lemma 8.2. *Let Λ be a lattice in $\mathbb{R}^d$ and let α be a primitive vector in Λ. Let $\psi : \mathbb{R}^d \to \mathbb{R}^d$ be an invertible linear map and consider $\Gamma = \psi(\Lambda)$ and $\beta = \psi(\alpha)$. Then $\widehat{\lambda}_1(\Lambda_{\alpha^\perp}) \asymp \widehat{\lambda}_1(\Gamma_{\beta^\perp})$ where the constants involved in $\asymp$ depend on the dimension and on the bilipschitz constant.*

Remark 8.3. Lemma 8.2 will be used with a map whose bilipschitz constant depends only on the dimension (see the proof of Lemma 8.7).

Proof. Observe that $\Gamma_{\beta^\perp}$ can be obtained in two steps with projections parallel to β. First project Γ onto the space $\psi(\alpha^\perp)$ and then project the image L of Γ onto $\beta^\perp$. Since $L = \psi(\Lambda_{\alpha^\perp})$, $\lambda_1(\Lambda_{\alpha^\perp}) \asymp \lambda_1(L)$. Next, the orthogonal projection on $\beta^\perp$ restricted to $\psi(\alpha^\perp)$ is also bilipschitz, hence

$$\lambda_1(\Gamma_{\beta^\perp}) \asymp \lambda_1(L) \asymp \lambda_1(\Lambda_{\alpha^\perp}).$$

Since $\mathrm{vol}(\Gamma) \asymp \mathrm{vol}(\Lambda)$ and $\|\beta\| \asymp \|\alpha\|$, it follows that $\mathrm{vol}(\Gamma_{\beta^\perp}) \asymp \mathrm{vol}(\Lambda_{\alpha^\perp})$. ☐

Lemma 8.4. *Let n and k be positive integers. Let $l_1, l_2, \ldots, l_k$ be positive real numbers. Consider the box B in $(\mathbb{Z}/n\mathbb{Z})^k$ defined by the product of real intervals $\prod_{i=1}^k [-l_i, l_i]$. Then the number of points z in $(\mathbb{Z}/n\mathbb{Z})^k$ of order n such that the subgroup $\langle z \rangle$ generated by z intersects the box B only at 0 is at least*

$$\phi(n)\left(n^{k-1} - 3^k\frac{D_1(n)}{n}\prod_{1\leq i\leq k} l_i\right).$$

Proof. We can suppose that the l_i's are arranged in increasing order: $l_1 \leq l_2 \leq \cdots \leq l_k$. Set

$$T = \left\{(x,y) \in (\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z})^{k-1} : order(x) = n, \langle (x,y) \rangle \cap B = \{0\}\right\}$$
$$Bad = \left\{(x,y) \in (\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z})^{k-1} : order(x) = n, \langle (x,y) \rangle \cap B \neq \{0\}\right\}$$

Since $\mathrm{card}\, T + \mathrm{card}\, Bad = \phi(n)n^{k-1}$, it is enough to bound the number of elements of Bad from above.

Let x be an element of order n in $\mathbb{Z}/n\mathbb{Z}$. Set

$$Bad(x) = \left\{y \in (\mathbb{Z}/n\mathbb{Z})^{k-1} : \langle (x,y) \rangle \cap B \neq \{0\}\right\}.$$

Note that $Bad(x) = xBad(1)$ so that $\operatorname{card} Bad = \phi(n)Bad(1)$. Hence, we have to bound $\operatorname{card} Bad(1)$ from above. Since

$$\begin{aligned} y \in Bad(1) &\Leftrightarrow \exists (a,b) \in B \setminus \{0\}, \exists\, k \in \mathbb{Z},\ a = k,\ b = ky, \\ &\Leftrightarrow \exists (a,b) \in B \setminus \{0\},\ b = ay, \end{aligned}$$

we have

$$\operatorname{card} Bad(1) \leq \sum_{(a,b)\in B\setminus\{0\}} \operatorname{card} E(a,b)$$

where $E(a,b) = \left\{ y \in (\mathbb{Z}/n\mathbb{Z})^{k-1} : b = ay \right\}$.

Let (a,b) be in $(\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z})^{k-1}$ such that $a \neq 0$ and $order(a) = p$ (we assume $a \neq 0$ because $a = 0$ and $b \neq 0$ implies $E(a,b) = \varnothing$). The number of solutions to the equation $b = az$ is $(n/p)^{k-1}$ if $b \in (\langle a \rangle)^{k-1}$ and 0 otherwise. Hence $\operatorname{card} E(a,b)$ is $(n/p)^{k-1}$ or 0. Since $\langle a \rangle = \langle n/p \rangle$, the number of b in $(\langle a \rangle)^{k-1}$ that are in the box

$$\prod_{2\leq i\leq k} [-l_i, l_i]$$

is at most $\prod_{2\leq i\leq k}\left(2\left\lfloor \frac{l_i}{n/p} \right\rfloor + 1\right)$. Therefore, the number of pairs (a,b) in B such that $a \neq 0$ and $order(a) = p$ is at most

$$2\left\lfloor \frac{l_1}{n/p} \right\rfloor \prod_{2\leq i\leq k}\left(2\left\lfloor \frac{l_i}{n/p} \right\rfloor + 1\right).$$

Thus,

$$\operatorname{card} Bad(1) \leq \sum_{p|n} (n/p)^{k-1} 2\left\lfloor \frac{l_1}{n/p} \right\rfloor \prod_{2\leq i\leq k}\left(2\left\lfloor \frac{l_i}{n/p} \right\rfloor + 1\right).$$

Since $\left\lfloor \frac{l_1}{n/p} \right\rfloor \neq 0$ implies $\left\lfloor \frac{l_i}{n/p} \right\rfloor \geq 1$ for $i \geq 1$, the product $2\left\lfloor \frac{l_1}{n/p} \right\rfloor \prod_{2\leq i\leq k}\left(2\left\lfloor \frac{l_i}{n/p} \right\rfloor + 1\right)$ is either 0 or, for each $i = 2, \ldots, k$, $2\left\lfloor \frac{l_i}{n/p} \right\rfloor + 1 \leq 3\frac{l_i}{n/p}$. Therefore,

$$2\left\lfloor \frac{l_1}{n/p} \right\rfloor \prod_{2\leq i\leq k}\left(2\left\lfloor \frac{l_i}{n/p} \right\rfloor + 1\right) \leq 3^k \prod_{1\leq i\leq k} \frac{l_i}{n/p}.$$

Hence,

$$\begin{aligned} \operatorname{card} Bad(1) &\leq \sum_{p|n} (n/p)^{k-1} 3^k \prod_{1\leq i\leq k} \frac{l_i}{n/p} \\ &= 3^k \prod_{1\leq i\leq k} l_i \sum_{p|n} \frac{p}{n} = 3^k \frac{D_1(n)}{n} \prod_{1\leq i\leq k} l_i \end{aligned}$$

and the lemma follows. □

Lemma 8.5. *Let y be in $D(x)$, let $\alpha_1, \dots, \alpha_{d-1}$ be a Minkowski-reduced basis for $\Lambda'_{xy} = \Lambda_y \cap H'_x$ and let α_d be the shortest vector in $\Lambda_y \setminus H'_x$. Then $\|\alpha_i\| \ll \lambda_i(y)$ for $i = 1, \dots, d$.*

Proof. Since $\lambda_1(\Lambda'_{xy}) \cdots \lambda_{d-1}(\Lambda'_{xy}) \ll \mathrm{vol}(\Lambda'_{xy})$ and $\lambda_1(y) \cdots \lambda_d(y) \gg \mathrm{vol}(\Lambda_y)$, we have for $i < d$

$$\frac{\lambda_i(\Lambda'_{xy})}{\lambda_i(y)} \ll \left(\frac{\prod_{j \in \{1,\dots,\hat{i},\dots,d-1\}} \lambda_j(y)}{\prod_{j \in \{1,\dots,\hat{i},\dots,d-1\}} \lambda_j(\Lambda'_{xy})} \right) \frac{\lambda_d(y)\,\mathrm{vol}(\Lambda'_{xy})}{\mathrm{vol}(\Lambda_y)}.$$

Recall that $\alpha_x^\perp$, which is defined in Section 6.1, is a vector orthogonal to H_x such that $H_x + \mathbb{Z}\alpha_x^\perp = H_x + \Lambda_x$. Since $\Lambda_x + H_x = \Lambda_y + H_x$, $\mathrm{vol}(\Lambda'_{xy})\|\alpha_x^\perp\| = \mathrm{vol}\,\Lambda_y$. Clearly $\lambda_j(\Lambda'_{xy}) \geq \lambda_j(y)$, $j = 1, \dots, d-1$, hence

$$\frac{\lambda_i(\Lambda'_{xy})}{\lambda_i(y)} \ll \frac{\lambda_d(y)}{\|\alpha_x^\perp\|}.$$

By Lemmas 5.6 and 2.2, $\lambda_d(y) \asymp \lambda_d(x) \asymp \|\alpha_x^\perp\|$, thus $\lambda_i(\Lambda'_{xy}) \ll \lambda_i(y)$, so that $\|\alpha_i\| \ll \lambda_i(\Lambda'_{xy}) \ll \lambda_i(y)$ for $i = 1, \dots, d-1$. Since α_d lies in the close ball $\overline{B}(\alpha_x^\perp, e(\Lambda'_{xy}))$ we have $\|\alpha_d\| \leq \|\alpha_x^\perp\| + e(\Lambda'_{xy}) \ll \lambda_d(x) + \lambda_{d-1}(\Lambda'_{xy}) \ll \lambda_d(y)$. □

Lemma 8.6. *The constant A_d can be chosen large enough so that the following property holds:*
Let n be a positive integer and let $\alpha = \sum_{i<d} m_i\alpha_i + n\alpha_d$ be in Λ_y. Then $|m_i| \leq \frac{\lambda_d(y)}{\lambda_i(y)}n$ for $i = 1, \dots, d-1$ implies $\alpha \in \mathcal{C}'_n(x)$. Conversely, $\alpha \in \mathcal{C}'_n(x)$ implies $|m_i| \ll A_d\frac{\lambda_d(y)}{\lambda_i(y)}n$.

Proof. Observe first that $\|\alpha_x^\perp\| \asymp \|\alpha_d\|$. Also, the condition $\alpha \in \mathcal{C}'_n(x)$ is equivalent to $\|\sum_{i<d} m_i\alpha_i + n(\alpha_d - \alpha_x^\perp)\| \leq A_d n\|\alpha_x^\perp\|$ and implies $\|\sum_{i<d} m_i\alpha_i\| \ll A_d n\|\alpha_x^\perp\|$. By Lemma 8.5, if $|m_i| \leq \frac{\lambda_d(y)}{\lambda_i(y)}n$ then $\|\sum_{i<d} m_i\alpha_i\| \ll n\|\alpha_d\|$ so that $\alpha \in \mathcal{C}'_n(x)$ if A_d is chosen large enough. On the other hand, if $v_i^\perp$ denotes the orthogonal projection of α_i onto the line orthogonal to the subspace spanned by $\alpha_j, j \neq i$, then by (M"), $\|\sum_{j<d} m_j\alpha_j\| \gg |m_i|\|v_i^\perp\| \gg |m_i|\lambda_i(y)$ so that if $\alpha \in \mathcal{C}'_n(x)$ then $|m_i|\lambda_i(y) \ll A_d n\|\alpha_d\|$, which, by Lemma 8.5 again, is $\ll A_d\lambda_d(y)n$. □

Let $^* : \mathbb{R}^d \to \mathbb{R}^d$ be the linear map that sends $z = \sum t_i\alpha_i$ to $z^* = \sum t_i\alpha_i^*$ where $\alpha_1^*, \dots, \alpha_d^*$ is the orthogonal basis obtained by applying the Gram-Schmidt orthogonalization process to the basis $\alpha_1, \dots, \alpha_d$. Since the basis $\alpha_1, \dots, \alpha_d$ is reduced, by (M"),

$$\sigma_i := \|\alpha_i^*\| \asymp \lambda_i(y).$$

Moreover, the linear map $^* : \mathbb{R}^d \to \mathbb{R}^d$ is bilipschitz with constant $\ll 1$.

Lemma 8.7. *Let $\alpha = \sum_{i<d} m_i\alpha_i + n\alpha_d$ be in $\mathcal{C}'_n(x)$. Then $\widehat{\lambda}_1(\alpha) \asymp \widehat{\lambda}_1(L_\alpha)$ where*

$$L_\alpha = \mathbb{Z}\alpha_1^* + \cdots + \mathbb{Z}\alpha_{d-1}^* + \mathbb{Z}\sum_{i<d} \frac{m_i}{n}\alpha_i^*.$$

Proof. Let U_α be the linear map defined by $U_\alpha(\alpha_i^*) = \alpha_i^*$ for $i < d$, and $U_\alpha(\alpha^*) = n\alpha_d^*$. In the orthonormal basis $e_i = \frac{1}{\sigma_i}\alpha_i^*$, we have $U_\alpha(e_i) = e_i$ for $i < d$ and

$$U_\alpha(e_d) = e_d - \frac{1}{\sigma_d}\sum_{i<d}\frac{m_i}{n}\alpha_i^*.$$

Since $|m_i| \ll \frac{\lambda_d(y)}{\lambda_i(y)}n$, U_α is bilipschitz with constant

$$\ll 1 + \frac{1}{\sigma_d}\sum_{i<d}\frac{|m_i|}{n}\sigma_i \asymp 1 + \frac{1}{\lambda_d(y)}\sum_{i<d}\frac{|m_i|}{n}\lambda_i(y) \ll 1.$$

It follows that the map $\psi : x \to U_\alpha(x^*)$ is bilipschitz as well. By Lemma 8.2, $\widehat{\lambda}_1(\Lambda_{\alpha^\perp}) \asymp \widehat{\lambda}_1(\psi(\Lambda_y)_{\psi(\alpha)^\perp})$ and since

$$\begin{aligned}\psi(\Lambda_y) &= \mathbb{Z}\psi(\alpha_1) + \cdots + \mathbb{Z}\psi(\alpha_d)\\ &= \mathbb{Z}\alpha_1^* + \cdots + \mathbb{Z}\alpha_{d-1}^* + \mathbb{Z}\sigma_d\left(e_d - \frac{1}{\sigma_d}\sum_{i<d}\frac{m_i}{n}\alpha_i^*\right)\end{aligned}$$

the orthogonal projection of $\psi(\Lambda_y)$ onto $\psi(\alpha)^\perp$ is the lattice L_α. □

Proof of Proposition 8.1. By Lemma 8.7, we are reduced to finding a lower bound, for a given n, on the number of primitive $\alpha = \sum_{i<d} m_i\alpha_i + n\alpha_d$ in $\mathcal{C}'_n(x)$ such that

$$\widehat{\lambda}_1(L_\alpha) > C\varepsilon$$

where C is a constant depending only on the dimension.

Observe that, when α is primitive, $\mathrm{vol}(L_\alpha) = \frac{\sigma_1\cdots\sigma_{d-1}}{n}$. Hence, the condition $\widehat{\lambda}_1(L_\alpha) > C\varepsilon$ is equivalent to

$$\lambda_1(L_\alpha) > C\varepsilon\left(\frac{\sigma_1\cdots\sigma_{d-1}}{n}\right)^{1/(d-1)}.$$

Changing C, we may assume that we deal with the sup norm instead of the Euclidean norm.

Set $m = (m_1, \ldots, m_{d-1})$ and

$$L(m,n) = \mathbb{Z}ne_1 + \cdots + \mathbb{Z}ne_{d-1} + \mathbb{Z}\sum_{i<d} m_i e_i.$$

$L(m,n)$ is the image of L_α by the linear map that sends $\sigma_i e_i$ to ne_i, $i = 1, \ldots, d-1$. So $\widehat{\lambda}_1(L_\alpha) > C\varepsilon$ holds if the lattice $L(m,n)$ has no nonzero points in the box

$$B(n,y,\varepsilon) = C\varepsilon\left(\frac{\sigma_1\cdots\sigma_{d-1}}{n}\right)^{1/(d-1)}\prod_{i<d}\left[-\frac{n}{\sigma_i},\frac{n}{\sigma_i}\right]$$

and if α is primitive, i.e., $(m_1, ..., m_{d-1}, n)$ is a primitive element of $\mathbb{Z}^d$.

Since the lattice $L(m,n)$ depends only on $m_1, \ldots, m_{d-1}$ mod n, we have a lower bound on the number of α with $\widehat{\lambda}_1(L_\alpha) > C\varepsilon$ by counting the number of α with $0 \le m_1, \ldots, m_{d-1} < n$

such that $L(m,n)$ has no nonzero point in $B(n,y,\varepsilon)$ and then multiplying by the number of disjoint translates of the parallelepipeds

$$\left\{ t = \sum_{i<d} t_i \alpha_i + n\alpha_d : t_i \in [0,n[, i = 1,\ldots,d-1 \right\}$$

that are contained in $\mathcal{C}'_n(x)$. Since the system of inequalities $|t_i| \le \frac{\lambda_d(y)}{\lambda_i(y)} n$ for $i = 1,\ldots,d-1$, implies $t \in \mathcal{C}'_n(x)$, by Lemma 8.6, the number of translates is $\gg \frac{\lambda_d(y)^{d-1}}{\lambda_1(y)\cdots\lambda_{d-1}(y)}$.

Finally, by Lemma 8.4, the number of α in $\mathcal{C}'_n(x) \cap \Lambda_y(\varepsilon)$ is

$$\gg \frac{\lambda_d^{d-1}}{\lambda_1 \cdots \lambda_{d-1}} \times \phi(n) \left(n^{d-2} - 3^{d-1} \frac{D_1(n)}{n} \prod_{i<d} l_i \right)$$

where $l_i = C\varepsilon \left(\frac{\sigma_1 \cdots \sigma_{d-1}}{n} \right)^{1/(d-1)} \frac{n}{\sigma_i}$ and λ_i is an abbreviation for $\lambda_i(y)$. Hence, with a new large enough constant C, it is

$$\begin{aligned} &\gg \frac{\lambda_d^{d-1}}{\lambda_1 \cdots \lambda_{d-1}} \times \phi(n) \left(n^{d-2} - C \frac{D_1(n)}{n} \varepsilon^{d-1} \left(\frac{\lambda_1 \cdots \lambda_{d-1}}{n} \right) \frac{n^{d-1}}{\lambda_1 \cdots \lambda_{d-1}} \right) \\ &= \frac{\lambda_d^{d-1}}{\lambda_1 \cdots \lambda_{d-1}} \times \phi(n) \left(n^{d-2} - C\varepsilon^{d-1} D_1(n) n^{d-3} \right). \end{aligned}$$

The last thing to do is bound $\operatorname{card}(\mathcal{C}'_n(x) \cap \Lambda_y)$ from above. By Lemma 2.3,

$$\operatorname{card}(\mathcal{C}'_n(x) \cap \Lambda_y) \ll \frac{\operatorname{diam} \mathcal{C}'_n(x)^{d-1}}{\operatorname{vol}(\Lambda'_y)} \ll \frac{(n\lambda_d)^{d-1}}{\lambda_1 \cdots \lambda_{d-1}}$$

hence,

$$\begin{aligned} \frac{\operatorname{card}(\mathcal{C}'_n(x) \cap \Lambda_y(\varepsilon))}{\operatorname{card}(\mathcal{C}'_n(x) \cap \Lambda_y)} &\gg \frac{\frac{\lambda_d^{d-1}}{\lambda_1 \cdots \lambda_{d-1}} \times \phi(n)(n^{d-2} - C\varepsilon^{d-1} D_1(n) n^{d-3})}{\frac{(n\lambda_d)^{d-1}}{\lambda_1 \cdots \lambda_{d-1}}} \\ &= \frac{\phi(n)}{n} \left(1 - C\varepsilon^{d-1} \frac{D_1(n)}{n} \right). \end{aligned}$$

□

We are now able to prove Proposition 6.11, i.e.

$$S_1(N,t) = \sum_{n=1}^{N} \frac{1}{n^{1+(t-d)\frac{d}{d-1}}} \frac{\operatorname{card}(\mathcal{C}'_n(x) \cap \Lambda_y(\varepsilon))}{\operatorname{card}(\mathcal{C}'_n(x) \cap \Lambda_y)} \gg \sum_{n=1}^{N} \frac{1}{n^{1+(t-d)\frac{d}{d-1}}}$$

when ε is small enough and $t > d$.

Proof. It is well known (see [13]) that

$$\Phi(n) \asymp D_s(n) \asymp n^2$$

where $\Phi(n) = \sum_{k \le n} \phi(k)$ and $D_s(n) = \sum_{k \le n} D_1(k)$. Therefore,

$$\begin{aligned}
\sum_{n=1}^{N} \frac{1}{n^{1+\delta}} \frac{\phi(n)}{n} &= \sum_{n=1}^{N} \frac{1}{n^{2+\delta}} (\Phi(n) - \Phi(n-1)) \\
&= \sum_{n=1}^{N-1} \left(\frac{1}{n^{2+\delta}} - \frac{1}{(n+1)^{2+\delta}} \right) \Phi(n) + \frac{\Phi(N)}{N^{2+\delta}} \\
&\gg \sum_{n=1}^{N-1} \frac{1}{n^{3+\delta}} n^2
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n=1}^{N} \frac{1}{n^{1+\delta}} \frac{D_1(n)}{n} &= \sum_{n=1}^{N} \frac{1}{n^{2+\delta}} (D_s(n) - D_s(n-1)) \\
&= \sum_{n=1}^{N-1} \left(\frac{1}{n^{2+\delta}} - \frac{1}{(n+1)^{2+\delta}} \right) D_s(n) + \frac{D_s(N)}{N^{2+\delta}} \\
&\ll \sum_{n=1}^{N-1} \frac{1}{n^{3+\delta}} n^2.
\end{aligned}$$

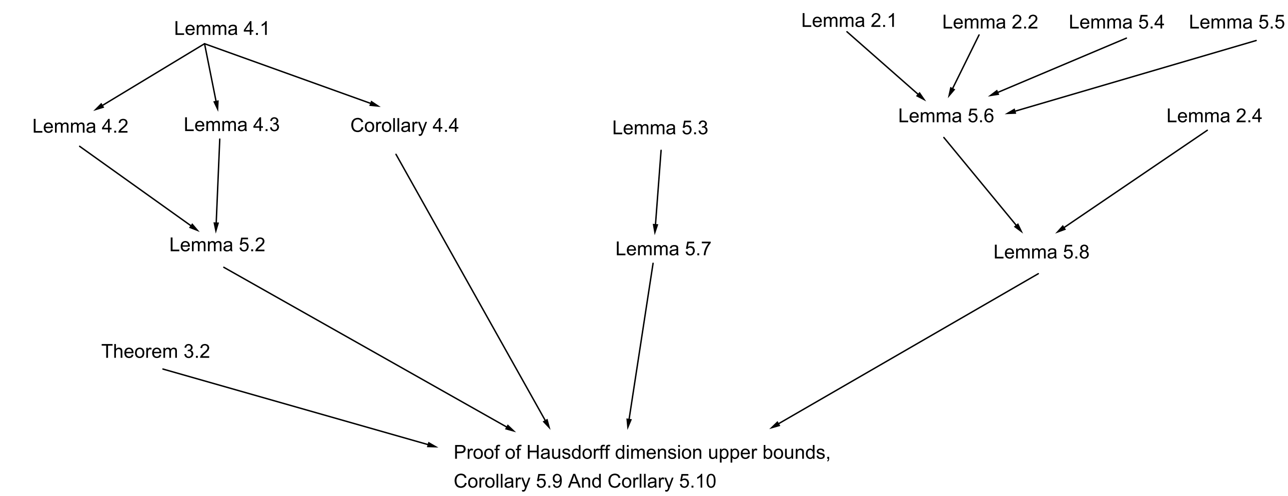
□

Remark 8.8. Precise asymptotics are known for counting problems such as

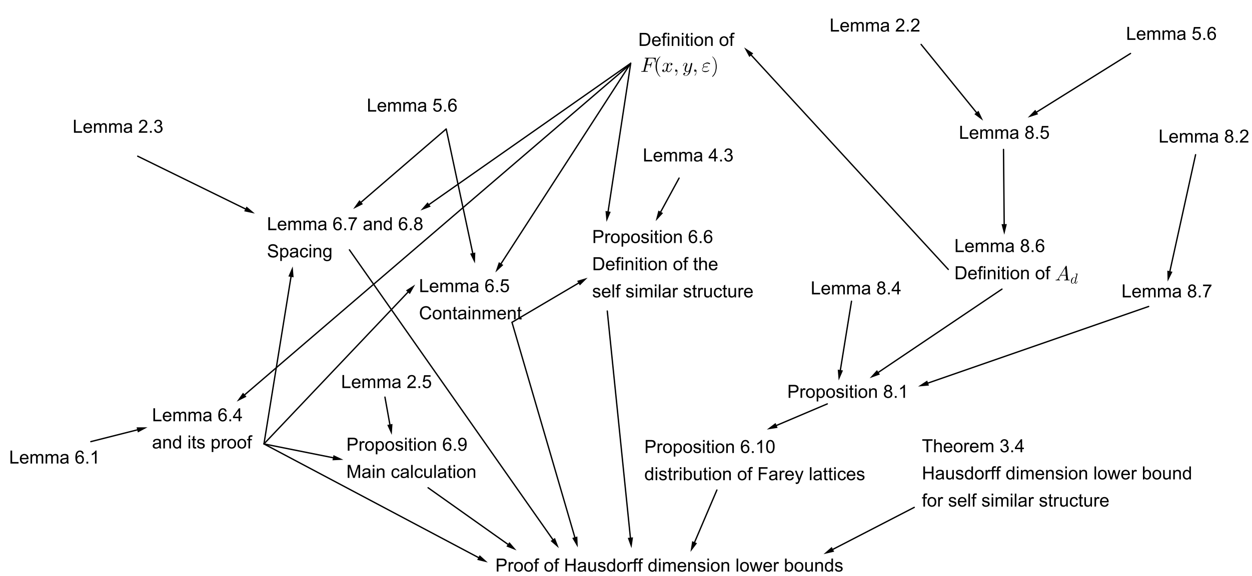
$$\lim_{T \to \infty} \frac{\operatorname{card} (\Lambda(\varepsilon) \cap B(0,T))}{\operatorname{card} (\Lambda \cap B(0,T))}$$

which can be computed using the results of Wolfgang Schmidt in [25]. Such estimates give good control when T is larger than some T_0 (which may depend on Λ), but provide little information for smaller values of T.

9. Appendix - Flowchart of the theorems



Upper bound proof.



Lower bound proof.

References

[1] J. S. Athreya and J. Chaika, *The Hausdorff Dimension of Non-Uniquely Ergodic directions in $H(2)$ is almost everywhere 1/2*, Geom. Top., to appear

[2] Baker, R. C. *Singular n-tuples and Hausdorff dimension*, Math. Proc. Cambridge Philos. Soc., **81** (1977), no 3, 377-385.

[3] Baker, R. C. *Singular n-tuples and Hausdorff dimension II*, Math. Proc. Cambridge Philos. Soc., **111** (1992), no 3, 577-384.

[4] Cassels, J. S. *An introduction to Diophantine approximation.* Cambridge Tracts in Mathematics and Mathematical Physics, No. 45. Cambridge University Press, New York, 1957, x+166 pp.

[5] Cheung, Y. *Hausdorff dimension of the set of points on divergent trajectories of a homogeneous flow on a product space*, Erg. Th. Dyn. Sys., **27** (2007), 65-85.

[6] Cheung, Y. *Hausdorff dimension of the set of singular pairs*, Ann. of Math., **173** (2011), 127-167.

[7] Cheung, Y., Hubert, P. and Masur, H., *Dichotomy of Hausdorff dimension for the set of nonergodic directions*, Inventiones, **183**, (2011), 337-383.

[8] Chevallier, N. *Best simultaneous Diophantine approximations and multidimensional continued fraction expansions*, Moscow J. of Comb. and Num. Th., **3** (2013), no 1.

[9] Chevallier, N. *Meilleures approximations diophantiennes d'un élément du tore T^d, (French)* Acta Arith. 97 (2001), no. 3, 219-240.

[10] Dani, S. G. *Divergent trajectories of flows on homogeneous spaces an Diophantine approximations,* J. Reine Angew. Math. **359** (1985), 55-89. (Corrections: J. Reine Angew. Math. **360** (1985), 214.)

[11] Davenport, H., Schmidt, W. *Dirichlet's theorem on diophantine approximation,* Symposia Mathematica, Vol. IV (INDAM, Rome, 1968/69), pp. 113-132, 1970.

[12] Davenport, H., Schmidt, W. *Dirichlet's theorem on diophantine approximation. II ,* Acta Arith. 16 (1969/1970) 413-424.

[13] Hardy, G.H., Wright, E.M. *An Introduction to the Theory of Numbers.* Fifth Edition, Oxford, 1979.

[14] Kadyrov, S., Kleinbock, D., Lindenstrauss, E., and Margulis, G.A., *Singular systems of linear forms and non-escape of mass in the space of lattices*, preprint.

[15] Kadyrov, S., and Pohl, A.D. *Amount of failure of upper-semicontinuity of entropy in noncompact rank one situations, and Hausdorff dimension*, preprint.

[16] Khintchine, A. *Über eine klasse linearer diophantische approximationen,* Rendiconti Circ. Mat. Soc. Palermo, 50 (1926), 170-195.

[17] Kleinbock, D., Weiss, Barak *Friendly measures, homogeneous flows, and singular vectors*, in Algebraic and Topological Dynamics, S. Kolyada, Y. Manin and T. Ward, eds, Cont. Math. **385** (2005).

[18] Kleinbock, D., Weiss, Barak *Dirichlet's theorem on Diophantine approximation and homogeneous flows,* J. Mod. Dyn. 2 (2008), no. 1, 43-62.

[19] Lagarias, J. C. *Best simultaneous Diophantine approximations. II. Behavior of consecutive best approximations,* Pacific J. Math. 102 (1982), no. 1, 61–88.

[20] Lagarias, J. C. *, Some new results in simultaneous Diophantine approximation,* Proc. Queens's University Number Theory Conference 1979, (P. Ribenboim, Ed.), Queen's Papers in Pure and Applied Math. No. 54, Queen's University, 1980, 453–474.

[21] Masur, H. *Hausdorff dimension of the set of nonergodic foliations of a quadratic differential,* Duke Math. J. **66** (1992), 387–442.

[22] Masur, H. and Smillie, J. *Hausdorff dimension of sets of nonergodic foliations,* Ann. of Math., **134** (1991), 455-543.

[23] Moshchevitin, N. G. *Best Diophantine approximations: the phenomenon of degenerate dimension, Surveys in geometry and number theory*, reports on contemporary Russian mathematics, 158–182, London Math. Soc. Lecture Note Ser., 338, Cambridge Univ. Press, Cambridge, 2007.

[24] Rynne, B.P. *A lower bound for the Hausdorff dimension of sets of singular n-tuples,* Math. Proc. Cambridge Philos. Soc., **107** (1990), no 2, 387-394.
[25] Schmidt, W. *Distribution of sub lattices of* $\mathbb{Z}^m$, Monasheft Math., **125** (1998), 37-81.
[26] Schmidt, W. *Asymptotic formulae for point lattices of bounded determinant and subspaces of bounded height*, Duke Math. J, **35** (1968), 327-340.
[27] Schmidt, W. *Diophantine Approximation*, Lecture Notes in Mathematics **(785)** Springer (1980).
[28] van der Waerden, B. L. *Die Reduktionstheorie der positiven quadratishen Formen*, Acta. Math., **96** (1956), 265-309.
[29] Weiss, Barak *Divergent trajectories on noncompact parameter spaces,* Geom. Funct. Anal., **14** (2004), no. 1, 94-149.
[30] Weiss, Barak *Divergent trajectories and* $\mathbb{Q}$*-rank*, Israel J. Math., **152** (2006), 221-227.
[31] Yang, L. *Hausdorff dimension of divergent trajectories of diagonal geodesic flow on product of hyperbolic spaces*, preprint.

San Francisco State University, 1600 Holloway Ave, San Francisco, CA 94132, U.S.A.
E-mail address: ycheung@sfsu.edu

Haute Alsace University, 4 Rue des Frères Lumière, 68093 Mulhouse Cedex, France
E-mail address: nicolas.chevallier@uha.fr