HAUSDORFF DIMENSIONS IN p-ADIC ANALYTIC GROUPS

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Abstract. Let G be a finitely generated pro-p group, equipped with the p-power series $\mathcal{P}\colon G_i = G^{p^i}$, $i \in \mathbb{N}_0$. The associated metric and Hausdorff dimension function $\mathrm{hdim}_G^{\mathcal{P}}\colon \{X \mid X \subseteq G\} \to [0,1]$ give rise to

$$\mathrm{hspec}^{\mathcal{P}}(G) = \{\mathrm{hdim}_G^{\mathcal{P}}(H) \mid H \le G\} \subseteq [0,1],$$

the Hausdorff spectrum of closed subgroups of G. In the case where G is p-adic analytic, the Hausdorff dimension function is well understood; in particular, $\mathrm{hspec}^{\mathcal{P}}(G)$ consists of finitely many rational numbers closely linked to the analytic dimensions of subgroups of G.

Conversely, it is a long-standing open question whether $|\mathrm{hspec}^{\mathcal{P}}(G)| < \infty$ implies that G is p-adic analytic. We prove that the answer is yes, in a strong sense, under the extra condition that G is soluble.

Furthermore, we explore the problem and related questions also for other filtration series, such as the lower p-series, the Frattini series, the modular dimension subgroup series and quite general filtration series. For instance, we prove, for $p > 2$, that every countably based pro-p group G with an open subgroup mapping onto $\mathbb{Z}_p \oplus \mathbb{Z}_p$ admits a filtration series $\mathcal{S}$ such that $\mathrm{hspec}^{\mathcal{S}}(G)$ contains an infinite real interval.

1. Introduction

The notion of Hausdorff dimension was pioneered by Hausdorff and developed systematically by Besicovitch and others starting from the 1930s. It is central to the subject of fractal geometry; compare [8]. More recently, the concept of Hausdorff dimension has led to fruitful applications in the context of profinite groups; e.g., see [4, 3, 6, 2, 12, 7, 10, 11, 9]. Let G be a countably based profinite group and fix a *filtration series* $\mathcal{S}$ of G, i.e., a descending chain $G = G_0 \supseteq G_1 \supseteq \dots$ of open normal subgroups $G_i \trianglelefteq G$ such that $\bigcap_i G_i = 1$. Such a chain forms a base of neighbourhoods of the identity and, if G is infinite, induces a translation-invariant metric on G such that the distance between $x, y \in G$ is $d^{\mathcal{S}}(x,y) = \inf\{|G : G_i|^{-1} \mid x \equiv y \pmod{G_i}\}$. This in turn yields the *Hausdorff dimension* $\mathrm{hdim}_G^{\mathcal{S}}(U) \in [0,1]$ of any subset $U \subseteq G$, with respect to $\mathcal{S}$.

Based on work of Abercrombie [1], Barnea and Shalev [4] gave the following 'algebraic' interpretation of the Hausdorff dimension of a closed subgroup H of G:

$$\mathrm{hdim}_G^{\mathcal{S}}(H) = \varliminf_{i\to\infty} \frac{\log|HG_i : G_i|}{\log|G : G_i|}; \tag{1.1}$$

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we are interested in Hausdorff dimension as a density function in this sense and the formula motivates the convenient backup definition $\mathrm{hdim}_G^{\mathcal{S}}(H) = \log|H|/\log|G|$ for finite groups G. The *Hausdorff spectrum* of G, with respect to $\mathcal{S}$, is

$$\mathrm{hspec}^{\mathcal{S}}(G) = \{\mathrm{hdim}_G^{\mathcal{S}}(H) \mid H \leq G\} \subseteq [0,1]$$

and reflects the range of Hausdorff dimensions of closed subgroups H of G.

Throughout we will be concerned with pro-p groups, where p denotes a fixed prime. Even for comparatively well behaved groups, such as p-adic analytic pro-p groups G, the Hausdorff dimension function and the Hausdorff spectrum of G are known to be sensitive to the choice of $\mathcal{S}$; see [4, Ex. 2.5]. However, as emphasised in [4], for every finitely generated pro-p group G there is a rather natural choice of filtration, namely the *p-power series*

$$\mathcal{P}\colon \pi_i(G) = G^{p^i} = \langle x^{p^i} \mid x \in G\rangle, \quad i \in \mathbb{N}_0.$$

This is substantiated by the following result, which has recently been generalised by Fernández-Alcober, Giannelli and González-Sánchez [9] to cover Hausdorff dimensions of analytic subgroups of R-analytic profinite groups over general pro-p domains R.

Theorem 1.1 (Barnea and Shalev [4])**.** *Let G be an infinite p-adic analytic pro-p group. Then every closed subgroup $H \leq G$ satisfies*

$$\mathrm{hdim}_G^{\mathcal{P}}(H) = \dim(H)/\dim(G),$$

where $\dim(X)$ *denotes the analytic dimension of a p-adic manifold X.*

In particular, $\mathrm{hspec}^{\mathcal{P}}(G) \subseteq \{0, 1/d, 2/d, \ldots, d-1/d, 1\}$*, where* $d = \dim(G) \geq 1$.

It remains an open problem whether this theorem can actually be turned into a characterisation of p-adic analytic pro-p groups in terms of Hausdorff spectra, in the spirit of [5, Interl. A]. Contrary to what is perhaps suggested in the introduction of [4], one has to be rather careful, as the assertion in Theorem 1.1 does not generally remain valid with respect to other standard filtration series. In passing, we mention another relevant result from [4], which in turn relies on a theorem of Zelmanov [16]: a finitely generated pro-p group G is p-adic analytic if and only if G contains no infinite closed subgroup H of Hausdorff dimension $\mathrm{hdim}_G^{\mathcal{P}}(H) = 0$ with respect to the p-power series $\mathcal{P}$.

In this paper we consider, in addition to the p-power filtration, three other natural and commonly used filtration series on finitely generated pro-p groups. We recall that the *lower p-series* (sometimes called lower p-central series) of a finitely generated pro-p group G is given recursively by

$$\mathcal{L}\colon P_1(G) = G, \quad \text{and} \quad P_i(G) = P_{i-1}(G)^p\,[P_{i-1}(G), G] \quad \text{for } i \geq 2,$$

while the *Frattini series* of G is given recursively by

$$\mathcal{F}\colon \Phi_0(G) = G, \quad \text{and} \quad \Phi_i(G) = \Phi_{i-1}(G)^p\,[\Phi_{i-1}(G), \Phi_{i-1}(G)] \quad \text{for } i \geq 1.$$

The (modular) *dimension subgroup series* (sometimes called Jennings or Zassenhaus series) of G is closely related to the filtration of the group ring $\mathbb{F}_pG$ by powers of its augmentation ideal; it can be defined recursively by

$$\mathcal{D}\colon D_1(G) = G, \quad \text{and} \quad D_i(G) = D_{\lceil i/p\rceil}(G)^p \prod_{1\leq j<i} [D_j(G), D_{i-j}(G)] \quad \text{for } i \geq 2.$$

Theorem 1.1 motivates a number of questions, related to, but more specific than Problem 1 in [4].

Problem 1.2. Given any two filtration series $\mathcal{S}_1, \mathcal{S}_2$ of a finitely generated pro-p group G, is it true that $\mathrm{hspec}^{\mathcal{S}_1}(G)$ is finite if and only if $\mathrm{hspec}^{\mathcal{S}_2}(G)$ is finite?

We solve this problem in the negative, by constructing for $p > 2$ a tailor-made filtration series $\mathcal{S}$ of the abelian pro-p group $G \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$ such that $\mathrm{hspec}^{\mathcal{S}}(G)$ contains an infinite real interval. By contrast, the Hausdorff spectrum $\mathrm{hspec}^{\mathcal{P}}(G) = \{0, 1/2, 1\}$ of the same group with respect to the p-power series and indeed its spectra with respect to other conventional filtration series are discrete; compare Proposition 3.2. More generally, we establish the following result.

Theorem 1.3. *Let G be a countably based pro-p group that has an open subgroup mapping surjectively onto $\mathbb{Z}_p \oplus \mathbb{Z}_p$. Then there exists a filtration series $\mathcal{S}$ of G such that* $\mathrm{hspec}^{\mathcal{S}}(G)$ *contains the complete real interval* $[1/p+1, p-1/p+1]$.

This unexpectedly erratic behaviour motivates us to focus on Hausdorff dimensions with respect to one of the four natural filtrations $\mathcal{P}, \mathcal{L}, \mathcal{F}, \mathcal{D}$.

Problems 1.4. Let G be a finitely generated pro-p group, and let $\mathcal{P}, \mathcal{L}, \mathcal{F}, \mathcal{D}$ denote the p-power series, the lower p-series, the Frattini series and the dimension subgroup series of G.

1. Suppose that $|\mathrm{hspec}^{\mathcal{S}}(G)| < \infty$ for at least one $\mathcal{S} \in \{\mathcal{P}, \mathcal{L}, \mathcal{F}, \mathcal{D}\}$. Does it follow that $|\mathrm{hspec}^{\mathcal{S}}(G)| < \infty$ for all $\mathcal{S} \in \{\mathcal{P}, \mathcal{L}, \mathcal{F}, \mathcal{D}\}$?
2. Suppose that $\mathcal{S} \in \{\mathcal{L}, \mathcal{F}, \mathcal{D}\}$. If G is p-adic analytic, does it follow that $\mathrm{hspec}^{\mathcal{S}}(G)$ is finite? What is $\mathrm{hspec}^{\mathcal{S}}(G)$?
3. Suppose that $\mathcal{S} \in \{\mathcal{P}, \mathcal{L}, \mathcal{F}, \mathcal{D}\}$ and $|\mathrm{hspec}^{\mathcal{S}}(G)| < \infty$. Does it follow that G is p-adic analytic?

Regarding Problems 1.4 (1) and (2) we obtain a positive partial solution.

Proposition 1.5. *Let G be a p-adic analytic pro-p group. Then the Hausdorff dimension functions with respect to the p-power series $\mathcal{P}$, the Frattini series $\mathcal{F}$ and the dimension subgroup series $\mathcal{D}$ coincide on closed subgroups $H \le G$, i.e.,*

$$\mathrm{hdim}_G^{\mathcal{P}}(H) = \mathrm{hdim}_G^{\mathcal{F}}(H) = \mathrm{hdim}_G^{\mathcal{D}}(H).$$

Consequently, $\mathrm{hspec}^{\mathcal{P}}(G) = \mathrm{hspec}^{\mathcal{F}}(G) = \mathrm{hspec}^{\mathcal{D}}(G)$.

The situation for the lower p-series, even for p-adic analytic pro-p groups, is less clear. In Example 4.1 we illustrate that there are p-adic analytic pro-p groups G for which $\mathrm{hspec}^{\mathcal{P}}(G) \neq \mathrm{hspec}^{\mathcal{L}}(G)$. In fact, we provide a concrete family of infinite p-adic analytic pro-p groups G such that $|\mathrm{hspec}^{\mathcal{L}}(G)|/\dim(G)$ is unbounded as $\dim(G) \to \infty$ within this family. This raises yet more interesting questions.

Problems 1.6. Does there exist, for any given p-adic analytic pro-p group G, a uniform bound $b(G)$ for $|\mathrm{hspec}^{\mathcal{S}}(G)|$, as $\mathcal{S}$ runs through all filtration series of G with $|\mathrm{hspec}^{\mathcal{S}}(G)| < \infty$?

If yes, does there exist, for any $n \in \mathbb{N}$, a uniform bound $b(n)$ for $b(G)$, as G runs through all p-adic analytic pro-p groups of dimension $\dim(G) \le n$?

Regarding Problem 1.4 (3) we obtain the following positive partial solution, in a strong sense. This gives a new characterisation of p-adic analytic pro-p groups amongst soluble pro-p groups in terms of Hausdorff dimension.

Theorem 1.7. *Let G be a finitely generated soluble pro-p group, and let $\mathcal{S}$ be any one of the p-power series $\mathcal{P}$, the Frattini series $\mathcal{F}$ or the dimension subgroup series $\mathcal{D}$. If G is not p-adic analytic then the Hausdorff spectrum* $\mathrm{hspec}^{\mathcal{S}}(G)$ *with respect to $\mathcal{S}$ contains an infinite real interval.*

Corollary 1.8. *Let G be a finitely generated soluble pro-p group, and let $\mathcal{S}$ be one of $\mathcal{P}$, $\mathcal{F}$ or $\mathcal{D}$. Then G is p-adic analytic if and only if* $\mathrm{hspec}^{\mathcal{S}}(G)$ *is finite.*

It remains an open question whether a statement similar to Theorem 1.7 holds true for the lower p-series.

Finally, we verify that Barnea and Shalev's characterisation of p-adic analytic pro-p groups in terms of Hausdorff dimension [4, Th. 1.3] remains partly valid with respect to the other filtration series considered here.

Theorem 1.9. *Let G be a finitely generated pro-p group, and let $\mathcal{S}$ be any one of the p-power series $\mathcal{P}$, the Frattini series $\mathcal{F}$ or the dimension subgroup series $\mathcal{D}$. Then the following are equivalent.*

(i) *The group G is p-adic analytic.*
(ii) *There exists a constant $c \in (0,1]$ such that every infinite closed subgroup $H \le G$ satisfies $\mathrm{hdim}^{\mathcal{S}}_G(H) \ge c$.*
(iii) *Every infinite closed subgroup $H \le G$ satisfies $\mathrm{hdim}^{\mathcal{S}}_G(H) > 0$.*
(iv) *The group G is finite, or there exists a closed subgroup $H \le G$ such that $H \cong \mathbb{Z}_p$ and $\mathrm{hdim}^{\mathcal{S}}_G(H) > 0$.*

Again, the situation for the lower p-series is less clear. In Example 4.3 we illustrate, for $\mathcal{S} = \mathcal{L}$, that (iv) does not generally imply (i), (ii) or (iii). On the other hand, we establish in Proposition 4.4, for $\mathcal{S} = \mathcal{L}$, that (i) still implies (ii), (iii) and (iv).

Notation and Organisation. Throughout, $\varliminf a_i = \varliminf_{i\to\infty} a_i$ denotes the lower limit (limes inferior) of a sequence $(a_i)_{i\in\mathbb{N}}$ in $\mathbb{R} \cup \{\pm\infty\}$. At times, some terms a_i may evaluate to $y/0$ for some $y \in \mathbb{R}_0$. For definiteness, we agree that $0/0 = 1$ and $y/0 = \infty$ for $y > 0$. Tacitly, subgroups of profinite groups are generally understood to be closed subgroups. As a default we set $P_0(G) = D_0(G) = G$ for every finitely generated pro-p group G.

Theorem 1.3 is proved in Section 3. Proposition 1.5 and Theorem 1.9 are established in Section 4. Theorem 1.7 is proved in Section 5.

2. Preliminaries

In this section we collect some auxiliary results for later use.

Proposition 2.1. *Let $\varphi \colon G \to \widetilde{G}$ be an epimorphism between infinite countably based profinite groups, and let $\widetilde{\mathcal{S}} \colon \widetilde{G}_0 \supseteq \widetilde{G}_1 \supseteq \ldots$ be a filtration series of $\widetilde{G}$. Then there exists a filtration series $\mathcal{S} \colon G_0 \supseteq G_1 \supseteq \ldots$ of G such that*

$$G_i\varphi = \widetilde{G}_i \quad \text{for all } i \in \mathbb{N}_0, \qquad \text{and} \qquad \mathrm{hspec}^{\mathcal{S}}(G) = \mathrm{hspec}^{\widetilde{\mathcal{S}}}(\widetilde{G}).$$

Proof. Clearly we can find a filtration series $G = G_0^* \supseteq G_1^* \supseteq \ldots$ of G such that $G_i^*\varphi \subseteq \widetilde{G}_i$ for all $i \in \mathbb{N}_0$. The function

$$f \colon \mathbb{N}_0 \to \mathbb{N}_0, \quad f(i) = \max\{j \in \mathbb{N}_0 \mid \widetilde{G}_i \subseteq G_j^*\varphi\}$$

is non-decreasing and $\lim_{i\to\infty} f(i) = \infty$. Choose, for each $i \in \mathbb{N}_0$, a suitable subset $X_i \subseteq G^*_{f(i)}$ such that $(\langle X_i\rangle^G G_i^*)\varphi = \widetilde{G}_i$. By putting $G_i^{**} = \langle X_i \cup X_{i+1} \cup \ldots\rangle^G G_i^*$,

$i \in \mathbb{N}_0$, we construct a filtration series $G = G_0^{**} \supseteq G_1^{**} \supseteq \dots$ of G such that $G_i^{**}\varphi = \widetilde{G}_i$ for all $i \in \mathbb{N}_0$.

Set $K = \ker(\varphi) \trianglelefteq G$. By putting $K_i = K \cap G_i^{**}$, $i \in \mathbb{N}_0$, we obtain a filtration series of K that consists of normal subgroups of G. Choosing a non-decreasing function $h\colon \mathbb{N}_0 \to \mathbb{N}_0$ such that

$$\lim_{i\to\infty} h(i) = \infty \qquad \text{and} \qquad \frac{\log|K : K_{h(i)}|}{\log|\widetilde{G} : \widetilde{G}_i|} \underset{i\to\infty}{\longrightarrow} 0,$$

we define the filtration series

$$\mathcal{S}\colon G_i = K_{h(i)} G_i^{**}, \quad i \in \mathbb{N}_0,$$

of G. The filtration series $\mathcal{S}$ satisfies

$$\frac{\log|KG_i : G_i|}{\log|G : KG_i|} = \frac{\log|K : K \cap G_i|}{\log|G : KG_i|} \leq \frac{\log|K : K_{h(i)}|}{\log|\widetilde{G} : \widetilde{G}_i|} \underset{i\to\infty}{\longrightarrow} 0.$$

Since

$$\frac{\log|KG_i : G_i|}{\log|G : G_i|} \leq \frac{\log|KG_i : G_i|}{\log|G : KG_i|},$$

this ensures, in particular, that $\mathrm{hdim}_G^{\mathcal{S}}(K) = 0$ is given by a proper limit, that is

$$\mathrm{hdim}_G^{\mathcal{S}}(K) = \lim_{i\to\infty} \frac{\log|KG_i : G_i|}{\log|G : G_i|} = 0.$$

Furthermore, for every $H \leq G$ we obtain

$$\begin{aligned}
\mathrm{hdim}_G^{\mathcal{S}}(H) &= \underline{\lim} \frac{\log|HG_i : G_i|}{\log|G : G_i|} \\
&= \underline{\lim} \frac{\log|HG_i : (HG_i \cap KG_i)|}{\log|G : KG_i| + \log|KG_i : G_i|} + \frac{\log|(HG_i \cap KG_i) : G_i|}{\log|G : G_i|} \\
&= \underline{\lim} \frac{\log|HKG_i : KG_i| / \log|G : KG_i|}{1 + \underbrace{\log|KG_i : G_i| / \log|G : KG_i|}_{\to 0 \text{ as } i\to\infty}} + \underbrace{\frac{\log|(HG_i \cap KG_i) : G_i|}{\log|G : G_i|}}_{\to 0 \text{ as } i\to\infty} \\
&= \mathrm{hdim}_{\widetilde{G}}^{\widetilde{\mathcal{S}}}(H\varphi).
\end{aligned}$$

We conclude that

$$G_i\varphi = \widetilde{G}_i \quad \text{for all } i \in \mathbb{N}_0, \qquad \text{and} \qquad \mathrm{hspec}^{\mathcal{S}}(G) = \mathrm{hspec}^{\widetilde{\mathcal{S}}}(\widetilde{G}). \qquad \square$$

The following lemma can be verified by routine arguments; e.g., see [13, Lem. 7.1].

Lemma 2.2. *Let G be a countably based profinite group, and let*

$$\mathcal{S}\colon G = G_0 \supseteq G_1 \supseteq \dots, \qquad \mathcal{S}^*\colon G = G_0^* \supseteq G_1^* \supseteq \dots$$

be filtration series of G. Suppose that

$$\lim_{i\to\infty} \frac{\log|G_iG_i^* : G_i|}{\log|G : G_iG_i^*|} = \lim_{i\to\infty} \frac{\log|G_iG_i^* : G_i^*|}{\log|G : G_iG_i^*|} = 0.$$

Then every closed subgroup $H \leq G$ satisfies $\mathrm{hdim}_G^{\mathcal{S}}(H) = \mathrm{hdim}_G^{\mathcal{S}^*}(H)$.

Corollary 2.3. *Let G be a countably based profinite group, and let*

$$\mathcal{X}\colon G = X_0 \supseteq X_1 \supseteq \dots, \qquad \mathcal{Y}\colon G = Y_0 \supseteq Y_1 \supseteq \dots$$

be filtration series of G. Suppose that

$$\mathbb{N}_0 \to \mathbb{N}_0, \quad i \mapsto i^* \qquad \textit{and} \qquad \mathbb{N}_0 \to \mathbb{N}_0, \quad j \mapsto j'$$

are non-decreasing functions such that

$$\lim_{i\to\infty} \frac{\log|X_i Y_{i^*} : X_i|}{\log|G : X_i Y_{i^*}|} = \lim_{i\to\infty} \frac{\log|X_i Y_{i^*} : Y_{i^*}|}{\log|G : X_i Y_{i^*}|} = 0$$

and

$$\lim_{j\to\infty} \frac{\log|X_{j'} Y_j : X_{j'}|}{\log|G : X_{j'} Y_j|} = \lim_{j\to\infty} \frac{\log|X_{j'} Y_j : Y_j|}{\log|G : X_{j'} Y_j|} = 0.$$

Then every closed subgroup $H \le G$ satisfies $\operatorname{hdim}_G^{\mathcal{X}}(H) = \operatorname{hdim}_G^{\mathcal{Y}}(H)$.

Proof. Observe that

$$\mathcal{X}' \colon G = X_{0'} \supseteq X_{1'} \supseteq \ldots, \qquad \mathcal{Y}^* \colon G = Y_{0^*} \supseteq Y_{1^*} \supseteq \ldots$$

are filtration series of G, obtained by thinning out $\mathcal{X}$ and $\mathcal{Y}$. Thus Lemma 2.2 implies, for every closed subgroup $H \le G$,

$$\operatorname{hdim}_G^{\mathcal{X}}(H) = \operatorname{hdim}_G^{\mathcal{Y}^*}(H) \ge \operatorname{hdim}_G^{\mathcal{Y}}(H) = \operatorname{hdim}_G^{\mathcal{X}'}(H) \ge \operatorname{hdim}_G^{\mathcal{X}}(H). \qquad \square$$

3. Filtration series and Hausdorff spectra for $\mathbb{Z}_p \oplus \mathbb{Z}_p$

In this section we prove Theorem 1.3. Observe that the Hausdorff spectrum of a procyclic pro-p group $G \cong \mathbb{Z}_p$ is $\operatorname{hspec}(G) = \{0, 1\}$, no matter which filtration series one chooses. In this section we study, nearly systematically, how the Hausdorff spectrum $\operatorname{hspec}^{\mathcal{S}}(G)$ for the abelian pro-p group $G = \mathbb{Z}_p \oplus \mathbb{Z}_p$ depends on the chosen filtration series $\mathcal{S}$. According to Theorem 1.1 we have $\operatorname{hspec}^{\mathcal{P}}(G) = \{0, 1/2, 1\}$ with respect to the p-power series $\mathcal{P}$.

Writing $v_p \colon \mathbb{Z}_p \to \mathbb{Z} \cup \{\infty\}$ for the standard p-adic valuation on $\mathbb{Z}_p$, we prove the following quite flexible result.

Lemma 3.1. *Let $G = \mathbb{Z}_p \oplus \mathbb{Z}_p$, and let*

$$\mathcal{S} \colon G_i = \langle (p^{a_i}, z_i), (0, p^{b_i}) \rangle, \quad i \in \mathbb{N}_0,$$

be an arbitrary filtration series of G; this means that $a_i, b_i \in \mathbb{N}_0$ and $z_i \in \mathbb{Z}_p$ satisfy the conditions: (i) *$(a_i)_{i\in\mathbb{N}_0}$ and $(b_i)_{i\in\mathbb{N}_0}$ are non-decreasing sequences starting at $a_0 = b_0 = 0$ and diverging to infinity;* (ii) *for each $i \in \mathbb{N}$,*

$$b_{i-1} \le v_p(z_i - p^{a_i - a_{i-1}} z_{i-1}).$$

Let H be a non-trivial, non-open closed subgroup of G. Then either there exists $\lambda \in \mathbb{Z}_p$ such that H is an open subgroup of $\langle (1, \lambda) \rangle$ and

$$\operatorname{hdim}_G^{\mathcal{S}}(H) = \varliminf_{i\to\infty} \max\left\{ \frac{a_i}{a_i + b_i}, 1 - \frac{v_p(z_i - p^{a_i}\lambda)}{a_i + b_i} \right\},$$

or there exists $\mu \in p\mathbb{Z}_p$ such that H is an open subgroup of $\langle (\mu, 1) \rangle$ and

$$\operatorname{hdim}_G^{\mathcal{S}}(H) = \varliminf_{i\to\infty} \max\left\{ \frac{a_i}{a_i + b_i}, 1 - \frac{v_p(p^{a_i} - z_i\mu)}{a_i + b_i} \right\}.$$

Proof. Clearly, a non-trivial, non-open closed subgroup H of G has analytic dimension $\dim(H) = 1$ and is procyclic. Consequently, H is open and hence of finite index in a maximal procyclic subgroup. The latter are the groups of the form $\langle (1, \lambda) \rangle$ with $\lambda \in \mathbb{Z}_p$ and $\langle (\mu, 1) \rangle$ with $\mu \in p\mathbb{Z}_p$. Since G is infinite, the Hausdorff dimension function is constant on commensurability classes of subgroups of G, and we may assume that $H = \langle (1, \lambda) \rangle$ or $H = \langle (\mu, 1) \rangle$ for suitable λ or μ.

Suppose first that $H = \langle (1,\lambda) \rangle$ with $\lambda \in \mathbb{Z}_p$. Observe that $\log_p |G : G_i| = a_i + b_i$ for $i \in \mathbb{N}_0$. Furthermore, setting $d_i = \min\{b_i, v_p(z_i - p^{a_i}\lambda)\}$, we see that the group HG_i is equal to $\langle (1,\lambda), (0,p^{d_i}) \rangle$ so that $\log_p |G : HG_i| = d_i$ and

$$\begin{aligned} \log_p |HG_i : G_i| &= \log_p |G : G_i| - \log_p |G : HG_i| \\ &= a_i + b_i - \min\{b_i, v_p(z_i - p^{a_i}\lambda)\} \\ &= \max\{a_i, a_i + b_i - v_p(z_i - p^{a_i}\lambda)\}. \end{aligned}$$

The result follows from (1.1).

Now suppose that $H = \langle (\mu, 1) \rangle$ with $\mu \in p\mathbb{Z}_p$. We argue in a similar way, using for $i \in \mathbb{N}_0$ that $\log_p |G : G_i| = a_i + b_i$ and $\log_p |G : HG_i| = d_i$, where $d_i = \min\{b_i + v_p(\mu), v_p(p^{a_i} - z_i\mu)\}$. ☐

Next we consider filtration series of $G = \mathbb{Z}_p \oplus \mathbb{Z}_p$, whose terms come from a fixed 'apartment', corresponding to a decomposition of $\mathbb{Q}_p^2$ into a direct sum of two lines. Such filtration series are somewhat close to the p-power series.

Proposition 3.2. *Let $G = \mathbb{Z}_p \oplus \mathbb{Z}_p$, and let*

$$\mathcal{S} \colon G_i = \langle (p^{a_i}, 0), (0, p^{b_i}) \rangle, \quad i \in \mathbb{N}_0,$$

be a filtration series of G; this means that $(a_i)_{i\in\mathbb{N}_0}$ and $(b_i)_{i\in\mathbb{N}_0}$ are non-decreasing integer sequences, starting at $a_0 = b_0 = 0$ and diverging to infinity.

Then, writing $x_i = a_i/(a_i + b_i)$ for $i \in \mathbb{N}_0$ and putting

$$\xi = \min\left\{ \varliminf_{i\to\infty} x_i,\ \varliminf_{i\to\infty} (1 - x_i) \right\}, \qquad \eta = \max\left\{ \varliminf_{i\to\infty} x_i,\ \varliminf_{i\to\infty} (1 - x_i) \right\},$$

$$\zeta = \varliminf_{i\to\infty} \max\{x_i, 1 - x_i\},$$

we have

$$\mathrm{hspec}^{\mathcal{S}}(G) = \{0, \xi, \eta, \zeta, 1\},$$

where

$$0 \le \xi \le \eta \le 1 - \zeta \le 1/2 \qquad \text{or} \qquad 0 \le \xi \le 1 - \zeta = 1 - \eta \le 1/2. \tag{3.1}$$

In particular, $\mathrm{hspec}^{\mathcal{S}}(G)$ *is discrete of size at most* 5.

Conversely, for any $\xi, \eta, \zeta \in [0,1]$ satisfying (3.1) *there exists a filtration $\mathcal{S}$ of the above form such that* $\mathrm{hspec}^{\mathcal{S}}(G) = \{0, \xi, \eta, \zeta, 1\}$.

Proof. The trivial subgroup has Hausdorff dimension 0, and open subgroups have Hausdorff dimension 1 in G. It remains to deal with non-trivial, non-open closed subgroups of G. By Lemma 3.1, it suffices to consider procyclic subgroups of the form $\langle (1,\lambda) \rangle$ and $\langle (\lambda, 1) \rangle$ for $\lambda \in \mathbb{Z}_p$. Furthermore, Lemma 3.1 yields

$$\mathrm{hdim}_G^{\mathcal{S}}(\langle (1,0) \rangle) = \varliminf x_i, \qquad \mathrm{hdim}_G^{\mathcal{S}}(\langle (0,1) \rangle) = \varliminf (1 - x_i)$$

and for $\lambda \in \mathbb{Z}_p \setminus \{0\}$,

$$\mathrm{hdim}_G^{\mathcal{S}}(\langle (1,\lambda) \rangle) = \mathrm{hdim}_G^{\mathcal{S}}(\langle (\lambda,1) \rangle) = \varliminf \max\{x_i, 1 - x_i\} = \zeta.$$

Thus $\mathrm{hspec}^{\mathcal{S}}(G) = \{0, \xi, \eta, \zeta, 1\}$ with ξ, η as defined in the statement of the proposition. We need to show that (3.1) holds. Without loss of generality we may assume $\xi = \varliminf x_i$ and $\eta = \varliminf (1 - x_i)$. We observe that $\xi \le \eta \le \zeta$ and $\zeta \ge 1/2$.

<u>Case 1.</u> Suppose $\eta \le 1/2$. Whenever $1 - x_i$ is close to η, then x_i is close to $1 - \eta$. Thus $\max\{x_i, 1 - x_i\}$ is close to $\max\{1 - \eta, \eta\} = 1 - \eta$ infinitely often, and hence $\zeta \le 1 - \eta$. This establishes $0 \le \xi \le \eta \le 1 - \zeta \le 1/2$.

Case 2. Suppose $\eta > 1/2$. Whenever x_i is close to ξ, then $1 - x_i$ is close to $1 - \xi$. Thus $1 - x_i$ is close to $1 - \xi$ infinitely often, and hence $\eta \leq 1 - \xi$. Whenever $1 - x_i$ is close to η, then x_i is close to $1 - \eta$. Thus $\max\{x_i, 1 - x_i\}$ is close to $\max\{1 - \eta, \eta\} = \eta$ infinitely often, and then $\zeta \leq \eta$ implies $\zeta = \eta$. This establishes $0 \leq \xi \leq 1 - \zeta = 1 - \eta \leq 1/2$.

For the converse statement, given any $\xi, \eta, \zeta \in [0, 1]$ satisfying the first condition in (3.1), we can choose non-decreasing integer sequences $(a_i)_{i \in \mathbb{N}_0}$ and $(b_i)_{i \in \mathbb{N}_0}$ with $a_0 = b_0 = 0$ such that $x_i = a_i/(a_i + b_i)$ satisfies

$$x_i \to \begin{cases} \xi & \text{as } i \to \infty \text{ subject to } i \equiv_3 0, \\ 1 - \eta & \text{as } i \to \infty \text{ subject to } i \equiv_3 1, \\ \zeta & \text{as } i \to \infty \text{ subject to } i \equiv_3 2. \end{cases}$$

This yields $\mathrm{hspec}^{\mathcal{S}}(G) = \{0, \xi, \eta, \zeta, 1\}$ for $\mathcal{S} : G_i = \langle (p^{a_i}, 0), (0, p^{b_i}) \rangle$, $i \in \mathbb{N}_0$.

Similarly, given any $\xi, \eta, \zeta \in [0, 1]$ satisfying the second condition in (3.1), we can arrange that

$$x_i \to \begin{cases} \xi & \text{as } i \to \infty \text{ subject to } i \equiv_2 0, \\ 1 - \zeta & \text{as } i \to \infty \text{ subject to } i \equiv_2 1, \end{cases}$$

and $\mathrm{hspec}^{\mathcal{S}}(G) = \{0, \xi, \zeta, 1\} = \{0, \xi, \eta, \zeta, 1\}$ for the corresponding filtration $\mathcal{S}$. □

As recorded in the introduction, it would be interesting to find out just how large the finite Hausdorff spectra $\mathrm{hspec}^{\mathcal{S}}(G)$ of a p-adic analytic pro-p group G can be in relation to the analytic dimension $\dim(G)$. Regarding infinite Hausdorff spectra we obtain, rather unexpectedly, the following result.

Proposition 3.3. *Let $G = \mathbb{Z}_p \oplus \mathbb{Z}_p$. Then*

$$\mathcal{S} \colon G_i = \langle (p^{a_i}, ip^{a_i}), (0, p^{b_i}) \rangle, \quad i \in \mathbb{N}_0,$$

where $a_0 = b_0 = 0$ and $a_i = p^i$, $b_i = p^{i+1}$ for $i \geq 1$, is a filtration series of G such that

$$[1/p+1, p-1/p+1] \subseteq \mathrm{hspec}^{\mathcal{S}}(G).$$

Proof. Let $\nu \in [1/p+1, p-1/p+1]$. Then

$$f \colon \mathbb{N} \to \mathbb{N}, \quad f(m) = \lceil p^{m+1} - p^m(p+1)\nu - 1 \rceil$$

is a strictly increasing function. Set $\lambda_0 = 1$ and, for $j \geq 1$, define $\lambda_j \in \mathbb{N}$ by

$$\lambda_j = \lambda_{j-1} + p^{f(\lambda_{j-1})}.$$

Observe that, as $j \to \infty$, the λ_j converge with respect to the p-adic topology to some $\lambda \in \mathbb{Z}_p$, and moreover $v_p(\lambda - \lambda_j) = v_p(\lambda_{j+1} - \lambda_j) = f(\lambda_j)$ for $j \geq 0$. We show that the closed subgroup $H_\lambda = \langle (1, \lambda) \rangle$ satisfies $\mathrm{hdim}_G^{\mathcal{S}}(H_\lambda) = \nu$.

Lemma 3.1 yields

$$\mathrm{hdim}_G^{\mathcal{S}}(H_\lambda) = \varliminf \max\{1/p+1, r_i\}, \qquad \text{where} \quad r_i = \frac{p^{i+1} - v_p(i - \lambda)}{p^i(p+1)}.$$

For $j \geq 0$ we observe that

$$r_{\lambda_j} = \frac{p^{\lambda_j+1} - v_p(\lambda_j - \lambda)}{p^{\lambda_j}(p+1)} = \frac{p^{\lambda_j+1} - f(\lambda_j)}{p^{\lambda_j}(p+1)} = \frac{p^{\lambda_j+1} - \lceil p^{\lambda_j+1} - p^{\lambda_j}(p+1)\nu - 1 \rceil}{p^{\lambda_j}(p+1)}.$$

As $\lceil x-1 \rceil \le x \le \lceil x \rceil$ for any $x \in \mathbb{R}$, we obtain

$$\nu \le r_{\lambda_j} \le \nu + \frac{1}{p^{\lambda_j}(p+1)}, \qquad \text{for } j \ge 0,$$

and thus

$$\mathrm{hdim}_G^{\mathcal{S}}(H_\lambda) \le \nu.$$

It suffices to show that

$$r_i \ge \nu \qquad \text{for all } i \in \mathbb{N} \smallsetminus \{\lambda_j \mid j \ge 0\}.$$

Let $i \in \mathbb{N}$ and $j \ge 0$ such that $\lambda_j < i < \lambda_{j+1}$. Since $i < \lambda_{j+1}$, we may write i as $i = i_0 + i_1 p + \ldots + i_s p^s$ in base p, where $0 \le i_0, \ldots, i_s < p$ and $s \le f(\lambda_j)$. By the construction of λ, this yields

$$v_p(i-\lambda) \le f(\lambda_j) = \lceil p^{\lambda_j+1} - p^{\lambda_j}(p+1)\nu - 1 \rceil \le p^{\lambda_j+1} - p^{\lambda_j}(p+1)\nu$$

and thus

$$r_i \ge \frac{p^{i+1} - (p^{\lambda_j+1} - p^{\lambda_j}(p+1)\nu)}{p^i(p+1)} = \frac{p - p^{\lambda_j - i+1} + p^{\lambda_j - i}(p+1)\nu}{p+1}.$$

Using $\lambda_j < i$ we deduce that

$$r_i \ge \frac{p-1+p^{\lambda_j-i}(p+1)\nu}{p+1} \ge \frac{p-1}{p+1} \ge \nu. \qquad \square$$

Proof of Theorem 1.3. The assertion follows from Propositions 3.3 and 2.1. $\square$

4. Hausdorff dimension with respect to the lower p-series, the Frattini series and the dimension subgroup series

In this section we establish Proposition 1.5 and Theorem 1.9. We begin, however, with an example illustrating that Hausdorff dimension with respect to the lower p-series is somewhat more delicate.

Example 4.1. Fix $m \in \mathbb{N}$ and consider the cyclotomic extension of the ring of p-adic integers, $\mathfrak{O} = \mathbb{Z}_p[\zeta]$, where ζ denotes a primitive p^mth root of unity. Recall that the cyclotomic field $\mathbb{Q}_p(\zeta)$ is a totally ramified extension of $\mathbb{Q}_p$ of degree $\varphi(p^m) = (p-1)p^{m-1}$. Indeed, $\pi = \zeta - 1$ is a uniformising element and

$$\mathfrak{O}_+ = \mathbb{Z}_p \oplus \pi\mathbb{Z}_p \oplus \ldots \oplus \pi^{\varphi(p^m)-1}\mathbb{Z}_p \cong \mathbb{Z}_p^{\varphi(p^m)}.$$

Furthermore, we have $\pi^{\varphi(p^m)}\mathfrak{O} = p\mathfrak{O}$; compare [15, IV, Prop. 17].

We choose $d \in \mathbb{N}$ and form the semidirect product $G = T \ltimes A$, where

- $T = \langle s_0, s_1, \ldots, s_{d-1} \rangle \cong \mathbb{Z}_p^d$,
- $A = \langle a_0, \ldots, a_{\varphi(p^m)-1} \rangle \cong \mathfrak{O}_+$ via $\psi \colon A \to \mathfrak{O}$, $\prod a_i^{\lambda_i} \mapsto \sum \lambda_i \pi^i$ and
- the action of T on A is given by

$$(b^{s_0})\psi = b\psi \cdot \zeta \quad \text{and} \quad [b, s_1] = \ldots = [b, s_{d-1}] = 1 \quad \text{for } b \in A.$$

Clearly, G has analytic dimension $\dim(G) = d + \varphi(p^m)$. It is straightforward to compute the terms of the lower p-series of G:

$$\mathcal{L} \colon P_i(G) = \langle s_0^{p^{i-1}}, \ldots, s_d^{p^{i-1}} \rangle \ltimes A_i, \quad \text{where } A_i\psi = \pi^{i-1}\mathfrak{O}.$$

In particular, this gives $\log_p |P_i(G) : P_{i+1}(G)| = d+1$. We deduce, for instance,

$$\mathrm{hdim}_G^{\mathcal{L}}(\langle s_0 \rangle) = {}^1\!/_{d+1} \quad \text{and} \quad \mathrm{hdim}_G^{\mathcal{L}}(\langle a_0 \rangle) = {}^1\!/_{(d+1)\varphi(p^m)}$$

in contrast to Theorem 1.1. A routine verification yields

$$\mathrm{hspec}^{\mathcal{L}}(G) = \big\{ {}^{j}\!/\!{}_{(d+1)\varphi(p^m)} \mid j \in \{0,1,\ldots,(d+1)\varphi(p^m)\} \big\}.$$

Indeed, one shows easily for any $H \le G$ that

$$\mathrm{hdim}_G^{\mathcal{L}}(H) = {}^{1}\!/\!{}_{d+1} \left(\dim(HA/A) + {}^{1}\!/\!{}_{\varphi(p^m)} \dim(H \cap A) \right).$$

In particular, for $0 \le j \le d$ and $0 \le k \le \varphi(p^m)$ the abelian groups

$$H_{j,k} = \langle s_0^{p^m}, \ldots, s_{j-1}^{p^m}, a_0, \ldots, a_{k-1} \rangle$$

have Hausdorff dimension

$$\mathrm{hdim}_G^{\mathcal{L}}(H_{j,k}) = {}^{1}\!/\!{}_{d+1} \left(j + {}^{k}\!/\!{}_{\varphi(p^m)} \right).$$

We observe that

$$\frac{|\mathrm{hspec}^{\mathcal{L}}(G)|}{\dim(G)} = \frac{(d+1)\varphi(p^m)+1}{d+\varphi(p^m)} \to d+1 \quad \text{as } m \to \infty$$

is unbounded as d tends to infinity. ◇

Next we show that the Hausdorff dimension functions with respect to the p-power series, the Frattini series and the dimension subgroup series coincide on p-adic analytic groups.

Proof of Proposition 1.5. Being p-adic analytic, the group G has finite rank. Let $H \le G$ be a closed subgroup, and let $\mathcal{P}\colon G_i = \pi_i(G) = G^{p^i}$, $i \in \mathbb{N}_0$, denote the p-power series of G.

First we compare $\mathcal{P}$ to the Frattini series $\mathcal{F}\colon \Phi_i(G)$, $i \in \mathbb{N}_0$. Clearly, $G_i \subseteq \Phi_i(G)$ for all $i \in \mathbb{N}_0$. On the other hand, [5, Prop. 3.9 and Th. 4.5] allow us to deduce that there exists $j \in \mathbb{N}_0$ such that G_i and $\Phi_i(G)$ are uniformly powerful for all $i \ge j$. Writing $d = \dim(G)$, we deduce, in particular, that for $i \ge j$ there are $x_1, \ldots, x_d \in G$ such that $G_i = \langle x_1^{p^i}, \ldots, x_d^{p^i} \rangle$, and consequently $G_{i+1} = \langle x_1^{p^{i+1}}, \ldots, x_d^{p^{i+1}} \rangle = G_i^{\,p}$. This gives $\log_p |G_i : G_{i+1}| = \dim(G) = \log_p |\Phi_i(G) : \Phi_{i+1}(G)|$ for all $i \ge j$, and thus

$$\log_p |\Phi_i(G) : G_i| = \log_p |\Phi_j(G) : G_j|$$

is constant for $i \ge j$. Therefore Lemma 2.2 yields $\mathrm{hdim}_G^{\mathcal{P}}(H) = \mathrm{hdim}_G^{\mathcal{F}}(H)$.

Next we compare $\mathcal{P}$ to the dimension subgroup series $\mathcal{D}\colon D_i(G)$, $i \in \mathbb{N}$. Clearly, $G_i \subseteq D_i(G)$ for all $i \in \mathbb{N}$. According to the argument used above and [5, Th. 11.4, 11.5; Lem. 11.22], we find $j \in \mathbb{N}$ such that: G_i and $D_i(G)$ are uniformly powerful, and $D_{pi}(G) = D_i(G)^p$ for all $i \ge j$. Consider the thinned out filtration series $\mathcal{D}^*\colon D_i^*(G) = D_{p^i j}(G)$, $i \in \mathbb{N}_0$. Then $G_i \subseteq D_{i-j}^*$ for $i \ge j$, with $\log_p |G_i : G_{i+1}| = \dim(G) = \log_p |D_{i-j}^*(G) : D_{i-j+1}^*(G)|$ and thus

$$\log_p |D_{i-j}^*(G) : G_i| = \log_p |D_0^*(G) : G_j|$$

is constant. We conclude from Lemma 2.2 that $\mathrm{hdim}_G^{\mathcal{P}}(H) = \mathrm{hdim}_G^{\mathcal{D}^*}(H)$. Furthermore, we note that $\log_p |D_i^*(G) : D_k(G)| \le \dim(G)$ for $p^i j \le k \le p^{i+1} j$. Hence, associating each term $D_i^*(G)$ of the series $\mathcal{D}^*$ with the terms $D_{p^i j}(G)$, $D_{p^i j+1}(G)$, $\ldots$, $D_{p^{i+1} j-1}(G)$ of the series $\mathcal{D}$, we employ Corollary 2.3 to deduce that $\mathrm{hdim}_G^{\mathcal{D}^*}(H) = \mathrm{hdim}_G^{\mathcal{D}}(H)$. □

In preparation for the proofs of Theorem 1.9 in this section and Theorem 1.7 in the next section we establish an auxiliary result of general interest.

Proposition 4.2. *Let G be a finitely generated pro-p group, and let $\mathcal{S}$ be any one of the p-power series $\mathcal{P}$, the Frattini series $\mathcal{F}$ or the dimension subgroup series $\mathcal{D}$ of G. Let $N \trianglelefteq H \le G$ be closed subgroups.*

Suppose that G is not p-adic analytic, but H/N is p-adic analytic. Then

$$\mathrm{hdim}_G^{\mathcal{S}}(H) = \mathrm{hdim}_G^{\mathcal{S}}(N).$$

Proof. For $i \in \mathbb{N}_0$ we denote the ith term of $\mathcal{S}$ by G_i and the ith term of the corresponding filtration of H by H_i, i.e.,

$$G_i = \begin{cases} \pi_i(G) = G^{p^i} & \text{if } \mathcal{S} = \mathcal{P}, \\ \Phi_i(G) & \text{if } \mathcal{S} = \mathcal{F}, \\ D_i(G) & \text{if } \mathcal{S} = \mathcal{D} \end{cases} \quad \text{and} \quad H_i = \begin{cases} \pi_i(H) = H^{p^i} & \text{if } \mathcal{S} = \mathcal{P}, \\ \Phi_i(H) & \text{if } \mathcal{S} = \mathcal{F}, \\ D_i(H) & \text{if } \mathcal{S} = \mathcal{D}. \end{cases}$$

We observe that, for $i \in \mathbb{N}_0$,

$$\frac{\log_p |HG_i : G_i|}{\log_p |G : G_i|} = \frac{\log_p |HG_i : NG_i|}{\log_p |G : G_i|} + \frac{\log_p |NG_i : G_i|}{\log_p |G : G_i|} \tag{4.1}$$

and

$$|HG_i : NG_i| = |H/N : (H \cap G_i)N/N| \le |H/N : H_iN/N|.$$

Below we show that

$$\frac{\log_p |H/N : H_iN/N|}{\log_p |G : G_i|} \underset{i\to\infty}{\longrightarrow} 0, \tag{4.2}$$

and hence, taking lower limits in (4.1), we obtain

$$\mathrm{hdim}_G^{\mathcal{S}}(H) = 0 + \varliminf \frac{\log_p |NG_i : G_i|}{\log_p |G : G_i|} = \mathrm{hdim}_G^{\mathcal{S}}(N).$$

Thus it remains to establish (4.2). Replacing H by an open subgroup, if necessary, we may assume that H/N is uniformly powerful. Let $d = d(H/N)$ denote the minimal number of generators of H/N.

First suppose $\mathcal{S} = \mathcal{P}$. Since H/N is uniformly powerful, we conclude that $\log_p |H/N : H_iN/N| = di$ for all $i \in \mathbb{N}_0$. Furthermore, a characterisation of p-adic analytic pro-p groups due to Lazard implies that $\log_p |G : G_i| \ge p^i$ for all $i \in \mathbb{N}_0$; see [14, App. A.3 (3.8.3)] and compare [5, Cor. 11.6 and 11.19]. This yields

$$\frac{\log_p |H/N : H_iN/N|}{\log_p |G : G_i|} \le \frac{di}{p^i} \underset{i\to\infty}{\longrightarrow} 0.$$

Next suppose $\mathcal{S} = \mathcal{F}$. By [5, Prop. 3.9], for $r \in \mathbb{N}$ there exists $i(r) \in \mathbb{N}$ such that for all $i \in \mathbb{N}$ with $i \ge i(r)$ we have $\log_p |G_i : G_{i+1}| = d(G_i) \ge r$; here $d(G_i)$ denotes the minimal number of generators of G_i. On the other hand, $H^{p^i} \subseteq H_i$ and hence $\log_p |H/N : H_iN/N| \le di$ for all $i \in \mathbb{N}_0$. This implies

$$\frac{\log_p |H/N : H_iN/N|}{\log_p |G : G_i|} \le \frac{di}{(i - i(r))r} \underset{i\to\infty}{\longrightarrow} d/r$$

and, letting $r \to \infty$, we conclude that (4.2) holds.

Finally suppose $\mathcal{S} = \mathcal{D}$. For $i \in \mathbb{N}$ set $k(i) = \lfloor \log_p i \rfloor \in \mathbb{N}_0$ so that $p^{k(i)} \le i < p^{k(i)+1}$. By [5, Th. 11.4], we have $\log_p |G : G_i| \ge i - 1$. On the other hand, $H^{p^{k(i)+1}} \subseteq H_{p^{k(i)+1}} \subseteq H_i$ and hence $\log_p |H/N : H_iN/N| \le d(k(i)+1)$. This implies

$$\frac{\log_p |H/N : H_iN/N|}{\log_p |G : G_i|} \le \frac{d(k(i)+1)}{i-1} \underset{i\to\infty}{\longrightarrow} 0. \qquad \square$$

Before moving on to the proof of Theorem 1.9, we remark that for $\mathcal{S} = \mathcal{P}$ the statement was already proved (though stated in a different way) by Barnea and Shalev [4], using a result of Zelmanov [16]. The latter implies that any infinite finitely generated pro-p group contains an element of infinite order and hence a subgroup isomorphic to $\mathbb{Z}_p$.

Proof of Theorem 1.9. First we show that (i) implies (ii). Suppose that G is p-adic analytic. Observe that a closed subgroup $H \le G$ is infinite if and only if it has analytic dimension $\dim(H) \ge 1$. Thus Theorem 1.1 and Proposition 1.5 imply that (ii) holds, with $c = {}^1/_{\dim(G)}$ if G is infinite.

Obviously (ii) implies (iii), and (iii) implies (iv) by Zelmanov's result [16]. It remains to show that (iv) implies (i). Arguing by contraposition, we suppose that G is not p-adic analytic. Then G is infinite. Let $H \le G$ be such that $H \cong \mathbb{Z}_p$. Then Proposition 4.2, for $N = 1$, implies $\mathrm{hdim}_G^{\mathcal{S}}(H) = 0$. ☐

The following example illustrates that extending Theorem 1.9 to the lower p-series requires more care: for $\mathcal{S} = \mathcal{L}$, condition (iv) does not generally imply (i), (ii) or (iii).

Example 4.3. Consider $G = G_1 \times G_2$, where $G_1 = \mathrm{SL}_3^1(\mathbb{F}_p[\![t]\!])$ and $G_2 = \mathrm{SL}_3^1(\mathbb{Z}_p)$. Observe that the lower p-series $\mathcal{L}$ of G satisfies

$$P_i(G) = P_i(G_1) \times P_i(G_2) = \mathrm{SL}_3^i(\mathbb{F}_p[\![t]\!]) \times \mathrm{SL}_3^i(\mathbb{Z}_p) \quad \text{for } i \in \mathbb{N},$$

where

$$\mathrm{SL}_3^i(\mathbb{F}_p[\![t]\!]) = \{g \in \mathrm{SL}_3(\mathbb{F}_p[\![t]\!]) \mid g \equiv_{t^i} 1\} \quad \text{and} \quad \mathrm{SL}_3^i(\mathbb{Z}_p) = \{g \in \mathrm{SL}_3(\mathbb{Z}_p) \mid g \equiv_{p^i} 1\}$$

denote the ith principal congruence subgroups of $\mathrm{SL}_3(\mathbb{F}_p[\![t]\!])$ and $\mathrm{SL}_3(\mathbb{Z}_p)$; compare [5, Prop. 13.29]. Thus $\log_p|P_i(G) : P_{i+1}(G)| = 8 + 8 = 16$ for all $i \in \mathbb{N}$.

For $j \in \{1, 2\}$ choose $H_j \le G_j$ with $H_j \cong \mathbb{Z}_p$. The p-power maps in $\mathrm{SL}_3(\mathbb{F}_p[\![t]\!])$ and $\mathrm{SL}_3(\mathbb{Z}_p)$ behave rather differently: for $g \in \mathrm{SL}_3^i(\mathbb{F}_p[\![t]\!])$ we have $g^p \in \mathrm{SL}_3^{pi}(\mathbb{F}_p[\![t]\!])$, whereas for $g \in \mathrm{SL}_3^i(\mathbb{Z}_p) \smallsetminus \mathrm{SL}_3^{i+1}(\mathbb{Z}_p)$ we have $g^p \in \mathrm{SL}_3^{i+1}(\mathbb{Z}_p) \smallsetminus \mathrm{SL}_3^{i+2}(\mathbb{Z}_p)$; compare [4, Sec. 4] and [5, Prop. 13.22]. This yields

$$\frac{\log_p|H_1 P_{i+1}(G) : P_{i+1}(G)|}{\log_p|G : P_{i+1}(G)|} \le \frac{\lfloor \log_p i \rfloor + 1}{16i} \xrightarrow[i \to \infty]{} 0$$

and hence

$$\mathrm{hdim}_G^{\mathcal{L}}(H_1) = 0, \quad \text{whereas} \quad \mathrm{hdim}_G^{\mathcal{L}}(H_2) = {}^1/_{16}. \qquad \diamond$$

Proposition 4.4. *Let G be an infinite p-adic analytic pro-p group. Then every closed subgroup $H \le G$ satisfies $\mathrm{hdim}_G^{\mathcal{L}}(H) \ge {}^{\dim(H)}/_{\dim(G)^2}$ with respect to the lower p-series $\mathcal{L}$.*

Proof. Set $d = \dim(G)$ and choose a uniformly powerful open normal subgroup $U \trianglelefteq G$. Let $j \in \mathbb{N}$ be such that $G^{p^j} \subseteq P_j(G) \subseteq U$. Writing $G_i = G^{p^i}$ and $U_i = U^{p^i}$ for the terms of the p-power filtrations of G and U, we observe that this implies

$$G_i \subseteq U_{i-j} \quad \text{and} \quad P_i(G) \subseteq U_{\lfloor (i-j)/d \rfloor} \quad \text{for all } i \in \mathbb{N} \text{ with } i \ge j.$$

Let $H \le G$ be any closed subgroup; without loss of generality we may assume that $H \le U$. Then there are constants $c_1, c_2 \in \mathbb{N}$ such that for all $i \in \mathbb{N}$ with $i \ge j$,

$$\log_p|HP_i(G) : P_i(G)| \ge \log_p|HU_{\lfloor (i-j)/d \rfloor} : U_{\lfloor (i-j)/d \rfloor}| \ge \dim(H) \cdot \lfloor (i-j)/d \rfloor - c_1$$

and

$$\log_p|G : P_i(G)| = \log_p|G : P_j(G)| + \log_p|P_j(G) : P_i(G)| \le c_2 + d(i-j).$$

This gives

$$\frac{\log_p|HP_i(G) : P_i(G)|}{\log_p|G : P_i(G)|} \ge \frac{\dim(H)\cdot\lfloor (i-j)/d\rfloor - c_1}{c_2 + d(i-j)} \underset{i\to\infty}{\longrightarrow} {}^{\dim(H)}/_{d^2},$$

and we conclude that $\mathrm{hdim}_G^{\mathcal{L}}(H) \ge {}^{\dim(H)}/_{d^2}$. □

Perhaps Proposition 4.4 can serve as a first step toward a positive solution of Problem 1.4 (2) for $\mathcal{S} = \mathcal{L}$.

5. Characterisation of soluble p-adic analytic groups

In this section we prove Theorem 1.7 and obtain as an immediate consequence Corollary 1.8.

Lemma 5.1. *Let $x, y, z \in \mathbb{R}_{>0}$ and $\eta \in [0,1]$.*

(i) *If ${}^x/_y \ge \eta - {}^1/_y$, then ${}^{x+z}/_{y+z} \ge \eta - {}^1/_{y+z}$.*
(ii) *If ${}^x/_y \ge \eta$, then ${}^{x+z}/_{y+z} \ge \eta$.*

Proof. Indeed, ${}^x/_y \ge \eta - {}^1/_y$ implies $x \ge \eta y - 1$ and, using $\eta \le 1$, we deduce that $x + z \ge \eta y + \eta z - 1$. This gives ${}^{x+z}/_{y+z} \ge \eta - {}^1/_{y+z}$.

Similarly, ${}^x/_y \ge \eta$ implies $x \ge \eta y$ and, using $\eta \le 1$, we deduce that $x + z \ge \eta y + \eta z$. This gives ${}^{x+z}/_{y+z} \ge \eta$. □

Proposition 5.2. *Let G be a countably based pro-p group which is not finitely generated, and let $\mathcal{S}\colon G_0 \supseteq G_1 \supseteq \ldots$ be a filtration series of G. Suppose that $\eta \in [0,1]$ is such that every finitely generated subgroup H of G satisfies $\mathrm{hdim}_G^{\mathcal{S}}(H) \le \eta$. Then there exists a closed subgroup $H \le G$ such that $\mathrm{hdim}_G^{\mathcal{S}}(H) = \eta$.*

Proof. Suppose that $\mathrm{hdim}_G^{\mathcal{S}}(H) < \eta$ for every finitely generated subgroup H of G. In particular this implies that $\eta > 0$. Furthermore, we may assume that $G_i \supsetneq G_{i+1}$ for all $i \in \mathbb{N}_0$. Thus

$$m(i) = \log_p|G : G_i|, \quad i \in \mathbb{N}_0,$$

is a strictly increasing sequence in $\mathbb{N}_0$, and G/G_i is a finite p-group of order $p^{m(i)}$ for each $i \in \mathbb{N}$.

Below we construct recursively an ascending sequence of finitely generated closed subgroups $1 = H_0 \subseteq H_1 \subseteq H_2 \subseteq \ldots$ such that

(i) $H_iG_i = H_{i'}G_i$ for all $i, i' \in \mathbb{N}$ with $i \le i'$;
(ii) for all $i \in \mathbb{N}$,
$$\tfrac{1}{m(i)} \log_p|H_iG_i : G_i| \ge \eta - {}^1/_{m(i)};$$
(iii) for infinitely many $i \in \mathbb{N}$,
$$\tfrac{1}{m(i)} \log_p|H_iG_i : G_i| \le \eta. \tag{5.1}$$

Setting $H = \langle H_0 \cup H_1 \cup \ldots\rangle \le G$, we obtain

$$\mathrm{hdim}_G^{\mathcal{S}}(H) = \underline{\lim}\ \tfrac{1}{m(i)} \log_p|HG_i : G_i| = \underline{\lim}\ \tfrac{1}{m(i)} \log_p|H_iG_i : G_i| = \eta.$$

Thus it remains to construct $H_1 \subseteq H_2 \subseteq \ldots$ so that (i), (ii) and (iii) hold. First we choose $l \in \mathbb{N}_0$ such that $\eta - {}^1/_{m(1)} \le {}^l/_{m(1)} \le \eta$ and pick a finitely generated closed subgroup $H_1 \le G$ such that H_1G_1/G_1 has order p^l. This guarantees, in particular, that (5.1) holds for $j = 1$.

Now suppose that finitely generated closed subgroups of G,

$$1 = H_0 \subseteq H_1 \subseteq \ldots \subseteq H_j, \qquad \text{for some } j \in \mathbb{N},$$

have been constructed such that (i) and (ii) already hold for indices $i, i' \leq j$ and such that (5.1) holds for $i = j$. We manufacture finitely generated closed subgroups $H_{j+1}, \ldots, H_k$ of G, for a suitable $k \in \mathbb{N}$ with $j < k$, to obtain an extended ascending chain

$$1 = H_0 \subseteq H_1 \subseteq \ldots \subseteq H_k$$

such that (i) and (ii) hold for indices $i, i' \leq k$ and such that (5.1) holds for $i = k$.

Case 1. Suppose that

$$\tfrac{1}{m(j+1)} \log_p|H_jG_{j+1} : G_{j+1}| \leq \eta.$$

Using Lemma 5.1(i), we observe that

$$\tfrac{1}{m(j+1)} \log_p|H_jG_j : G_{j+1}| = \frac{\log_p|H_jG_j : G_j| + \log_p|G_j : G_{j+1}|}{\log_p|G : G_j| + \log_p|G_j : G_{j+1}|} \geq \eta - {}^{1}\!/_{m(j+1)}.$$

We may now take $k = j+1$ and finish the proof as follows. Writing $l' = \log_p|H_jG_k : G_k|$ and $l'' = \log_p|H_jG_{k-1} : G_k|$, we find $l \in \mathbb{N}_0$ such that

$$l' \leq l \leq l'' \qquad \text{and} \qquad \eta - {}^{1}\!/_{m(j+1)} \leq {}^{l}\!/_{m(j+1)} \leq \eta.$$

Since G/G_k is a finite p-group, we further find a finitely generated closed subgroup $H_k \leq G$ with $H_j \subseteq H_k$ satisfying

$$H_jG_k \subseteq H_kG_k \subseteq H_jG_{k-1} \qquad \text{and} \qquad \log_p|H_kG_k : G_k| = l.$$

Case 2. Suppose that

$$\tfrac{1}{m(j+1)} \log_p|H_jG_{j+1} : G_{j+1}| > \eta.$$

Since H_j is finitely generated, our hypotheses give $\operatorname{hdim}_G^{\mathcal{S}}(H_j) < \eta$. Choose $k \in \mathbb{N}$ with $k > j+1$ minimal subject to the condition

$$\tfrac{1}{m(k)} \log_p|H_jG_k : G_k| < \eta.$$

Setting $H_{j+1} = H_{j+2} = \ldots = H_{k-1} = H_j$, we observe that (i) and (ii) certainly hold for indices $i, i' \leq k-1$. Using Lemma 5.1(ii), we observe that

$$\tfrac{1}{m(k)} \log_p|H_jG_{k-1} : G_k| = \frac{\log_p|H_jG_{k-1} : G_{k-1}| + \log_p|G_{k-1} : G_k|}{\log_p|G : G_{k-1}| + \log_p|G_{k-1} : G_k|} \geq \eta \geq \eta - {}^{1}\!/_{m(k)}.$$

We may now conclude the proof exactly as in Case 1. □

Lemma 5.3. *Let G be a countably based profinite group, and let $\mathcal{S}\colon G_0 \supseteq G_1 \supseteq \ldots$ be a filtration series of G. Let $H \leq G$ be a closed subgroup such that* $\operatorname{hspec}_G^{\mathcal{S}}(H)$ *is given by a proper limit, that is*

$$\operatorname{hdim}_G^{\mathcal{S}}(H) = \lim_{i \to \infty} \frac{\log_p|HG_i : G_i|}{\log_p|G : G_i|}.$$

Let $\mathcal{S}|_H\colon H = H_0 \supseteq H_1 \supseteq \ldots$, where $H_i = H \cap G_i$ for $i \in \mathbb{N}_0$, denote the induced filtration series of H. Then for every $B \leq H$,

$$\operatorname{hdim}_G^{\mathcal{S}}(B) = \operatorname{hdim}_G^{\mathcal{S}}(H) \cdot \operatorname{hdim}_H^{\mathcal{S}|_H}(B).$$

Proof. We observe that for $i \in \mathbb{N}_0$,

$$\frac{\log_p|BG_i : G_i|}{\log_p|G : G_i|} = \frac{\log_p|BH_i : H_i|}{\log_p|H : H_i|} \frac{\log_p|H : H_i|}{\log_p|G : G_i|}.$$

The claim follows by taking lower limits. □

Theorem 5.4. *Let G be a countably based pro-p group, and let $\mathcal{S}\colon G_0 \supseteq G_1 \supseteq \ldots$ be a filtration series of G. Let $H \le G$ be a closed subgroup such that $\xi = \mathrm{hdim}_G^{\mathcal{S}}(H) > 0$ is given by a proper limit, that is*

$$\xi = \lim_{i\to\infty} \frac{\log_p|HG_i : G_i|}{\log_p|G : G_i|}.$$

Suppose further that $\eta \in [0, \xi)$ is such that every finitely generated closed subgroup $K \le H$ satisfies $\mathrm{hdim}_G^{\mathcal{S}}(K) \le \eta$. Then

$$(\eta, \xi] \subseteq \mathrm{hspec}^{\mathcal{S}}(G).$$

Proof. Let $\vartheta \in \mathbb{R}$ with $\eta < \vartheta < \xi$. Let $\mathcal{S}|_H\colon H_0 \supseteq H_1 \supseteq \ldots$, where $H_i = H \cap G_i$ for $i \in \mathbb{N}_0$, denote the induced filtration series of H. By Lemma 5.3, every finitely generated closed subgroup K of H satisfies $\mathrm{hdim}_H^{\mathcal{S}|_H}(K) \le \eta/\xi < \vartheta/\xi$. As H is not finitely generated, Proposition 5.2 shows: there is a closed subgroup $B \le H$ such that $\mathrm{hdim}_H^{\mathcal{S}|_H}(B) = \vartheta/\xi$. Using Lemma 5.3, we deduce that

$$\mathrm{hdim}_G^{\mathcal{S}}(B) = \mathrm{hdim}_G^{\mathcal{S}}(H) \cdot \mathrm{hdim}_H^{\mathcal{S}|_H}(B) = \vartheta.$$ □

Proof of Theorem 1.7. Suppose that the finitely generated soluble pro-p group G is not p-adic analytic. Consider the derived series of G, consisting of $G^{(0)} = G$ and $G^{(j)} = [G^{(j-1)}, G^{(j-1)}]$ for $j \in \mathbb{N}$.

Observe that $\mathrm{hdim}_G^{\mathcal{S}}(G^{(0)}) = 1$ and $\mathrm{hdim}_G^{\mathcal{S}}(G^{(j)}) = 0$ for all sufficiently large j. Hence there is a maximal integer $k \ge 0$ such that $\mathrm{hdim}_G^{\mathcal{S}}(G^{(k)}) = 1$. We want to apply Theorem 5.4 for the closed subgroup $G^{(k)}$. Quite trivially,

$$\frac{\log_p|G^{(k)}G_i : G_i|}{\log_p|G : G_i|} \le 1 \qquad \text{for all } i \in \mathbb{N}_0,$$

where G_i denotes the ith term of the given filtration series $\mathcal{S}$. Hence $\mathrm{hdim}\,\mathcal{S}_G(G^{(k)})$ is actually given by a proper limit, that is

$$\mathrm{hdim}_G^{\mathcal{S}}(G^{(k)}) = \lim_{i\to\infty} \frac{\log_p|G^{(k)}G_i : G_i|}{\log_p|G : G_i|}.$$

Put $\eta = \mathrm{hdim}_G^{\mathcal{S}}(G^{(k+1)}) < 1$, and let $K \le G^{(k)}$ be any finitely generated closed subgroup. Then Proposition 4.2, applied to $N = G^{(k+1)}$ and $H = KN$, yields

$$\mathrm{hdim}_G^{\mathcal{S}}(K) \le \mathrm{hdim}_G^{\mathcal{S}}(H) = \mathrm{hdim}_G^{\mathcal{S}}(N) = \eta.$$

Hence all conditions in Theorem 5.4 are satisfied. □

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