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Hazard rate for credit risk and hedging defaultable contingent claims

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Abstract. We provide a concise exposition of theoretical results that appear in modeling default time as a random time, we study in details the invariance martingale property and we establish a representation theorem which leads, in a complete market setting, to the hedging portfolio of a vulnerable claim. Our main result is that, to hedge a defaultable claim one has to invest the value of this contingent claim in the defaultable zero-coupon.

Key words: Default risk, representation theorem, hedging

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1 Introduction

We study an arbitrage free financial market, where a risk-free asset S^0 and a risky asset S are traded. The filtration generated by the discounted prices S/S^0 is denoted by $\mathbf{F}^S$. The agents have some information on the asset prices modeled by a filtration $\mathbf{F}$. In the *structural approach*, the default time τ is a stopping time in the filtration $\mathbf{F}^S$ and it is assumed that the agents have all the information contained on the prices, i.e., $\mathbf{F} = \mathbf{F}^S$, whereas in the *reduced-form approach*, the default arrives "by surprise", as in Cox modeling (see Lando [16]). An intermediary case is when τ is a $\mathbf{F}^S$-stopping time and $\mathbf{F} \subset \mathbf{F}^S$, for example when the filtration $\mathbf{F}$ is the trivial

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one or when $\mathbf{F}$ is the filtration generated by the information of prices observed at discrete times, as in Duffie and Lando [10]. The reader can refer to Bielecki and Rutkowski [4] and Cossin and Pirotte [6] for a full treatment. Crucial problems of hedging are studied for example in Bélanger et al. [3]

In a first section, we present the model and we establish a general representation theorem. In a second section, we discuss the role of the hypothesis of invariance of martingales and establish that this hypothesis holds as soon as the default-free market is complete and arbitrage free and the defaultable market is arbitrage free. We show that under some regularity condition, the default time is the first time when a stochastic barrier is reached, as in Cox process modeling. We give the hedging of vulnerable contingent claims using defaultable zero-coupon (DZC in short) and default-free assets, when the default-free market is complete. Our main result shows that to hedge a defaultable claim one has to invest the value of this contingent claim in the defaultable zero-coupon. This result is not surprising and is linked with the hazard process approach: the default arrives by surprise so that the jump in the hedging portfolio value is equal to the jump in the DZC.

2 The model

All the processes and random variables are constructed on a given probability space $(\Omega, \mathcal{G}, P)$. In what follows, we limit our study, mainly for simplicity of notation, to the case where there is only one risky financial asset, whose price at time t is denoted by S_t. As in Musiela and Rutkowski [17], the interest rate is supposed to be a non-negative process r, we denote by $R_t = \exp\left(-\int_0^t r_s ds\right)$ the discounted factor and by $S^0_t = \exp\left(\int_0^t r_s ds\right)$ the savings account. The filtration generated by the discounted price process $\widetilde{S}_t = S_t/S^0_t$ is denoted by $\mathbf{F}^S = (\mathcal{F}^S_t = \sigma(\widetilde{S}_s, s \leq t); t \geq 0)$. We refer to the market where the savings account and the risky asset S are traded up to time T as the *default-free market.* We assume that there exists at least one probability Q equivalent to P on $\mathcal{F}^S_T$ such that $(\widetilde{S}_t, t \leq T)$ is a $\mathbf{F}^S$-martingale, so that the default-free market is arbitrage free.

A default occurs at a random time τ (i.e., a non-negative random variable). In the defaultable world, the payment of a contingent claim depends whether or not the default has appeared before the maturity. In particular, a defaultable zero-coupon bond (DZC) with maturity T pays 1 unit at maturity if and only if the default has not appeared before T. More generally, we investigate the case where a promised payoff $X \in \mathcal{F}^S_T$ is paid at maturity if the default has not appeared and where a compensation is paid at hit (at the default time) if the default occurs before maturity.

2.1 Filtrations and equivalent martingale measures

We assume, as in [12] that the t-time information available to the agents in the default-free market is a σ-algebra $\mathcal{F}_t$. We do not assume that the agents have complete information on the prices: the filtration $\mathbf{F} = (\mathcal{F}_t, t \geq 0)$ can be smaller than $\mathbf{F}^S$.

We denote by $\mathbf{D}$ the filtration $\mathbf{D} = (\mathcal{D}_t; t \geq 0)$ with $\mathcal{D}_t = \sigma(D_s, s \leq t)$ where D is the default process defined as $D_t = 1\!\!1_{\{\tau \leq t\}}$. At time t, the agent's information on the prices and on default time is $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t$: at any time the agent knows whether or not the default has appeared. Hence, the default time τ is a $\mathbf{G}$-stopping time where $\mathbf{G} = (\mathcal{G}_t,\, t \geq 0)$. In fact, $\mathbf{G}$ is the smallest filtration which contains $\mathbf{F}$, satisfying the usual hypotheses, such that τ is a $\mathbf{G}$-stopping time.

2.2 Hazard process

In this section, we work under a reference probability P. Later on, this probability will be either the historical probability, or a risk neutral one.
Let F be the right-continuous version of the submartingale $F_t = P(\tau \leq t|\mathcal{F}_t)$ and G the conditional survival probability $G_t = 1 - F_t$. We assume that $F_t < 1$ a.s. for any t (In particular, τ is not a $\mathbf{F}$-stopping time. See Andreasen [1] and Bélanger et al. [3] for generalization). We introduce the $\mathbb{R}^+$-valued *hazard process* $\varGamma_t = -\ln(G_t)$. We assume for simplicity that $F_0 = 0$ so that $\varGamma_0 = 0$. Obviously, the hazard process depends of the reference probability. For typographical reasons, we shall sometimes use F, G and $\varGamma$ in the same formula.
We recall a key lemma stated as an example in Dellacherie [7] (p. 64) and its corollary.

Lemma 1 *Let $X \in \mathcal{F}_T$ be integrable. Then,*

$$E(X 1\!\!1_{T<\tau}|\mathcal{G}_t) = 1\!\!1_{\{t<\tau\}} \frac{1}{G_t} E(X G_T|\mathcal{F}_t)\,. \tag{1}$$

Corollary 1 *Let h be a $\mathbf{F}$-predictable bounded process. Then,*

a)

$$E(h_\tau|\mathcal{G}_t) = 1\!\!1_{\{\tau \leq t\}} h_\tau - 1\!\!1_{\{t<\tau\}} \frac{1}{G_t} E\Big(\int_t^\infty h_u dG_u\,|\mathcal{F}_t\Big)\,. \tag{2}$$

b) In particular, if F is increasing and continuous,

$$E(h_\tau|\mathcal{G}_t) = 1\!\!1_{\{\tau \leq t\}} h_\tau + 1\!\!1_{\{t<\tau\}}\, E\Big(\int_t^\infty h_u \exp\big(\varGamma_t - \varGamma_u\big)\, d\varGamma_u\,|\mathcal{F}_t\Big)\,. \tag{3}$$

The proof of this lemma and its corollary is based on the important remark that any $\mathcal{G}_t$-measurable random variable is equal, on the set $\{t < \tau\}$, to an $\mathcal{F}_t$-measurable random variable.
It is also useful to note that for any $\mathbf{G}$-predictable process h, there exists an $\mathbf{F}$-predictable process h^* such that both processes are equal on the set $\{t \leq \tau\}$, i.e., $h_t 1\!\!1_{\{t\leq\tau\}} = h^*_t 1\!\!1_{\{t\leq\tau\}}$. Moreover, under the hypothesis $\forall t, F_t < 1$, the process $(h^*_t, t \geq 0)$ is unique (see [8], p. 186).

As an application of Lemma 1, it is easy to check that the discontinuous process $(L_t, t \geq 0)$ where

$$L_t = 1\!\!1_{t<\tau} e^{\varGamma_t} = (1 - D_t)\, e^{\varGamma_t}$$

is a $\mathbf{G}$-martingale.

2.3 Decomposition of the **F***-martingales as* **G***-semi-martingales*

One major question is to describe the dynamics of the assets in the filtration **G**. As mentioned in Hull and White [13] "When we move from the vulnerable world to a default-free world, the stochastic processes followed by the underlying state variables may change." We study here the reverse case, i.e. we move from the default-free world to the vulnerable one.

The submartingale F admits a unique Doob-Meyer decomposition of the from $F_t = Z_t + A_t$, where Z is an **F**-martingale and A an **F**-predictable increasing process. Moreover, the process $M_t = D_t - \Lambda_{t\wedge\tau}$ where $d\Lambda_t = dA_t/G_{t-}$ is a **G**-martingale.
We require now, as in [2] that (C) holds, where
(C) One of the following conditions is satisfied

(i) Any **F**-martingale is continuous
(ii) For any **F**-stopping time θ, $P(\tau = \theta) = 0$.

Under this condition, an **F**-martingale has no common jump with the hazard process, τ is a **G**-totally inaccessible stopping time and for any **F**-martingale m, the process

$$\widehat{m}_{t\wedge\tau} = m_{t\wedge\tau} + \int_0^{t\wedge\tau} e^{\Gamma_s} d[m, Z]_s$$

is a stopped **G**-martingale, where $[X, Y]$ is the quadratic covariation of two martingales X and Y (see Azéma et al. [2]; Dellacherie et al. [8], p. 188; Yor [18], p. 41 for proof of this result and comments on the condition (C)). We shall refer to $\widehat{m}$ as the **G**-martingale part of the **F**-martingale m; in particular $\widehat{Z}$ is defined as

$$\widehat{Z}_{t\wedge\tau} = Z_{t\wedge\tau} + \int_0^{t\wedge\tau} e^{\Gamma_s} d[Z, Z]_s \,.$$

2.4 Representation theorem

Our aim is now to establish a representation theorem for **G**-martingales. Such a representation theorem is difficult to obtain for any **G**-martingale. Azéma et al. [2] have studied the particular case where **F** is the natural filtration of a Brownian motion W and τ is an honest time in $\mathcal{F}_\infty$, that is the end of a predictable set (see Dellacherie et al. [8], p.188; Yor [18], p.41). They established that any **G**-martingale can be written as a sum of a stochastic integral with respect to the **G**-martingale $\widehat{W}$, a stochastic integral with respect to M and a third martingale of the form $v\mathbb{1}_{(\tau\le t)}$ where $v \in \mathcal{F}_\tau^+$ such that $E(v|\mathcal{F}_\tau) = 0$, where $\mathcal{F}_\tau^+$ is generated by the random variables h_τ where h is **F**-progressively measurable. For example, if $\tau = \sup\{t \le 1 \,:\, W_t = 0\}$, then $v = V\,\mathrm{sgne}(W_1)$ with $V \in L^2(\mathcal{F}_\tau)$ (see Yor [18], p. 74).

In what follows, we restrict our attention to the class of **G**-martingales of the form $E(h_\tau|\,\mathcal{G}_t)$, or $E(X\mathbb{1}_{T<\tau}|\,\mathcal{G}_t)$.

Theorem 1 *Suppose that* (C) *holds and let* $F = Z + A$ *be the Doob-Meyer decomposition of* F*. Let* h *be an* $\mathbf{F}$*-predictable process such that* h_τ *is integrable, and* $H_t = E(h_\tau | \mathcal{G}_t)$*. Then, the* $\mathbf{G}$*-martingale* H *admits a decomposition in martingales as follows*

$$H_t = m_0^h + \int_0^{t\wedge\tau} e^{\Gamma_{s-}} (d\widehat{m}_s^h - (h_s - J_{s-}^h) d\widehat{Z}_s) + \int_{]0,t\wedge\tau]} e^{\Delta\Gamma_s} (h_s - J_{s-}^h)\, dM_s\,. \tag{4}$$

Here m^h *is the* $\mathbf{F}$*-martingale*

$$m_t^h = E\Big(\int_0^\infty h_u dF_u | \mathcal{F}_t\Big) = E\Big(\int_0^\infty h_u dA_u | \mathcal{F}_t\Big)\,.$$

The processes $\widehat{m}^h$ *and* $\widehat{Z}$ *are the* $\mathbf{G}$*-martingale parts of the* $\mathbf{F}$*-martingales* m^h *and* Z*,* $J_t^h = e^{\Gamma_t}(m_t^h - \int_0^t h_u dF_u)$ *and* M *is the discontinuous* $\mathbf{G}$*-martingale* $M_t = D_t - \Lambda_{t\wedge\tau}$ *where* $d\Lambda_t = \dfrac{dA_t}{1 - F_{t-}}$*.*
Furthermore,

$$J_t^h \mathbb{1}_{t<\tau} = H_t \mathbb{1}_{t<\tau}\,.$$

Proof The rather technical proof is based on Itô's calculus and property (C). We give the proof in the Appendix. □

Corollary 2 *Suppose that* (C) *holds and that* F *is continuous. Let* $Y \in \mathcal{F}_T$ *be integrable and*

$$Y_t = E(Y \mathbb{1}_{T<\tau} | \mathcal{G}_t) = \mathbb{1}_{t<\tau} e^{\Gamma_t} E(Y G_T | \mathcal{F}_t) = \mathbb{1}_{t<\tau} e^{\Gamma_t} m_t^Y\,,$$

where m^Y *is the* $\mathbf{F}$*-martingale*

$$m_t^Y = E(Y G_T | \mathcal{F}_t)\,.$$

Then,

$$Y_t = m_0^Y + \int_0^{t\wedge\tau} e^{\Gamma_s} \left(d\widehat{m}_s^Y + Y_{s-} d\widehat{Z}_s\right) - \int_{]0,t\wedge\tau]} Y_{s-}\, dM_s\,. \tag{5}$$

Proof The proof follows from Theorem 1 with $h_t = Y \mathbb{1}_{T<t}$. Nevertheless, when F is continuous, a direct proof can be established from the remark that $Y_t = L_t m_t^Y$ and some Itô's calculus which leads in particular to

$$\begin{aligned} dL_t &= -\frac{dD_t}{G_t} + (1 - D_{t-})\left(\frac{dF_t}{G_t^2} + \frac{d\langle F\rangle_t}{G_t^3}\right) \\ &= -\frac{1}{G_t} dM_t + \frac{(1 - D_{t-})}{G_t^2} d\widehat{Z}_t\,. \end{aligned}$$

It remains to apply integration by parts formula. □

3 Invariance of martingale hypothesis

We assume that the knowledge of the default time does not induce arbitrages in the market,[1] hence there exists a $\mathbf{G}$-equivalent martingale measure, i.e., a probability Q^* on $\mathcal{G}_T$, equivalent to P, such that $(\widetilde{S}_t, t \leq T)$ is a martingale. Let us remark that, if $\mathbf{F} = \mathbf{F}^S$, the restriction of any $\mathbf{G}$ e.m.m. to $\mathcal{F}_T^S$ is an $\mathbf{F}^S$-e.m.m. Indeed, if

$$E_{Q^*}(\widetilde{S}_T | \mathcal{G}_t) = \widetilde{S}_t$$

taking the expectation with respect to $\mathcal{F}_t^S$ of both members, we get

$$E_{Q^*}(\widetilde{S}_T | \mathcal{F}_t^S) = \widetilde{S}_t \,.$$

We introduce an invariance of martingale hypothesis denoted by (H), which implies that the dynamics of the asset price are the same in the default-free world and in the defaultable world. We discuss the meaning of that hypothesis, its stability under a change of probability measure, its links with absence of arbitrage opportunities in the defaultable world, and we study the hedging of defaultable contingent claims in that setting.

(H) *Any* $\mathbf{F}$*-square integrable martingale is a* $\mathbf{G}$*-square integrable martingale.*

3.1 Arbitrage free markets

We discuss now the hypothesis on the modeling of default time that we require in order to avoid arbitrages in the defaultable market. We show that under completion of the default-free market, (H) hypothesis holds under any $\mathbf{G}$-e.m.m.

Proposition 1 *Assume that there exists a unique probability Q, equivalent to P on $\mathcal{F}_T^S$ such that the discounted price process $(\widetilde{S}_t, t \leq T)$ is an $\mathbf{F}^S$-martingale under the probability Q. Assume moreover that there exists at least one probability Q^*, equivalent to P on $\mathcal{G}_T$ such that $(\widetilde{S}_t, t \leq T)$ is a $\mathbf{G}$-martingale under the probability Q^*. Then,* (H) *holds under Q^*.*

Proof We give a "financial proof" of this obvious and important result. From the hypothesis, any square integrable r.v. X, such that $XR_T \in \mathcal{F}_T^S$ (any contingent claim) can be written as a stochastic integral with respect to the discounted price, i.e., there exists x and a square integrable $\mathbf{F}^S$ predictable process θ such that $R_T X = x + \int_0^T \theta_s d\widetilde{S}_s$. The t-time price of the contingent claim is $E_Q(XR_T/R_t | \mathcal{F}_t^S)$. Obviously, X is hedgeable by the $\mathbf{G}$-adapted strategy θ and, from the uniqueness of price for hedgeable claims, for any $\mathbf{G}$-e.m.m. Q^*,

$$E_Q(XR_T | \mathcal{F}_t^S) = E_{Q^*}(XR_T | \mathcal{G}_t) \,.$$

Hence, $E_Q(Z | \mathcal{F}_t^S) = E_{Q^*}(Z | \mathcal{G}_t)$ for any square integrable $\mathcal{F}_T$-measurable r.v. Z, therefore any square integrable $\mathbf{F}^S$-Q^*-martingale is a $\mathbf{G}$-Q^*-martingale.

[1] This is a restrictive assumption on the time τ. See [12] for examples where this assumption is not satisfied.

Remark 1 It is easy to construct arbitrage free models where (H) does not hold. For example, in a incomplete information case as in Duffie and Lando [10], a straightforward computation establishes that the hazard process is not increasing, hence (H) does not hold (see also Sect. 3.3).

3.2 Characterization of (H) hypothesis

It is well known [9] that (H) hypothesis holds under Q^* if and only if

$$\forall\, t, \quad Q^*(\tau \le t | \mathcal{F}_\infty) = Q^*(\tau \le t | \mathcal{F}_t). \tag{6}$$

In particular, F and $\varGamma$, evaluated under Q^* are increasing processes. This is in particular the case for Cox processes (see e.g., Lando [16]) where τ is defined via a given non-negative $\mathbf{F}$-adapted process γ as

$$\tau = \inf \left\{ t \ge 0\,,\, \int_0^t \gamma_s ds \ge \varTheta \right\}$$

where $\varTheta$ is a given random variable, independent of $\mathbf{F}$, generally chosen with an exponential law.

The following interesting lemma, which establishes that working under (H) hypothesis is equivalent to a Cox process modeling is proved in [11].

Lemma 2 *If* (H) *hypothesis holds and F is continuous, then the random variable $\varGamma_\tau$ is exponentially distributed and independent of $\mathcal{F}_\infty$. Hence,*

$$\tau = \inf\{t \,:\, \varGamma_t \ge \varTheta\}$$

where $\varTheta$ is an exponential random variable, independent of $\mathcal{F}_\infty$.

Proof We suppose that (H) holds, which implies that

$$Q^*(\tau \le t | \mathcal{F}_\infty) = e^{-\varGamma_t}\,.$$

Setting $\varTheta \overset{def}{=} \varGamma_\tau$ leads to

$$\{t < \varTheta\} = \{t < \varGamma_\tau\} = \{C_t < \tau\},$$

where C is the right inverse of $\varGamma$, so that $\varGamma_{C_t} = t.$ Therefore

$$Q^*(\varTheta > u | \mathcal{F}_\infty) = e^{-\varGamma_{C_u}} = e^{-u}.$$

We have thus established that $\varTheta$ is an exponential random variable, independent of the σ-field $\mathcal{F}_\infty$. Furthermore, $\tau = \inf\{t : \varGamma_t > \varGamma_\tau\} = \inf\{t : \varGamma_t > \varTheta\}$. □

3.3 Brownian filtration, stability of (H) hypothesis

We have seen that if the default-free market is complete, and both markets are arbitrage free (H) holds under any e.m.m. However, hypothesis on τ are generally done under the historical probability measure. In general, (H) hypothesis is not stable under a change of probability (see Kusuoka [15] for a counterexample). We now check that if (H) holds under the historical probability P, and if the $\mathbf{F}$-market is complete and arbitrage free, then the defaultable market is arbitrage free. Under our hypotheses, denoting by Q the e.m.m. for the $\mathbf{F}$-market and by $(\zeta_t, t \geq 0)$ its Radon-Nikodym density, the process $(\zeta_t \widetilde{S}_t, t \geq 0)$ is a P-$\mathbf{F}$ martingale, hence a P-$\mathbf{G}$ martingale. Then, there exists at least a $\mathbf{G}$-e.m.m. defined as $dQ^*|_{\mathcal{G}_t} = \zeta_t dP|_{\mathcal{G}_t}$ and the $\mathbf{G}$-market is arbitrage free.

3.4 Dynamics of defaultable zero-coupon

We assume now that a defaultable zero-coupon bond of maturity T is traded on the market and we denote by $\rho(t,T)$ its t-time price.

We assume that the market where the asset, a default-free zero-coupon and the DZC are traded, is arbitrage free. Then, there exists at least one $\mathbf{G}$-e.m.m. Q^* such that the discounted price of the DZC is a $\mathbf{G}$-martingale, that is, $\rho(t,T)R_t = E_{Q^*}(R_T 1\!\!1_{T<\tau}|\mathcal{G}_t)$. We emphasize that the e.m.m. is chosen by the market which trades the defaultable zero-coupon at the market price $\rho(t,T)$. We do not assume the uniqueness of the e.m.m. Q^*; the DZC being traded, for any e.m.m., the equality

$$\rho(t,T)R_t = E_{Q^*}(R_T 1\!\!1_{T<\tau}|\mathcal{G}_t) = 1\!\!1_{t<\tau} e^{\Gamma_t} \widetilde{m}_t = L_t m_t R_t \tag{7}$$

holds, where $\widetilde{m}_t = m_t R_t = E_{Q^*}(R_T G_T|\mathcal{F}_t)$ and where the hazard process is $\Gamma_t = -\ln(G_t) = -\ln Q^*(\tau > t|\mathcal{F}_t)$.

Proposition 2 *Suppose that* (H) *holds under* Q^* *and that* F *is continuous. Then*

$$d\rho(t,T) = L_{t-} dm_t - \rho(t-,T) dM_t \,.$$

Proof The result follows from $\rho(t,T) = L_t m_t$ and $dL_t = -L_{t-} dM_t$. □

3.5 Representation theorem

In the case where F is continuous and (H) holds, the representation Theorem 1 and the Corollary 2 write

Proposition 3 *Suppose that* (H) *holds under* Q^* *and* F *is continuous, and let* $M_t = D_t - \Gamma_{t\wedge\tau}$ *where* $\Gamma_t = -\ln Q^*(\tau > t|\mathcal{F}_t)$. *Let* h *be an* $\mathbf{F}$*-predictable process such that* h_τ *is integrable, and* $H_t = E(h_\tau|\,\mathcal{G}_t)$. *Then,*

$$H_t = m_0^h + \int_0^{t\wedge\tau} e^{\Gamma_s} dm_s^h + \int_{]0,t\wedge\tau]} (h_s - H_{s-}^h)\, dM_s \,. \tag{8}$$

Here m^h is the $\mathbf{F}$*-martingale*

$$m_t^h = E_Q\Big(\int_0^\infty h_u dF_u | \mathcal{F}_t\Big),$$

Let $Y \in \mathcal{F}_T$ be integrable. Then, the $\mathbf{G}$*-martingale $Y_t = E_{Q^*}(Y \mathbb{1}_{T<\tau} | \mathcal{G}_t)$ admits a decomposition as follows*

$$Y_t = m_0^Y + \int_0^{t\wedge\tau} e^{\Gamma_u} dm_u^Y - \int_{]0,t]} Y_{u-} dM_u$$

where m^Y is the $\mathbf{F}$*-martingale*

$$m_t^Y = E_Q\Big(Y G_T | \mathcal{F}_t\Big).$$

3.6 Hedging strategies

We suppose that $\mathbf{F} = \mathbf{F}^S$, and that the default-free market is complete and arbitrage free. We assume that a defaultable zero-coupon is available on the market and (H) holds under the $\mathbf{G}$-e.m.m. Q^*. We also assume that the process F is continuous.

We now make precise the hedging of a vulnerable claim.

3.6.1 Terminal payoff

We study in a first step the hedging strategy for $X \mathbb{1}_{T<\tau}$ based on savings account, risky asset and defaultable zero-coupon.

We recall that a pair (α, β) of $\mathbf{F}^S$-adapted processes is an hedging strategy for the contingent claim Y with $Y R_T \in \mathcal{F}_T^S$ if, denoting by $V_t = \alpha_t S_t^0 + \beta_t S_t$ the value of this strategy, the self-financing relation $dV_t = \alpha_t dS_t^0 + \beta_t dS_t$ holds and $Y = \alpha_T S_T^0 + \beta_T S_T$. A self-financing strategy is characterized by its initial value x and the parameter β via $R_t V_t = x + \int_0^t \beta_s d\widetilde{S}_s = E_Q(R_T Y | \, cF_t^S)$. The number of shares of riskless asset for this strategy is $\alpha_t = R_t(V_t - \beta_t S_t)$. We shall call “hedging portfolio" of Y the number β of shares of the asset held in the self-financing portfolio.

In the same way, a triple $(a_t, b_t, c_t; t \geq 0)$ of $\mathbf{G}$-adapted processes is an hedging strategy for Y with $Y R_T \in \mathcal{G}_T$ if the value process $V_t = a_t S_t^0 + b_t S_t + c_t \rho_t$ satisfies $V_T = Y$ and the self-financing relation $dV_t = a_t dS_t^0 + a_t dS_t + c_t d\rho_t$.

The default-free market being complete and $X G_T R_T \in \mathcal{F}_T^S$, there exists a predictable process $(\mu_t^X, t \geq 0)$ (the hedging portfolio) and a constant m_0^X (the price) such that

$$X G_T R_T = m_0^X + \int_0^T \mu_s^X d\widetilde{S}_s\,, \tag{9}$$

Let Q be the $\mathbf{F}$-e.m.m. and $\widetilde{m}_t^X = E_Q(X G_T R_T | \mathcal{F}_t)$ the discounted t-time price of $X G_T$. The strategy $(R_t(m_t^X - \mu_t^X S_t), \mu_t^X)$ is a self-financing strategy hedging the contingent claim $X G_T$.

We denote, as in (7)

$$m_t R_t = \widetilde{m}_t^1 = E_Q(G_T R_T | \mathcal{F}_t) = m_0 + \int_0^t \mu_s d\widetilde{S}_s \,, \tag{10}$$

the discounted price of G_T.

Theorem 2 *Assume that $\mathbf{F} = \mathbf{F}^S$, the $\mathbf{F}$-market is complete, the $\mathbf{F}$ and $\mathbf{G}$-markets are arbitrage free, and that $F_t = Q^*(\tau \leq t | \mathcal{F}_t^S)$ is continuous. Let X^d be the value of the defaultable contingent claim $X \mathbb{1}_{T<\tau}$, i.e.*

$$X_t^d R_t = E_{Q^*}(X R_T \mathbb{1}_{T<\tau} | \mathcal{G}_t) \,.$$

The self-financing hedging strategy $(a_t, b_t, c_t; t \geq 0)$ for the defaultable contingent claim $X \mathbb{1}_{T<\tau}$, based on the riskless bond, the asset and the defaultable zero-coupon satisfies

$$a_t S_t^0 + b_t S_t + c_t \rho(t,T) = X_t^d$$

and consists of, on $t < \tau$

(i) $c_t = \dfrac{X_t^d}{\rho(t,T)}$,

(ii) a position on the savings account and the risky asset such that

$$a_t S_t^0 + b_t S_t = 0 \,.$$

More precisely, the self financing hedging strategy is made of

(i) a long position of $\dfrac{m_t^X}{m_t}$ *defaultable zero-coupon,*

(ii) a number of asset' shares equal to $e^{\Gamma_t} \left(\mu_t^X - \dfrac{m_t^X}{m_t} \mu_t \right)$

(iii) an amount in the riskless bond equal to

$$-e^{\Gamma_t} \left(\mu_t^X - \frac{m_t^X}{m_t} \mu_t \right) S_t \,,$$

where the different values (m_t, μ_t) and (m_t^X, μ_t^X) are defined in (10) and (9).

Proof One can prove this theorem as an application of the representation theorem. However, we prefer to give a direct proof. The price of the defaultable claim $X \mathbb{1}_{T<\tau}$ is X_t^d defined by

$$\begin{aligned} R_t X_t^d &= E_{Q^*}(X R_T \mathbb{1}_{T<\tau} | \mathcal{G}_t) = \mathbb{1}_{t<\tau} e^{\Gamma_t} E_Q(X R_T G_T | \mathcal{F}_t) \\ &= \mathbb{1}_{t<\tau} e^{\Gamma_t} \widetilde{m}_t^X = L_t \widetilde{m}_t^X \end{aligned}$$

and the price of the DZC satisfies

$$R_t \rho(t,T) = \mathbb{1}_{t<\tau} e^{\Gamma_t} \widetilde{m}_t \,.$$

Hence

$$X_t^d = \frac{m_t^X}{m_t} \rho(t,T)$$

We now check that there exists a triple (a, b, c) determining a self-financing portfolio such that $c_t = \frac{m_t^X}{m_t}$ and $a_t S_t^0 + b_t S_t = 0$. From Proposition 2 the self financing condition reads

$$a_t S_t^0 r_t dt + b_t dS_t + c_t (L_{t-} dm_t + m_t dL_t) = dX_t^d = L_{t-} dm_t^X + m_t^X dL_t \,. \tag{11}$$

Using $a_t S_t^0 + b_t S_t = 0$, equality (11) reduces to

$$-b_t S_t r_t dt + b_t dS_t + c_t L_{t-} dm_t = L_{t-} dm_t^X \,.$$

or, in terms of discounted processes

$$b_t d\widetilde{S}_t + c_t L_{t-} \mu_t d\widetilde{S}_t = L_{t-} \mu_t^X d\widetilde{S}_t \,.$$

Hence the form of b given in the theorem. □

It is difficult to make explicit the hedging, since it requires the hedging of claims of the form $\exp\left(\int_0^T \gamma_s ds\right)$ where γ is a random process. This problem is similar to hedging under a stochastic interest rate. However, in the very particular case where F and r are deterministic we get

$$c_t = E_Q(X|\mathcal{F}_t), \mu_t = 0 \,.$$

Let Δ be the hedging strategy of X:

$$X R_T = x + \int_0^T \Delta_s d\widetilde{S}_s \,.$$

Then,

$$R_T X 1\!\!1_{T<\tau} = h + \int_0^{T\wedge\tau} e^{\Gamma_s - \Gamma_T} \Delta_s d\widetilde{S}_s + \int_{]0, T\wedge\tau]} E_Q(X|\mathcal{F}_s) d\widetilde{\rho}_s \,.$$

and the hedging strategy of the vulnerable claim $X 1\!\!1_{T<\tau}$ consists of holding

(a) $E_Q(X|\mathcal{F}_t)$ DZC, so that the wealth invested in DZC is $E_Q(X|\mathcal{F}_t)\rho(t, T)$

(b) $e^{\Gamma_t - \Gamma_T} \Delta_t$ shares of risky asset.

3.6.2 Rebate part

We compute the quantity $E_{Q^*}(h_\tau 1\!\!1_{\tau \le T} R_\tau | \mathcal{G}_t)$, which corresponds to the discounted price of the rebate, when the compensation is payed at hit.

Proposition 4 *Let C_t^h be the value of the rebate, i.e.*

$$C_t^h R_t = E_{Q^*}\left(h_\tau R_\tau 1\!\!1_{\tau<T} | \mathcal{G}_t\right) = h_\tau R_\tau 1\!\!1_{\tau \le t} + 1\!\!1_{t<\tau} e^{\Gamma_t} E_Q \left(\int_t^T R_u h_u dF_u | \mathcal{F}_t \right) . \tag{12}$$

The hedging strategy before default time of the rebate part consists of *(i)* $c_t = \frac{1}{\rho_t}(C^h_t - h_t)$ *defaultable zero-coupon*

(ii) a position on savings account and risky asset such that

$$a_t S^0_t + b_t S_t = h_t$$

and the strategy (a_t, b_t, c_t) *is self financing.*

More precisely the hedging strategy of the rebate part is made of (before the default time)

(i) $\frac{1}{\rho(t,T)}(C^h_t - h_t)$ *defaultable zero-coupon*

(ii) $e^{\Gamma_t}\left[\mu^h_t - \frac{1}{m_t}\mu_t C^h_t\right] + \frac{1}{m_t}\mu_t h_t$ *shares of the asset*

(iii) a cash amount of $h_t + S_t\left[e^{\Gamma_t}\frac{1}{m_t}\mu_t C^h_t - \left[\frac{1}{m_t}\mu_t h_t + e^{\Gamma_t}\mu^h_t\right]\right]$, *where* μ^h *is defined by*

$$E\left(\int_0^T R_u h_u dF_u | \mathcal{F}_t\right) = m^h_0 + \int_0^t \mu^h_s d\widetilde{S}_s \,. \tag{13}$$

Proof We denote by C^h_t the price of the rebate defined by (12) and by μ^h, defined in (13), the hedging strategy for the discounted claim $\int_0^T R_u h_u dF_u$ in the default-free world. The representation theorem states that

$$E_{Q^*}(h_\tau R_\tau \mathbb{1}_{\tau \le T}|\mathcal{G}_t) = C^h_0 + \int_0^{t\wedge\tau} e^{\Gamma_u}\mu^h_u d\widetilde{S}_u + \int_{[0,t\wedge\tau[} R_u(h_u - C^h_{u-})dM_u \,.$$

Hence, introducing $\widetilde{m}_t$, the discounted price of G_T

$$\begin{aligned} E_{Q^*}\left(h_\tau R_\tau \mathbb{1}_{\tau \le T}|\mathcal{G}_t\right) = C^h_0 &+ \int_0^{t\wedge\tau} e^{\Gamma_u}\mu^h_u d\widetilde{S}_u \\ &- \int_{[0,t\wedge\tau[} (h_u R_u - \widetilde{C}^h_u)\frac{1}{L_u \widetilde{m}_u}[d\widetilde{\rho}_u - L_u d\widetilde{m}_u] \end{aligned}$$

which leads to

$$\begin{aligned} E_{Q^*}\left(h_\tau R_\tau \mathbb{1}_{\tau \le T}|\mathcal{G}_t\right) = m^h_0 &+ \int_0^{t\wedge\tau}\left[e^{\Gamma_u}\left(\mu^h_u - C^h_u\frac{\mu^h_u}{m_u}\right) + \frac{\mu_u h_u}{m_u}\right]d\widetilde{S}_u \\ &- \int_{[0,t\wedge\tau[}(h_u - C^h_{u-})\frac{1}{L_u m_u}d\widetilde{\rho}_u \end{aligned}$$

□

Corollary 3 *Under* (H), *if F is continuous, the defaultable-market is complete as soon as a defaultable zero-coupon is traded.*

4 Conclusion

We have proved that, if the vulnerable market is complete, hedging strategies are trivial and give a great importance to the DZC. It remains to study the case with several correlated defaults. In the case where defaults are independents, the result has the same form. The general case is a work in progress.

Appendix

Proof of theorem

As usual, G^c is the martingale continuous part of the semi-martingale G. We recall that $d[G]_t = d[G^c]_t + (\Delta G_t)^2$. Then, Itô's formula leads to

$$\begin{aligned} d(G_t^{-1}) &= -\frac{1}{(G_{t-})^2} dG_t + \frac{1}{(G_{t-})^3} d[G^c]_t + \left(e^{\Gamma_t} - e^{\Gamma_{t-}} + \frac{1}{(G_{t-})^2} \Delta G_t \right) \\ &= \frac{1}{(G_{t-})^2} \left(-dG_t + \frac{1}{G_{t-}} d[G^c]_t \right) + \frac{1}{G_t \, (G_{t-})^2} (\Delta G_t)^2 \\ &= \frac{1}{(G_{t-})^2} \left(-dG_t + \frac{1}{G_{t-}} d[G^c]_t + \frac{1}{G_t} (\Delta G_t)^2 \right) \end{aligned} \tag{14}$$

The quadratic variation of the processes

$$Y_t = m_t^h - \int_0^t h_u dF_u = m_t^h + \int_0^t h_u dG_u$$

and $e^{\Gamma_t} = G_t^{-1}$ is

$$\begin{aligned} d[e^\Gamma, Y]_t &= d[e^\Gamma, m^h]_t + h_t d[e^\Gamma, G]_t \\ &= \frac{1}{(G_{t-})^2} \left[-d[G, m^h]_t + \frac{1}{G_t} (\Delta G_t)^2 \Delta m_t^h - h_t d[G]_t + \frac{h_t}{G_t} (\Delta G_t)^3 \right] \\ &= \frac{1}{(G_{t-})^2} \left[-d[G, m^h]_t + \frac{1}{G_t} (\Delta G_t)^2 \Delta m_t^h - h_t d[G^c]_t \right. \\ &\qquad \left. - h_t (\Delta G_t)^2 + \frac{h_t}{G_t} (\Delta G_t)^3 \right] \end{aligned}$$

From integration by part formula, the dynamics of $J_t^h = Y_t e^{\Gamma_t}$ are

$$\begin{aligned} dJ_t^h &= e^{\Gamma_{t-}} dY_t + Y_{t-} de^{\Gamma_t} + d[e^\Gamma, Y]_t \\ &= -e^{\Gamma_{t-}} (J_{t-}^h - h_t) dG_t + \frac{J_{t-}^h - h_t}{(G_{t-})^2} d[G^c]_t + \frac{J_{t-}^h - h_t}{G_t G_{t-}} (\Delta G_t)^2 \\ &\qquad + e^{\Gamma_{t-}} \left(dm_t^h + \frac{1}{G_t G_{t-}} (\Delta G_t)^2 \Delta m_t^h \right) - \frac{1}{(G_{t-})^2} d[G, m^h]_t \\ &= e^{\Gamma_{t-}} (J_{t-}^h - h_t) \left(-dG_t + e^{\Gamma_{t-}} d[G^c]_t + \frac{1}{G_t} (\Delta G_t)^2 \right) \\ &\qquad + e^{\Gamma_{t-}} \left(dm_t^h + \frac{1}{G_t G_{t-}} (\Delta G_t)^2 \Delta m_t^h - \frac{1}{G_{t-}} d[G, m^h]_t \right) \end{aligned}$$

The decomposition of F in the filtration $\mathbf{G}$ is

$$F_{t\wedge\tau} = Z_{t\wedge\tau} + A_{t\wedge\tau} = \widehat{Z}_{t\wedge\tau} - \int_0^{t\wedge\tau} \frac{1}{G_s} d[Z]_s + A_{t\wedge\tau}$$

Then, on the set $\{\tau > t\}$

$$dJ_t^h = e^{\Gamma_{t-}} \left[(J_{t-}^h - h_t) d\widehat{Z}_t + d\widehat{m}_t^h + (J_{t-}^h - h_t) dC_t + dK_t \right]$$

where

$$\begin{aligned} dC_t &= \frac{1}{G_t}(\Delta G_t)^2 + e^{\Gamma_{t-}} d[G^c] + dA_t - \frac{1}{G_t} d[Z]_t \\ dK_t &= \frac{1}{G_t G_{t-}} (\Delta G_t)^2 \Delta m_t^h - \frac{1}{G_{t-}} d[G, m^h]_t + \frac{1}{G_t} d[G, m^h]_t \end{aligned}$$

If G has no jump at time t, $dK_t = 0$, if G has a jump

$$dK_t = -(\Delta G_t)\,(\Delta m_t^h)\,\frac{1}{G_t G_{t-}}\big(-\Delta G_t + G_t - G_{t-}\big) = 0\,.$$

The first term is equal to

$$\begin{aligned} dC_t &= \frac{1}{G_t}(\Delta G_t)^2 + e^{\Gamma_t} d[G^c]_t + dA_t - \frac{1}{G_t} d[Z]_t \\ &= \frac{1}{G_t} d[G]_t + dA_t - \frac{1}{G_t} d[Z]_t \\ &= \frac{1}{G_t} d[A]_t + dA_t = \frac{1}{G_t}(\Delta A_t)^2 + dA_t \\ &= e^{\Delta \Gamma_t} dA_t \end{aligned}$$

where we have used that, if the $\mathbf{F}$ martingales are continuous $\Delta A = \Delta F$ and that A is continuous in the case $P(\tau = \theta) = 0$ (see Jeulin [14], p. 65). It remains to compensate the jump at time τ in order to obtain the result. □

References

1. Andreasen, J.: Credit explosives. Preprint, 2001
2. Azéma, J., Jeulin, T., Knight, F., Yor, M.: Le théorème d'arrêt en une fin d'ensemble prévisible. In: Séminaire de Probabilités XXVII (Lecture Notes in Mathematics, vol. 1557). Berlin Heidelberg New York: Springer 1993, pp. 133–158
3. Bélanger, A., Shreve, S.E., Wong, D.: A unified model for credit derivatives. Preprint, Carnegie Mellon, 2001
4. Bielecki, T., Rutkowski, M.: Credit risk: modeling, valuation and hedging. Berlin Heidelberg New York: Springer 2001
5. Blanchet-Scalliet, C.: Processus à sauts et risque de défaut. Thèse, 2001, Université d'Evry
6. Cossin, D., Pirotte, H.: Advanced credit risk analysis. Chichester: Wiley 2001
7. Dellacherie, C.: Un exemple de la théorie générale des processus. In: Meyer, P.A. (ed.) Séminaire de Probabilités IV (Lecture Notes in Mathematics, vol. 124). Berlin Heidelberg New York: Springer 1970, pp. 60–70

8. Dellacherie, C., Maisonneuve, B., Meyer, P-A.: Probabilités et potentiel, chapitres XVII-XXIV, Processus de Markov (fin). Compléments de calcul stochastique. Paris: Hermann 1992
9. Dellacherie, C., Meyer, P-A.: A propos du travail de Yor sur les grossissements des tribus. In: Meyer, P.A. (ed.) Séminaire de probabilités XII (Lecture Notes in Mathematics, vol. 649). Berlin Heidelberg New York: Springer 1978, pp. 69–78
10. Duffie, D., Lando, D.: Term structure of credit spreads with incomplete accounting information. Econometrica **69**, 633–664 (2000)
11. El Karoui, N.: Modélisation de l'information. CEA-EDF-INRIA, Ecole d'été, 1999. Unpubl. manuscript
12. Elliott, R.J., Jeanblanc, M., Yor, M.: On models of default risk. Math. Finance **10**, 179–196 (2000)
13. Hull, A., White, J.: The impact of default risk on the price of options and other derivative securities. Journal of banking and finance **19**, 299–322 (1995)
14. Jeulin, T.: Semi-martingales et grossissement de filtration. (Lecture Notes in Mathematics, vol. 833). Berlin Heidelberg New York: Springer 1980
15. Kusuoka, S.: A remark on default risk models. Adv. Math. Econ. **1**, 69–82 (1999)
16. Lando, D.: On Cox processes and credit risky securities. Review of Derivatives Research **2**, 99–120 (1998)
17. Musiela, M., Rutkowski, M.: Arbitrage pricing. Martingale methods in financial modelling. Berlin Heidelberg New York: Springer 1997
18. Yor, M.: Some aspects of Brownian motion. Lectures in Math., ETH Zürich: Birkhäuser 1997 (Part II: Some recent martingale problems)