Hindawi Publishing Corporation
International Journal of Differential Equations
Volume 2010, Article ID 764738, 8 pages
doi:10.1155/2010/764738

Research Article

He's Variational Iteration Method for Solving Fractional Riccati Differential Equation

H. Jafari and H. Tajadodi

Department of Mathematics and Computer Science, University of Mazandaran, P.O. Box 47416-95447, Babolsar, Iran

Correspondence should be addressed to H. Jafari, jafari@umz.ac.ir

Received 10 August 2009; Accepted 28 January 2010

Academic Editor: Shaher M. Momani

We will consider He's variational iteration method for solving fractional Riccati differential equation. This method is based on the use of Lagrange multipliers for identification of optimal value of a parameter in a functional. This technique provides a sequence of functions which converges to the exact solution of the problem. The present method performs extremely well in terms of efficiency and simplicity.

1. Introduction

The fractional calculus has found diverse applications in various scientific and technological fields [1, 2], such as thermal engineering, acoustics, electromagnetism, control, robotics, viscoelasticity, diffusion, edge detection, turbulence, signal processing, and many other physical processes. Fractional differential equations (FDEs) have also been applied in modeling many physical, engineering problems, and fractional differential equations in nonlinear dynamics [3, 4].

The variational iteration method was proposed by He [5] and was successfully applied to autonomous ordinary differential equation [6], to nonlinear partial differential equations with variable coefficients [7], to Schrodinger-KdV, generalized Kd and shallow water equations [8], to linear Helmholtz partial differential equation [9], recently to nonlinear fractional differential equations with Caputo differential derivative [10, 11], and to other fields, [12]. The variational iteration method gives rapidly convergent successive approximations of the exact solution if such a solution exists; otherwise a few approximations can be used for numerical purposes. The method is effectively used in [6–8, 13–15] and the references therein. Jafari et al. applied the variational iteration method to the Gas Dynamics

Equation and Stefan problem [13, 14]. We consider here the following nonlinear fractional Riccati differential equation:

$$D_*^{\alpha} y(t) = A(t) + B(t)y + C(t)y^2, \tag{1.1}$$

subject to the initial conditions

$$y^{(k)}(0) = c_k, \quad k = 0, 1, \ldots, n-1, \tag{1.2}$$

where α is fractional derivative order, n is an integer, $A(t)$, $B(t)$, and $C(t)$ are known real functions, and c_k is a constant. There are several definitions of a fractional derivative of order $\alpha > 0$. The two most commonly used definitions are the Riemann-Liouville and Caputo. Each definition uses Riemann-Liouville fractional integration and derivatives of whole order. The difference between the two definitions is in the order of evaluation. Riemann-Liouville fractional integration of order a is defined as

$$I^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \quad \alpha > 0, \ x > 0. \tag{1.3}$$

The following two equations define Riemann-Liouville and Caputo fractional derivatives of order α, respectively:

$$D^{\alpha} f(x) = \frac{d^m}{dx^m} \left(I^{m-\alpha} f(x) \right), \tag{1.4}$$

$$D_*^{\alpha} f(x) = I^{m-\alpha} \left(\frac{d^m}{dx^m} f(x) \right), \tag{1.5}$$

where $m - 1 < \alpha \leqslant m$ and $m \in N$. We have chosen to use the Caputo fractional derivative because it allows traditional initial and boundary conditions to be included in the formulation of the problem, but for homogeneous initial condition assumption, these two operators coincide. For more details on the geometric and physical interpretation for fractional derivatives of both the Riemann-Liouville and Caputo types, see [1].

2. Analysis of the Variational Iteration Method

We consider the fractional differential equation

$$D_*^{\alpha} y(t) = A(t) + B(t)y + C(t)y^2, \quad 0 < \alpha \leqslant 1, \tag{2.1}$$

with initial condition $y(0) = 0$, where $D^{\alpha} = d^{\alpha}/dt^{\alpha}$. According to the variational iteration method [5], we construct a correction functional for (2.1) which reads

$$y_{n+1} = y_n + I^{\alpha} \lambda(\xi) \left[\frac{d^{\alpha} y_n}{d\xi^{\alpha}} - A(t) - B(t) y_n - C(t) y_n^2 \right]. \tag{2.2}$$

To identify the multiplier, we approximately write (2.2) in the form

$$y_{n+1} = y_n + \int_0^t \lambda(\xi)\left[\frac{d^\alpha y_n}{d\xi^\alpha} - A(t) - B(t)\tilde{y}_n - C(t)\tilde{y}_n^2\right]d\xi, \tag{2.3}$$

where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory, and $\tilde{y}_n$ is a restricted variation, that is, $\delta\tilde{y}_n = 0$.

The successive approximation y_{n+1}, $n \geqslant 0$ of the solution $y(t)$ will be readily obtained upon using Lagrange's multiplier, and by using any selective function y_0. The initial value $y(0)$ and $y_t(0)$ are usually used for selecting the zeroth approximation y_0. To calculate the optimal value of λ, we have

$$\delta y_{n+1} = \delta y_n + \delta\int_0^t \lambda(\xi)\frac{dy_n}{d\xi}d\xi = 0. \tag{2.4}$$

This yields the stationary conditions $\lambda'(\xi) = 0$, and $1 + \lambda(\xi) = 0$, which gives

$$\lambda = -1. \tag{2.5}$$

Substituting this value of Lagrangian multiplier in (2.3), we get the following iteration formula

$$y_{n+1} = y_n - I^\alpha\left[\frac{d^\alpha y_n}{d\xi^\alpha} - A(t) - B(t)y_n - C(t)y_n^2\right], \tag{2.6}$$

and finally the exact solution is obtained by

$$y(t) = \lim_{n\to\infty} y_n(t). \tag{2.7}$$

3. Applications and Numerical Results

To give a clear overview of this method, we present the following illustrative examples.

Example 3.1. Consider the following fractional Riccati differential equation:

$$\frac{d^\alpha y}{dt^\alpha} = -y^2(t) + 1, \quad 0 < \alpha \leqslant 1, \tag{3.1}$$

subject to the initial condition $y(0) = 0$.

The exact solution of (3.1) is $y(t) = (e^{2t} - 1)/(e^{2t} + 1)$, when $\alpha = 1$.

In view of (2.6) the correction functional for (3.1) turns out to be

$$y_{n+1} = y_n - I^\alpha\left(\frac{d^\alpha y_n}{d\xi^\alpha} + y_n^2 - 1\right)d\xi. \tag{3.2}$$

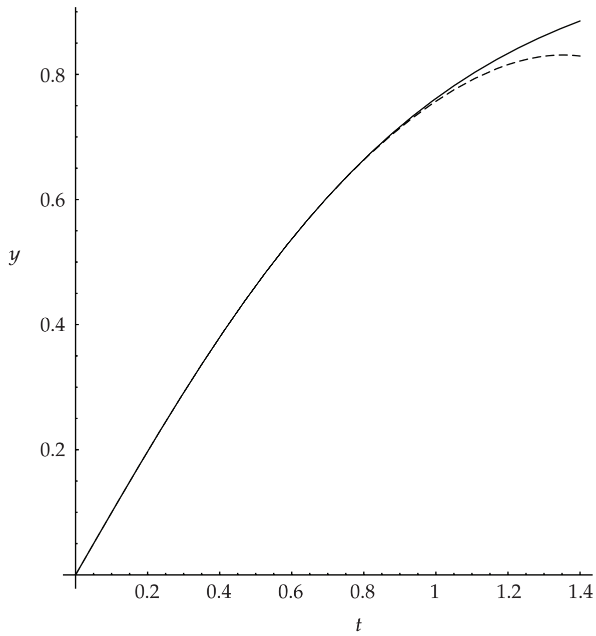

Figure 1: Dashed line: Approximate solution.

Beginning with $y_0(t) = t^\alpha/\Gamma(1+\alpha)$, by the iteration formulation (3.2), we can obtain directly the other components as

$$\begin{aligned} y_1(t) &= \frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{\Gamma(1+2\alpha)t^{3\alpha}}{(\alpha+1)^2\Gamma(1+3\alpha)}, \\ y_2(t) &= \frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{\Gamma(1+2\alpha)t^{3\alpha}}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)} + \frac{2^{3+2\alpha}\Gamma(4\alpha)\Gamma(1/2+\alpha)t^{5\alpha}}{\sqrt{\pi}\Gamma(\alpha)\Gamma(1+\alpha)\Gamma(1+3\alpha)\Gamma(1+5\alpha)} \\ &\quad - \frac{64^\alpha\Gamma(1+2\alpha)^2\Gamma(1/2+3\alpha)t^{7\alpha}}{\sqrt{\pi}\Gamma(1+\alpha)^4\Gamma(1+3\alpha)\Gamma(1+7\alpha)}, \\ &\ \vdots \end{aligned} \tag{3.3}$$

and so on. The nth Approximate solution of the variational iteration method converges to the exact series solution. So, we approximate the solution $y(t) = \lim_{n\to\infty} y_n(t)$.

In Figure 1, Approximate solution $(y(t) \cong y_3(t))$ of (3.4) using VIM and the exact solution have been plotted for $\alpha = 1$. In Figure 2, Approximate solution $(y(t) \cong y_3(t))$ of (3.4) using VIM and the exact solution have been plotted for $\alpha = 0.98$.

Comment. This example has been solved using HAM, ADM, and HPM in [16–18]. It should be noted that these methods have given same result after applying the *Padé* approximants on $y(t)$.

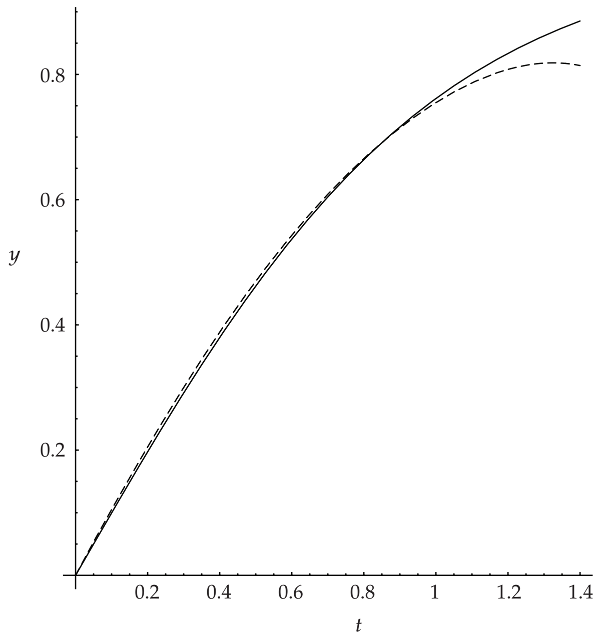

Figure 2: Dashed line: Approximate solution.

Example 3.2. Consider the following fractional Riccati differential equation:

$$\frac{d^\alpha y}{dt^\alpha} = 2y(t) - y^2(t) + 1, \quad 0 < \alpha \leqslant 1, \tag{3.4}$$

subject to the initial condition $y(0) = 0$.

The exact solution of (3.4) is $y(t) = 1 + \sqrt{2}\tanh(\sqrt{2}t + (1/2)\log((\sqrt{2}-1)/(\sqrt{2}+1)))$, when $\alpha = 1$.

Expanding $y(t)$ using Taylor expansion about $t = 0$ gives

$$y(t) = t + t^2 + \frac{t^3}{3} - \frac{t^4}{3} - \frac{7t^5}{15} - \frac{7t^6}{45} + \frac{53t^7}{315} + \frac{71t^8}{315} + \cdots . \tag{3.5}$$

The correction functional for (3.4) turns out to be

$$y_{n+1} = y_n - I^\alpha\left(\frac{d^\alpha y_n}{d\xi^\alpha} - 2y_n + y_n^2 - 1\right)d\xi. \tag{3.6}$$

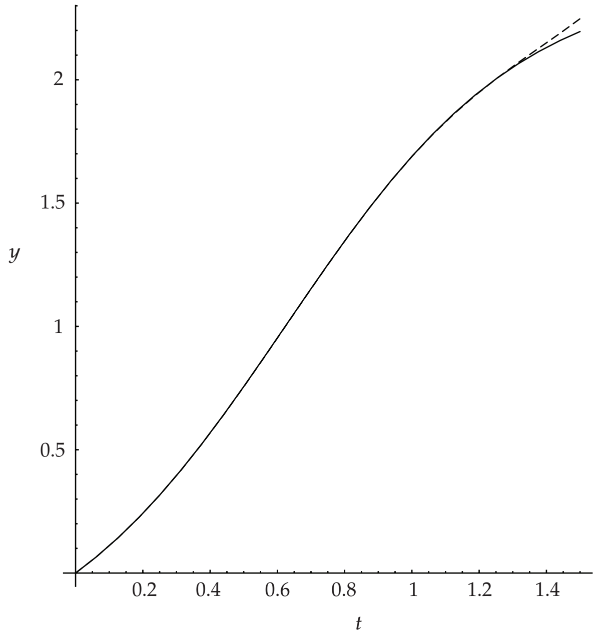

Figure 3: Dashed line: Approximate solution.

Beginning with $y_0(t) = t^\alpha/\Gamma(1+\alpha)$, by the iteration formulation (3.6), we can obtain directly the other components as

$$\begin{aligned}
y_1(t) &= \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{\sqrt{\pi}2^{1-2\alpha}t^{2\alpha}}{\Gamma(\alpha+1)\Gamma(\alpha+1/2)} - \frac{4^\alpha t^{3\alpha}\Gamma(\alpha+1/2)}{\sqrt{\pi}\Gamma(\alpha+1)\Gamma(3\alpha+1)}, \\
y_2(t) &= \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{\sqrt{\pi}2^{1-2\alpha}t^{2\alpha}}{\Gamma(\alpha+1)\Gamma(\alpha+1/2)} - \frac{2^{-1+2\alpha}\Gamma(\alpha+1/2)t^{4\alpha}}{\sqrt{\pi}\alpha\Gamma(\alpha+1)\Gamma(4\alpha)} + \frac{4^{3\alpha}}{\Gamma(3\alpha+1)} \\
&\quad - \frac{\Gamma(1+2\alpha)t^3\alpha}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)} - \frac{12\Gamma(3\alpha)t^4\alpha}{\Gamma(\alpha)\Gamma(2\alpha+1)\Gamma(4\alpha+1)} \\
&\quad - \frac{\sqrt{\pi}2^{4-2\alpha}\Gamma(4\alpha)t^5\alpha}{\Gamma(\alpha)\Gamma(\alpha+1/2)\Gamma(2\alpha+1)\Gamma(5\alpha+1)} + \frac{2^{3+2\alpha}\Gamma(4\alpha)\Gamma(\alpha+1/2)t^5\alpha}{\sqrt{\pi}\Gamma(\alpha)\Gamma(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)} \\
&\quad + \frac{20\Gamma(5\alpha)t^6\alpha}{\Gamma(\alpha)\Gamma(\alpha+1)\Gamma(3\alpha+1)\Gamma(6\alpha+1)} - \frac{1024^\alpha\Gamma(\alpha+1/2)^2\Gamma(3\alpha+1/2)t^7\alpha}{\sqrt{\pi^3}\Gamma(\alpha+1)^2\Gamma(3\alpha+1)\Gamma(7\alpha+1)} \\
&\ \vdots
\end{aligned} \tag{3.7}$$

and so on. In Figure 3, Approximate solution $(y(t) \cong y_3(t))$ of (3.4) using VIM and the exact solution have been plotted for $\alpha = 1$. In Figure 4, Approximate solution $(y(t) \cong y_3(t))$ of (3.4) using VIM and the exact solution have been plotted for $\alpha = 0.98$.

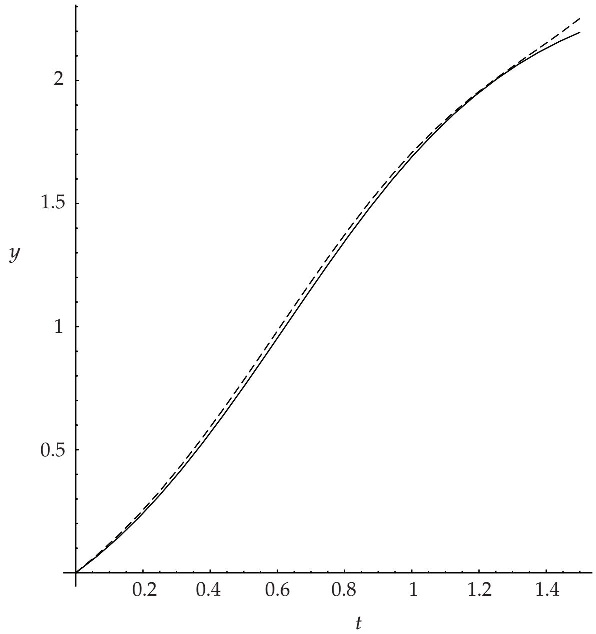

Figure 4: Dashed line: Approximate solution.

4. Conclusion

In this paper the variational iteration method is used to solve the fractional Riccati differential equations. We described the method, used it on two test problems, and compared the results with their exact solutions in order to demonstrate the validity and applicability of the method.

Acknowledgments

The authors express their gratitude to the referees for their valuable suggestions and corrections for improvement of this paper. *Mathematica* has been used for computations in this paper.

References

[1] I. Podlubny, *Fractional Differential Equations*, vol. 198 of *Mathematics in Science and Engineering*, Academic Press, San Diego, Calif, USA, 1999.

[2] K. B. Oldham and J. Spanier, *The Fractional Calculus*, Academic Press, New York, NY, USA, 1974.

[3] H. Jafari and V. Daftardar-Gejji, “Solving a system of nonlinear fractional differential equations using Adomian decomposition,” *Journal of Computational and Applied Mathematics*, vol. 196, no. 2, pp. 644–651, 2006.

[4] J. G. Lu and G. Chen, “A note on the fractional-order Chen system,” *Chaos, Solitons & Fractals*, vol. 27, no. 3, pp. 685–688, 2006.

[5] J. He, “A new approach to nonlinear partial differential equations,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 2, no. 4, pp. 230–235, 1997.

[6] J.-H. He, “Variational iteration method for autonomous ordinary differential systems,” *Applied Mathematics and Computation*, vol. 114, no. 2-3, pp. 115–123, 2000.

[7] J.-H. He, "Variational principles for some nonlinear partial differential equations with variable coefficients," *Chaos, Solitons & Fractals*, vol. 19, no. 4, pp. 847–851, 2004.

[8] M. A. Abdou and A. A. Soliman, "New applications of variational iteration method," *Physica D*, vol. 211, no. 1-2, pp. 1–8, 2005.

[9] S. Momani and S. Abuasad, "Application of He's variational iteration method to Helmholtz equation," *Chaos, Solitons & Fractals*, vol. 27, no. 5, pp. 1119–1123, 2006.

[10] Z. M. Odibat and S. Momani, "Application of variational iteration method to nonlinear differential equations of fractional order," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 7, no. 1, pp. 27–34, 2006.

[11] S. Momani and Z. Odibat, "Numerical approach to differential equations of fractional order," *Journal of Computational and Applied Mathematics*, vol. 207, no. 1, pp. 96–110, 2007.

[12] J.-H. He, "Some asymptotic methods for strongly nonlinear equations," *International Journal of Modern Physics B*, vol. 20, no. 10, pp. 1141–1199, 2006.

[13] H. Jafari, H. Hosseinzadeh, and E. Salehpoor, "A new approach to the gas dynamics equation: an application of the variational iteration method," *Applied Mathematical Sciences*, vol. 2, no. 48, pp. 2397–2400, 2008.

[14] H. Jafari, A. Golbabai, E. Salehpoor, and Kh. Sayehvand, "Application of variational iteration method for Stefan problem," *Applied Mathematical Sciences*, vol. 2, no. 60, pp. 3001–3004, 2008.

[15] H. Jafari and A. Alipoor, "A new method for calculating General Lagrange's multiplier in the variational iteration method," *Numerical Method for Partial Differential Equations*, In press, 2010.

[16] J. Cang, Y. Tan, H. Xu, and S.-J. Liao, "Series solutions of non-linear Riccati differential equations with fractional order," *Chaos, Solitons & Fractals*, vol. 40, no. 1, pp. 1–9, 2009.

[17] S. Momani and N. Shawagfeh, "Decomposition method for solving fractional Riccati differential equations," *Applied Mathematics and Computation*, vol. 182, no. 2, pp. 1083–1092, 2006.

[18] Z. Odibat and S. Momani, "Modified homotopy perturbation method: application to quadratic Riccati differential equation of fractional order," *Chaos, Solitons & Fractals*, vol. 36, no. 1, pp. 167–174, 2008.