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Key Points:
- There exist multiple regimes of the heat and momentum transport scalings in horizontal convection
- The limiting scaling laws, derived here, include various known and new limiting scaling laws
- The smooth transitions between the limiting regimes lead to different effective scaling exponents

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Heat and momentum transport scalings in horizontal convection

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Abstract In a horizontal convection (HC) system heat is supplied and removed exclusively through a single, top, or bottom, surface of a fluid layer. It is commonly agreed that in the studied Rayleigh number (*Ra*) range, the convective heat transport, measured by the Nusselt number, follows the Rossby (1965) scaling, which is based on the assumptions that the HC flows are laminar and determined by their boundary layers. However, the universality of this scaling is questionable, as these flows are observed to become more turbulent with increasing *Ra*. Here we propose a theoretical model for heat and momentum transport scalings with *Ra*, which is based on the Grossmann and Lohse (2000) ideas, applied to HC flows. The obtained multiple scaling regimes include in particular the Rossby scaling and the ultimate scaling by Siggers et al. (2004). Our results have bearing on the understanding of the convective processes in many geophysical systems and engineering applications.

Among other mechanisms of the large-scale ocean circulation, including atmospheric pressure, Coriolis force, and shoreline configuration, seawater density inhomogeneity plays an important role [*Cushman-Roisin and Beckers*, 2011]. The density gradients, which are routed in differences of the temperature and salinity distributions, influence the global thermohaline circulation of the ocean [*Whitehead*, 1995]. One of the most important features of heat and mass transport of the ocean is that heat is supplied to and removed from the ocean predominantly through its upper surface, where the ocean contacts the atmosphere [*Rossby*, 1965]. Apart from the ocean convection, such flow configurations are relevant in many other geophysical systems, in planetary atmospheres, like in the atmosphere of Venus [*Houghton*, 1977; *Scotti and White*, 2011], and also in process engineering, as, for example, in glass-melting furnaces [*Chiu-Webster et al.*, 2008].

Horizontal convection (HC) [*Stern*, 1975; *Hughes and Griffiths*, 2008; *Griffiths et al.*, 2013] may serve as a paradigm system for the development of the quantitative scaling theory for heat and momentum transport in the above flow configurations, as it captures their most relevant features. In a HC system heat is supplied and removed exclusively through the bottom of a horizontal fluid layer, while the other boundaries are adiabatic (see Figure 1 for the HC setup scheme and nomenclature). Once the scaling theory for HC flows has been developed, it can be further extended to the cases of stratified flows and more realistic geometries and fluid properties.

In his seminal work *Rossby* [1965] studied HC for Prandtl numbers $Pr \equiv \nu/\kappa$ between 10 and 10^4 and Rayleigh numbers $Ra \equiv \alpha g \Delta L^3/(\kappa\nu)$ between 10^7 and 10^{10}. Here ν denotes the kinematic viscosity, κ is the thermal diffusivity, α is the isobaric thermal expansion coefficient of the fluid, g is the acceleration due to gravity, L is the length of the cell, and $\Delta \equiv (T_+ - T_-)$ where T_+ is the temperature of the heated part of the bottom and T_- is the temperature of the cooled part of the bottom. *Rossby* [1965] found the scaling of the mean heat flux, measured by the dimensionless Nusselt number $Nu \equiv -\langle \partial T/\partial z\rangle_+/(\Delta/L) = \langle \partial T/\partial z\rangle_-/(\Delta/L)$, as $Nu \propto Ra^\beta$ with the scaling exponent $\beta = 1/5$. This is nowadays referred as Rossby scaling. Here z is the vertical coordinate, T is the temperature, and $\langle\cdot\rangle_+$ and $\langle\cdot\rangle_-$ denote the averaging in time and over the heated and cooled halves of the bottom.

The Rossby scaling is based on the assumptions that the HC flows are laminar and determined by their boundary layers (BLs). This scaling is supported by several numerical simulations [e.g., by *Chiu-Webster et al.*, 2008; *Gayen et al.*, 2014, 2012; *Mullarney et al.*, 2004; *Rossby*, 1998] and laboratory experiments [e.g., by *Griffiths et al.*, 2013; *Hughes et al.*, 2007; *Mullarney et al.*, 2004; *Wang and Huang*, 2005]. Nevertheless, such a universality in

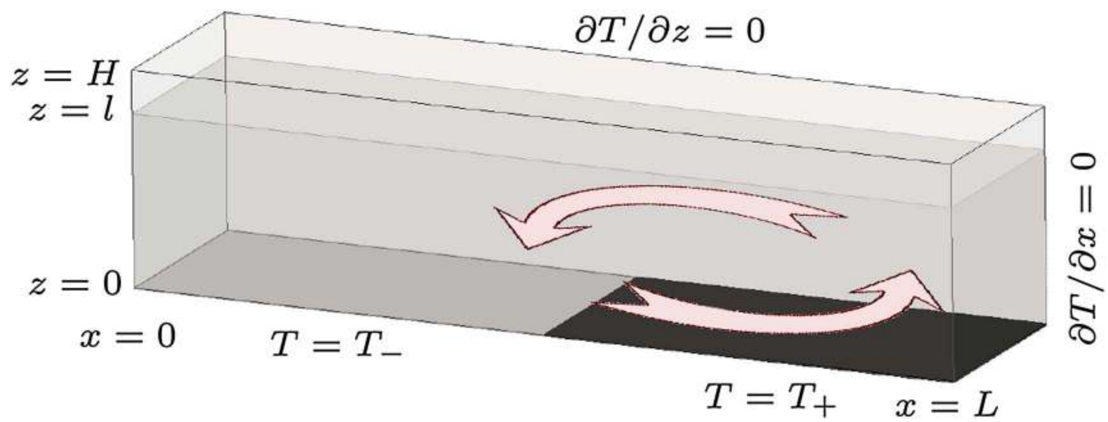

Figure 1. Scheme of a HC setup. The right half of the bottom plate is heated, $T = T_+$, while the left half is cooled, $T = T_- < T_+$. The top and side walls are adiabatic, $\partial T/\partial \mathbf{n} = 0$. The location of the clustered thermal plumes activity up to the height $z = l$ and the direction of the large-scale flow for high *Ra* are sketched with the arrows.

the scaling seems to be very questionable, since the HC flows are observed to become more turbulent with increasing *Ra*, as it has been shown by *Mullarney et al.* [2004]; *Paparella and Young* [2002]; *Scotti and White* [2011]; *Sheard and King* [2011]; *Wang and Huang* [2005]. Thus, *Siggers et al.* [2004] showed with variational analysis that the upper bound of the scaling exponent β in HC equals 1/3, and this allows scalings different from that by Rossby. The theoretical result by *Siggers et al.* [2004] is also consistent with the estimate by *Winters and Young* [2009] for the upper bound of the mean thermal dissipation rate in HC. We refer the regime with the upper bound limiting scaling $Nu \propto Ra^{1/3}$ as the ultimate regime. To date, neither simulations nor experiments have reported such ultimate scaling. We attribute this to the very limited *Ra* range of the conducted numerical and experimental investigations.

In the well-investigated Rayleigh-Bénard convection (RBC) [see, e.g., *Ahlers et al.*, 2009, 2012; *Chillà and Schumacher*, 2012; *Lohse and Xia*, 2010; *Castaing et al.*, 1989; *Grossmann and Lohse*, 2000, 2011; *Siggia*, 1994; *Shishkina et al.*, 2015] the situation is quantitatively different but qualitatively similar. In RBC the temperature T_+ is imposed at the whole bottom, the top temperature is set to T_-, and the reference distance L is the height of the cell. In contrast to RBC, the flow structure in HC is strongly asymmetrical, being more unstable along the heated part of the bottom and close to the vertical wall (right side in Figure 1). Further, as we show below, the exponent β in the limiting scalings $Nu \propto Ra^{\beta}$ behaves differently, being in HC $\beta = 1/5$ in the laminar *Rossby* [1965] scaling and at most $\beta = 1/3$ in the ultimate regime predicted by *Siggers et al.* [2004], while in RBC $\beta = 1/4$ in the laminar case (found in experiments by *Davis* [1922]), $\beta = 1/3$ in the *Malkus* [1954] regime and $\beta = 1/2$ in the ultimate *Kraichnan* [1962] regime [see also *Doering et al.*, 2006]. In particular, there exist extended transitional ranges connecting the various regimes in RBC [see, e.g., *He et al.*, 2012] with intermediate effective scaling exponents [*Ahlers et al.*, 2009]. *Grossmann and Lohse* [2000, 2001, 2002, 2004, 2011] developed a theory (GL) for the effective scaling of the Nusselt number *Nu* and Reynolds number *Re* with *Ra*, which shows that there is no universal exponent β in the scaling law $Nu \propto Ra^{\beta}$. A simplified schematic sketch of the resulting various regimes is shown in Figure 2a. (The full phase diagram with the five prefactors of the theory properly adopted to experimental data is shown in Figure 1 of *Stevens et al.* [2013].) Applying their ideas to the case of HC, we should be able to predict all possible limiting *Nu* versus *Ra* scaling regimes also in HC.

We consider the following governing equations in a Cartesian coordinate system (x, y, z), for HC in Boussinesq approximation: $\nabla \cdot \mathbf{u} = 0$ and

$$\partial \mathbf{u}/\partial t + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \nu \nabla^2 \mathbf{u} + \alpha g \theta \mathbf{e}_z, \quad (1)$$

$$\partial \theta/\partial t + \mathbf{u} \cdot \nabla \theta = \kappa \nabla^2 \theta, \quad (2)$$

where $\mathbf{u} \equiv (u_x, u_y, u_z)$ is the velocity vector function, θ is the reduced temperature, $\theta \equiv T - 0.5(T_+ + T_-)$, p is the kinetic pressure, and $\mathbf{e}_z \equiv (0, 0, 1)^T$. On the domain boundaries, $\mathbf{u} = 0$; at the top and side walls, $\partial \theta/\partial \mathbf{n} = 0$; $\theta = \Delta/2$ on S_+, and $\theta = -\Delta/2$ on S_-. Here $\mathbf{n}$ is the unit normal vector; S_+ and S_- are, respectively, the right and left halves of the bottom $S = S_+ \cup S_-$.

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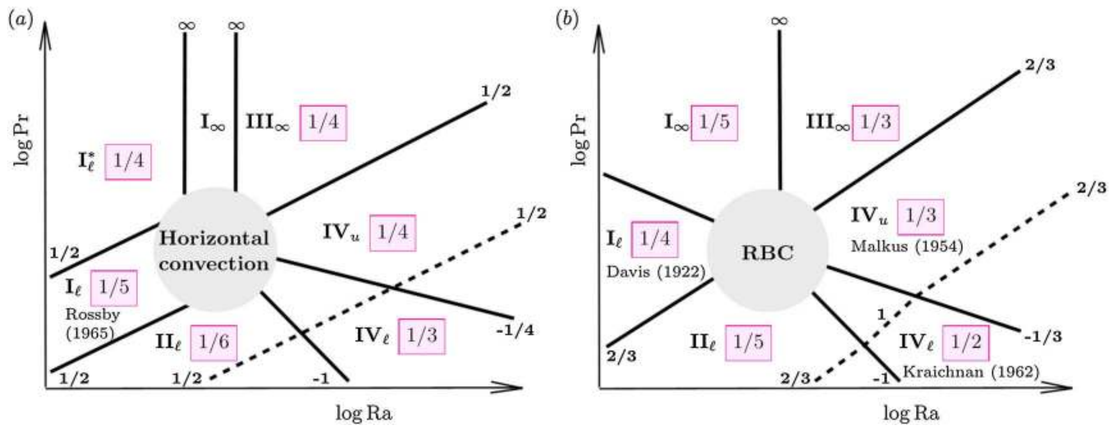

Figure 2. Schematic sketch of the phase diagram in (Ra, Pr) plane of main possible regimes in the scaling $Nu \sim Ra^{\beta}$ in (a) horizontal convection as suggested here and (b) Rayleigh-Bénard convection [*Grossmann and Lohse*, 2000, 2001, 2002, 2004, 2011]. The scaling exponent β for each regime is given in a magenta box. The boundaries between neighbor regimes, $Pr \sim Ra^{\gamma}$, are determined by matching Nu in these regimes; the exponent γ is written close to each corresponding boundary. Dash lines denote the boundaries between the laminar and turbulent viscous BLs. Only slopes of the regime boundaries are relevant in these diagrams, not their exact locations. For the full phase diagram for the RB case as it results from the adoption of the five prefactors of the theory to experimental data we refer to Figure 1 of *Stevens et al.* [2013].

Averaging in time (denoted by the bar) of (2) yields

$$\nabla \cdot \overline{\mathbf{F}} = 0, \quad F_i \equiv \frac{u_i\theta - \kappa\partial\theta/\partial x_i}{\kappa\Delta/L}, \; i = x, y, z. \tag{3}$$

Integration of (3) in the whole HC cell V gives $\langle F_z \rangle_{z=0} = 0$, which in the case $|S_+| = |S_-|$ means $\langle F_z \rangle_- = -\langle F_z \rangle_+$. Here $\langle\cdot\rangle_+$, $\langle\cdot\rangle_-$, and $\langle\cdot\rangle_z$ denote averaging in time and over S_+, S_- and a horizontal cross section at the height z, respectively. Integration of (3) in $S \times [0, z]$ leads to a conclusion that the mean vertical heat flux at any height z equals zero: $\langle F_z \rangle_z = 0$. Averaging of $\langle u_z\theta \rangle_z = \kappa\langle \partial\theta/\partial z \rangle_z$ over $z \in [0, H]$ and taking into account $\langle\theta\rangle_{z=0} = 0$ yield [*Paparella and Young*, 2002]:

$$\langle u_z\theta \rangle_V = \kappa(\langle\theta\rangle_{z=H} - \langle\theta\rangle_{z=0})/H \leq \kappa\Delta/(2H). \tag{4}$$

Here $\langle\cdot\rangle_V$ denotes the time and volume average.

In thermal convection the Nu and Re scalings versus Ra, Pr are determined by the fundamental quantities of the kinetic dissipation rate $\epsilon_u \equiv \nu \sum_i (\nabla u_i)^2$ and thermal dissipation rate $\epsilon_\theta \equiv \kappa(\nabla\theta)^2$; see *Grossmann and Lohse* [2000]. Multiplying (2) by θ and integrating in time and V yields

$$\langle\epsilon_\theta\rangle_V = -\frac{\kappa}{H}\langle\theta\frac{\partial\theta}{\partial z}\rangle_{z=0} = -\frac{\kappa\Delta}{2H}\langle\frac{\partial\theta}{\partial z}\rangle_+ = \frac{\Gamma}{2}\frac{\kappa\Delta^2}{L^2}Nu, \tag{5}$$

where $\Gamma \equiv L/H$ is the HC cell aspect ratio. The estimate (5) of $\langle\epsilon_\theta\rangle_V$ is similar to that in RBC (up to $\Gamma/2$).

Multiplying (1) by $\mathbf{u}$ and further integrating in time and V and taking into account (4), we obtain

$$\langle\epsilon_u\rangle_V = \alpha g\langle u_z\theta\rangle_V \leq \frac{\alpha g\kappa\Delta}{2H} = \frac{\Gamma}{2}\frac{\nu^3}{L^4}Ra\,Pr^{-2}, \tag{6}$$

which is very different from the RBC case, where a similar equality holds and an extra factor ($Nu - 1$) is present in the right-hand side. As the mean kinetic dissipation rate in HC is generally smaller than in RBC, one can understand now why for the same considered Ra in HC and RBC, one obtains generally smaller Nu and Re in the case of HC (the absolute values and also the scaling exponents). Some authors say in this respect that HC is not truly turbulent and refer to the estimate of $\langle\epsilon_u\rangle_V$ (6) (presented here in a different, but equivalent, form as in *Paparella and Young* [2002]) as the "antiturbulence theorem." However, *Scotti and White* [2011] and some other authors, e.g., *Mullarney et al.* [2004], *Sheard and King* [2011], and *Wang and Huang* [2005], found that with increasing Ra, the HC flows become more turbulent. From relation (6) it follows that in HC, in contrast to RBC, $\langle\epsilon_u\rangle_V$ cannot grow faster than $\propto Ra$ as $Ra \to \infty$, but this does not mean that the HC flows cannot be truly turbulent, as proposed by *Paparella and Young* [2002]. Also, the aspect ratio Γ can influence transition to turbulence much stronger than in RBC, since $\langle\epsilon_u\rangle_V \propto \Gamma$.

Another consequence from (6) is the fact that in HC, the mean temperature at the top, $\langle T\rangle_{z=H}$, is larger than that at the bottom, $\langle T\rangle_{z=0}$. This follows from relations (6) and (4), namely, $\langle\epsilon_u\rangle_V \propto \langle u_z\theta\rangle_V \propto (\langle\theta\rangle_{z=H} - \langle\theta\rangle_{z=0})$, and the fact that the mean kinetic dissipation rate is positive for any nonzero flow, $\langle\epsilon_u\rangle_V > 0$.

Following *Grossmann and Lohse* [2000], we decompose the globally averaged dissipation rates (6) and (5) in a HC flow into their BL and bulk contributions as

$$\begin{aligned}\langle\epsilon_u\rangle_V &= \epsilon_{u,\mathrm{BL}} + \epsilon_{u,\mathrm{bulk}},\\ \langle\epsilon_\theta\rangle_V &= \underbrace{\epsilon_{\theta,\mathrm{BL}}}_{\text{BL contributions}} + \underbrace{\epsilon_{\theta,\mathrm{bulk}}}_{\text{bulk contributions}}.\end{aligned}$$

Here $\epsilon_{u,\mathrm{BL}}$ is the kinetic dissipation rate, which is averaged in time and over the viscous boundary layers near all rigid walls and further multiplied by the relative volume of all viscous boundary layers (i.e., by the ratio of the viscous boundary layers volume and the volume of the whole convection cell). Analogously, $\epsilon_{u,\mathrm{bulk}}$ is the kinetic dissipation rate, which is averaged in time and in the bulk part of the domain outside the viscous boundary layers and further multiplied by the relative volume of the bulk region. In a similar way the boundary layer contribution $\epsilon_{\theta,\mathrm{BL}}$ and the bulk contribution $\epsilon_{\theta,\mathrm{bulk}}$ to the globally averaged thermal dissipation rate are defined. The thicknesses of the corresponding BLs can be estimated with the standard slope velocity BL thickness λ_u and slope temperature BL thickness λ_θ, respectively; see *Grossmann and Lohse* [2000].

Further, we define regimes I–IV as BL-BL, bulk-BL, BL-bulk, and bulk-bulk dominance in $\langle\epsilon_u\rangle_V$ and $\langle\epsilon_\theta\rangle_V$, respectively. As the cases $\lambda_\theta \ll \lambda_u$ (large *Pr*) and $\lambda_\theta \gg \lambda_u$ (small *Pr*) can lead to different scalings, we assign the subscripts u and ℓ to the regimes I–IV, which indicate the upper Pr and lower Pr cases, respectively. While equating $\langle\epsilon_u\rangle_V$ and $\langle\epsilon_\theta\rangle_V$ to their estimated either bulk or BL contributions and taking into account the balance between the thermal and viscous BL thicknesses, we obtain eight theoretically possible limiting scaling regimes.

Note that regimes II_u and III_ℓ are less important than the other regimes in HC by the following reasons. On the one hand, the thermal BL in II_u is expected to be thicker than the kinetic one due to the BL dominance in $\langle\epsilon_\theta\rangle_V$. On the other hand, the thermal BL in II_u should be thinner than the kinetic one because of the large *Pr*. By similar argumentation, the regime III_ℓ is also small, if it exists at all.

To derive the limiting scalings, the following assumptions are made with respect to the BL thicknesses: $\lambda_\theta \sim l/Nu$ and $\lambda_u \sim l/\sqrt{Re}$. As it was derived for 2-D thermal BLs in *Shishkina et al.* [2015] (see equations (13) and (14), and explanations there), the latter relation must be fulfilled for the existence of a similarity solution of the thermal BL equation, even if the BLs are strongly fluctuating.

Since the equations (1) and (2) imply

$$\begin{aligned}\frac{1}{2}\left[\frac{\partial\mathbf{u}^2}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}^2\right] &= \nu\nabla(\mathbf{u}\cdot\nabla\mathbf{u}) - \epsilon_u - \mathbf{u}\cdot\nabla p + \alpha g\theta u_z,\\ \frac{1}{2}\left[\frac{\partial\theta^2}{\partial t} + \mathbf{u}\cdot\nabla\theta^2\right] &= \kappa\nabla(\theta\cdot\nabla\theta) - \epsilon_\theta,\end{aligned}$$

the value of ϵ_u in the bulk is of a similar order of magnitude as $(\mathbf{u}\cdot\nabla)\mathbf{u}^2$. Analogously, ϵ_θ in the bulk is of a similar order of magnitude as $(\mathbf{u}\cdot\nabla)\theta^2$. As a result, in the ϵ_u bulk dominating regimes II_ℓ, IV_ℓ, and IV_u, the value of $\epsilon_{u,\mathrm{bulk}}$ is estimated as

$$\epsilon_{u,\mathrm{bulk}} \sim U\frac{U^2}{l}\frac{l-\lambda_u}{l} \approx \frac{U^3}{l} = \frac{\nu^3}{l^4}Re^3.$$

Here U is the reference velocity of the large-scale flow, l is the height of the fluid layer, which is involved in the large-scale flow, and $(l-\lambda_u)$ represents the thicknesses of the bulk. Similarly, in the ϵ_θ bulk dominating regime IV_ℓ, the value of $\epsilon_{\theta,\mathrm{bulk}}$ is estimated as

$$\epsilon_{\theta,\mathrm{bulk}} \sim U\frac{\Delta^2}{l}\frac{l-\lambda_\theta}{l} \approx \frac{U\Delta^2}{l} = \frac{\kappa\Delta^2}{l^2}\mathrm{Pr}\,Re.$$

In the case of large Pr (regimes III_u and IV_u), the thermal BL is embedded into the kinetic one, and therefore, in the above formula the magnitude of the velocity of the flow, which carries the temperature in the bulk, should be reduced from U to $(\lambda_\theta/\lambda_u)U$, which yields

$$\epsilon_{\theta,\text{bulk}} \sim \frac{\lambda_\theta}{\lambda_u}\frac{U\Delta^2}{l}\frac{l-\lambda_\theta}{l} \approx \frac{\lambda_\theta}{\lambda_u}\frac{U\Delta^2}{l} = \frac{\kappa\Delta^2}{l^2}\frac{\text{Pr}\,Re^{3/2}}{\text{Nu}}. \tag{7}$$

In the ϵ_u BL dominating regimes I_ℓ, I_u, and III_u the kinetic dissipation rate in the BL is estimated as $\sim \nu(U/\lambda_u)^2$, and therefore

$$\epsilon_{u,\text{BL}} \sim \nu\frac{U^2}{\lambda_u^2}\frac{\lambda_u}{l} = \frac{\nu^3}{l^4}Re^{5/2}. \tag{8}$$

With increasing Pr, the BL thickness λ_u cannot increase to infinity and saturates at a certain value of order l. In that case $\epsilon_{u,\text{BL}}$ scales not according to (8) but as

$$\epsilon_{u,\text{BL}} \sim \nu\frac{U^2}{\lambda_u^2} = \frac{\nu^3}{l^4}Re^2. \tag{9}$$

For small Ra or very large Pr, this leads to special regimes I_ℓ^*, I_∞, and III_∞ "above," respectively, I_ℓ, I_u, and III_u; see *Grossmann and Lohse* [2001]. Analogously, $\epsilon_{\theta,\text{bulk}}$ is estimated in III_∞ differently from (7), namely, as

$$\epsilon_{\theta,\text{bulk}} \sim \frac{\lambda_\theta}{l}U\frac{\Delta^2}{l}\frac{l-\lambda_\theta}{l} \approx \frac{\lambda_\theta}{l}\frac{U\Delta^2}{l} = \frac{\kappa\Delta^2}{l^2}Pr\,Re\,Nu^{-1}.$$

In the ϵ_θ-BL dominating regimes I_ℓ, I_u, and II_ℓ, the thermal dissipation rate in the BL is estimated as $\sim \kappa(\Delta/\lambda_\theta)^2$, which leads to $\epsilon_{\theta,\text{BL}} \sim \kappa\frac{\Delta^2}{\lambda_\theta^2}\frac{\lambda_\theta}{l} = \kappa\frac{\Delta^2}{l^2}\frac{\lambda_u}{\lambda_\theta}Re^{1/2}$. For small Pr, i.e., in the regimes I_ℓ and II_ℓ, holds $\lambda_\theta/\lambda_u \sim \text{Pr}^{-1/2}$, while and for large Pr, i.e., in the regime I_u, holds $\lambda_\theta/\lambda_u \sim \text{Pr}^{-1/3}$ [*Schlichting and Gersten*, 2000; *Grossmann and Lohse*, 2000; *Shishkina et al.*, 2013, 2014]. Note that in the ϵ_θ-BL dominating regimes I_ℓ and II_ℓ, where $\lambda_u \ll \lambda_\theta$ (small Pr), the scaling of Nu with Pr and Re can be easily estimated from the heat transfer balance in the BL heat equation $u_x\partial_x\theta + u_z\partial_z\theta = \kappa\partial_z^2\theta$, which implies $U\Delta/l \sim \kappa\Delta/\lambda_\theta^2$. This is equivalent to $(Ul/\nu)(\nu/\kappa) \sim (l/\lambda_\theta)^2$, which yields

$$Nu \sim Re^{1/2}Pr^{1/2}. \tag{10}$$

The scalings of $\langle\epsilon_u\rangle_V$ and $\langle\epsilon_\theta\rangle_V$ within the different regimes are summarized in Table 1. By equating $\langle\epsilon_u\rangle_V$ and $\langle\epsilon_\theta\rangle_V$ to their estimated either bulk or BL contributions, we obtain the limiting scalings of Nu and Re in HC, which are also presented in Table 1 and in addition schematically sketched in Figure 2b (compare with the corresponding schematic sketch for RBC in Figure 2a). One can see that the *Grossmann and Lohse* [2000] ansatz applied to HC suggests different scaling regimes, including the *Rossby* [1965] scaling, which is the laminar BL-dominated regime I_ℓ, and the regime IV_ℓ, which is the limiting scaling proposed by *Siggers et al.* [2004]. Note that in HC, the flows become turbulent for larger Ra than in RBC: thus, for Ra about 10^9 the bulk RBC flows are already turbulent [see, e.g., *He et al.*, 2011; *Shi et al.*, 2012; *Kaczorowski et al.*, 2011; *Wagner and Shishkina*, 2013], while the HC flows are still laminar [*Gayen et al.*, 2014].

The critical Rayleigh number Ra_{cr} for the transition to the ultimate regime one can estimate following *Grossmann and Lohse* [2000, 2002]. As the shear Reynolds number Re_s (based on λ_u) exceeds a certain critical value $Re_{s,cr} \sim 400$ [*Landau and Lifshitz*, 1987; *Grossmann and Lohse*, 2000, 2002], the viscous BL becomes turbulent, while for $Re_s < Re_{s,cr}$ holds $\lambda_u/H \sim Re^{-1/2}$ [*Grossmann and Lohse*, 2000; *Shishkina et al.*, 2015]. This together with a balance of $\langle\epsilon_u\rangle_V$ with its turbulent bulk contribution $\sim (\nu^3/l^4)Re^3$ gives $Ra_{cr} \sim Pr^2Re_{s,cr}^6$, which for $Pr \sim 1$ leads to the following estimate for the critical Ra for the transition to the ultimate regime: $Ra_{cr} \sim 4\times10^{15}$.

In conclusion, we have applied the ideas of *Grossmann and Lohse* [2000] for Rayleigh-Bénard convection (RBC) to horizontal convection (HC), revealing various known and new limiting scaling laws for that case. The theory also implies that there are no sharp transitions between the various regimes, but smooth ones, leading to effective scaling exponents different from those of the limiting cases, just as in the case of RB flow. The next step would be to provide sufficiently many and precise numerical and/or experimental data on $Nu(Ra, Pr)$ and

Table 1. Scalings of $\langle\epsilon_u\rangle_V$, $\langle\epsilon_\theta\rangle_V$, Nu and Re in Different Limiting Regimes in HC

Regime	$\langle\epsilon_u\rangle_V \sim Ra\,Pr^{-2}$	$\langle\epsilon_\theta\rangle_V \sim Nu$	Re	Nu
I_∞	$\sim Re^2$	$\sim Re^{1/3}$	$\sim Ra^{1/2}$	$\sim Ra^{1/6}$
I_ℓ	$\sim Re^{5/2}$	$\sim Re^{1/2}$	$\sim Ra^{2/5}$	$\sim Ra^{1/5}$
I^*_ℓ	$\sim Re^2$	$\sim Re^{1/2}$	$\sim Ra^{1/2}$	$\sim Ra^{1/4}$
II_ℓ	$\sim Re^3$	$\sim Re^{1/2}$	$\sim Ra^{1/3}$	$\sim Ra^{1/6}$
III_∞	$\sim Re^2$	$\sim Re\,Nu^{-1}$	$\sim Ra^{1/2}$	$\sim Ra^{1/4}$
IV_u	$\sim Re^3$	$\sim Re^{3/2}Nu^{-1}$	$\sim Ra^{1/3}$	$\sim Ra^{1/4}$
IV_ℓ	$\sim Re^3$	$\sim Re$	$\sim Ra^{1/3}$	$\sim Ra^{1/3}$

$Re(Ra, Pr)$ in HC to allow for the adoption of the five prefactors of the theory, namely those in the four scaling relations for $\epsilon_{u,\mathrm{BL}}$, $\epsilon_{u,\mathrm{bulk}}$, $\epsilon_{\theta,\mathrm{BL}}$, and $\epsilon_{\theta,\mathrm{bulk}}$, and one prefactor for the absolute strength of the wind. For RBC this was done by *Grossmann and Lohse* [2001] based on the available RBC data those days, with a slight revision in *Stevens et al.* [2013], based on the then available data. After this is done, one will be able to predict Nu and Re for any Ra and Pr in HC, as it is now already possible in RBC. The advancement of the theory to the case of other classical boundary conditions, like in vertical convection [*Ng et al.*, 2015], and to more complicated geometries, see, e.g., *Bailon-Cuba et al.* [2012], *Koerner et al.* [2013], and *Wagner and Shishkina* [2015], is the subject of future theoretical studies.

Further, we would like to comment on the applicability of the so-called "zeroth law of turbulence" [*Frisch*, 1995; *Sreenivasan*, 1984] to turbulent HC flows. According to the zeroth law of turbulence, the dimensionless dissipation factor β, which is the mean kinetic dissipation rate measured in nonviscous units, i.e., $\beta \equiv \langle\epsilon_u\rangle_V L/U^3$, should tend to a finite positive constant as $Re \to \infty$ (or as $\nu \to 0$). From (6) and the definition of β one obtains that $\beta \propto Ra\,Re^{-3}Pr^{-2}$. Numerical simulations [*Shishkina and Wagner*, 2016] show that the proportionality coefficient here is independent from Ra and Pr and is determined by the cell geometry. As follows from our theory, in turbulent regimes II_ℓ, IV_ℓ, and IV_u, the Reynolds number scales as $Re \sim Ra^{1/3}Pr^{-2/3}$. The last two relations give $\beta \to$ constant > 0 as $Re \to \infty$, which is fully consistent with the zeroth law of turbulence. Note that taking the free-fall velocity $\sqrt{\alpha g \Delta H}$ instead of the wind velocity U in the definition of β (as in *Scotti and White* [2011]) would lead to the dissipation factor that vanishes as $\sim Ra^{-1/2}Pr^{-1/2}$ and hence would made turbulent RBC and turbulent HC somewhat special among other turbulent flows. Furthermore, this would imply that Re scales as $\sim\sqrt{Ra/Pr}$ in all turbulent regimes in RBC and HC, which is in conflict to experiments; see *Ahlers et al.* [2009]. Indeed, for the case of turbulent RBC it has been now well established that there are different scaling regimes with different scalings of Re with Ra. Thus, with our theory we have clarified the zeroth law of turbulence issue in HC.

Finally, we stress again one important difference between RBC and HC: While in RBC the height L of the sample is the relevant length scale, in HC it is l, which is a priori not known. For relatively flat samples one will have $l = L$; however, if the sample size gets very large, the upper fluid layers in the sample may be unaffected by the HC at the bottom part of the cell, and $l \ll L$.

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