Commun. math. Phys. 48, 195—197 (1976)

Communications in
Mathematical
Physics

Heat Equation on Phase Space and the Classical Limit of Quantum Mechanical Expectation Values

A. Grossmann
Centre de Physique Théorique, C.N.R.S., F-13274 Marseille Cedex 2, France

R. Seiler
Freie Universität Berlin, FB 20, Physik, D-1000 Berlin 33

Abstract. The expectation value of a quantum mechanical operator, taken in coherent states and suitably rescaled, is the solution of an initial value problem for the heat equation on phase space, in which $\hbar$ plays the role of time, and the classical observable is the distribution of temperature at $\hbar=0$.

Introduction

A recent paper by Hepp [1] is devoted to the classical limit of (rescaled) expectation values in coherent states and to their time evolution. Here we sharpen some results of [1] by relating the classical limit to an initial value problem in $\hbar$. This is done with the help of a quantization formula derived in [2].

Notations

Denote by E a 2ν-dimensional real vector space with a symplectic form σ. (Phase space for $\nu<\infty$ degrees of freedom.) Elements of E will be denoted by $a, b, v\ldots$. Fix on E a σ-allowed complex structure J, i.e. a linear map satisfying $J^2=-1$, $\sigma(Ja, Jv)=\sigma(a, v)$ and $\sigma(a, Ja)>0$ for $a\neq 0$. Introduce the orthogonal form $s(a, v)=\sigma(a, Jv)$, and the (phase space) Gaussian $\Omega(v)=e^{-\pi s(v,v)}$. Normalize the invariant measure dv on E by the requirement $\int \Omega(v)dv=1$. This is equivalent to the requirement $F^2=1$ where F is the symplectic Fourier transform:

$$Ff)(v)=\tilde{f}(v)=\int e^{2i\pi\sigma(v,v')}f(v')dv' .$$

In the Hilbert space $L^2(E; dv)$ consider the family of functions Ω^a:

$$\Omega^a(v)=e^{-2i\pi\sigma(a,v)}\Omega(v+a) .$$

Denote by $\mathscr{H}$ the closed linear span of the family Ω^a, with the scalar product inherited from $L^2(E; dv)$. For any $\Phi\in\mathscr{H}$ one has $(\Omega^a, \Phi)=k\Phi(-a)$, with

$$k=(\Omega, \Omega)=2^{-\nu} .$$

Also, for every $\Phi \in \mathscr{H}$ one has $F\Phi = M\Phi$, where M is the parity operator:

$$(M\Phi)(v) = \Phi(-v).$$

Define $(W(a)\Phi)(v) = e^{-2i\pi\sigma(a,v)}\Phi(v+a)$. The Weyl operators $W(a)$ act irreducibly in $\mathscr{H}$.

A convenient way of writing (even very unbounded) linear operators A in $\mathscr{H}$ is to consider the associated kernel: $A(a,b) = (\Omega^a, A\Omega^b)$. One proves then

$$(A\Phi)(a) = (1/k^2) \int A(-a, -b)\Phi(b)db.$$

Weyl Quantization

It consists in associating, to a function f_c on phase space, the operator $Q(f_c)$ defined formally by

$$Q(f_c) = \int \tilde{f}_c(v) W(-v/2) dv = \int f_c(v/2) W(v) M dv. \tag{1}$$

It has been shown in [2] that the two expressions coincide. In order to avoid a discussion of the convergence of the operator-valued integrals, we replace (1) by the kernel

$$(\Omega^a, Q(f_c)\Omega^b) = \int \tilde{f}_c(v)\Omega^{(a,b)}(-v/2)dv = \int f_c(v/2)\Omega^{(a,-b)}(v)dv \tag{2}$$

where

$$\Omega^{(a,b)}(v) = (\Omega^a, W(v)\Omega^b) = ke^{2i\pi\sigma(b,a)}e^{-2i\pi\sigma(a+b,v)}\Omega(v-a+b). \tag{3}$$

Heat Equation on Phase Space

Define the Laplacian, Δ, on E, by $\Delta = -FsF$, where s is the operator of multiplication by $s(v,v)$. Consider on E the heat equation:

$$\partial f/\partial \hbar = (\pi/4)\Delta f. \tag{4}$$

Let f_c be a function in the uniqueness and correctness class for (4); this is only a very mild requirement. Define $f(\hbar, v)$ as the solution of (4), with initial data $f_c(v)$. In other words, $f(\hbar, v)$ is the distribution of "temperature" at "time" $\hbar$, resulting from an initial distribution $f(0,v) = f_c(v)$.

Theorem. *One has*

$$f(\hbar, v) = (1/k)\left(\Omega^{\hbar^{-\frac{1}{2}}v}, \quad Q(f_c(\hbar^{\frac{1}{2}}\cdot))\Omega^{\hbar^{-\frac{1}{2}}v}\right) \tag{5}$$

where $f_c(\hbar^{\frac{1}{2}}\cdot)$ is the function $v \to f_c(\hbar^{\frac{1}{2}}v)$. □

Equation (5) describes very intuitively the way in which a (suitably rescaled) matrix element tends to a classical function. We shall apply it in a forthcoming paper to the study of time evolution.

In order to prove (5), specialize (2) to

$$(\Omega^a, Q(f)\Omega^a) = k \int f(v/2)\Omega(v-2a)dv$$

and notice that $G_\lambda(v) = \lambda^{-\nu}\Omega(\lambda^{-\frac{1}{2}}v)$ is the elementary solution of the heat equation $\partial G/\partial\lambda = \pi\Delta G$.

It is possible to derive analogous equations for off-diagonal matrix elements, and equations in which the initial data are given by $\tilde{f}_c$.

Acknowledgement. This work was partially supported by contract SE 2871 of the Deutsche Forschungsgemeinschaft.

References

1. Hepp, K.: Commun. math. Phys. **35**, 265 (1974)
2. Grossmann, A.: Commun. math. Phys. **48**, 191—194 (1976)

Communicated by H. Araki

Received November 10, 1975