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RESEARCH PAPER

ON FRACTIONAL HEAT EQUATION

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Abstract

In this paper, the long-time behavior of the Cesaro mean of the fundamental solution for fractional Heat equation corresponding to random time changes in the Brownian motion is studied. We consider both stable subordinators leading to equations with the Caputo-Djrbashian fractional derivative and more general cases corresponding to differential-convolution operators, in particular, distributed order derivatives.

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1. Introduction

Let $\{X_t, t \geq 0; P_x, x \in E\}$ be a strong Markov process in the phase space $\mathbb{R}^d$. Denote by T_t its transition semigroup (in an appropriate Banach space) and by L the generator of this semigroup. Let $S_t, t \geq 0$ be a subordinator (i.e., a non-decreasing real-valued Lévy process) with $S_0 = 0$ and the Laplace exponent Φ:

$$\mathbf{E}[e^{-\lambda S_t}] = e^{-t\Phi(\lambda)} \ t, \quad \lambda > 0.$$

We assume that S_t is independent of X_t.

Denote by $E_t, t > 0$ the inverse subordinator, and introduce the time-changed process $Y_t = X_{E_t}$. Our goal is to analyze the properties of Y_t depending on the given Markov process X_t and the particular choice of subordinator S_t, see Theorem 3.1 and Theorem 4.1 below. There is a lot

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of interest on this kind of problem in diverse disciplines. In addition to purely stochastic motivations, the transform of the Markov process X_t in the non-Markov one Y_t implies the presence of effects in the corresponding dynamics. This feature is important in a number of physical models. In particular, progress in the understanding of this process may lead to the realization of useful models of biological time in the evolution of species and ecological systems. Currently, there exist rather complete studies of such problems in the case of so-called θ-stable subordinators $(0 < \theta < 1)$ [6, 15] and in special examples of initial processes X_t (see, e.g., [17], [12], [13]).

As a basic characteristic of the new process Y_t, we may study the time evolution

$$u(t,x) = \mathbf{E}^x[f(Y_t)]$$

for a given initial data f.

As it was pointed out in several works, see e.g. [20], [7] and references therein, $u(t,x)$ is the unique strong solution (in some appropriate sense) to the following Cauchy problem

$$\mathbb{D}_t^{(k)} u(t,x) = Lu(t,x) \quad u(0,x) = f(x). \tag{1.1}$$

Here we have a generalized fractional derivative (GFD for short)

$$\mathbb{D}_t^{(k)} \phi(t) = \frac{d}{dt} \int_0^t k(t-s)(\phi(s) - \phi(0))ds$$

with a kernel k uniquely defined by Φ.

Let $u_0(t,x)$ be the solution to a similar Cauchy problem but with ordinary time derivative

$$\frac{\partial}{\partial t} u(t,x) = Lu(t,x) \quad u(0,x) = f(x). \tag{1.2}$$

In stochastic terminology, it is the solution to the forward Kolmogorov equation corresponding to the process X_t. Under quite general assumptions there is a convenient and essentially obvious relation between these evolutions that is known as the subordination principle:

$$u(t,x) = \int_0^\infty u_0(\tau,x) G_t(\tau) d\tau,$$

where $G_t(\tau)$ is the density of E_t.

A similar relation holds for fundamental solutions (or heat kernels in another terminology) $v(x,t)$ and $v^E(x,t)$ of equations (1.2) and (1.1), respectively. For certain classes of *a priori* bounds for fundamental solutions $v(x,t)$, the properties of the subordinated kernels were studied in [8]. The main technical tool used in this work involves a scaling property assumed for Φ [8] that is a global condition on the Lévy characteristic $\Phi(\lambda)$. It is nevertheless difficult to give an interpretation of this scaling assumption in terms of the subordinator.

The aim of the present work is to extend the class of random times for which one may obtain information about the time asymptotic of $v^E(x,t)$. We consider the following three classes of admissible kernels $k \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$, characterized in terms of the Laplace transforms $\mathcal{K}(\lambda)$ as $\lambda \to 0$ (i.e., as local conditions):

$$\mathcal{K}(\lambda) \sim \lambda^{\theta-1}, \quad 0 < \theta < 1. \tag{C1}$$

$$\mathcal{K}(\lambda) \sim \lambda^{-1} L\left(\frac{1}{\lambda}\right), \quad L(y) := \mu(0)\log(y)^{-1}. \tag{C2}$$

$$\mathcal{K}(\lambda) \sim \lambda^{-1} L\left(\frac{1}{\lambda}\right), \quad L(y) := C\log(y)^{-1-s}, \;\; s > 0, \; C > 0. \tag{C3}$$

We would like to emphasize that these classes of kernels leads to differential-convolution operators, in particular, the Caputo-Djrbashian fractional derivative (C1) and distributed order derivatives (C2),(C3). For each kernel of this type, we establish the asymptotic properties of the subordinated heat kernels from different classes of a priory bounds. It is important to stress that in working with much more general random times (i.e., without the scaling property), a price must be paid for such an extension, namely the replacement of $v^E(x,t)$ by its Cesaro mean. This is the key technical observation that underlies the analysis of several different model situations.

2. Preliminaries

Let $S = \{S(t), \; t \geq 0\}$ be a subordinator, that is a process with stationary and independent non-negative increments starting from 0. They form a special class of Lévy processes taking values in $[0,\infty)$ and their sample paths are non-decreasing. In addition we assume that S has no drift (see [3] for more details). The infinite divisibility of the law of S implies that its Laplace transform can be expressed in the form

$$\mathbb{E}(e^{-\lambda S(t)}) = e^{-t\Phi(\lambda)}, \quad \lambda \geq 0,$$

where $\Phi : [0,\infty) \longrightarrow [0,\infty)$, called the *Laplace exponent* (or *cumulant*), is a *Bernstein function*. The associated Lévy measure σ has support in $[0,\infty)$ and fulfills

$$\int_{(0,\infty)} (1 \wedge \tau)\, d\sigma(\tau) < \infty \tag{2.1}$$

such that the Laplace exponent Φ can be expressed as

$$\Phi(\lambda) = \int_{(0,\infty)} (1 - e^{-\lambda\tau})\, d\sigma(\tau), \tag{2.2}$$

which is known as the *Lévy-Khintchine formula* for the subordinator S. In addition we assume that the Lévy measure σ satisfy

$$\sigma(0,\infty)=\infty. \tag{2.3}$$

For the given Lévy measure σ, we define the function k by

$$k:(0,\infty)\longrightarrow(0,\infty),\ t\mapsto k(t):=\sigma\big((t,\infty)\big) \tag{2.4}$$

and denote its Laplace transform by $\mathcal{K}$; i.e., for any $\lambda\geq 0$ one has

$$\mathcal{K}(\lambda):=\int_0^\infty e^{-\lambda t}k(t)\,dt. \tag{2.5}$$

We note that by the Fubini theorem, the function $\mathcal{K}$ is given in terms of the Laplace exponent. Specifically,

$$\mathcal{K}(\lambda)=\int_0^\infty e^{-\lambda t}\int_{(t,\infty)}d\sigma(s)\,dt=\int_{(0,\infty)}\int_0^s e^{-\lambda t}\,dt\,d\sigma(s)=\frac{1}{\lambda}\Phi(\lambda),$$

i.e.,

$$\Phi(\lambda)=\lambda\mathcal{K}(\lambda),\quad \forall\lambda\geq 0. \tag{2.6}$$

Given the inverse process E of the subordinator S, namely

$$E(t):=\inf\{s\geq 0:\ S(s)\geq t\}=\sup\{s\geq 0:\ S(s)\leq t\}, \tag{2.7}$$

the marginal density of $E(t)$ will be denoted by $G_t(\tau)$, $t,\tau\geq 0$, more explicitly

$$G_t(\tau)\,d\tau=\partial_\tau P(E(t)\leq\tau)=\partial_\tau P(S(\tau)\geq t)=-\partial_\tau P(S(\tau)<t).$$

Example 1. θ-stable subordinator and Gamma processes.

(1) Let S be a θ-stable subordinator $\theta\in(0,1)$ with Laplace exponent

$$\Phi_\theta(\lambda)=\lambda^\theta=\frac{\theta}{\Gamma(1-\theta)}\int_0^\infty(1-e^{-\lambda\tau})\tau^{-1-\theta}\,d\tau,$$

from which it follows that the Lévy measure σ is given by

$$d\sigma(\tau)=\frac{\theta}{\Gamma(1-\theta)}\tau^{-1-\theta}\,d\tau.$$

The restriction $\theta\in(0,1)$ and not $\theta\in(0,2)$ is due to the requirement (2.1). The boundary $\theta=1$ corresponds to a degenerate case since $S(t)=t$.

We have $\mathcal{K}(\lambda)=\lambda^{\theta-1}$ and $k(t)=t^{-\theta}/\Gamma(1-\theta)$. The corresponding GFD $\mathbb{D}_t^{(k)}$ is the *Caputo-Djrbashian fractional derivative* $\mathbb{D}_t^{(\theta)}$ of order $\theta\in(0,1)$.

(2) The Gamma process $Y^{(a,b)}$ with parameters $a, b > 0$ is given by its Laplace exponent $\Phi_{(a,b)}$ as

$$\Phi_{(a,b)}(\lambda) = a\log\left(1+\frac{\lambda}{b}\right) = \int_0^\infty (1-e^{-\lambda\tau})a\tau^{-1}e^{-b\tau}\,d\tau,$$

where the second equality is known as the Frullani integral [1]. The corresponding Lévy measure is given by

$$d\sigma(\tau) = a\tau^{-1}e^{-b\tau}\,d\tau.$$

We have $\mathcal{K}(\lambda) = \lambda^{-1}a\log\left(1+\frac{\lambda}{b}\right)$ and $k(t) = a\Gamma(0, bt)$. The corresponding GFD is given by

$$(\mathbb{D}_t^{(a,b)}f)(t) = \frac{d}{dt}\int_0^t \Gamma(0, b(t-s))(f(s)-f(0))\,ds.$$

An important characteristic of the density $G_t(\tau)$ is given by its Laplace transform. More precisely, does the τ-Laplace (or t-Laplace) transform of $G_t(\tau)$ are known for an arbitrary subordinator? Thus, we would like to compute the following integrals

$$\int_0^\infty e^{-\lambda\tau}G_t(\tau)\,d\tau \quad \text{or} \quad \int_0^\infty e^{-\lambda t}G_t(\tau)\,dt.$$

The answer for the t-Laplace transform is affirmative and the result is given in (2.14) below. On the other hand, for the τ-Laplace transform a partial answer has been given for the class of θ-stable processes; namely

Example 2. (cf. Prop. 1(a) in [5]).

If S is a θ-stable process, then the inverse process $E(t)$ has the Mittag-Leffler distribution, as follows,

$$\mathbb{E}(e^{-\lambda E(t)}) = \sum_{n=0}^{\infty}\frac{(-\lambda t^\theta)^n}{\Gamma(n\theta+1)} = E_\theta(-\lambda t^\theta). \tag{2.8}$$

It follows from the asymptotic behavior of the Mittag-Leffler function E_θ that

$$\mathbb{E}(e^{-\lambda E(t)}) \sim \frac{C}{t^\theta}, \quad \text{as} \quad t\to\infty.$$

In addition, using the fact that

$$E_\theta(-x) = \int_0^\infty e^{-x\tau}M_\theta(\tau)\,d\tau, \quad \forall x\geq 0, \tag{2.9}$$

where M_θ is the so-called M-Wright (cf. [14] for more details and properties), it follows that

$$\mathbb{E}(e^{-\lambda E(t)}) = \int_0^\infty e^{-\lambda t^\theta \tau}M_\theta(\tau)\,d\tau = \int_0^\infty e^{-\lambda\tau}t^{-\theta}M_\theta(\tau t^{-\theta})\,d\tau$$

from which we obtain the density of $E(t)$ explicitly as

$$G_t(\tau) = t^{-\theta} M_\theta(\tau t^{-\theta}). \tag{2.10}$$

For a general subordinator, the following lemma determines the t-Laplace transform of $G_t(\tau)$, with k and $\mathcal{K}$ given in (2.4) and (2.5), respectively.

LEMMA 2.1. *The t-Laplace transform of the density $G_t(\tau)$ is given by*

$$\int_0^\infty e^{-\lambda t} G_t(\tau)\, dt = \mathcal{K}(\lambda) e^{-\tau\lambda\mathcal{K}(\lambda)}. \tag{2.11}$$

In addition, the double (τ, t)-Laplace transform of $G_t(\tau)$ is given by

$$\int_0^\infty \int_0^\infty e^{-p\tau} e^{-\lambda t} G_t(\tau)\, dt\, d\tau = \frac{\mathcal{K}(\lambda)}{\lambda\mathcal{K}(\lambda) + p}.$$

P r o o f. For any $\tau \geq 0$ let η_τ be the distribution of $S(\tau)$, that is

$$\mathbb{E}(e^{-\lambda S(\tau)}) = e^{-\tau\Phi(\lambda)} = \int_0^\infty e^{-\lambda s}\, d\eta_\tau(s). \tag{2.12}$$

Defining

$$g(\lambda, \tau) := \mathcal{K}(\lambda) e^{-\tau\Phi(\lambda)}, \quad \tau, \lambda > 0 \tag{2.13}$$

under assumption (2.3), for all $t > 0$ it follows from Theorem 3.1 in [16] that the density $G_t(\tau)$ of the random variable $E(t)$ if given by

$$G_t(\tau) = \int_0^t k(t-s)\, d\eta_\tau(s).$$

It follows then that

$$\int_0^\infty e^{-\lambda t} G_t(\tau)\, dt = g(\lambda, \tau) = \mathcal{K}(\lambda) e^{-\tau\Phi(\lambda)}. \tag{2.14}$$

In fact, by the Fubini's theorem we obtain

$$\begin{aligned}
\int_0^\infty e^{-\lambda t} G_t(\tau)\, dt &= \int_0^\infty e^{-\lambda t} \int_0^t k(t-s)\, d\eta_\tau(s)\, dt \\
&= \int_0^\infty \int_s^\infty e^{-\lambda t} k(t-s)\, dt\, d\eta_\tau(s) \\
&= \mathcal{K}(\lambda) \int_0^\infty e^{-\lambda s}\, d\eta_\tau(s) \\
&= g(\lambda, \tau).
\end{aligned}$$

In addition, it follows easily from (2.13) that

$$\int_0^\infty g(\lambda, \tau)\, d\tau = \frac{1}{\lambda}$$

so that (2.14) may be written as

$$\int_0^\infty e^{-\lambda t}\,dt \int_0^\infty G_t(\tau)\,d\tau = \frac{1}{\lambda}$$

which implies that $G_t(\tau)$ is a τ-density on $\mathbb{R}_+$:

$$\int_0^\infty G_t(\tau)\,d\tau = 1.$$

Finally, the double (τ, t)-Laplace transform follows from

$$\begin{aligned}\int_0^\infty \int_0^\infty e^{-p\tau} e^{-\lambda t} G_t(\tau)\,dt\,d\tau &= \int_0^\infty e^{-p\tau} g(\lambda,\tau)\,d\tau \\ &= \mathcal{K}(\lambda)\int_0^\infty e^{-p\tau} e^{-\tau\lambda\mathcal{K}(\lambda)}\,d\tau \\ &= \frac{\mathcal{K}(\lambda)}{\lambda\mathcal{K}(\lambda)+p}. \end{aligned} \tag{2.15}$$

□

3. **Subordinated heat kernel**

In this section, we investigate the long-time behavior of the fundamental solutions for fractional evolution equations corresponding to random time changes in the Brownian motion by the inverse process E_t, $t \geq 0$. We consider three classes of time change, namely those corresponding to the θ-stable subordinator, $0 < \theta < 1$, the distributed order derivative, and the class of Stieltjes functions. Henceforth L will always denotes a slowly varying function at infinity (SVF), that is,

$$\lim_{x\to\infty} \frac{L(\lambda x)}{L(x)} = 1,$$

see for instance [4] and [19]) for more details, while C, C' are constants whose values are unimportant, and which may change from line to line.

Let $v(x,t)$ be the fundamental solution of the heat equation

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} &= \frac{1}{2}\Delta u(x,t) \\ u(x,0) &= \delta(x), \end{cases} \tag{3.1}$$

where Δ denotes the Laplacian in $\mathbb{R}^d$. It is well known that the solution $v(x,t)$ of (3.1), called heat kernel (also known as Green function), is given by

$$v(x,t) = \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|x|^2}{2t}} \tag{3.2}$$

and the associated stochastic process is the classical Brownian motion in $\mathbb{R}^d$. Notice that the heat kernel $v(x,t)$ has the following long-time behavior

$$v(x,t) \sim Ct^{-d/2}, \text{ as } t \to \infty.$$

We are interested in studying the long-time behavior of the subordination of the solution $v(x,t)$ by the density $G_t(\tau)$, that is,

$$v^E(x,t) := \int_0^\infty v(x,\tau)G_t(\tau)\,d\tau = \frac{1}{(2\pi)^{d/2}}\int_0^\infty \tau^{-d/2}e^{-\frac{|x|^2}{2\tau}}G_t(\tau)\,d\tau. \quad (3.3)$$

Then $v^E(x,t)$ is the fundamental solution of the general fractional time differential equation, that is,

$$\begin{cases} \mathbb{D}_t^{(k)}u(x,t) &= \frac{1}{2}\Delta u(x,t) \\ u(x,0) &= \delta(x). \end{cases} \quad (3.4)$$

Here $\mathbb{D}_t^{(k)}$ are differential-convolution operators defined, for any nonnegative kernel $k \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$, by

$$\big(\mathbb{D}_t^{(k)}u\big)(t) := \frac{d}{dt}\int_0^t k(t-\tau)u(\tau)\,d\tau - k(t)u(0), \ t>0. \quad (3.5)$$

(See [11] for more details and examples.)

In order to study the time evolution of $v^E(x,t)$, one possibility is to define its Cesaro mean

$$M_t\big(v^E(x,t)\big) := \frac{1}{t}\int_0^t v^E(x,s)\,ds,$$

which may be written as

$$M_t\big(v^E(x,t)\big) = \int_0^\infty v(x,\tau)M_t\big(G_t(\tau)\big)d\tau. \quad (3.6)$$

The long-time behavior of the Cesaro mean $M_t\big(v^E(x,t)\big)$ was investigated in [10, Sec. 3] for the three classes of admissible kernels and $d \geq 3$. The method was based on the ratio Tauberian theorem from [12]. More precisely, the following theorem was shown.

Theorem 3.1. *Let $v^E(x,t)$ be the subordination of $v(x,t)$ by the kernel $G_t(\tau)$. Then the long-time behavior of the Cesaro mean of $v^E(x,t)$ as $t \to \infty$ is given by*

$$M_t(v^E(x,t)) \sim \begin{cases} Ct^{-\theta}, & k \in (C1), \\ C\log(t)^{-1}, & k \in (C2), \\ C\log(t)^{-1-s}, & k \in (C3). \end{cases}$$

In the next section we use an alternative method to find the long-time behavior of the Cesaro mean $M_t\big(v^E(x,t)\big)$.

4. **Alternative method for subordinated heat kernel**

The Laplace transform method is based on the result of Lemma 2.1 wherein the t-Laplace transform of the subordination $v^E(x,t)$ is explicitly given by

$$
\begin{aligned}
(\mathscr{L}v^E(x,\cdot))(\lambda) &= C\int_0^\infty \tau^{-d/2}e^{-\frac{|x|^2}{2\tau}}(\mathscr{L}G_\cdot(\tau))(\lambda)\,d\tau\\
&= C\mathcal{K}(\lambda)\int_0^\infty \tau^{-d/2}e^{-\frac{|x|^2}{2\tau}-\tau\lambda\mathcal{K}(\lambda)}\,d\tau. \qquad (4.1)
\end{aligned}
$$

The integral in (4.1) is computed according to the formula,

$$
\int_0^\infty \tau^{-d/2}e^{-\frac{a}{\tau}-b\tau}\,d\tau = \begin{cases} \frac{\sqrt{\pi}e^{-2\sqrt{ab}}}{\sqrt{b}}, & d=1,\\ 2K_0\left(2\sqrt{ab}\right), & d=2,\\ 2\left(\frac{a}{b}\right)^{(2-d)/4}K_{d/2-1}\left(2\sqrt{ab}\right), & d\geq 3, \end{cases}
$$

(see for instance [9, page 146, eqs. (27), (29)]), where $a=\frac{|x|^2}{2}$, $b=\lambda\mathcal{K}(\lambda)$, and $K_\nu(z)$ is the modified Bessel function of the second kind [2, Sec. 9.6]. The asymptotic of the Bessel function $K_\nu(z)$ as $z\to 0$ is well known (e.g., see [2, Eqs. (9.6.8) and (9.6.9)]) and is given by

$$
K_0(z)\sim -\ln(z), \qquad (4.2)
$$

$$
K_\nu(z)\sim \frac{1}{2}\Gamma(\nu)\left(\frac{z}{2}\right)^{-\nu}\sim Cz^{-\nu}, \quad \Re(\nu)>0. \qquad (4.3)
$$

With these explicit formulas, we study each class (C1), (C2), and (C3) separately which constitutes the main contribution of this paper.

THEOREM 4.1. *Let $v(x,t)$ be the fundamental solution of the heat equation (3.1) and $v^E(x,t)$ its subordination by the density $G_t(\tau)$. Then the long-time asymptotic of $v^E(x,t)$ is given according the admissible classes of kernels k by*

Class (C1):

$$
M_t(v^E(x,t))\sim \begin{cases} Ct^{-\theta/2}, & d=1,\\ Ct^{-\theta}\log(\sqrt{2}|x|t^{-\theta/2}), & d=2,\\ C|x|^{(\theta+1)(2-d)/2}t^{-\theta}, & d\geq 3. \end{cases}
$$

Class (C2):

$$
M_t(v^E(x,t))\sim \begin{cases} C\log(t)^{-1/2}e^{-\sqrt{2\mu(0)}|x|\log(t)^{-1/2}}, & d=1,\\ C\log(t)^{-1}\ln\left(\sqrt{2\mu(0)}|x|\log(t)^{-1/2}\right), & d=2,\\ C|x|^{2-d}\log(t)^{-1}, & d\geq 3. \end{cases}
$$

Class (C3):

$$M_t(v^E(x,t)) \sim \begin{cases} C\log(t)^{-(1+s)/2} e^{-C'\sqrt{2}|x|\log(t)^{-(1+s)/2}}, & d=1,\\ C\log(t)^{-1-s}\ln\left(C'|x|\log(t)^{-(1+s)/2}\right), & d=2,\\ C|x|^{2-d}\log(t)^{-1-s}, & d\geq 3.\end{cases}$$

P r o o f. **(C1):** For this class, $\mathcal{K}(\lambda)=\lambda^{\theta-1}$, $0<\theta<1$.

(1) For $d=1$, we obtain

$$(\mathscr{L}v^E(x,\cdot))(\lambda) = C\lambda^{-1+\theta/2}e^{-\sqrt{2}|x|\lambda^{\theta/2}} = \lambda^{-(1-\theta/2)}L\left(\frac{1}{\lambda}\right),$$

where $L(y)=Ce^{-\sqrt{2}|x|y^{-\theta/2}}$ is a SVF. An application of the Karamata Tauberian theorem (see for example [18, Sec. 2.2] or [4, Sec. 1.7]) gives

$$M_t\big(v^E(x,t)\big) \sim Ct^{-\theta/2}e^{-\sqrt{2}|x|t^{-\theta/2}} \sim Ct^{-\theta/2}, \quad t\to\infty.$$

(2) For $d=2$, we have

$$(\mathscr{L}v^E(x,\cdot))(\lambda) = C\lambda^{-(1-\theta)}K_0(\sqrt{2}|x|\lambda^{\theta/2}) = \lambda^{-(1-\theta)}L\left(\frac{1}{\lambda}\right),$$

where $L(y)=CK_0(\sqrt{2}|x|y^{-\theta/2})$ is a SVF. Invoking the Karamata Tauberian theorem and (4.2) yields, for $t\to\infty$,

$$M_t\big(v^E(x,t)\big) \sim Ct^{-\theta}K_0\big(\sqrt{2}|x|t^{-\theta/2}\big) \sim Ct^{-\theta}\ln\big(\sqrt{2}|x|t^{-\theta/2}\big).$$

(3) For $d\geq 3$, the Laplace transform of $v^E(x,t)$ has the form

$$\begin{aligned}(\mathscr{L}v^E(x,\cdot))(\lambda) &= C|x|^{(2-d)/2}\lambda^{-(1-\theta)}\left(\frac{1}{\lambda}\right)^{\theta(2-d)/4}K_{\frac{d}{2}-1}(\sqrt{2}|x|\lambda^{\theta/2})\\ &= \lambda^{-(1-\theta)}L\left(\frac{1}{\lambda}\right),\end{aligned}$$

where $L(y)=C|x|^{(2-d)/2}y^{\theta(2-d)/4}K_{\frac{d}{2}-1}(\sqrt{2}|x|y^{-\theta/2})$ is a SVF. It follows from the Karamata Tauberian theorem and (4.3) that

$$M_t\big(v^E(x,t)\big) \sim Ct^{-\theta}L(t) \sim C|x|^{(\theta+1)(2-d)/2}t^{-\theta}, \quad t\to\infty.$$

(C2): Here we have $\mathcal{K}(\lambda)\sim\lambda^{-1}L(\lambda^{-1})$ as $\lambda\to 0$, where $L(y)=\mu(0)\log(y)^{-1}$, $\mu(0)\neq 0$. Again we study the three different cases $d=1$, $d=2$ and $d\geq 3$.

(1) For $d = 1$, the t-Laplace transform of $v^E(x,t)$ can be written, for $\lambda \to 0$, as

$$(\mathscr{L}v^E(x,\cdot))(\lambda) = C\lambda^{-1}\log(\lambda^{-1})^{-1/2}e^{-\sqrt{2\mu(0)}|x|\log(\lambda^{-1})^{-1/2}} = \lambda^{-1}L\left(\frac{1}{\lambda}\right),$$

where $L(y) = C\log(y)^{-1/2}e^{-\sqrt{2\mu(0)}|x|\log(y)^{-1/2}}$ is a SVF. An application of the Karamata Tauberian theorem gives

$$M_t\big(v^E(x,t)\big) \sim CL(t) \sim C\log(t)^{-1/2}e^{-\sqrt{2\mu(0)}|x|\log(t)^{-1/2}}, \quad t \to \infty.$$

(2) If $d = 2$, we have

$$(\mathscr{L}v^E(x,\cdot))(\lambda) = C\lambda^{-1}\log(\lambda^{-1})^{-1}K_0\left(\sqrt{2\mu(0)}|x|\log(\lambda^{-1})^{-1/2}\right) = \lambda^{-1}L\left(\frac{1}{\lambda}\right),$$

where $L(y) = C\log(y)^{-1}K_0\left(\sqrt{2\mu(0)}|x|\log(y)^{-1/2}\right)$ is a SVF. As $t \to \infty$ then by the Karamata Tauberian theorem and (4.2) we obtain

$$M_t\big(v^E(x,t)\big) \sim CL(t) \sim C\log(t)^{-1}\ln\big(\sqrt{2\mu(0)}|x|\log(t)^{-1/2}\big).$$

(3) For $d \geq 3$, it follows that, as $\lambda \to 0$,

$$\begin{aligned}(\mathscr{L}v^E(x,\cdot))(\lambda) &= C|x|^{(2-d)/2}\lambda^{-1}\log(\lambda^{-1})^{-1+(2-d)/4} \\ &\quad \times K_{\frac{d}{2}-1}\left(C'|x|\log(\lambda^{-1})^{-1/2}\right) \\ &= \lambda^{-1}L\left(\frac{1}{\lambda}\right),\end{aligned}$$

where

$$L(y) = C|x|^{(2-d)/2}\log(y)^{-1+(2-d)/4}K_{\frac{d}{2}-1}\left(C'|x|\log(y)^{-1/2}\right)$$

is a SVF. To verify that $L(y)$ is a SVF, one may note that $\log(y)^{-1+(2-d)/4}$ as well as is $K_{\frac{d}{2}-1}\big(C'|x|\log(y)^{-1/2}\big)$ according to (4.3); the stated result then follows from Prop. 1.3.6 in [4]. It follows from the Karamata Tauberian theorem and (4.3) that

$$M_t\big(v^E(x,t)\big) \sim CL(t) \sim C|x|^{2-d}\log(t)^{-1}, \quad t \to \infty.$$

(C3): We now have $\mathcal{K}(\lambda) \sim C\lambda^{-1}L(\lambda^{-1})^{-1-s}$, as $\lambda \to 0$ and $s > 0$, $C > 0$.

(1) For $d=1$, the t-Laplace transform of $v^E(x,t)$ can be written, for $\lambda\to 0$, as

$$(\mathscr{L}v^E(x,\cdot))(\lambda) = C\lambda^{-1}\log(\lambda^{-1})^{-(1+s)/2}e^{-C'\sqrt{2}|x|\log(\lambda^{-1})^{-(1+s)/2}}$$
$$= \lambda^{-1}L\left(\frac{1}{\lambda}\right),$$

where $L(y) = C\log(y)^{-(1+s)/2}e^{-C'\sqrt{2}|x|\log(y)^{-(1+s)/2}}$ is a SVF, as is easily seen. An application of the Karamata Tauberian theorem gives, as $t\to\infty$,

$$M_t\big(v^E(x,t)\big)\sim CL(t)\sim C\log(t)^{-(1+s)/2}e^{-C'\sqrt{2}|x|\log(t)^{-(1+s)/2}}.$$

(2) For $d=2$, we have

$$(\mathscr{L}v^E(x,\cdot))(\lambda) = C\lambda^{-1}\log(\lambda^{-1})^{-1-s}K_0\left(C'|x|\log(\lambda^{-1})^{-(1+s)/2}\right)$$
$$= \lambda^{-1}L\left(\frac{1}{\lambda}\right),$$

where

$$L(y) = C\log(y)^{-1-s}K_0\left(C'|x|\log(y)^{-(1+s)/2}\right)$$

is a SVF. Use the Karamata Tauberian theorem and (4.2) now yield the behavior as $t\to\infty$

$$M_t\big(v^E(x,t)\big)\sim CL(t)\sim C\log(t)^{-1-s}\ln\left(C'|x|\log(t)^{-(1+s)/2}\right).$$

(3) For $d\geq 3$, it follows that

$$\begin{aligned}(\mathscr{L}v^E(x,\cdot))(\lambda) &= C|x|^{(2-d)/2}\lambda^{-1}\log(\lambda^{-1})^{-(1+s)(1-(2-d)/4)}\\ &\quad\times K_{\frac{d}{2}-1}\left(C'\sqrt{2}|x|\log(\lambda^{-1})^{-(1+s)/2}\right)\\ &= \lambda^{-1}L\left(\frac{1}{\lambda}\right),\end{aligned}$$

as $t\to\infty$, where

$$L(y) = C|x|^{(2-d)/2}\log(y)^{-(1+s)(2+d)/4}K_{\frac{d}{2}-1}\left(C'\sqrt{2}|x|\log(y)^{-(1+s)/2}\right)$$

is a SVF. We note that $L(y)$ is the product of two SVF's which is a SVF (see Prop. 1.3.6 in [4]). It the follows from the Karamata Tauberian theorem and (4.3) that

$$M_t\big(v^E(x,t)\big)\sim CL(t)\sim C|x|^{2-d}\log(t)^{-1-s},\quad t\to\infty.$$

□

Remark. (Gaussian convolution kernel).

We consider the nonlocal operator $\mathcal{L}$ on functions $u : \mathbb{R}^d \longrightarrow \mathbb{R}$ defined in integral form by

$$(\mathcal{L}u)(x) := (a * u)(x) - u(x) = \int_{\mathbb{R}^d} a(x-y)[u(y) - u(x)]\, dy, \tag{4.4}$$

where the convolution kernel a is non-negative, symmetric, bounded, and integrable, i.e.,

$$a(x) \geq 0, \qquad a(x) = a(-x), \qquad a(x) \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d). \tag{4.5}$$

In addition, the kernel a is a density in $\mathbb{R}^d$ with finite second moment; explicitly

$$\langle a \rangle := \int_{\mathbb{R}^d} a(x)\, dx = 1, \quad \int_{\mathbb{R}^d} |x|^2 a(x)\, dx < \infty. \tag{4.6}$$

Since $\mathcal{L}$ is a bounded operator in $L^2(\mathbb{R}^d)$, its heat semigroup $e^{t\mathcal{L}}$ can be easily computed by using the exponential series according to

$$e^{t\mathcal{L}} = e^{-t} e^{ta*} = e^{-t} \sum_{k=0}^{\infty} t^k \frac{a^{*k}}{k!} = e^{-t}\mathrm{Id} + e^{-t} \sum_{k=1}^{\infty} t^k \frac{a^{*k}}{k!}.$$

By removing the singular part $e^{-t}\mathrm{Id}$ of the heat semigroup, we obtain the *regularized* heat kernel

$$v(x,t) = e^{-t} \sum_{k=1}^{\infty} t^k \frac{a^{*k}\,(x)}{k!} \tag{4.7}$$

with the source at the origin. In other words, for any $f \in L^2(\mathbb{R}^d)$, a solution to the nonlocal Cauchy problem

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} & = \mathcal{L}u(x,t), \\ u(x,0) & = f(x), \end{cases} \tag{4.8}$$

has the form $u(x,t) = e^{-t} f(x) + (v * f)(x,t)$ with v given by (4.7). In particular, the fundamental solution of the problem (4.8) is

$$u(x,t) = e^{-t}\delta(x) + v(x,t).$$

If we denote by $v^E(x,t)$ the subordination of $v(x,t)$ (the regular part of $u(x,t)$) by the density $G_t(\tau)$, then it turns out that the Cesaro mean of $v^E(x,t)$ long time behavior depends crucially on the ratio between $|x|$ and t. The details of this investigation we postpone for a forthcoming paper.

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