HEAT KERNEL ESTIMATES FOR THE FRACTIONAL LAPLACIAN WITH DIRICHLET CONDITIONS

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We give sharp estimates for the heat kernel of the fractional Laplacian with Dirichlet condition for a general class of domains including Lipschitz domains.

1. Introduction. Explicit sharp estimates for the Green function of the Laplacian in $C^{1,1}$ domains were completed in 1986 by Zhao [43]. Sharp estimates of the Green function of Lipschitz domains were given in 2000 by Bogdan [6]. Explicit qualitatively sharp estimates for the classical heat kernel in $C^{1,1}$ domains were established in 2002 by Zhang [42]. Qualitatively sharp heat kernel estimates in Lipschitz domains were given in 2003 by Varopulous [41]. The development of the boundary potential theory of the fractional Laplacian follows a parallel path. Green function estimates were obtained in 1997 and 1998 by Kulczycki [29] and Chen and Song [21] for $C^{1,1}$ domains, and in 2002 by Jakubowski for Lipschitz domains [28]. In 2008 Chen, Kim and Song [19] gave sharp explicit estimates for the heat kernel $p_D(t, x, y)$ of the fractional Laplacian on $C^{1,1}$ domains D. The main contribution of the present paper is the following result.

THEOREM 1. *If D is κ-fat, then there is $C = C(\alpha, D)$ such that*

$$(1) \quad C^{-1} P^x(\tau_D > t) P^y(\tau_D > t) \leq \frac{p_D(t, x, y)}{p(t, x, y)} \leq C P^x(\tau_D > t) P^y(\tau_D > t)$$

for $0 < t \leq 1$ and $x, y \in D$.

Here $p(t, x, y)$ is the heat kernel of the fractional Laplacian on $\mathbb{R}^d$, and

$$P^x(\tau_D > t) = \int_{\mathbb{R}^d} p_D(t, x, y) \, dy$$

defines the *survival probability* of the corresponding isotropic α-stable Lévy process in D. The result applies also to unbounded domains, in particular, to domains above the graph of a Lipschitz function, where we can take *arbitrary $t > 0$*.

In fact, (1) holds with $C = C(\alpha, d, \kappa)$ under the mere condition that D is $(\kappa, t^{1/\alpha})$-fat at x and at y; see Sections 3 and 4 for definitions and results. For *exterior* domains we have a result free from local geometric assumptions:

COROLLARY 1. *If* $\mathrm{diam}(D^c) < \infty$, *then* (1) *holds with* $C = C(\alpha, d)$ *for all* $t > \mathrm{diam}(D^c)^\alpha$ *and* $x, y \in D$.

For exterior domains of class $C^{1,1}$ a more explicit estimate is given in Theorem 3 below. We also like to note that a useful variant of Theorem 1 is given in Theorem 2.

Expression (1) is motivated by these applications of the semigroup property of p_D:

$$p_D(2t, x, y) = \int_{\mathbb{R}^d} p_D(t, x, z) p_D(t, z, y)\, dz \le P^x(\tau_D > t) c(t),$$

where $c(t) = \sup_{z, y \in \mathbb{R}^d} p(t, z, y) \ge \sup_{z, y \in \mathbb{R}^d} p_D(t, z, y)$ [see (12)], and

$$p_D(3t, x, y) = \int \int p_D(t, x, z) p_D(t, z, w) p_D(t, w, y)\, dw\, dz$$

$$\le P^x(\tau_D > t) c(t) P^y(\tau_D > t).$$

The latter inequality is quite satisfactory for $x = y$, because $c(t) = p(t, x, x)$. Off-diagonal (x, y) in (1) require, however, a deeper analysis. Our proof of (1) is based on the boundary Harnack principle (BHP) [14] (see also earlier [40]), a version of the Ikeda–Watanabe [27] formula (18), scaling (14) and comparability of p with its Lévy measure (5); see (28). Counterparts of these are important in view of possible generalizations.

In what follows (1) and analogous *sharp estimates* will be written as

$$p_D(t, x, y) \overset{C}{\approx} P^x(\tau_D > t) p(t, x, y) P^y(\tau_D > t),$$

meaning that either ratio of the sides is bounded by a number $C \in (0, \infty)$, and C does not depend on the variables shown (here: t, x, y). We will skip C from notation if unimportant for our goals.

Let $\delta_D(x) = \mathrm{dist}(x, D^c)$. As mentioned above, domains D of class $C^{1,1}$ enjoy the following sharp and explicit estimate of Chen, Kim and Song [19]:

$$(2) \quad \frac{p_D(t, x, y)}{p(t, x, y)} \approx \left(1 \wedge \frac{\delta_D^{\alpha/2}(x)}{t^{1/2}}\right)\left(1 \wedge \frac{\delta_D^{\alpha/2}(y)}{t^{1/2}}\right), \qquad 0 < t \le 1, x, y \in \mathbb{R}^d.$$

We note that (2) agrees with (1) because by [10], Corollary 1,

$$P^x(\tau_D > t) \approx 1 \wedge \frac{\delta_D^{\alpha/2}(x)}{t^{1/2}} \qquad \text{for } 0 < t \le 1, x, y \in \mathbb{R}^d.$$

In fact, starting with (1), we are able to recover and strengthen (2), with a simpler proof; see Example 5 and Proposition 1 below. We note that (1) was conjectured in

[10] based on the cases of $C^{1,1}$ domains [19] and circular cones [10]. We should also mention that the Gaussian estimates of Varopoulous [41] have a shape similar to (1), in particular, they involve the survival probability. Thus, the present paper builds on the evidence accumulated in [19, 41] and [10]. We also note that the upper bound in (2) was proved in 2006 by Siudeja for semibounded convex domains ([39], Theorem 1.6), and stated for general convex domains in [39], Remark 1.7. Some of our present techniques were inspired by [32], Theorem 4.2, of Kulczycki and Siudeja, [2], Proposition 2.9, of Bañuelos and Kulczycki, and [1], Section 4, of Bogdan and Bañuelos.

It is a consequence of Lemma 1 below that we can apply BHP [14, 40] to conveniently estimate $P^x(\tau_D > 1)$ by some kernel functions of D, namely, by the Martin kernel with the pole at infinity *or* the expected survival time [we use *scaling* to estimate $P^x(\tau_D > t)$ for general $t > 0$]. The estimate and the resulting bounds for the heat kernel are collected in Theorem 2, followed by a number of applications. In particular, we give a simple proof of the main result of [10] for the circular cones V:

$$(3) \qquad \frac{p_V(t, x, y)}{p(t, x, y)} \approx \frac{(1 \wedge \delta_V(x)/t^{1/\alpha})^{\alpha/2}}{(1 \wedge |x|/t^{1/\alpha})^{\alpha/2-\beta}} \frac{(1 \wedge \delta_V(y)/t^{1/\alpha})^{\alpha/2}}{(1 \wedge |y|/t^{1/\alpha})^{\alpha/2-\beta}}.$$

Here $\beta \in [0, \alpha)$ is a characteristic of the cone, and *all* $t > 0$ and $x, y \in \mathbb{R}^d$ are allowed. We should add to (1), (2) and (3) that [4, 16]

$$(4) \qquad p_t(x) \overset{c}{\approx} \frac{t}{|x|^{d+\alpha}} \wedge t^{-d/\alpha}, \qquad t > 0, x \in \mathbb{R}^d.$$

Here $c = c(\alpha, d)$, meaning that $c \in (0, \infty)$ may be so chosen to depend only on d and α. We like to note that the estimates for general κ-fat domains cannot be as explicit as those for $C^{1,1}$ domains. In particular, the decay rate β at the vertex of a cone delicately depends on the aperture of the cone; see [1, 10, 35] (see also [6]). Nevertheless, Lipschitz domains offer a natural setting for studying the boundary behavior of the Green function and the heat kernel for both the Brownian motion and the isotropic α-stable Lévy processes. This is due to the scaling, the rich range of asymptotic behaviors depending on the local geometry of the domain's boundary, connections to the boundary Harnack principle and approximate factorization of the Green function, and applications in the perturbation theory of generators, in particular, via the 3G Theorem [1, 6, 7, 26, 43] and 3P Theorem [13]. The κ-fat sets are a convenient generalization of Lipschitz domains, with similar features. It is noteworthy that (1) is an approximate factorization of the heat kernel (see also [6, 14] in this connection).

We should add that the $C^{1,1}$ condition specifies the geometry of a domain only in bounded scales (see Definition 3). This renders the range of time in (2) restricted to $0 < t \leq 1$. In what follows we will also study the probability of survival for large times (and unbounded domains). This is straightforward for special Lipschitz domains (thus for circular cones), but less so for general κ-fat or $C^{1,1}$ domains. As

an interesting case study we consider domains with bounded complement (i.e., exterior domains) of class $C^{1,1}$. These have distinctive geometries at infinity and at the boundary, resulting in nontrivial completion of (2). We remark that exterior $C^{1,1}$ domains in dimension $d > \alpha$ have been recently studied in [22], too. We also remark that [25], Theorem 4.4 bounds the survival probability of the relativistic process in a half-line, and [31] gives an explicit formula for the transition density of the killed Cauchy process ($\alpha = 1$) on the half-line. Regarding other recent estimates [3, 17, 20, 23, 36] for the transition density and potential kernel of jump-type processes, we need to point out that generally these only concern processes without killing. Killing corresponds to the Dirichlet "boundary" condition (analogous to the negative Schrödinger perturbation [8, 12]) and it severely influences the asymptotics of the transition density and Green function. Needless to say, the asymptotics are crucial for solving the Dirichlet problem [24, 25].

We like to mention possible applications and further directions of research. The estimate (1) fits well into the technique of Schrödinger perturbations of [12], which should produce straightforward consequences. Also, the distribution of τ_D, given by (18) below, can be estimated by using (1). Further, we conjecture that for certain domains D, $\lim p_D(t, x, y)/P^x(\tau_D > t)$ exists as x approaches a boundary point of D. This may lead to representation theorems for nonnegative parabolic functions of the fractional Laplacian (compare [14], Theorems 2 and 3) and construction of excursion laws. We need to remark here that our estimates are inconclusive about the (irregular [14]) boundary points of D, but we conjecture that (1) indeed extends to ∂D. Finally, it seems important to understand the behavior of $p_D(t, x, y)$ for domains which are rather small at a boundary point or at infinity. In this connection we refer the interested reader to the recent study of *intrinsic ultracontractivity* by Kwaśnicki [33]; see also [10, 19, 30] and the notion of *inaccessibility* in [14].

Our general references to the boundary potential theory of the fractional Laplacian are [7] and [14]. We also refer the reader to [9] for a broad non-technical overview of the methods and goals of the theory.

The paper is composed as follows. In Section 2 we recall basic facts about the killed isotropic α-stable Lévy processes. In Section 3 we prove Theorem 1 and Corollary 1. In Section 4 we state and prove Theorem 2 and give applications to specific domains. In particular, we strengthen (2) and part of the results of [19] (see Proposition 1, Theorem 3 and Corollary 2), and we discuss exterior $C^{1,1}$ domains in dimension $d = 1 < \alpha$.

2. Preliminaries. In what follows, $\mathbb{R}^d$ denotes the Euclidean space of dimension $d \geq 1$, dy is the Lebesgue measure on $\mathbb{R}^d$, and $0 < \alpha < 2$. Our primary analytic data are as follows: a nonempty open set $D \subset \mathbb{R}^d$ and the Lévy measure given by density function

$$(5) \qquad \nu(y) = \frac{2^\alpha \Gamma((d+\alpha)/2)}{\pi^{d/2}|\Gamma(-\alpha/2)|}|y|^{-d-\alpha}.$$

The coefficient in (5) is such that

$$(6) \qquad \int_{\mathbb{R}^d} [1 - \cos(\xi \cdot y)] \nu(y) \, dy = |\xi|^\alpha, \qquad \xi \in \mathbb{R}^d.$$

For (smooth compactly supported) $\phi \in C_c^\infty(\mathbb{R}^d)$, the fractional Laplacian is

$$\Delta^{\alpha/2} \phi(x) = \lim_{\varepsilon \downarrow 0} \int_{|y| > \varepsilon} [\phi(x + y) - \phi(x)] \nu(y) \, dy, \qquad x \in \mathbb{R}^d$$

(see [7, 9] for a broader setup). If $r > 0$ and $\phi_r(x) = \phi(rx)$, then

$$(7) \qquad \Delta^{\alpha/2} \phi_r(x) = r^\alpha \Delta^{\alpha/2} \phi(rx), \qquad x \in \mathbb{R}^d.$$

We let p_t be the smooth real-valued function on $\mathbb{R}^d$ with Fourier transform,

$$(8) \qquad \int_{\mathbb{R}^d} p_t(x) e^{ix \cdot \xi} \, dx = e^{-t|\xi|^\alpha}, \qquad t > 0, \xi \in \mathbb{R}^d.$$

In particular, the maximum of p_t is $p_t(0) = 2^{1-\alpha} \pi^{-d/2} \alpha^{-1} \Gamma(d/\alpha) / \Gamma(d/2) t^{-d/\alpha}$. According to (6) and the Lévy–Khinchine formula, $\{p_t\}$ is a probabilistic convolution semigroup with Lévy measure $\nu(y) \, dy$; see [16, 38] or [9]. We have the following scaling property,

$$(9) \qquad p_t(x) = t^{-d/\alpha} p_1(t^{-1/\alpha} x), \qquad t > 0, x \in \mathbb{R}^d,$$

which may be considered a consequence of (8). It is noteworthy that by (4) we have

$$(10) \qquad p_t(x) \approx p_{2t}(x), \qquad t > 0, x \in \mathbb{R}^d.$$

We denote

$$p(t, x, y) = p_t(y - x),$$

and we have

$$(11) \quad \int_s^\infty \int_{\mathbb{R}^d} p(u - s, x, z)[\partial_u \phi(u, z) + \Delta_z^{\alpha/2} \phi(u, z)] \, dz \, du = -\phi(s, x),$$

where $s \in \mathbb{R}$, $x \in \mathbb{R}^d$, and $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$; see, for example, [12], (36).

We define the isotropic α-stable Lévy process (X_t, P^x) by stipulating transition probability

$$P_t(x, A) = \int_A p(t, x, y) \, dy, \qquad t > 0, x \in \mathbb{R}^d, A \subset \mathbb{R}^d,$$

initial distribution $P^x(X(0) = x) = 1$, and cádlág paths. Thus, P^x, E^x denote the distribution and expectation for the process starting at x. We define the *time of the first exit* from D, or *survival time*,

$$\tau_D = \inf\{t > 0 : X_t \notin D\},$$

and the *time of first hitting D*,

$$T_D = \inf\{t > 0 : X_t \in D\}.$$

We define, as usual,

$$p_D(t, x, y) = p(t, x, y) - E^x[\tau_D < t; p(t - \tau_D, X_{\tau_D}, y)], \qquad t > 0, x, y \in \mathbb{R}^d.$$

We have that

(12) $$0 \le p_D(t, x, y) = p_D(t, y, x) \le p(t, x, y),$$

hence,

(13) $$\int p_D(t, x, y)\,dy = \int p_D(t, x, y)\,dx \le 1.$$

If $x \in D^c$ is *regular* for the Dirichlet problem on D [14], that is, $P^x(\tau_D = 0) = 1$, then $p_D(t, x, y) = 0$ and (1) is trivially satisfied. By this remark, if all the points of ∂D are regular for D, then we can write $x, y \in \mathbb{R}^d$ in Theorem 1, instead of $x, y \in D$. The remark also applies to Examples 1–8 in Section 4. By the strong Markov property, p_D is the transition density of the isotropic stable process *killed* when leaving D, meaning that we have the following Chapman–Kolmogorov equation,

$$\int_{\mathbb{R}^d} p_D(s, x, z) p_D(t, z, y)\,dz = p_D(s + t, x, y), \qquad s, t > 0, x, y \in \mathbb{R}^d,$$

and for nonnegative or bounded (Borel) functions $f : \mathbb{R}^d \to \mathbb{R}$,

$$\int_{\mathbb{R}^d} f(y) p_D(t, x, y)\,dy = E^x[\tau_D < t; f(X_t)], \qquad t > 0, x \in \mathbb{R}^d.$$

For $s \in \mathbb{R}$, $x \in \mathbb{R}^d$, and $\phi \in C_c^\infty(\mathbb{R} \times D)$, we have

$$\int_s^\infty \int_D p_D(u - s, x, z)[\partial_u \phi(u, z) + \Delta_z^{\alpha/2} \phi(u, z)]\,dz\,du = -\phi(s, x),$$

which extends (11) and justifies calling p_D the heat kernel of the (Dirichlet) fractional Laplacian *on D*. It is well known that p_D is jointly continuous and positive for $(t, x, y) \in (0, \infty) \times D \times D$. We have a scaling property, $p_{rD}(r^\alpha t, rx, ry) = r^d p_D(t, x, y), r > 0$, or

(14) $$p_D(t, x, y) = t^{-d/\alpha} p_{t^{-1/\alpha}D}(1, t^{-1/\alpha}x, t^{-1/\alpha}y), \qquad t > 0, x, y \in \mathbb{R}^d,$$

in agreement with (9) and (7). Thus, $P^{rx}(\tau_{rD} > r^\alpha t) = P^x(\tau_D > t)$, or

(15) $$P^x(\tau_D > t) = \int_{\mathbb{R}^d} p_D(t, x, y)\,dy = P^{t^{-1/\alpha}x}(\tau_{t^{-1/\alpha}D} > 1).$$

REMARK 1. For $c > 0$ consider $\tilde{\nu} = c\nu$, the corresponding heat kernels $\tilde{p}, \tilde{p}_D$, probability and expectation $\tilde{P}^x, \tilde{E}^x$. Clearly, $\tilde{p}_D(t, x, y) = p_D(ct, x, y)$.

The Green function of D is defined as

$$(16) \qquad G_D(x, y) = \int_0^\infty p_D(t, x, y)\, dt,$$

and scaling of p_D yields the following scaling of G_D,

$$(17) \qquad G_{rD}(rx, ry) = r^{\alpha-d} G_D(x, y).$$

A result of Ikeda and Watanabe [27] asserts that for $x \in D$ the P^x-distribution of $(\tau_D, X_{\tau_D-}, X_{\tau_D})$ restricted to $X_{\tau_D-} \neq X_{\tau_D}$ is given by the density function

$$(18) \qquad (s, u, z) \mapsto p_D(s, x, u)\nu(z - u).$$

For geometrically nice domains, for example, for the ball, $P^x(X_{\tau_D-} \neq X_{\tau_D}) = 1$ for $x \in D$ [14], and then by (16) and (18) the P^x-distribution of X_{τ_D} has the density function given by the Poisson kernel,

$$(19) \qquad P_D(x, z) = \int_D G_D(x, u)\nu(z - u)\, du.$$

For $x_0 \in \mathbb{R}^d$ and $r > 0$ we consider the ball $B(x_0, r) = \{x \in \mathbb{R}^d : |x - x_0| < r\}$ and $B^c(x_0, r) = \{x \in \mathbb{R}^d : |x - x_0| > r\}$ (open complement of a ball).

There is a constant C depending only on d, α and p, such that

$$(20) \qquad P_U(x_1, y_1) P_U(x_2, y_2) \overset{C}{\approx} P_U(x_1, y_2) P_U(x_2, y_1),$$

whenever $U \subset B(x_0, r) \subset \mathbb{R}^d$ is open, $0 < p < 1$, $r > 0$, $x_0 \in \mathbb{R}^d$, $x_1, x_2 \in U \cap B(x_0, rp)$, and $y_1, y_2 \in B(x_0, r)^c$. This boundary Harnack principle (BHP) follows from [14], Lemma 7 and the proof of Theorem 1, and it is essentially an approximate factorization of P_U. We encourage the interested reader to directly verify the estimate in the special case of (22) below.

The Green function and Poisson kernel of $B(x_0, r)$ are known explicitly:

$$(21) \qquad G_{B(x_0,r)}(x, v) = \mathcal{B}_{d,\alpha}|x - v|^{\alpha-d} \int_0^w \frac{s^{\alpha/2-1}}{(s+1)^{d/2}}\, ds,$$

$$(22) \qquad P_{B(x_0,r)}(x, y) = \mathcal{C}_{d,\alpha} \left[\frac{r^2 - |x - x_0|^2}{|y - x_0|^2 - r^2}\right]^{\alpha/2} \frac{1}{|x - y|^d},$$

where $\mathcal{B}_{d,\alpha} = \Gamma(d/2)/(2^\alpha \pi^{d/2}[\Gamma(\alpha/2)]^2)$, $\mathcal{C}_{d,\alpha} = \Gamma(d/2)\pi^{-1-d/2}\sin(\pi\alpha/2)$,

$$w = (r^2 - |x - x_0|^2)(r^2 - |v - x_0|^2)/|x - v|^2,$$

$|x - x_0| < r$, $|v - x_0| < r$, and $|y - x_0| \geq r$; see [5, 37]. Thus,

$$(23) \qquad P^x\big(|X_{\tau_{B(0,1)}}| > R\big) = \int_{|y| \geq R} P_{B(0,1)}(x, y)\, dy \approx \frac{(1 - |x|)^{\alpha/2}}{R^\alpha},$$

where $x \in B(0, 1)$ and $R \geq 2$. Also, for $|x - x_0| \leq r$ we have [8]

$$(24) \qquad E^x \tau_{B(x_0, r)}(x) = \frac{2^{1-\alpha} \Gamma(d/2)}{\alpha \Gamma((d+\alpha)/2) \Gamma(\alpha/2)} (r^2 - |x - x_0|^2)^{\alpha/2}.$$

All the sets and functions considered below are Borelian. *Positive* means *strictly positive*. *Domain* means a nonempty open set (connectedness need not be assumed in this theory).

3. Factorization.
We consider nonempty open set $D \subset \mathbb{R}^d$.

DEFINITION 1.   Let $x \in D$, $r > 0$ and $0 < \kappa \leq 1$. We say that D is (κ, r)-fat at x if there is a ball $B(A, \kappa r) \subset D \cap B(x, r)$. If this is true for every $x \in D$, then we say that D is (κ, r)-fat. We say that D is κ-fat if there is $R > 0$ such that D is (κ, r)-fat for all $r \in (0, R]$.

REMARK 2.   The ball is $1/2$-fat.

DEFINITION 2.   Given $B(A, \kappa) \subset D \cap B(x, 1)$, we consider $U = D \cap B(x, |x - A| + \kappa/3)$, $B_1 = B(A, \kappa/3) \subset U$ and $B_2 = B(A', \kappa/6)$ such that $B(A', \kappa/3) \subset B(A, \kappa) \setminus U$; see the picture:
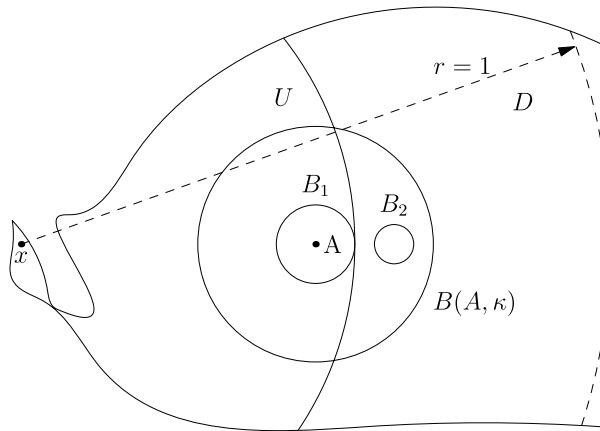
LEMMA 1.   *There is $C = C(\alpha, d, \kappa)$ such that if D is $(\kappa, 1)$-fat at x, then*

$$(25) \qquad P^x(\tau_D > 1/3) \leq C P^x(\tau_D > 3).$$

PROOF.   Consider $x \in D$ and $B(A, \kappa)$ and U as above. For $|x - A| < \kappa/2$,

$$1 \geq P^x(\tau_D > 1/3) \geq P^x(\tau_D > 3)$$
$$\geq P^x\big(\tau_{B(x,\kappa/2)} > 3\big) = P^0\big(\tau_{B(0,\kappa/2)} > 3\big) > 0,$$

and (25) is proved. We will now assume that $|x - A| \geq \kappa/2$. We note that

(26) $$P^x(\tau_D > 1/3) \leq P^x(\tau_U > 1/3) + P^x(X_{\tau_U} \in D).$$

We have $P^x(X_{\tau_U} \in D) = \int_D P_U(x, y)\,dy$. Indeed, if $B = B(x, |x - A| + \kappa/3)$ as in the definition of U, then $P^x(X_{\tau_U} \in \partial U \cap D) \leq P^x(X_{\tau_B} \in \partial B) = 0$; see the discussion preceding (19) above. Similarly, $P^x(X_{\tau_U} \in B_2)$ is an integral of the Poisson kernel P_U. We consider BHP for $x_1 = x$, $x_2 = A$, $p = 1 - \kappa/3 > (1 - \kappa)/(1 - \kappa + \kappa/3)$. Integrating (20) on D and B_2, we obtain

$$\frac{P^x(X_{\tau_U} \in D)}{P^A(X_{\tau_U} \in D)} \leq c\frac{P^x(X_{\tau_U} \in B_2)}{P^A(X_{\tau_U} \in B_2)}.$$

We note that (the denominator) $P^A(X_{\tau_U} \in B_2) \geq P^A(X_{\tau_{B_1}} \in B_2) \geq c > 0$ [see (22)], therefore, $P^x(X_{\tau_U} \in D) \leq c P^x(X_{\tau_U} \in B_2)$. We also observe that $u \mapsto \int_{B_2} \nu(y - u)\,dy$ is bounded away from zero and infinity on U. By (19),

$$P^x(X_{\tau_U} \in B_2) = \int_U G_U(x, u) \int_{B_2} \nu(y - u)\,dy\,du \approx \int_U G_U(x, u)\,du = E^x\tau_U.$$

Clearly, $P^x(\tau_U > 1/3) \leq 3E^x\tau_U$. By (26), $P^x(\tau_D > 1/3) \leq cE^x\tau_U$. By the strong Markov property,

$$E^x\tau_U \leq cP^x(X_{\tau_U} \in B_2) \leq cE^x\big[X_{\tau_U} \in B_2; P^{X_{\tau_U}}\big(\tau_{B(X_{\tau_U},\kappa/6)} > 3\big)\big]$$
$$\leq cP^x(\tau_D > 3). \qquad \square$$

REMARK 3.   If D is $(\kappa, 1)$-fat at x, then by the above proof we have

(27) $$P^x(\tau_D > 1/3) \approx P^x(\tau_D > 3) \approx P^x(\tau_D > 1) \approx P^x(X_{\tau_U} \in D) \approx E^x\tau_U.$$

In fact, we can replace 3 by any finite $\mathcal{E} \geq 1$, at the expense of having the comparability between *each* pair of expressions in (27) holding with a constant $C = C(\alpha, d, \kappa, \mathcal{E})$.

LEMMA 2.   *Consider open $D_1, D_3 \subset D$ such that $\mathrm{dist}(D_1, D_3) > 0$. Let $D_2 = D \setminus (D_1 \cup D_3)$. If $x \in D_1$ and $y \in D_3$, then*

$$p_D(1, x, y) \leq P^x(X_{\tau_{D_1}} \in D_2) \sup_{s<1, z\in D_2} p(s, z, y) + E^x\tau_{D_1} \sup_{u\in D_1, z\in D_3} \nu(z - u)$$

and

$$p_D(1, x, y) \geq P^x(\tau_{D_1} > 1)P^y(\tau_{D_3} > 1) \inf_{u\in D_1, z\in D_3} \nu(z - u).$$

PROOF.   By the strong Markov property,

$$p_D(1, x, y) = E^x[p_D(1 - \tau_{D_1}, X_{\tau_{D_1}}, y), \tau_{D_1} < 1],$$

which is

$$E^x[p_D(1 - \tau_{D_1}, X_{\tau_{D_1}}, y), \tau_{D_1} < 1, X_{\tau_{D_1}} \in D_2]$$
$$+ E^x[p_D(1 - \tau_{D_1}, X_{\tau_{D_1}}, y), \tau_{D_1} < 1, X_{\tau_{D_1}} \in D_3] = I + II.$$

Clearly,

$$I \leq P^x(X_{\tau_{D_1}} \in D_2) \sup_{s<1, z \in D_2} p(s, z, y).$$

Consider D_1 such that $P^x(X_{\tau_{D_1}} \in \partial D_1 \cap D) = 0$, for example, D_1 being an inter-section of D with a Lipschitz domain. By (18), the density function of $(\tau_{D_1}, X_{\tau_{D_1}})$ at (s, z) for $z \in D$ equals

$$f^x(s, z) = \int_{D_1} p_{D_1}(s, x, u)\nu(z - u)\, du.$$

For $z \in D_3$,

$$f^x(s, z) = \int_{D_1} p_{D_1}(s, x, u)\nu(z - u)\, du \leq P^x(\tau_{D_1} > s) \sup_{u \in D_1, z \in D_3} \nu(z - u),$$

hence, by (13),

$$II = \int_0^1 \int_{D_3} p_D(1 - s, z, y) f^x(s, z)\, dz\, ds$$

$$\leq \sup_{u \in D_1, z \in D_3} \nu(z - u) \int_0^1 \int_{D_3} p_D(1 - s, z, y) P^x(\tau_{D_1} > s)\, dz\, ds$$

$$\leq \int_0^1 P^x(\tau_{D_1} > s)\, ds \sup_{u \in D_1, z \in D_3} \nu(z - u) \leq E^x \tau_{D_1} \sup_{u \in D_1, z \in D_3} \nu(z - u).$$

The upper bound follows. The case of general D_1 follows by approximating from below, and continuity of p and ν. The lower bound obtains analogously

$$II \geq \inf_{u \in D_1, z \in D_3} \nu(z - u) \int_0^1 \int_{D_3} p_D(1 - s, z, y) P^x(\tau_{D_1} > s)\, dz\, ds$$

$$\geq P^x(\tau_{D_1} > 1) \inf_{u \in D_1, z \in D_3} \nu(z - u) \int_0^1 \int_{D_3} p_{D_3}(1 - s, z, y)\, dz\, ds. \qquad \square$$

REMARK 4.   Lemma 2 also holds for $\tilde{\nu}$, $\tilde{p}$, $\tilde{P}^x$ and $\tilde{E}^x$ of Remark 1.

In what follows we will often use the fact that

$$(28) \qquad\qquad\qquad 1 \wedge \nu(z - u) \approx p(1, u, z).$$

LEMMA 3.   *If D is $(\kappa, 1)$-fat at x and y, then*

$$p_D(2, x, y) \le C(\alpha, d, \kappa) P^x(\tau_D > 2) P^y(\tau_D > 2) p(2, x, y).$$

PROOF.   If $|x - y| \le 8$, then $p(1, x, y) \approx 1$, and by the semigroup property, (10) and Lemma 1,

$$
\begin{aligned}
p_D(1, x, y) &= \int_{\mathbb{R}^d} p_D(1/2, x, z) p_D(1/2, z, y) \, dz \\
&\le \sup_z p(1/2, z, y) P^x(\tau_D > 1/2) \\
&\le c P^x(\tau_D > 1) p(1, x, y).
\end{aligned}
$$
(29)

Here $c = c(\alpha, d, \kappa)$. If $|x - y| > 8$, then we will apply Lemma 2 with $D_1 = U = D \cap B(A, |x - A| + \kappa/3)$, as in Definition 2, and $D_3 = \{z \in D : |z - x| > |x - y|/2\}$. Since $\sup_{s<1, z \in D_2} p(s, z, y) \le c p(1, x, y)$, and $\sup_{u \in D_1, z \in D_3} \nu(z - u) \le c p(1, x, y)$ [see (28)], by Remark 3, we obtain

$$
\begin{aligned}
p_D(1, x, y) &\le c p(1, x, y)[P^x(X_{\tau_U} \in D) + E^x \tau_U] \\
&\le c P^x(\tau_D > 1) p(1, x, y),
\end{aligned}
$$
(30)

hence, by (29), (30), symmetry, the semigroup property and Lemma 1,

$$
\begin{aligned}
p_D(2, x, y) &= \int p_D(1, x, z) p_D(1, z, y) \, dz \\
&\le c P^x(\tau_D > 1) P^y(\tau_D > 1) \int p(1, x, z) p(1, z, y) \, dz \\
&\le c P^x(\tau_D > 2) P^y(\tau_D > 2) p(2, x, y). \qquad \square
\end{aligned}
$$

Under the assumptions of Lemma 3, $\tilde{C} = \tilde{C}(\alpha, d, \kappa)$ exists such that

$$p_D(1, x, y) \le \tilde{C} P^x(\tau_D > 1) P^x(\tau_D > 1) p(1, x, y).$$
(31)

Indeed, according to Remark 1, we consider $\tilde{\nu} = \frac{1}{2}\nu$ and the corresponding $\tilde{p}, \tilde{p}_D, \tilde{P}^x$, obtaining

$$
\begin{aligned}
p_D(1, x, y) &= \tilde{p}_D(2, x, y) \le \tilde{C} \tilde{P}^x(\tau_D > 2) \tilde{P}^x(\tau_D > 2) \tilde{p}(2, x, y) \\
&= \tilde{C} P^x(\tau_D > 1) P^x(\tau_D > 1) p(1, x, y).
\end{aligned}
$$

LEMMA 4.   *If $r > 0$, then there is a constant $C = C(\alpha, d, r)$ such that*

$$p_{B(u,r) \cup B(v,r)}(1, u, v) \ge C p(1, u, v), \qquad u, v \in \mathbb{R}^d.$$

PROOF. For $|u - v| \geq r/2$ we use (28) and Lemma 2 with $D = B(u, r) \cup B(v, r)$, $D_1 = B(u, r/8)$ and $D_3 = B(v, r/8)$:

$$p_{B(u,r) \cup B(v,r)}(1, u, v) \geq P^u(\tau_{D_1} > 1) P^v(\tau_{D_3} > 1) \inf_{u \in D_1, z \in D_3} \nu(z - u)$$

$$\geq c\big[P^0\big(\tau_{B(0,r/8)} > 1\big)\big]^2 p(1, u, v).$$

For $|u - v| \leq r/2$, by (4), we simply have

$$p_{B(u,r) \cup B(v,r)}(1, u, v) \geq \inf_{|z| < r/2} p_{B(0,r)}(1, 0, z) \geq c \geq cp(1, u, v). \qquad \square$$

LEMMA 5. *If D is $(\kappa, 1)$-fat at x and y, then*

$$p_D(3, x, y) \geq C(\alpha, d, \kappa) P^x(\tau_D > 3) P^y(\tau_D > 3) p(3, x, y).$$

PROOF. Consider U^x, B_2^x, and U^y, B_2^y, selected according to Definition 2 for x and y, correspondingly. By the semigroup property, Lemma 4 with $r = \kappa/6$, and (4),

$$p_D(3, x, y) \geq \int_{B_2^y} \int_{B_2^x} p_D(1, x, u) p_D(1, u, v) p_D(1, v, y) \, du \, dv$$

$$\geq cp(1, x, y) \int_{B_2^x} p_D(1, x, u) \, du \int_{B_2^y} p_D(1, v, y) \, dv.$$

For $u \in B_2^x = B(A', \kappa/6)$, by Lemma 2 with $D_1 = U^x = U$ and $D_3 = B(A', \kappa/4)$, and by Remark 3, we obtain

$$p_D(1, x, u) \geq P^x(\tau_U > 1) P^0\big(\tau_{B(0, \kappa/12)} > 1\big) \inf_{w \in U, z \in D_3} \nu(z - w)$$

$$\geq c P^x(\tau_U > 1) \geq c P^x(\tau_D > 1).$$

Similarly, $p_D(1, v, y) \geq c P^y(\tau_D > 1)$, hence, by Lemma 1, we have

$$p_D(3, x, y) \geq c P^y(\tau_D > 1) p(1, x, y) P^x(\tau_D > 1)$$

$$\geq c P^y(\tau_D > 3) p(3, x, y) P^x(\tau_D > 3). \qquad \square$$

Under the assumptions of Lemma 5 we also have that

$$(32) \qquad p_D(1, x, y) \geq \tilde{C}(\alpha, d, \kappa) P^x(\tau_D > 1) P^y(\tau_D > 1) p(1, x, y).$$

This is proved analogously to (31).

PROOF OF THEOREM 1. Assume that $R \geq 1$ and D is (κ, r)-fat for $0 < r \leq R$. If $t^{1/\alpha} \in (0, R]$, then $t^{-1/\alpha} D$ is $(\kappa, 1)$-fat. The estimate (1) follows from (31), (32) and scaling; see (14) and (15). In fact, we have $C = C(\alpha, d, \kappa)$ in (1). If $R < 1$, then we argue as in the case of (31) $C = C(\alpha, d, \kappa, R)$ or, alternatively, we use Remark 6 below. $\square$

PROOF OF COROLLARY 1.   Note that D is $(1/4, r)$-fat for $r \geq 2\,\mathrm{diam}(D^c)$, and so we obtain (1) for $t \geq 2^\alpha \mathrm{diam}(D^c)$ with the same constant C. If we consider $\tilde{v} = 2^{-\alpha}v$ and argue like in the case of (31), then we obtain the wider range of t, as in the statement of Corollary 1. $\square$

REMARK 5.   Since the κ-fatness condition is more restrictive when κ is bigger, the above constants $C = C(\alpha, d, \kappa)$ may be chosen decreasing with respect to κ. Also, if D has a *tangent* inner ball of radius 1 at every boundary point, then the constants in Lemmas 3 and 5 depend only on α and d.

REMARK 6.   If D is (κ, r)-fat at x and $1 \leq K < \infty$, then D is $(\kappa/K, rK)$-fat at x. This observation together with scaling allows to easily *increase time*, compare (31) or (32), at the expense of enlarging the constants of comparability. The argument, however, does not allow to decrease time. Remark 1 is more flexible in this respect.

4. Applications.   We let $s_D(x) = E_x \tau_D = \int G_D(x, y)\, dv$ if this expectation is finite for $x \in D$, otherwise we let $s_D(x) = M_D(x)$, the Martin kernel with the pole at infinity for D,

$$M_D(x) = \lim_{D \ni y, |y| \to \infty} \frac{G_D(x, y)}{G_D(x_0, y)}.$$

We should note that this (alternative) definition of s_D is natural in view of [14], Theorem 2. The choice of $x_0 \in D$ is merely a normalization, $M_D(x_0) = 1$, and will not be reflected in the notation. By the scaling of the Green function (17), we obtain

$$(33) \qquad \frac{s_{rD}(rx)}{s_{rD}(ry)} = \frac{s_D(x)}{s_D(y)}, \qquad x, y \in D, r > 0.$$

We denote by $A_r(x)$ or $A_r(x, \kappa, D)$ every point A such that $B(A, \kappa r) \subset D \cap B(x, r)$, as in Definition 1. It is noteworthy that $A_r(x)$ approximately dominates x in terms of the distance to ∂D:

$$(34) \qquad \delta_D(A_r(x)) \approx r \vee \delta_D(x).$$

If D is $(\kappa, 1)$-fat at x, then rD is (κ, r)-fat at rx, and (every) $rA_1(x, \kappa, D)$ may serve as $A_r(rx, \kappa, rD)$.

THEOREM 2.   *If D is $(\kappa, t^{1/\alpha})$-fat at x and y, then*

$$(35) \qquad P^x(\tau_D > t) \overset{C}{\approx} \frac{s_D(x)}{s_D(A_{t^{1/\alpha}}(x))},$$

where $C = C(d, \alpha, \kappa)$ and, furthermore,

$$(36) \qquad p_D(t, x, y) \overset{C}{\approx} \frac{s_D(x)}{s_D(A_{t^{1/\alpha}}(x))} p(t, x, y) \frac{s_D(y)}{s_D(A_{t^{1/\alpha}}(y))}.$$

PROOF. To verify (35), we first let $t = 1$ and assume that D is $(\kappa, 1)$-fat at x. Let $A = A_1(x)$. If $E^x \tau_D < \infty$, then we consider the set $U \subset D$ of Definition 2, and we obtain

$$E^x \tau_D = E^x \tau_U + E^x s_D(X_{\tau_U}).$$

By Remark 3, $E^x \tau_U \approx P^x(\tau_D > 1)$. Since $E^A \tau_U \approx 1$, we trivially have

$$\frac{E^x \tau_U}{E^A \tau_U} \approx P^x(\tau_D > 1).$$

Similarly, $P^A(X_{\tau_U} \in D) \approx 1$. By BHP and Remark 3, we obtain

$$(37) \qquad \frac{E^x s_D(X_{\tau_U})}{E^A s_D(X_{\tau_U})} \approx \frac{P^x(X_{\tau_U} \in D)}{P^A(X_{\tau_U} \in D)} \approx P^x(X_{\tau_U} \in D) \approx P^x(\tau_D > 1).$$

This yields (35) in the considered case. If $E^x \tau_D = \infty$, then s_D is harmonic and we have $s_D(x) = E^x s_D(X_{\tau_U})$ (see [14], Theorem 2 and (77)) and we proceed directly via (37). The case of general t in (35) is obtained by the scaling of (33) and (15). Finally, (36) follows from (35) and Theorem 1. The resulting comparability constants depend only on α, d and κ. □

REMARK 7. Assume that D is κ-fat, so that there is $R > 0$ such that D is (κ, r)-fat for every $r \leq R$. Then (35) and (36) hold with $C = C(d, \alpha, \kappa)$ for all $x, y \in D$ and $t \leq R^\alpha$.

Below we give a number of applications.

EXAMPLE 1. We let $R > 0$ and $D = B(0, R) \subset \mathbb{R}^d$. By (24), the expected survival time is $s_D(x) \overset{C}{\approx} \delta_D^{\alpha/2}(x) R^{\alpha/2}$, where $C = C(d, \alpha)$. By (34), $s_D(A_{t^{1/\alpha}}(x)) \overset{C}{\approx} (t^{1/\alpha} \vee \delta_D(x))^{\alpha/2} R^{\alpha/2}$, therefore, for all $t \leq R^\alpha$ and $x, y \in \mathbb{R}^d$,

$$(38) \qquad P^x(\tau_D > t) \overset{C}{\approx} \frac{\delta_D^{\alpha/2}(x)}{(t^{1/\alpha} \vee \delta_D(x))^{\alpha/2}} = \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2}$$

and

$$(39) \qquad p_D(t, x, y) \overset{C}{\approx} \left(1 \wedge \frac{\delta_D^{\alpha/2}(x)}{t^{1/2}}\right) p(t, x, y) \left(1 \wedge \frac{\delta_D^{\alpha/2}(y)}{t^{1/2}}\right).$$

To be explicit, $\delta_{B(0,R)}(x) = (R - |x|) \vee 0$, and $\delta_{B(0,R)^c}(x) = (|x| - R) \vee 0$, and (38), (39) on D^c follow because all $x \in D^c$ are regular for D.

EXAMPLE 2. Let $D \subset \mathbb{R}^d$ be a half-space. The Martin kernel with the pole at infinity for D is $s_D(x) = \delta_D^{\alpha/2}(x)$ [1]. We see that (38) and (39) hold with $C = C(d, \alpha)$ for *all* $t \in (0, \infty)$ and $x, y \in \mathbb{R}^d$.

EXAMPLE 3. Let $D = B^c(0, 1) \subset \mathbb{R}^d$ and $d \geq \alpha$. By the Kelvin transform ([18] or [14]) and (21),

$$M_D(x) = \lim_{y \to \infty} \frac{|x|^{\alpha-d}|y|^{\alpha-d}G_B(x/|x|^2, y/|y|^2)}{|x_0|^{\alpha-d}|y|^{\alpha-d}G_B(x_0/|x_0|^2, y/|y|^2)} = \frac{|x|^{\alpha-d}G_B(x/|x|^2, 0)}{|x_0|^{\alpha-d}G_B(x_0/|x_0|^2, 0)},$$

where

$$G_B(z, 0) = \mathcal{B}_{d,\alpha}|z|^{\alpha-d}\int_0^{|z|^{-2}-1} \frac{s^{\alpha/2-1}}{(s+1)^{d/2}}\,ds, \qquad 0 < |z| < 1.$$

Thus, there is $c = c(x_0, d, \alpha)$ such that

$$(40) \qquad M_D(x) = c\int_0^{|x|^2-1} \frac{s^{\alpha/2-1}}{(s+1)^{d/2}}\,ds, \qquad |x| \geq 1.$$

If $d > \alpha$, then $s_D(x) \approx 1 \wedge \delta_D^{\alpha/2}(x)$, $s_D(A_{t^{1/\alpha}}(x)) \approx 1 \wedge (t^{1/\alpha} \vee \delta_D(x))^{\alpha/2}$, thus,

$$(41) \qquad P^x(\tau_D > t) \overset{C}{\approx} \frac{1 \wedge \delta_D^{\alpha/2}(x)}{1 \wedge (t^{1/\alpha} \vee \delta_D(x))^{\alpha/2}} = 1 \wedge \frac{\delta_D^{\alpha/2}(x)}{(1 \wedge t^{1/\alpha})^{\alpha/2}}$$

and

$$p_D(t, x, y) \overset{C}{\approx} \left(1 \wedge \frac{\delta_D^{\alpha/2}(x)}{1 \wedge t^{1/2}}\right) p(t, x, y)\left(1 \wedge \frac{\delta_D^{\alpha/2}(y)}{1 \wedge t^{1/2}}\right)$$

for all $0 < t < \infty$ and $x, y \in \mathbb{R}^d$. Here $C = C(d, \alpha)$.

For $\alpha = d = 1$, (40) yields $s_D(x) \approx \log(1 + \delta_D^{1/2}(x))$, $s_D(A_{t^{1/\alpha}}(x)) \approx \log(1 + (t \vee \delta_D(x))^{1/2})$, thus, for all $0 < t < \infty$ and $x, y \in \mathbb{R}^d$ we have

$$(42) \qquad P^x(\tau_D > t) \approx \frac{\log(1 + \delta_D^{1/2}(x))}{\log(1 + (t \vee \delta_D(x))^{1/2})} = 1 \wedge \frac{\log(1 + \delta_D^{1/2}(x))}{\log(1 + t^{1/2})}$$

and

$$\frac{p_D(t, x, y)}{p(t, x, y)} \approx \left(1 \wedge \frac{\log(1 + \delta_D^{1/2}(x))}{\log(1 + t^{1/2})}\right)\left(1 \wedge \frac{\log(1 + \delta^{1/2}(y))}{\log(1 + t^{1/2})}\right).$$

Sharp explicit estimates for $p_{B^c(0,R)}$ with arbitrary $R > 0$ follow by scaling.

EXAMPLE 4. Let $D = B^c(0, 1) \subset \mathbb{R}^d$ and $1 = d < \alpha$. We have that

$$G_{\{0\}^c}(x, y) = G_D(x, y) + E^x G_{\{0\}^c}(X_{T_B}, y).$$

Let $c_\alpha = [-2\Gamma(\alpha)\cos(\pi\alpha/2)]^{-1}$. By [18], Lemma 4, for $x, y \in \mathbb{R}$,

$$G_{\{0\}^c}(x, y) = c_\alpha(|y|^{\alpha-1} + |x|^{\alpha-1} - |y - x|^{\alpha-1}).$$

If follows that

$$G_D(x, y) = c_\alpha\big(|x|^{\alpha-1} - |x - y|^{\alpha-1} - E^x(|X_{\tau_D}|^{\alpha-1} - |X_{\tau_D} - y|^{\alpha-1})\big).$$

Since $|X_{\tau_D}| \leq 1$ a.s., $\lim_{y\to\infty}(-|x-y|^{\alpha-1} + E^x|X_{\tau_D} - y|^{\alpha-1}) = 0$, for every $x \in \mathbb{R}$. If $|x| \geq 2$, then we can find $c = c(\alpha, x_0)$ such that

$$M_D(x) = \frac{|x|^{\alpha-1} - E^x|X_{\tau_D}|^{\alpha-1}}{|x_0|^{\alpha-1} - E^{x_0}|X_{\tau_D}|^{\alpha-1}} = c(|x|^{\alpha-1} - E^x|X_{\tau_D}|^{\alpha-1})$$

$$\approx |x|^{\alpha-1} \approx \delta_D(x)^{\alpha-1}.$$

On the other hand, by BHP, $M_D(x) \approx \delta_D^{\alpha/2}(x)$ if $\delta(x) \leq 1$ (compare Example 2). We thus have $s_D(x) \approx \delta_D^{\alpha-1}(x) \wedge \delta_D^{\alpha/2}(x)$, $s_D(A_{t^{1/\alpha}}(x)) \approx (t^{1/\alpha} \vee \delta_D(x))^{\alpha-1} \wedge (t^{1/\alpha} \vee \delta_D(x))^{\alpha/2}$, and for all $0 < t < \infty$, $x, y \in \mathbb{R}^d$, we obtain

$$(43) \qquad P^x(\tau_D > t) \stackrel{C}{\approx} \frac{\delta_D^{\alpha-1}(x) \wedge \delta_D^{\alpha/2}(x)}{(t^{1/\alpha} \vee \delta_D(x))^{\alpha-1} \wedge (t^{1/\alpha} \vee \delta_D(x))^{\alpha/2}},$$

hence,

$$p_D(t, x, y) \stackrel{C}{\approx} \left(1 \wedge \frac{\delta_D^{\alpha-1}(x) \wedge \delta_D^{\alpha/2}(x)}{t^{1-1/\alpha} \wedge t^{1/2}}\right) p(t, x, y) \left(1 \wedge \frac{\delta_D^{\alpha-1}(y) \wedge \delta_D^{\alpha/2}(y)}{t^{1-1/\alpha} \wedge t^{1/2}}\right).$$

Here $C = C(\alpha)$. To estimate $p_{B^c(0,R)}$ with arbitrary $R > 0$, we use scaling.

DEFINITION 3. We say that (open) D is of class $C^{1,1}$ at scale $r > 0$ if for every $Q \in \partial D$ there exist balls $B(x', r) \subset D$ and $B(x'', r) \subset D^c$ tangent at Q. If D is $C^{1,1}$ at some (unspecified) positive scale (hence also at smaller scales), then we simply say D is $C^{1,1}$.

$C^{1,1}$ domains may be equivalently defined using local coordinates [34].

REMARK 8. If D is $C^{1,1}$ at scale r, then it is $(1/2, p)$-fat for all $p \in (0, r]$.

REMARK 9. Let D be $C^{1,1}$ at scale r. Let $x \in D$, and let $Q \in \partial D$ be such that $\delta_D(x) = |x - Q|$. Consider the above balls $B(x', r)$ and $B(x'', r)$. If $\delta_D(x) < r$, then let $B_x = B(x', r)$, otherwise $B_x = B(x, \delta_D(x))$. Thus, $\delta_{B_x}(x) = \delta_D(x)$, and the radius of B_x is $r \vee \delta_D(x)$.

EXAMPLE 5. We will verify (2) for $C^{1,1}$ domains D. For the proof we initially assume that $D \neq \mathbb{R}^d$ is $C^{1,1}$ at scale $r = 1$. Let $x \in D$. We adopt the notation of Remark 9 and consider (the ball) B_x and (the open complement of a ball) $B^c(x'', 1)$ tangent at $Q \in \partial D$. Since $B_x \subset D \subset B^c(x'', 1)$, we have

$$P^x(\tau_{B_x} > 1) \leq P^x(\tau_D > 1) \leq P^x(\tau_{B^c(x'',1)} > 1).$$

Clearly, $\delta_{B_x}(x) = \delta_D(x) = |Q - x| = \delta_{B^c(x'',1)}(x)$. By (38) and (41)–(43),

$$P^x(\tau_D > t) \approx \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2}, \qquad t \leq 1.$$

By Remark 8 and Theorem 1, there is $C = C(d, \alpha)$ such that, for all $x, y \in \mathbb{R}^d$,

$$p_D(t, x, y) \overset{C}{\approx} \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{t^{1/2}} \right) \left(\frac{t}{|x|^{d+\alpha}} \wedge t^{-d/\alpha} \right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{t^{1/2}} \right), \qquad t \leq 1.$$

If D is $C^{1,1}$ at a scale $r < 1$, then $r^{-1}D$ is $C^{1,1}$ at scale 1. This yields (2) in time range $0 < t \leq r^\alpha$. Remark 3 allows for an extension to all $t \in (0, 1]$, with a constant depending on d, α and r. The case of $D = \mathbb{R}^d$ is trivial.

Further estimates for $C^{1,1}$ domains will be given in Proposition 1, Theorem 3 and Corollary 2.

EXAMPLE 6. Let $d \geq 2$. For $x = (x_1, \ldots, x_{d-1}, x_d) \in \mathbb{R}^d$ we denote $\tilde{x} = (x_1, \ldots, x_{d-1})$, so that $x = (\tilde{x}, x_d)$. Let $\lambda < \infty$. We consider a Lipschitz function $\gamma : \mathbb{R}^{d-1} \to \mathbb{R}$, that is, $|\gamma(\tilde{x}) - \gamma(\tilde{y})| \leq \lambda|\tilde{x} - \tilde{y}|$. We define a special Lipschitz domain $D = \{x = (\tilde{x}, x_d) \in \mathbb{R}^d : x_d > \gamma(\tilde{x})\}$. For such D the geometric notions of Theorem 2 become more explicit as we will see below. We note that D is $((2\sqrt{1 + \lambda^2})^{-1}, r)$-fat for all $r > 0$ ([16], Remark 1). For $x = (\tilde{x}, x_d) \in D$ and $r > 0$ we define $x^{(r)} = (\tilde{x}, \gamma(\tilde{x}) + r)$. If x is close to ∂D, then $x^{(1)}$ dominates x in the direction of the last coordinate. We note that $P^{x^{(1)}}(\tau_D > 1) \geq c > 0$. Here $c = c(d, \alpha, \lambda)$. By Remark 3 and BHP,

$$(44) \qquad\qquad P^x(\tau_D > 1) \overset{C}{\approx} 1 \wedge \frac{M_D(x)}{M_D(x^{(1)})}, \qquad x \in D,$$

where $C = C(\alpha, d, \lambda)$. By scaling, the Martin kernel with the pole at infinity for rD is a constant multiple of $M_D(x/r)$. By (44), we obtain

$$(45) \quad P^x(\tau_D > t) = P^{t^{-1/\alpha}x}(\tau_{t^{-1/\alpha}D} > 1) \overset{C}{\approx} 1 \wedge \frac{M_D(x)}{M_D(x^{(t^{1/\alpha})})}, \qquad x \in D.$$

We note in passing that (45) agrees with (35) because $r \mapsto M_D(x^{(r)})$ is increasing [15]. Or, in our previous notation we can take $A_r(x, \kappa, D) = x^{(r \vee (x_d - \gamma(\tilde{x})))}$. We substitute (45) into (1) so that for all $0 < t < \infty$ and $x, y \in D$ (in fact, by regularity, for $x, y \in \mathbb{R}^d$) we have

$$p_D(t, x, y) \overset{C}{\approx} \left(1 \wedge \frac{M_D(x)}{M_D(x^{(t^{1/\alpha})})} \right) p(t, x, y) \left(1 \wedge \frac{M_D(y)}{M_D(y^{(t^{1/\alpha})})} \right).$$

EXAMPLE 7. For circular cones V [10] we have

$$(46) \qquad\qquad M_V(x) = |x|^\beta M_V(x/|x|), \qquad x \neq 0,$$

where $0 \leq \beta < \alpha$ is a characteristic of the cone; see [1]. By [35], Lemma 3.3,

$$M_V(x) \approx \delta_V(x)^{\alpha/2}|x|^{\beta - \alpha/2}, \qquad x \in \mathbb{R}^d;$$

see also [10] and [35]. Considering (44), by simple manipulations, we obtain

$$(47) \qquad 1 \wedge \frac{\delta_V(x)^{\alpha/2}|x|^{\beta-\alpha/2}}{(1 \vee |\tilde{x}|)^{\beta-\alpha/2}} \overset{C}{\approx} (1 \wedge \delta_V(x)^{\alpha/2})(1 \wedge |x|)^{\beta-\alpha/2},$$

where $C = C(\lambda)$. By (1) and scaling, we get (3).

The interested reader may find more references on stable processes and Brownian motion in cones in [10]. Note that (46) holds for generalized open cones, that is, open sets $\varnothing \neq V \subset \mathbb{R}^d$ such that $kV = V$ for all $k > 0$ [1].

EXAMPLE 8.   Let $d = 1, 2, \ldots$ and $V = \mathbb{R}^d \setminus \{x_d = 0\}$. This generalized cone is non-Lipschitz but it is $(1/2, r)$-fat for every $r > 0$. Let $1 < \alpha < 2$. From [1], Example 3.3, we have $M_V(x) = |x_d|^{\alpha-1}$ (the decay near a hyperplane is slower than near a half-space). We consider $t = 1$ in (36). We let $A_1(x) = (\tilde{x}, x_d + 1/2)$ if $x_d > 0$ and $A_1(x) = (\tilde{x}, x_d - 1/2)$ otherwise. Thus,

$$\frac{M_V(x)}{M_V(A_1(x))} = \frac{|x_d|^{\alpha-1}}{(|x_d| + 1/2)^{\alpha-1}} \approx (1 \wedge |x_d|)^{\alpha-1}.$$

By (1) and scaling, we obtain the following analogue of (3):

$$(48) \qquad \frac{p_V(t, x, y)}{p(t, x, y)} \approx \left(1 \wedge \frac{\delta_V(x)}{t^{1/\alpha}}\right)^{\alpha-1} \left(1 \wedge \frac{\delta_V(y)}{t^{1/\alpha}}\right)^{\alpha-1}, \qquad t > 0, x, y \in \mathbb{R}^d.$$

We note that V is the complement of a point if $d = 1$.

If D is bounded and $\kappa > 0$ is fixed, then D is *not* (κ, r) at large scales r, and the asymptotics of the probability of survival are exponential. Indeed, for the fractional Laplacian with Dirichlet condition on D^c we let $\lambda_1 > 0$ be its first eigenvalue and $\phi_1 > 0$ the corresponding eigenfunction [normalized in $L^2(D, dx)$]; see [30]. The following approximation results from the intrinsic ultracontractivity of every bounded domain [30]:

$$p_D(t, x, y) \approx \phi_1(x)\phi_1(y)e^{-\lambda_1 t}, \qquad t \geq 1, x, y \in \mathbb{R}^d.$$

Here comparability constants depend on D and α (see also Proposition 1 below). Given that infinity is *inaccessible* [14] from bounded D, it is of considerable interest to understand the behavior of the heat kernel related to accessible and inaccessible points of D (see also [33] in this connection).

In the remainder of the paper we will study $C^{1,1}$ domains in more detail. We focus on unbounded domains, large times and dependence of the comparability constants on global geometry of the domains.

Example 1 and intrinsic ultracontractivity yield the following result.

LEMMA 6.   *There exist $\lambda_1 = \lambda_1(\alpha, d) > 0$ and $C = C(\alpha, d)$ such that for all $r > 0, t > 0$ and $x \in \mathbb{R}^d$ we have*

$$P^x\big(\tau_{B(0,r)} > t\big) \overset{C}{\approx} \left[1 \wedge \left(\frac{\delta_{B(0,r)}(x)}{r \wedge t^{1/\alpha}}\right)^{\alpha/2}\right] e^{-\lambda_1 t/r^\alpha}.$$

LEMMA 7.   *Let $d > \alpha$, $0 < r < R$, $W = B(0, r) \cup B^c(0, R)$. There is $c = c(\alpha, d)$ such that for all $t > 0$ and $x \in \mathbb{R}^d$ we have*

$$P^x(\tau_W > t) \ge c\left(\frac{r}{R}\right)^\alpha \left[1 \wedge \left(\frac{\delta_{B(0,r)}(x)}{r \wedge t^{1/\alpha}}\right)^{\alpha/2}\right].$$

PROOF.   By scaling, we only need to consider $r = 1 < R$. By [5], we obtain

$$P^x\big(T_{B(0,1)} = \infty\big) = \frac{\Gamma(d/2)}{\Gamma((d-\alpha)/2)\Gamma(\alpha/2)} \int_0^{|x|^2 - 1} \frac{u^{\alpha/2 - 1}}{(u+1)^{d/2}} \, du$$

$$\approx 1 \wedge \delta_{B^c(0,1)}^{\alpha/2}(x)$$

[compare (40)]. Thus, there is $c = c(d, \alpha)$ such that

$$P^y\big(T_{B(0,R)} = \infty\big) \ge c > 0, \qquad |y| > 2R.$$

Let $x \in B(0, 1)$. For $t \ge 1$ we use (23) to obtain

$$P^x(\tau_W > t) \ge P^x(\tau_W = \infty)$$

$$\ge E^x\big\{|X_{\tau_{B(0,1)}}| \ge 2R; \, P^{X_{\tau_{B(0,1)}}}\big(T_{B(0,R)} = \infty\big)\big\}$$

$$\ge c P^x\big(|X_{\tau_{B(0,1)}}| \ge 2R\big) \ge c\frac{1}{R^\alpha}\delta_{B(0,1)}^{\alpha/2}(x).$$

By (38), for $t \le 1$ we even have

$$P^x(\tau_W > t) \ge P^x\big(\tau_{B(0,1)} > t\big) \approx 1 \wedge \left(\frac{\delta_{B(0,1)}(x)}{1 \wedge t^{1/\alpha}}\right)^{\alpha/2}. \qquad \square$$

The $C^{1,1}$ condition at a given scale fails to determine the fatness of D at larger scales and, consequently, the exact asymptotics of the survival probability. The following is a substitute.

PROPOSITION 1.   *If D is $C^{1,1}$ at some scale $r > 0$, then*

(49)
$$C^{-1} e^{-\lambda_1 t/(r \vee \delta_D(x))^\alpha} \left[1 \wedge \left(\frac{\delta_D(x)}{r \wedge t^{1/\alpha}}\right)^{\alpha/2}\right]$$

$$\le P^x(\tau_D > t) \le C\left[1 \wedge \left(\frac{\delta_D(x)}{r \wedge t^{1/\alpha}}\right)^{\alpha/2}\right]$$

for all $t > 0$ and $x \in \mathbb{R}^d$. Here $C = C(\alpha, d)$ and $\lambda_1 = \lambda_1(\alpha, d)$.

If also $d > \alpha$ and $\mathrm{diam}(D^c) < \infty$, then for all $t > 0$ and $x \in \mathbb{R}^d$,

$$(50) \qquad P^x(\tau_D > t) \geq C^{-1} \left(\frac{r}{\mathrm{diam}(D^c)} \right)^\alpha \left[1 \wedge \left(\frac{\delta_D(x)}{r \wedge t^{1/\alpha}} \right)^{\alpha/2} \right].$$

PROOF.   Consider $x \in D$, $B_x \subset D$ and $B(x'', r) \subset D^c$ of Remark 9. Clearly, $\tau_{B_x} \leq \tau_D \leq T_{B(x'',r)}$, thus,

$$P^x(\tau_{B_x} > t) \leq P^x(\tau_D > t) \leq P^x(T_{B(x'',r)} > t).$$

Lemma 6 yields the estimate

$$C^{-1} e^{-\lambda_1 t/(r \vee \delta_D(x))^\alpha} \left[1 \wedge \left(\frac{\delta_D(x)}{(r \vee \delta_D(x)) \wedge t^{1/\alpha}} \right)^{\alpha/2} \right] \leq P^x(\tau_D > t)$$

and

$$P^x(\tau_D > t) \leq C \left[1 \wedge \left(\frac{\delta_D(x)}{r \wedge t^{1/\alpha}} \right)^{\alpha/2} \right],$$

which simplifies to (49) as $\delta_D(x) > r$ yields $\delta_D(x)/[(r \vee \delta_D(x)) \wedge t^{1/\alpha}] \geq 1$. To prove (50), we consider $\rho = \mathrm{diam}(D^c) \geq 2r$, the center, say, x_0, of B_x, and $W := B_x \cup B^c(x_0, \rho + r \vee \delta_D(x)) \subset D$. By Lemma 7 and Remark 9,

$$\begin{aligned} P^x(\tau_D > t) &\geq P^x(\tau_W > t) \\ &\geq c \left(\frac{r \vee \delta_D(x)}{\rho + r \vee \delta_D(x)} \right)^\alpha \left[1 \wedge \left(\frac{\delta_D(x)}{(r \vee \delta_D(x)) \wedge t^{1/\alpha}} \right)^{\alpha/2} \right] \\ &\geq c \left(\frac{r}{\rho} \right)^\alpha \left[1 \wedge \left(\frac{\delta_D(x)}{r \wedge t^{1/\alpha}} \right)^{\alpha/2} \right]. \qquad \Box \end{aligned}$$

In view of Theorem 1, (49) mildly strengthens [19], Theorem 1.1(i) [i.e., (2) above]. We also get the following result.

THEOREM 3.   *Let $d > \alpha$. If D is $C^{1,1}$ at scale r and $\mathrm{diam}(D^c) < \infty$, then*

$$C^{-1} \left(\frac{r}{\mathrm{diam}(D^c)} \right)^{2\alpha}$$

$$\leq \frac{p_D(t, x, y)}{[1 \wedge (\delta_D(x)/(r \wedge t^{1/\alpha}))^{\alpha/2}] p(t, x, y) [1 \wedge (\delta_D(y)/(r \wedge t^{1/\alpha}))^{\alpha/2}]}$$

$$\leq C$$

for all $t > 0$ and $x, y \in \mathbb{R}^d$. Here $C = C(\alpha, d)$.

PROOF.   The result follows from (50) and Corollary 1. $\Box$

A similar result (with less control of the constants) is given in [22].[1]

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[1]Paper [22] appeared on arXiv after the first draft [11] of the present paper.

REMARK 10. We consider the recurrent case $\alpha \geq d = 1$. If $D \subset \mathbb{R}$ is the complement of a finite union of bounded closed intervals, then

$$P^x(\tau_D > t) \overset{C}{\approx} 1 \wedge \frac{\delta_D(x)^{\alpha-1} \wedge \delta_D(x)^{\alpha/2}}{t^{1-1/\alpha} \wedge t^{1/2}}, \qquad t > 0, x \in \mathbb{R}^d, \text{ if } \alpha > 1,$$

$$P^x(\tau_D > t) \overset{C}{\approx} 1 \wedge \frac{\log(1 + \delta_D(x)^{1/2})}{\log(1 + t^{1/2})}, \qquad t > 0, x \in \mathbb{R}^d, \text{ if } \alpha = 1,$$

where $C = C(D, \alpha)$. The estimates follow easily from Examples 2 and 3.

COROLLARY 2. *If $D \subset \mathbb{R}$ is the complement to a finite union of bounded closed intervals, then $C = C(D, \alpha)$ exists such that for all $t > 0$ and $x, y \in \mathbb{R}$,*

$$\frac{p_D(t, x, y)}{p(t, x, y)} \overset{C}{\approx} \left[1 \wedge \frac{\delta_D(x)^{\alpha-1} \wedge \delta_D(x)^{\alpha/2}}{t^{1-1/\alpha} \wedge t^{1/2}} \right]\left[1 \wedge \frac{\delta_D(y)^{\alpha-1} \wedge \delta_D(y)^{\alpha/2}}{t^{1-1/\alpha} \wedge t^{1/2}} \right]$$

for $\alpha > 1$, while for $\alpha = 1$ we have

$$\frac{p_D(t, x, y)}{p(t, x, y)} \overset{C}{\approx} \left[1 \wedge \frac{\log(1 + \delta_D(x)^{1/2})}{\log(1 + t^{1/2})} \right]\left[1 \wedge \frac{\log(1 + \delta_D(y)^{1/2})}{\log(1 + t^{1/2})} \right].$$

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