

Heat kernel estimates on spaces with varying dimension

Takumu Ooi*

Abstract

We obtain sharp two-sided heat kernel estimates on spaces with varying dimension, in which two spaces of general dimension are connected at one point. On these spaces, if the dimensions of the two constituent parts are different, the volume doubling property fails with respect to the measure induced by the associated Lebesgue measures. Thus the parabolic Harnack inequalities fail and the heat kernels do not enjoy Aronson type estimates. Our estimates show that the on-diagonal estimates are independent of the dimensions of the two parts of the space for small time, whereas they depend on their transience or recurrence for large time. These are multidimensional version of a space considered by Z.-Q. Chen and S. Lou (Ann. Probab. 2019), in which a 1-dimensional space and a 2-dimensional space are connected at one point.

Key words Space of varying dimension, Brownian motion, transition density, heat kernel estimates

MSC(2010) 60J60, 60J35, 31C25, 60H30, 60J45

1 Introduction

The heat kernel, the fundamental solution of the heat equation, has been studied in many areas, both for mathematical interest and for its importance in applications. The heat kernel is the transition density of Brownian motion, and it is difficult to determine its explicit form except in some special cases, such as on Euclidean spaces. Thus, heat kernel estimates have been studied on various spaces, see for example, [3, 8, 12, 14, 11, 19]. In a remarkable series of result, Grigor'yan [8], Saloff-Coste [19] and Sturm [20, 21] proved that the following are equivalent on a metric measure space : (i) the volume doubling property and scaled Poincaré inequalities, (ii) the parabolic Harnack inequalities, (iii) Aronson type estimates of the heat kernel. These results were extended to the setting of graphs in [6].

In studies of heat kernel estimates, the volume doubling property is a natural assumption. However, there are many spaces that do not satisfy this property.

*Research Institute for Mathematical Sciences, Kyoto University, Kyoto, 606-8502, JAPAN. E-mail:ooitaku@kurims.kyoto-u.ac.jp

One such example is a space with varying dimension given as following: for fixed $\varepsilon > 0$,

$$\mathbb{R}_\varepsilon^2 \cup \mathbb{R}_+ \cup \{a^*\} := \{(x, 0) \mid x \in \mathbb{R}^2, |x| > \varepsilon\} \cup \{(0, 0, x) \mid x > 0\} \cup \{a^*\}.$$

Here, we identify $\{x \in \mathbb{R}^2 \mid |x| \leq \varepsilon\}$ and $0 \in \mathbb{R}$ with a point a^* . Z.-Q. Chen and S. Lou [5] constructed a stochastic process on $\mathbb{R}_\varepsilon^2 \cup \mathbb{R}_+ \cup \{a^*\}$ that they called Brownian motion with varying dimension (BMVD). Note that $\mathbb{R}_\varepsilon^2 \cup \mathbb{R}_+ \cup \{a^*\}$ was considered instead of $\mathbb{R}^2 \cup \mathbb{R}_+$ because 2-dimensional Brownian motion never hits 0. For BMVD, the following heat kernel estimates were given. To state the result, let ρ be the shortest path metric derived from the Euclidean metric on the two parts of the space (the precise definition is denoted below) and $|x|_\rho$ be the distance between x and a^* with respect to ρ .

Theorem 1.1. ([5, Theorem 1.3, 1.4]) [I] *Let $T > 0$ be fixed. The transition density $p(t, x, y)$ of BMVD satisfies the following estimates when $t \in (0, T)$.*

(i) *For $x \in \mathbb{R}_+$ and $y \in \mathbb{R}_\varepsilon^2 \cup \mathbb{R}_+ \cup \{a^*\}$,*

$$p(t, x, y) \asymp \frac{1}{\sqrt{t}} e^{-\rho(x, y)^2/t}.$$

(ii) *For $x, y \in \mathbb{R}_\varepsilon^2 \cup \{a^*\}$ with $|x|_\rho \vee |y|_\rho < 1$,*

$$p(t, x, y) \asymp \frac{1}{\sqrt{t}} e^{-\rho(x, y)^2/t} + \frac{1}{t} \left(1 \wedge \frac{|x|_\rho}{\sqrt{t}}\right) \left(1 \wedge \frac{|y|_\rho}{\sqrt{t}}\right) e^{-|x-y|^2/t},$$

and for $x, y \in \mathbb{R}_\varepsilon^2 \cup \{a^\}$ with $|x|_\rho \vee |y|_\rho \geq 1$,*

$$p(t, x, y) \asymp \frac{1}{t} e^{-\rho(x, y)^2/t}.$$

[II] *The transition density $p(t, x, y)$ of BMVD satisfies the following estimates for $t \geq 8$.*

(i) *For $x, y \in \mathbb{R}_\varepsilon^2 \cup \{a^*\}$,*

$$p(t, x, y) \asymp \frac{1}{t} e^{-\rho(x, y)^2/t}.$$

(ii) *For $x \in \mathbb{R}_+$ and $y \in \mathbb{R}_\varepsilon^2 \cup \{a^*\}$, when $|y|_\rho \leq 1$,*

$$p(t, x, y) \asymp \frac{1}{t} \left(1 + \frac{|x| \log t}{\sqrt{t}}\right) e^{-\rho(x, y)^2/t},$$

and when $|y|_\rho > 1$,

$$p(t, x, y) \asymp \frac{1}{t} \left(1 + \frac{|x|}{\sqrt{t}} \log \left(1 + \frac{\sqrt{t}}{|y|_\rho}\right)\right) e^{-\rho(x, y)^2/t}.$$

(iii) *For $x, y \in \mathbb{R}_+$*

$$p(t, x, y) \asymp \frac{e^{-|x-y|^2/t}}{\sqrt{t}} \left(1 \wedge \frac{|x|}{\sqrt{t}}\right) \left(1 \wedge \frac{|y|}{\sqrt{t}}\right) + \frac{e^{-(|x|_\rho + |y|_\rho)^2/t}}{t} \left(1 + \frac{(|x| + |y|) \log t}{\sqrt{t}}\right).$$

Here and throughout this paper, we use the notation $a \wedge b := \min\{a, b\}$, $a \vee b := \max\{a, b\}$, and for $(t, x, y) \in A \subset [0, \infty) \times (\mathbb{R}_\varepsilon^d \cup \mathbb{R}_{\varepsilon'}^{d'} \cup \{a^*\}) \times (\mathbb{R}_\varepsilon^d \cup \mathbb{R}_{\varepsilon'}^{d'} \cup \{a^*\})$ and non-negative functions $f(t, x, y), g(t, x, y), h(t, x, y)$,

$$fe^{-h} \lesssim ge^{-h}$$

(\gtrsim , respectively) means that there exist $C > 0, c_1 > 0, c_2 > 0$, independent of $(t, x, y) \in A$, such that $fe^{-c_1 h} \leq Cge^{-c_2 h}$ for $(t, x, y) \in A$ (\geq , respectively). Moreover,

$$fe^{-h} \succsim ge^{-h}$$

means that $fe^{-h} \lesssim ge^{-h}$ and $fe^{-h} \gtrsim ge^{-h}$. In computations, constants C, c may change from line to line.

Concerning other work for heat kernel estimates on spaces with varying dimension, S. Lou deduced such for Brownian motion with drift on $\mathbb{R}_\varepsilon^2 \cup \mathbb{R}_+ \cup \{a^*\}$ in [17] and obtained an explicit expression for the heat kernel of distorted Brownian motion on $\mathbb{R}^3 \cup \mathbb{R}_+$ in [18].

In this paper, we estimate the heat kernel for Brownian motion on spaces with general varying dimension. To introduce the setting more precisely, let $d \geq d' \geq 1$ and $\varepsilon, \varepsilon' > 0$. We define

$$\mathbb{R}_\varepsilon^d := \{x \in \mathbb{R}^d ; |x| > \varepsilon\}, \quad \mathbb{R}_{\varepsilon'}^{d'} := \{x \in \mathbb{R}^{d'} ; |x| > \varepsilon'\},$$

where $|\cdot|$ is the Euclidean norm. For simplicity, set $\mathbb{R}_\varepsilon^1 := \mathbb{R}_+ := (0, \infty)$ for all ε . For \mathbb{R}^d and $\mathbb{R}^{d'}$, we identify $\{x \in \mathbb{R}^d ; |x| \leq \varepsilon\}$ and $\{x \in \mathbb{R}^{d'} ; |x| \leq \varepsilon'\}$ with a point a^* . We will establish heat kernel estimates for Brownian motion on

$\mathbb{R}_\varepsilon^d \cup \mathbb{R}_{\varepsilon'}^{d'} \cup \{a^*\}$, where $\mathbb{R}_\varepsilon^d \cup \mathbb{R}_{\varepsilon'}^{d'}$ means $\{(x, \overbrace{0, \dots, 0}^{d'} \mid x \in \mathbb{R}_\varepsilon^d\} \cup \{(0, \overbrace{\dots, 0}^d, y) \mid y \in \mathbb{R}_{\varepsilon'}^{d'}\}$.

We define a neighborhood of a^* as $\{a^*\} \cup (U_1 \cap \mathbb{R}_\varepsilon^d) \cup (U_2 \cap \mathbb{R}_{\varepsilon'}^{d'})$ for some neighborhoods U_1 of $\{x \in \mathbb{R}^d ; |x| \leq \varepsilon\}$ and U_2 of $\{x \in \mathbb{R}^{d'} ; |x| \leq \varepsilon'\}$. Moreover, we consider the topology on $\mathbb{R}_\varepsilon^d \cup \mathbb{R}_{\varepsilon'}^{d'} \cup \{a^*\}$ induced by the neighborhoods. We denote the Borel σ -field by $\mathcal{B} := \mathcal{B}(\mathbb{R}_\varepsilon^d \cup \mathbb{R}_{\varepsilon'}^{d'} \cup \{a^*\})$.

For a constant $p > 0$, we define $m_p(A) := m^{(d)}(A \cap \mathbb{R}^d) + p m^{(d')}(A \cap \mathbb{R}^{d'})$ for $A \in \mathcal{B}$. Here, $m^{(d)}$ is the Lebesgue measure on \mathbb{R}^d . In particular, $m_p(\{a^*\}) = 0$.

We extend the definition of Brownian motion with varying dimension as follows. In Theorem 2.1, we will describe the existence and the uniqueness of a process satisfying the following definition.

Definition 1.2. *Let $d \geq d' \geq 1$, $\varepsilon, \varepsilon' > 0$ and $p > 0$. Brownian motion with varying dimension (BMVD in abbreviation) with parameters $(\varepsilon, \varepsilon', p)$ on $\mathbb{R}_\varepsilon^d \cup \mathbb{R}_{\varepsilon'}^{d'} \cup \{a^*\}$ is an m_p -symmetric diffusion $X = (\{X_t\}, \{\mathbb{P}_x\})$ on $\mathbb{R}_\varepsilon^d \cup \mathbb{R}_{\varepsilon'}^{d'} \cup \{a^*\}$ such that:*

- (i) *its part process on \mathbb{R}_ε^d or $\mathbb{R}_{\varepsilon'}^{d'}$ has the same law as Brownian motion killed upon leaving \mathbb{R}_ε^d or $\mathbb{R}_{\varepsilon'}^{d'}$, respectively,*
- (ii) *it admits no killings on a^* .*

Throughout the paper, $X = (\{X_t\}, \{\mathbb{P}_x\})$ denotes BMVD, \mathbb{E}_x denotes the expectation corresponding to \mathbb{P}_x and $P_t f(x) := \mathbb{E}_x(f(X_t))$ for a bounded Borel measurable function f . Let $p(t, x, y)$ be the heat kernel with respect to m_ρ whose existence will be proved in Proposition 2.2. Let C_c^∞ be the set of all smooth functions with compact support and $\sigma_K := \inf \{t > 0 \mid X_t \in K\}$ be the hitting time of $K \in \mathcal{B}$.

Next, we introduce a distance ρ on $\mathbb{R}_\varepsilon^d \cup \mathbb{R}_{\varepsilon'}^{d'} \cup \{a^*\}$, as follows,

$$|x|_\rho := \begin{cases} |x| - \varepsilon & \text{for } x \in \mathbb{R}_\varepsilon^d, \\ |x| - \varepsilon' & \text{for } x \in \mathbb{R}_{\varepsilon'}^{d'}, \\ 0 & \text{for } x = a^*, \end{cases}$$

$$\rho(x, y) := (|x|_\rho + |y|_\rho) \wedge |x - y| \text{ for } x, y \in \mathbb{R}_\varepsilon^d \cup \mathbb{R}_{\varepsilon'}^{d'} \cup \{a^*\}.$$

Here, for $x \in \mathbb{R}_\varepsilon^d \cup \{a^*\}, y \in \mathbb{R}_{\varepsilon'}^{d'} \cup \{a^*\}$ or $x \in \mathbb{R}_{\varepsilon'}^{d'} \cup \{a^*\}, y \in \mathbb{R}_\varepsilon^d \cup \{a^*\}$, we define $|x - y| := \infty$.

The following theorems are the main results in this paper.

Theorem 1.3 (Small time estimates). *Let $d \geq d' \geq 1$ and $T \geq 1$ be fixed. The heat kernel $p(t, x, y)$ satisfies the following estimates when $t \in (0, T]$.*

(i) For $x, y \in \mathbb{R}_{\varepsilon'}^{d'}$ with $|x|_\rho \vee |y|_\rho \leq 1$,

$$p(t, x, y) \asymp \frac{1}{\sqrt{t}} e^{-\rho(x, y)^2/t} + \frac{1}{t^{d'/2}} \left(1 \wedge \frac{|x|_\rho}{\sqrt{t}}\right) \left(1 \wedge \frac{|y|_\rho}{\sqrt{t}}\right) e^{-|x-y|^2/t}.$$

For $x, y \in \mathbb{R}_{\varepsilon'}^{d'}$ with $|x|_\rho \vee |y|_\rho > 1$,

$$p(t, x, y) \asymp \frac{1}{t^{d'/2}} e^{-\rho(x, y)^2/t}$$

(ii) For $x, y \in \mathbb{R}_\varepsilon^d$ with $|x|_\rho \vee |y|_\rho \leq 1$,

$$p(t, x, y) \asymp \frac{1}{\sqrt{t}} e^{-\rho(x, y)^2/t} + \frac{1}{t^{d/2}} \left(1 \wedge \frac{|x|_\rho}{\sqrt{t}}\right) \left(1 \wedge \frac{|y|_\rho}{\sqrt{t}}\right) e^{-|x-y|^2/t}.$$

For $x, y \in \mathbb{R}_\varepsilon^d$ with $|x|_\rho \vee |y|_\rho > 1$,

$$p(t, x, y) \asymp \frac{1}{t^{d/2}} e^{-\rho(x, y)^2/t}$$

(iii) For $x \in \mathbb{R}_\varepsilon^d \cup \{a^*\}, y \in \mathbb{R}_{\varepsilon'}^{d'} \cup \{a^*\}$,

$$p(t, x, y) \asymp \frac{1}{\sqrt{t}} e^{-\rho(x, y)^2/t}.$$

Note that when $d' = 1$ and $d = 2$, the estimates are the same as those in Theorem 1.1 [I]. Intuitively, if BMVD hits a^* , or both x and y are close to a^* , a 1-dimensional effect appears in the heat kernel. If either x or y is far from a^* , the dimension on which BMVD lives affects the heat kernel. We will prove Theorem 1.3 in Section 3.

Concerning large time estimates, we give four theorems depending on the dimensions of the two parts of the space.

Theorem 1.4 (Large time estimates I). *Let $d \geq 3, d' = 1$ and $T > 0$ be large. The heat kernel $p(t, x, y)$ satisfies the following estimates when $T \leq t$.*

- (i) For $x, y \in \mathbb{R}_+$ with $|x|_\rho \wedge |y|_\rho > 1$,

$$p(t, x, y) \asymp \frac{|x||y|}{\sqrt{t}(|x| + \sqrt{t})(|y| + \sqrt{t})} e^{-\rho(x,y)^2/t}.$$

- (ii) For $x, y \in \mathbb{R}_\varepsilon^d$ with $|x|_\rho \wedge |y|_\rho > 1$,

$$p(t, x, y) \asymp \frac{1}{t^{3/2}|x|^{d-2}|y|^{d-2}} e^{-(|x|_\rho + |y|_\rho)^2/t} + \frac{1}{t^{d/2}} e^{-\rho(x,y)^2/t}.$$

- (iii) For $x \in \mathbb{R}_+ \cup \{a^*\}, y \in \mathbb{R}_\varepsilon^d \cup \{a^*\}$ or $x, y \in \mathbb{R}_+$ with $|y|_\rho \leq 1$ or $x, y \in \mathbb{R}_\varepsilon^d$ with $|x|_\rho \leq 1$,

$$p(t, x, y) \asymp \left(\frac{1}{t^{d/2}} + \frac{|x|}{t^{3/2}|y|^{d-2}} \right) e^{-\rho(x,y)^2/t}.$$

Since 1-dimensional Brownian motion is recurrent and d -dimensional Brownian motion is transient for $d \geq 3$, if BMVD starting from a point in \mathbb{R}_+ enters \mathbb{R}_ε^d and stays there for a long time, it is likely to escape to infinity. Thus, intuitively \mathbb{R}_+ affects the heat kernel more than \mathbb{R}^d . We will prove Theorem 1.4 in Section 4 using the projection.

Theorem 1.5 (Large time estimates II). *Let $d = d' = 2$ and T be large. The heat kernel $p(t, x, y)$ satisfies the following estimates when $T \leq t$.*

- (i) For $x, y \in \mathbb{R}_\varepsilon^2$ or $x, y \in \mathbb{R}_\varepsilon^2$,

$$p(t, x, y) \asymp \frac{1}{t} e^{-\rho(x,y)^2/t}.$$

- (ii) For $x \in \mathbb{R}_\varepsilon^2 \cup \{a^*\}$ and $y \in \mathbb{R}_\varepsilon^2 \cup \{a^*\}$,

$$p(t, x, y) \asymp \frac{1}{t} \left(U_t(x)U_t(y) + \frac{U_t(x) \log |y|}{\log(1+t|y|)} + \frac{U_t(y) \log |x|}{\log(1+t|x|)} \right) e^{-\rho(x,y)^2/t}.$$

Here, $U_t(x) := \frac{1}{\log(t+|x|)} + \left(1 - \frac{\log |x|}{\log \sqrt{t}}\right)_+$.

Theorem 1.6 (Large time estimates III). *Let $d \geq 3, d' = 2$ and T be large. The heat kernel $p(t, x, y)$ satisfies the following estimates when $T \leq t$.*

(i) For $x, y \in \mathbb{R}_\varepsilon^d$,

$$p(t, x, y) \asymp \frac{1}{t(\log t)^2 |x|^{d-2} |y|^{d-2}} e^{-(|x|_\rho + |y|_\rho)^2/t} + \frac{1}{t^{d/2}} e^{-\rho(x, y)^2/t}.$$

(ii) For $x, y \in \mathbb{R}_{\varepsilon'}^2$,

$$p(t, x, y) \asymp \frac{\log(1 + |x|) \log(1 + |y|)}{t \log(1 + t|y|) \log(1 + t|x|)} e^{-\rho(x, y)^2/t}.$$

(iii) For $x \in \mathbb{R}_\varepsilon^d \cup \{a^*\}, y \in \mathbb{R}_{\varepsilon'}^2 \cup \{a^*\}$,

$$p(t, x, y) \asymp \left(\frac{1}{t(\log t)^2 |x|^{d-2}} + \frac{H_t(y)}{t^{d/2}} \right) e^{-\rho(x, y)^2/t}.$$

Here, $H_t(y) := \frac{1}{(\log(1 + |y|))^2} + \left(\frac{1}{2 \log(1 + |y|)} - \frac{1}{\log t} \right)_+$.

For $d = d' = 2$, BMVD is recurrent and a 2-dimensional effect appears in the large time estimates. For $d \geq 3, d' = 2$, we have a mixed case of recurrent and transient parts of the space. In this case, \mathbb{R}^2 affects the heat kernel more than \mathbb{R}^d for a similar reason as in the case of $d \geq 3, d' = 1$. We will prove Theorem 1.5 and 1.6 in Section 6 using Doob's h -transform and the relative Faber-Krahn inequality.

Theorem 1.7 (Large time estimates IV). *Let $d \geq d' \geq 3$ and T be large. The heat kernel $p(t, x, y)$ satisfies the following estimates when $T \leq t$.*

(i) For $x, y \in \mathbb{R}_{\varepsilon'}^{d'}$,

$$p(t, x, y) \asymp \frac{1}{t^{d'/2}} e^{-\rho(x, y)^2/t}.$$

(ii) For $x, y \in \mathbb{R}_\varepsilon^d$ with $|x|_\rho \vee |y|_\rho \leq 1$,

$$p(t, x, y) \asymp \frac{1}{t^{d'/2}} e^{-\rho(x, y)^2/t}.$$

For $x, y \in \mathbb{R}_\varepsilon^d$ with $|x|_\rho \vee |y|_\rho > 1$,

$$p(t, x, y) \asymp \frac{1}{t^{d'/2} |x|^{d-2} |y|^{d-2}} e^{-(|x|_\rho + |y|_\rho)^2/t} + \frac{1}{t^{d/2}} e^{-\rho(x, y)^2/t}.$$

(iii) For $x \in \mathbb{R}_\varepsilon^d \cup \{a^*\}, x \in \mathbb{R}_{\varepsilon'}^{d'} \cup \{a^*\}$,

$$p(t, x, y) \asymp \left(\frac{1}{t^{d'/2} |x|^{d-2}} + \frac{1}{t^{d/2} |y|^{d-2}} \right) e^{-\rho(x, y)^2/t}.$$

For $d \geq d' \geq 3$, both Brownian motion on \mathbb{R}^d and $\mathbb{R}^{d'}$ are transient. Intuitively, $d \geq d'$ yields that d -dimensional Brownian motion escape to infinity faster than d' -dimensional Brownian motion. Thus, \mathbb{R}^d affects the large time heat kernel more than $\mathbb{R}^{d'}$ if BMVD starts near a^* . We will prove Theorem 1.7 in Section 5 by estimating $p(t, a^*, a^*)$ and using $\mathbb{P}_x(\sigma_{a^*} \in ds)$.

In related works, A. Grigor'yan, L. Saloff-Coste and S. Ishiwata obtained heat kernel estimates for Brownian motion on the connected sum of manifolds [14, 11]. To explain their results, we present the following definition.

Definition 1.8. *Let M_1 and M_2 be n -dimensional manifolds. A connected sum $M := M_1 \# M_2$ is a manifold constructed by removing a ball inside each manifold and gluing together these boundary spheres. A non-empty compact set $K \subset M$ is a central part of M if the exterior $M \setminus K$ is a disjoint union of open sets E_1 and E_2 such that each E_i is homeomorphic to $M_i \setminus K_i$ for some compact $K_i \subset M_i$.*

Let $S^{d-d'}$ be the $d - d'$ dimensional unit sphere. For $d > d' \geq 1$, $\mathbb{R}^d \# (\mathbb{R}^{d'} \times S^{d-d'})$ is not a space with varying dimension but, by considering the ball to have large radius, it looks similar to a space with varying dimension. Furthermore, our large time heat kernel estimates for BMVD are, up to the distances with which the results are stated, of the same form as those for Brownian motion on $\mathbb{R}^d \# (\mathbb{R}^{d'} \times S^{d-d'})$ given in [14, 11]. In fact, in order to prove Theorem 1.4, 1.6, we borrow some techniques from [14].

Finally, we give some remark about the approach by Chen and Lou ([5]). For large time, they estimated $p(t, a^*, a^*)$ by using the estimate of $\mathbb{P}_x(\sigma_{a^*} \in ds)$ and the Markov property $p(t, a^*, a^*) = \int p(t/2, a^*, x)^2 dm_p(x)$, and obtained desired off-diagonal estimates. By careful calculations, their method also works for general dimensions. However, our method gives relations between the behaviour of BMVD and that of Brownian motion on the connected sum of manifolds studied by [14, 11], which is of independent interest. So we take this approach.

Acknowledgements I would like to thank Professor Takashi Kumagai, my supervisor, for helpful discussions and for carefully reading this paper, and Professor Ryoki Fukushima for useful comments about Brownian motion with darning. I also thank Professor Laurent Saloff-Coste for giving me important advice about the relationship between the problem studied here and the estimation of heat kernels on the connected sum of manifolds, Professor David A. Croydon for checking the introduction of this paper. After the manuscript was written, Professor Zhen-Qing Chen pointed me out that the approach in [5] should work for general dimensions as well, and it was indeed true. He also gave me valuable comments including one concerning the proof of Proposition 5.1. I would deeply thank him for the comments.

2 Preliminary

Throughout the paper, we fix $\varepsilon, \varepsilon', p > 0$. In this section, we first prove the existence and the uniqueness of BMVD. We then show the existence and some properties of the heat kernel for BMVD. We also prove the space with varying dimension fails to the volume doubling property and we give some lemmas that will be used in Section 4-6.

Theorem 2.1. *For $d \geq d' \geq 1$, $\varepsilon, \varepsilon' > 0$ and $p > 0$, BMVD with parameters $(\varepsilon, \varepsilon', p)$ on $\mathbb{R}_\varepsilon^d \cup \mathbb{R}_{\varepsilon'}^{d'} \cup \{a^*\}$ exists and is unique in law. Furthermore, its associated Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathbb{R}_\varepsilon^d \cup \mathbb{R}_{\varepsilon'}^{d'} \cup \{a^*\}; m_p)$ is given by*

$$\mathcal{F} := \left\{ f \in L^2(\mathbb{R}_\varepsilon^d \cup \mathbb{R}_{\varepsilon'}^{d'} \cup \{a^*\}; m_p) \left| \begin{array}{l} f|_{\mathbb{R}_\varepsilon^d} \in H^1(\mathbb{R}_\varepsilon^d), f|_{\mathbb{R}_{\varepsilon'}^{d'}} \in H^1(\mathbb{R}_{\varepsilon'}^{d'}) \\ f(x) = f(a^*) \text{ q.e. on } \partial\mathbb{R}_\varepsilon^d \cup \partial\mathbb{R}_{\varepsilon'}^{d'} \end{array} \right. \right\},$$

$$\mathcal{E}(f, g) := \frac{1}{2} \int_{\mathbb{R}_\varepsilon^d} \nabla f \cdot \nabla g \, dm_p + \frac{1}{2} \int_{\mathbb{R}_{\varepsilon'}^{d'}} \nabla f \cdot \nabla g \, dm_p \text{ for } f, g \in \mathcal{F}.$$

Proof. The proof is the same as that of [5, Theorem 2.2]. \square

Proposition 2.2. *There exists a heat kernel $p(t, x, y)$ with respect to m_p which is continuous for each $t > 0$. Moreover, for all $t > 0$, it holds that $p(t, a^*, a^*) \lesssim t^{-d/2} \vee t^{-d'/2}$.*

Proof. $\|\cdot\|_{L^i}$ denotes L^i -norm with respect to m_p . Since \mathbb{R}_ε^d and $\mathbb{R}_{\varepsilon'}^{d'}$ have smooth boundaries, for all $f \in \mathcal{F} \cap L^1(\mathbb{R}_\varepsilon^d \cup \mathbb{R}_{\varepsilon'}^{d'} \cup \{a^*\})$, by classical Nash's inequality, there is $C > 0$ such that

$$\|f|_{\mathbb{R}_\varepsilon^d}\|_{L^2}^{1+2/d} \leq C \|f|_{\mathbb{R}_\varepsilon^d}\|_{L^1}^{2/d} \cdot \|\nabla f|_{\mathbb{R}_\varepsilon^d}\|_{L^2},$$

$$\|f|_{\mathbb{R}_{\varepsilon'}^{d'}}\|_{L^2}^{1+2/d'} \leq C \|f|_{\mathbb{R}_{\varepsilon'}^{d'}}\|_{L^1}^{2/d'} \cdot \|\nabla f|_{\mathbb{R}_{\varepsilon'}^{d'}}\|_{L^2}.$$

Then, for all $f \in \mathcal{F}$, we have

$$\|f\|_{L^2}^2 \leq C \left(\mathcal{E}(f, f)^{d/d+2} \|f\|_{L^1}^{4/d+2} + \mathcal{E}(f, f)^{d'/d'+2} \|f\|_{L^1}^{4/d'+2} \right).$$

By [3, Corollary 2.12], the heat kernel $p(t, x, y)$ with respect to m_p exists and the desired inequality holds for a.e. x, y , so it is sufficient to prove the continuity of $p(t, \cdot, \cdot)$. By Definition 1.2 (i), $p(t, \cdot, \cdot)$ is continuous on $(\mathbb{R}_\varepsilon^d \cup \mathbb{R}_{\varepsilon'}^{d'}) \times (\mathbb{R}_\varepsilon^d \cup \mathbb{R}_{\varepsilon'}^{d'})$. For fixed t, y , $p(t, x, y) = \int p(t/2, x, z) p(t/2, z, y) dm_p(z) = P_{t/2} p(t/2, \cdot, y)$ is quasi-continuous ([4, Proposition 3.1.9]) and, since a^* is nonpolar for X , $p(t, \cdot, y)$ is continuous. By the symmetry, $p(t, \cdot, \cdot)$ is continuous. \square

In this paper, for x and $r > 0$, we define $B(x; r) := \{y \in \mathbb{R}_\varepsilon^d \cup \mathbb{R}_{\varepsilon'}^{d'} \cup \{a^*\} \mid \rho(x, y) < r\}$.

Proposition 2.3. *For $d > d' \geq 1$, the volume doubling property fails on $\mathbb{R}_\varepsilon^d \cup \mathbb{R}_{\varepsilon'}^{d'} \cup \{a^*\}$ for m_p .*

Proof. For $r > 0$, we take $x \in \mathbb{R}_\varepsilon^d$ with $|x| = r + \varepsilon$, see Figure 1, then we have

$$m_p(B(x; r)) = \frac{\pi^{d/2}}{\Gamma(d/2 + 1)} r^d \quad \text{and}$$

$$m_p(B(x; 2r)) \geq \frac{\pi^{d/2} r^d}{\Gamma(d/2 + 1)} + \frac{p\pi^{d'/2} \left((r + \varepsilon')^{d'} - \varepsilon'^{d'} \right)}{\Gamma(d'/2 + 1)} \geq c(r^d + r^{d'}).$$

Now, if there exists $C > 0$ such that $m_p(B(x; 2r)) \leq C m_p(B(x; r))$ for all x , then we obtain $r^d + r^{d'} \leq cr^d$, so $1 + r^{d'-d} \leq c$. $1 + r^{d'-d} \rightarrow \infty$ as $r \rightarrow 0$ and this is a contradiction. \square

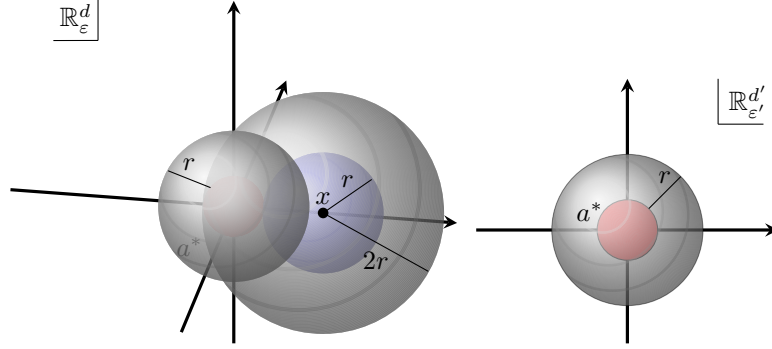


Figure 1: $B(x; r)$ and $B(x; 2r)$

Let $p_{\mathbb{R}_\varepsilon^d}(t, x, y)$ be the transition density of the part process of BMVD killed upon exiting \mathbb{R}_ε^d . According to [23], the following proposition holds.

Proposition 2.4. *Let $d \geq 2$. For $x, y \in \mathbb{R}_\varepsilon^d$ and $t > 0$, it holds that*

$$p_{\mathbb{R}_\varepsilon^d}(t, x, y) \asymp \frac{1}{t^{d/2}} \left(1 \wedge \frac{|x|_\rho}{\sqrt{t} \wedge 1} \right) \left(1 \wedge \frac{|y|_\rho}{\sqrt{t} \wedge 1} \right) e^{-|x-y|^2/t}. \quad (2.1)$$

Let $\bar{p}_{\mathbb{R}_\varepsilon^d}(t, x, y) := \int_0^t p(t-s, a^*, y) \mathbb{P}_x(\sigma_{a^*} \in ds)$ for $x, y \in \mathbb{R}_\varepsilon^d$. In order to estimate $\bar{p}_{\mathbb{R}_\varepsilon^d}(t, x, y)$, we prepare some lemmas for σ_{a^*} . According to [2, Theorem 3], the following two lemmas hold when $\varepsilon = 1$. By the scaling, they hold for every $\varepsilon > 0$.

Lemma 2.5. *For $d \geq 3$ and $x \in \mathbb{R}_\varepsilon^d$, it holds that*

$$\mathbb{P}(\sigma_{a^*} \in ds) \asymp \frac{|x|_\rho}{|x|} \frac{e^{-|x|_\rho^2/s}}{s^{d/2} + s^{3/2}|x|^{(d-3)/2}} ds.$$

Lemma 2.6. *For $d = 2$ and $x \in \mathbb{R}_\varepsilon^2$, it holds that*

$$\mathbb{P}(\sigma_{a^*} \in ds) \asymp \frac{|x|_\rho}{|x|} \frac{1 + \log|x|}{(1 + \log(1 + s/|x|))(1 + \log(s + |x|))} \frac{(|x| + s)^{1/2}}{s^{3/2}} e^{-|x|_\rho/s} ds.$$

We will use the following elementary estimate.

Lemma 2.7. *Let $d \geq 3$. Then for $t \geq 1$ and $x \in \mathbb{R}_\varepsilon^d$, we have*

$$\frac{e^{-|x|_\rho^2/t}}{t^{d/2} + t^{3/2}|x|^{(d-3)/2}} \gtrsim \frac{e^{-|x|_\rho^2/t}}{t^{d/2}}. \quad (2.2)$$

Proof. When $|x| \leq \sqrt{t}/2$, (2.2) follows from $t^{d/2} + t^{3/2}|x|^{(d-3)/2} \lesssim t^{d/2} + t^{3/2}t^{(d-3)/4} \lesssim t^{d/2}$.

When $|x| > \sqrt{t}/2$,

$$\frac{e^{-c|x|_\rho^2/t}}{t^{d/2} + t^{3/2}|x|^{(d-3)/2}} \geq \frac{e^{-(c+1)|x|_\rho^2/t} \left(\frac{|x|_\rho^2}{t}\right)^{(d-3)/4}}{t^{d/2} + t^{3/2}|x|^{(d-3)/2}} \gtrsim \frac{e^{-|x|_\rho^2/t}}{t^{d/2} + t^{(d+3)/4}} \gtrsim \frac{e^{-|x|_\rho^2/t}}{t^{d/2}}.$$

□

In the next two lemmas, we obtain the estimates of hitting distribution.

Lemma 2.8. *Let $d \geq 3$. Then for $x \in \mathbb{R}_\varepsilon^d$ and $t > 1$, we have*

$$\mathbb{P}_x(\sigma_{a^*} \leq t) \asymp \frac{1}{|x|^{d-2}} e^{-|x|_\rho^2/t}. \quad (2.3)$$

Proof. When $|x|_\rho \geq 1$, (2.3) follows from [13, Theorem 4.4 (1)].

When $|x|_\rho < 1$, $\mathbb{P}_x(\sigma_{a^*} \leq t) \leq 1$ and there is some $C > 0$ with $\mathbb{P}_x(\sigma_{a^*} \leq t) \geq \mathbb{P}_x(\sigma_{a^*} \leq 1) \geq C$, so (2.3) holds. □

Lemma 2.9. *Let $d = 2$. Then for $x \in \mathbb{R}_\varepsilon^2$ with $|x|_\rho \geq 1$, we have*

$$(i) \mathbb{P}_x(\sigma_{a^*} \leq t) \asymp \frac{1}{\log|x|} e^{-|x|_\rho^2/t} \text{ for } 0 < t < 2|x|^2,$$

$$(ii) \mathbb{P}_x(\sigma_{a^*} \leq t) \asymp 1 - \frac{\log|x|}{\log\sqrt{t}} \text{ for } 2|x|^2 \leq t.$$

Proof. See [13, Theorem 4.11]. □

The next lemma gives the relations between $e^{-\rho(x,y)^2/t}$, $e^{-(|x|_\rho+|y|_\rho)^2/t}$ and $e^{-|x-y|^2/t}$ for large time.

Lemma 2.10. *Let $T > 0$ be fixed and $d \geq 1$. For $T \leq t$ and $x, y \in \mathbb{R}_\varepsilon^d$, we have*

$$(i) e^{-\rho(x,y)^2/t} \asymp e^{-|x-y|^2/t} \gtrsim e^{-(|x|_\rho+|y|_\rho)^2/t} \text{ if } |x|_\rho \vee |y|_\rho > 1,$$

$$(ii) e^{-\rho(x,y)^2/t} \asymp e^{-|x-y|^2/t} \asymp e^{-(|x|_\rho+|y|_\rho)^2/t} \text{ if } |x|_\rho \vee |y|_\rho \leq 1,$$

$$(iii) e^{-\rho(x,y)^2/t} \asymp e^{-|x-y|^2/t} \asymp e^{-(|x|_\rho+|y|_\rho)^2/t}$$

if $|x|_\rho > 1 > b \geq |y|_\rho$ for some b .

Proof. (i) When $|x|_\rho \vee |y|_\rho > 1$, we may assume $|x|_\rho > 1$ without loss of generality. If $\rho(x, y) = |x - y|$, there is nothing to prove. If $\rho(x, y) = |x|_\rho + |y|_\rho$, then we have

$$\rho(x, y) \leq |x - y| \leq |x| + |y| = |x|_\rho + |y|_\rho + 2\varepsilon \leq (2\varepsilon + 1)(|x|_\rho + |y|_\rho) = (2\varepsilon + 1)\rho(x, y).$$

Hence, the desired estimate holds.

(ii) When $|x|_\rho \vee |y|_\rho \leq 1$, we have

$$\frac{(|x|_\rho + |y|_\rho)^2}{t} \leq \frac{4}{T} \leq \frac{4}{T} + \frac{|x - y|^2}{t} \leq \frac{4}{T} + \frac{(|x| + |y|)^2}{t} \leq \frac{4}{T} + \frac{2(2\varepsilon)^2}{T} + \frac{2(|x|_\rho + |y|_\rho)^2}{t}.$$

Hence, desired estimate holds.

(iii) When $|x|_\rho \geq 1 > b \geq |y|_\rho$, we have

$$\begin{aligned} & \frac{(|x|_\rho + |y|_\rho)^2}{t} - 2\frac{|x - y|^2}{t} = \frac{(|x| + |y| - 2\varepsilon)^2 - 2|x - y|^2}{t} \\ & \leq \frac{(|x - y| + |y| + |y| - 2\varepsilon)^2 - 2|x - y|^2}{t} = \frac{(2|y|_\rho + |x - y|)^2 - 2|x - y|^2}{t} \\ & \leq \frac{2(2|y|_\rho)^2 + 2|x - y|^2 - 2|x - y|^2}{t} = \frac{8|y|_\rho^2}{t} \leq \frac{8b^2}{T}. \end{aligned}$$

Hence, we have $e^{-|x - y|^2/t} \lesssim e^{-(|x|_\rho + |y|_\rho)^2/2t} e^{4b^2/T}$ and by (i), the desired estimate holds. \square

3 Small time estimate

In this section, we prove Theorem 1.3 in the same way as [5, section 4]. First, we define

$$u(x) := \begin{cases} -|x|_\rho & : x \in \mathbb{R}_\varepsilon^{d'} \\ |x|_\rho & : x \in \mathbb{R}_\varepsilon^d \end{cases} \quad (3.1)$$

and $Y_t := u(X_t)$. Then $u \in \mathcal{F}^{loc}$, where \mathcal{F}^{loc} denotes the local Dirichlet space of $(\mathcal{E}, \mathcal{F})$. We will prove that the heat kernel for Y enjoys 1-dimensional Gaussian estimates. Combining this with the fact that $\bar{p}_{\mathbb{R}_\varepsilon^d}(t, x, y)$ (resp. $p(t, x, y)$) depends only on $|x|_\rho$ and $|y|_\rho$ for $x, y \in \mathbb{R}_\varepsilon^d$ (resp. $x \in \mathbb{R}_\varepsilon^d, y \in \mathbb{R}_\varepsilon^{d'}$), we prove Theorem 1.3.

We first derive the stochastic differential equation that Y satisfies, and then use it to obtain Gaussian heat kernel estimates of Y .

Proposition 3.1.

$$dY_t = dB_t + \frac{(d - 1)\mathbf{1}_{\{Y_t > 0\}}}{2(Y_t + \varepsilon)} dt + \frac{p(d' - 1)\mathbf{1}_{\{Y_t < 0\}}}{2(Y_t - \varepsilon')} dt + \frac{|\partial\mathbb{R}_\varepsilon^d| - p|\partial\mathbb{R}_\varepsilon^{d'}|}{|\partial\mathbb{R}_\varepsilon^d| + p|\partial\mathbb{R}_\varepsilon^{d'}|} d\hat{L}_t^0(Y),$$

where $|\cdot|$ is the Lebesgue measure, B is one-dimensional Brownian motion and $\hat{L}^0(Y)$ is symmetric semimartingale local time of Y at 0. Here, for convenience, we define $|\partial\mathbb{R}_\varepsilon^{d'}| = 1$ when $d' = 1$.

Proof. By the Fukushima decomposition ([4, Chapter 4]), there exist local martingale additive functional $M^{[u]}$ and continuous additive functional locally having zero energy $N^{[u]}$ such that $Y_t - Y_0 = M_t^{[u]} + N^{[u]}$, \mathbb{P}_x -a.s. for q.e. $x \in \mathbb{R}_\varepsilon^d \cup \mathbb{R}_{\varepsilon'}^{d'} \cup \{a^*\}$. For any $\psi \in C_c^\infty(\mathbb{R}_\varepsilon^d \cup \mathbb{R}_{\varepsilon'}^{d'} \cup \{a^*\})$, it holds that

$$\begin{aligned}
\mathcal{E}(u, \psi) &= \frac{1}{2} \int_{\mathbb{R}_\varepsilon^d} \nabla |x|_\rho \cdot \nabla \psi dx + \frac{p}{2} \int_{\mathbb{R}_{\varepsilon'}^{d'}} \nabla(-|x|_\rho) \cdot \nabla \psi dx & (3.2) \\
&= -\frac{1}{2} \int_{\mathbb{R}_\varepsilon^d} \frac{d-1}{|x|} \psi dx + \frac{1}{2} \int_{\partial \mathbb{R}_\varepsilon^d} \psi(a^*) \frac{\partial |x|_\rho}{\partial \mathbf{n}} \sigma(dx) \\
&\quad + \frac{p}{2} \int_{\mathbb{R}_{\varepsilon'}^{d'}} \frac{d'-1}{|x|} \psi dx - \frac{p}{2} \int_{\partial \mathbb{R}_{\varepsilon'}^{d'}} \psi(a^*) \frac{\partial |x|_\rho}{\partial \mathbf{n}} \sigma(dx) \\
&= -\frac{1}{2} \int_{\mathbb{R}_\varepsilon^d} \frac{d-1}{|x|} \psi dx + \frac{p}{2} \int_{\mathbb{R}_{\varepsilon'}^{d'}} \frac{d'-1}{|x|} \psi dx - \frac{1}{2} (|\partial \mathbb{R}_\varepsilon^d| - p|\partial \mathbb{R}_{\varepsilon'}^{d'}|) \psi(a^*) \\
&= - \int_{\mathbb{R}_\varepsilon^d \cup \mathbb{R}_{\varepsilon'}^{d'} \cup \{a^*\}} \psi(x) \nu(dx),
\end{aligned}$$

where \mathbf{n} is the outward normal vector of the surface $\partial \mathbb{R}_\varepsilon^d \cup \partial \mathbb{R}_{\varepsilon'}^{d'}$, σ is the surface measure on $\partial \mathbb{R}_\varepsilon^d \cup \partial \mathbb{R}_{\varepsilon'}^{d'}$, and

$$\nu(dx) := \frac{d-1}{2|x|} \mathbf{1}_{\mathbb{R}_\varepsilon^d}(x) dx - \frac{p(d'-1)}{2|x|} \mathbf{1}_{\mathbb{R}_{\varepsilon'}^{d'}}(x) dx + \frac{(|\partial \mathbb{R}_\varepsilon^d| - p|\partial \mathbb{R}_{\varepsilon'}^{d'}|)}{2} \delta_{\{a^*\}}.$$

By [7, Theorem 5.5.5], it holds that

$$dN_t^{[u]} = \frac{(d-1)\mathbf{1}_{\{X_t \in \mathbb{R}_\varepsilon^d\}}}{2(u(X_t) + \varepsilon)} dt - \frac{p(d'-1)\mathbf{1}_{\{X_t \in \mathbb{R}_{\varepsilon'}^{d'}\}}}{2(-u(X_t) + \varepsilon')} dt + (|\partial \mathbb{R}_\varepsilon^d| - p|\partial \mathbb{R}_{\varepsilon'}^{d'}|) dL_t^0(X).$$

Here, $L_t^0(X)$ is the positive continuous additive functional of X whose Revuz measure is $\frac{1}{2}\delta_{\{a^*\}}$.

Let $u_n := (-n \vee u) \wedge n$, then $u_n \in \mathcal{F}$. By [4, Theorem 4.3.11] and strongly locality of $(\mathcal{E}, \mathcal{F})$, for any $\varphi \in \mathcal{F} \cap C_c(\mathbb{R}_\varepsilon^d \cup \mathbb{R}_{\varepsilon'}^{d'} \cup \{a^*\})$, we have $\int \varphi d\mu_{\langle u_n \rangle} = 2\mathcal{E}(u_n \varphi, u_n) - \mathcal{E}(u_n^2, \varphi) = \int \varphi |\nabla u_n|^2 dm_p$. Here, $d\mu_{\langle u_n \rangle}$ is the Revuz measure corresponding to $\langle M^{[u_n]} \rangle$. Then we obtain $d\mu_{\langle u_n \rangle} = |\nabla u_n|^2 dm_p = \mathbf{1}_{\{|x|_\rho \leq n\}} dm_p$. It yields $d\mu_{\langle u \rangle} = dm_p$. By [4, Theorem 4.1.8], $\langle M^{[u]} \rangle_t = t$ and $B_t := M_t^{[u]}$ is one-dimensional Brownian motion. Thus it holds that

$$\begin{aligned}
dY_t &= dB_t + \frac{(d-1)\mathbf{1}_{\{Y_t > 0\}}}{2(Y_t + \varepsilon)} dt + \frac{p(d'-1)\mathbf{1}_{\{Y_t < 0\}}}{2(Y_t - \varepsilon')} dt \\
&\quad + (|\partial \mathbb{R}_\varepsilon^d| - p|\partial \mathbb{R}_{\varepsilon'}^{d'}|) dL_t^0(X). & (3.3)
\end{aligned}$$

Next, we show $d\hat{L}_t^0(Y) = (|\partial \mathbb{R}_\varepsilon^d| + p|\partial \mathbb{R}_{\varepsilon'}^{d'}|) dL_t^0(X)$.

Let $v(x) := |x|_\rho$, so $|Y_t| = v(X_t)$ holds. Then, by the similar computation as above, for one-dimensional Brownian motion \tilde{B} , we have

$$d|Y_t| = d\tilde{B}_t + \frac{(d-1)\mathbf{1}_{\{Y_t > 0\}}}{2(Y_t + \varepsilon)} dt - \frac{p(d'-1)\mathbf{1}_{\{Y_t < 0\}}}{2(Y_t - \varepsilon')} dt + (|\partial \mathbb{R}_\varepsilon^d| + p|\partial \mathbb{R}_{\varepsilon'}^{d'}|) dL_t^0(X).$$

While, by Tanaka's formula and (3.3), we have

$$\begin{aligned}
d|Y_t| &= \text{sign}(Y_t)dY_t + dL_t^0(Y) \\
&= \text{sign}(Y_t)dB_t + \frac{(d-1)\mathbf{1}_{\{Y_t>0\}}}{2(Y_t+\varepsilon)}dt - \frac{p(d'-1)\mathbf{1}_{\{Y_t<0\}}}{2(Y_t-\varepsilon')}dt \\
&\quad - (|\partial\mathbb{R}_\varepsilon^d| - p|\partial\mathbb{R}_{\varepsilon'}^{d'}|)dL_t^0(X) + dL_t^0(Y),
\end{aligned} \tag{3.4}$$

where $\text{sign}(x) := \mathbf{1}_{\{x>0\}} - \mathbf{1}_{\{x\leq 0\}}$. By the uniqueness of the decomposition of a continuous semi-martingale to a continuous local martingale and a continuous bounded variation process, we have $dL_t^0(Y) = 2|\partial\mathbb{R}_\varepsilon^d|dL_t^0(X)$.

By the similar computation as above for $-Y$ and $|Y_t| = |-Y_t|$, it holds that $dL_t^0(-Y) = dL_t^0(Y) - 2(|\partial\mathbb{R}_\varepsilon^d| - p|\partial\mathbb{R}_{\varepsilon'}^{d'}|)dL_t^0(X)$. Then we have

$$d\hat{L}_t^0(Y) = \frac{dL_t^0(Y) + dL_t^0(-Y)}{2} = (|\partial\mathbb{R}_\varepsilon^d| + p|\partial\mathbb{R}_{\varepsilon'}^{d'}|)dL_t^0(X). \tag{3.5}$$

By (3.3) and (3.5), the desired SDE follows. \square

Proposition 3.2. *Y has a jointly continuous density function $p^{(Y)}(t, x, y)$ with respect to the Lebesgue measure on \mathbb{R} . Furthermore, for any $T \geq 1$, $p^{(Y)}(t, x, y) \asymp \frac{1}{\sqrt{t}}e^{-|x-y|^2/t}$ for $(t, x, y) \in (0, T] \times \mathbb{R} \times \mathbb{R}$.*

Proof. This follows from the proof of [5, Proposition 4.4]. \square

In the following propositions, we prove Theorem 1.3.

Proposition 3.3 (Theorem 1.3(iii)). *Fix $T \geq 1$. Then it holds that*

$$p(t, x, y) \asymp \frac{1}{\sqrt{t}}e^{-\rho(x,y)^2/t} \text{ for } t \in (0, T], x \in \mathbb{R}_{\varepsilon'}^{d'} \cup \{a^*\}, y \in \mathbb{R}_\varepsilon^d \cup \{a^*\}.$$

Proof. Since BMVD hits a^* , we have

$$\begin{aligned}
p(t, x, y) &= \int_0^t p(t-s, a^*, x)\mathbb{P}_y(\sigma_{a^*} \in ds) \\
&= \int_0^t \int_0^{t-s} p(t-s-w, a^*, a^*)\mathbb{P}_x(\sigma_{a^*} \in dw)\mathbb{P}_y(\sigma_{a^*} \in ds).
\end{aligned}$$

Thus $(x, y) \mapsto p(t, x, y)$ depends only on $|x|_\rho$ and $|y|_\rho$. For $a > b > 0$, we have

$$\begin{aligned}
\int_a^b p^{(Y)}(t, -|x|_\rho, |y|_\rho)d|y|_\rho &= \mathbb{P}_{-|x|_\rho}(a \leq Y_t \leq b) \\
&= \mathbb{P}_x(X_t \in \mathbb{R}_\varepsilon^d, a \leq |X_t|_\rho \leq b) \\
&= \int_{\{y \in \mathbb{R}_\varepsilon^d, a \leq |y|_\rho \leq b\}} p(t, x, y)m_p(dy) \\
&= \int_a^b |\partial B(0; |y|_\rho + \varepsilon)|p(t, x, y)d|y|_\rho.
\end{aligned} \tag{3.6}$$

Thus, we have $p^{(Y)}(t, -|x|_\rho, |y|_\rho)d|y|_\rho = |\partial B(0; |y|)|p(t, x, y) \asymp |y|^{d-1}p(t, x, y)$.
By Proposition 3.2, it holds that

$$p(t, x, y) \asymp \frac{1}{|y|^{d-1}} \frac{1}{\sqrt{t}} e^{-(-|x|_\rho - |y|_\rho)^2/t} = \frac{1}{|y|^{d-1}\sqrt{t}} e^{-\rho(x, y)^2/t}. \quad (3.7)$$

Since $\varepsilon \leq |y|$ and (3.7), we have

$$p(t, x, y) \asymp \frac{1}{|y|^{d-1}\sqrt{t}} e^{-\rho(x, y)^2/t} \lesssim \frac{1}{\sqrt{t}} e^{-\rho(x, y)^2/t}.$$

Moreover, if $|y|_\rho \leq 1$ we have $p(t, x, y) \gtrsim \frac{1}{\sqrt{t}} e^{-\rho(x, y)^2/t}$ and if $|y|_\rho > 1$ we have

$$\begin{aligned} p(t, x, y) &\asymp \frac{1}{|y|^{d-1}\sqrt{t}} e^{-\rho(x, y)^2/t} \geq \frac{1}{|y|^{d-1}\sqrt{t}} \left(\frac{t}{T}\right)^{(d-1)/2} e^{-\rho(x, y)^2/t} \\ &\gtrsim \frac{1}{\rho(x, y)^{d-1}\sqrt{t}} \left(\frac{t}{T}\right)^{(d-1)/2} e^{-c\rho(x, y)^2/t} \gtrsim \frac{1}{\sqrt{t}} e^{-(c+1)\rho(x, y)^2/t}. \end{aligned}$$

□

Proposition 3.4 (Theorem 1.3(ii)). *Fix $T \geq 1$, then for all $t \leq T, x, y \in \mathbb{R}_\varepsilon^d$, it holds that*

$$p(t, x, y) \asymp \frac{e^{-\rho(x, y)^2/t}}{\sqrt{t}} + \frac{e^{-|x-y|^2/t}}{t^{d/2}} \left(1 \wedge \frac{|x|_\rho}{\sqrt{t}}\right) \left(1 \wedge \frac{|y|_\rho}{\sqrt{t}}\right) \text{ if } |x|_\rho \vee |y|_\rho \leq 1,$$

$$p(t, x, y) \asymp \frac{1}{t^{d/2}} e^{-\rho(x, y)^2/t} \text{ if } |x|_\rho \vee |y|_\rho > 1.$$

Proof. When $d = 1$, the statement holds from Proposition 3.2, so we assume $d \geq 2$.

For $x, y \in \mathbb{R}_\varepsilon^d$, it holds that $p(t, x, y) = p_{\mathbb{R}_\varepsilon^d}(t, x, y) + \bar{p}_{\mathbb{R}_\varepsilon^d}(t, x, y)$. Since $\bar{p}_{\mathbb{R}_\varepsilon^d}(t, x, y)$ depends only on $|x|_\rho$ and $|y|_\rho$, for $0 < a < b$, we have

$$\begin{aligned} \mathbb{P}_x(\sigma_{a^*} < t, X_t \in \mathbb{R}_\varepsilon^d, a \leq |X_t|_\rho \leq b) &= \int_{\{a \leq |y|_\rho \leq b\}} \bar{p}_{\mathbb{R}_\varepsilon^d}(t, x, y) m_p(dy) \quad (3.8) \\ &\asymp \int_a^b (|y|_\rho + \varepsilon)^{d-1} \bar{p}_{\mathbb{R}_\varepsilon^d}(t, x, y) d|y|_\rho. \end{aligned}$$

The left hand side of (3.8) is equal to

$$\mathbb{P}_{|x|_\rho}^{(Y)}(\sigma_0 < t, Y_t > 0, a \leq Y_t \leq b) = \int_a^b \int_0^t p^{(Y)}(t-s, 0, |y|_\rho) \mathbb{P}_{|x|_\rho}^{(Y)}(\sigma_0 \in ds) d|y|_\rho.$$

Here, $\mathbb{P}^{(Y)}$ is a probability measure with respect to Y . Thus, by using Proposi-

tion 3.2, it follows that

$$\begin{aligned}
(|y|_\rho + \varepsilon)^{d-1} \bar{p}_{\mathbb{R}_\varepsilon^d}(t, x, y) &\asymp \int_0^t p^{(Y)}(t-s, 0, |y|_\rho) \mathbb{P}_{|x|_\rho}^{(Y)}(\sigma_0 \in ds) d|y|_\rho \\
&\asymp \int_0^t p^{(Y)}(t-s, 0, |y|_\rho) \mathbb{P}_{-|x|_\rho}^{(Y)}(\sigma_0 \in ds) d|y|_\rho \\
&= p^{(Y)}(t, -|x|_\rho, |y|_\rho) \\
&\asymp \frac{1}{\sqrt{t}} e^{-(|x|_\rho + |y|_\rho)^2/t}. \tag{3.9}
\end{aligned}$$

Case 1 $|x|_\rho \vee |y|_\rho \leq 1$: Since $\varepsilon \leq |y|_\rho + \varepsilon \leq 1 + \varepsilon$, we have by (3.9),

$$\bar{p}_{\mathbb{R}_\varepsilon^d}(t, x, y) \asymp \frac{1}{\sqrt{t}} e^{-(|x|_\rho + |y|_\rho)^2/t}. \tag{3.10}$$

If $\rho(x, y) \asymp |x|_\rho + |y|_\rho$, we obtain $\bar{p}_{\mathbb{R}_\varepsilon^d}(t, x, y) \asymp \frac{1}{\sqrt{t}} e^{-\rho(x, y)^2/t}$.

If $\rho(x, y) = |x - y|$ and $|x|_\rho \wedge |y|_\rho \leq \sqrt{t}$, we may assume $|x|_\rho \leq \sqrt{t}$ without loss of generality. Then, it holds that

$$\begin{aligned}
\rho(x, y) &\leq |x|_\rho + |y|_\rho \leq \sqrt{t} + |y|_\rho \leq \sqrt{t} + |x| + |x - y| - \varepsilon \\
&= \sqrt{t} + |x|_\rho + |x - y| \leq 2\sqrt{t} + |x - y|. \tag{3.11}
\end{aligned}$$

By (3.11), it holds that $e^{-\rho(x, y)^2/t} \geq e^{-(|x|_\rho + |y|_\rho)^2/t} \geq e^{-2(2\sqrt{t})^2/t} e^{-\rho(x, y)^2/t}$. Thus, by (3.10), we have $\bar{p}_{\mathbb{R}_\varepsilon^d}(t, x, y) \asymp \frac{1}{\sqrt{t}} e^{-\rho(x, y)^2/t}$.

If $\rho(x, y) = |x - y|$ and $|x|_\rho \wedge |y|_\rho > \sqrt{t}$, by (3.10) and (2.1), we have

$$p(t, x, y) \asymp \frac{1}{\sqrt{t}} e^{-(|x|_\rho + |y|_\rho)^2/t} + \frac{1}{t^{d/2}} e^{-|x-y|^2/t} \lesssim \frac{1}{\sqrt{t}} e^{-\rho(x, y)^2/t} + \frac{1}{t^{d/2}} e^{-|x-y|^2/t},$$

$$p(t, x, y) \gtrsim \frac{1}{t^{d/2}} e^{-|x-y|^2/t} \gtrsim \frac{1}{\sqrt{t}} e^{-\rho(x, y)^2/t} + \frac{1}{t^{d/2}} e^{-|x-y|^2/t}.$$

Case 2 $|x|_\rho \vee |y|_\rho > 1$: Without loss of generality, we may assume $|y|_\rho > 1$. By (3.9), it holds that

$$\begin{aligned}
\bar{p}_{\mathbb{R}_\varepsilon^d}(t, x, y) &\asymp \frac{1}{(|y|_\rho + \varepsilon)^{d-1}} \frac{1}{\sqrt{t}} e^{-(|x|_\rho + |y|_\rho)^2/t} \\
&\geq \frac{1}{2(\varepsilon + 1)(|x|_\rho + |y|_\rho)^{d-1}} \frac{1}{\sqrt{t}} e^{-(|x|_\rho + |y|_\rho)^2/t} \\
&\gtrsim \frac{1}{(|x|_\rho + |y|_\rho)^{d-1}} \frac{1}{\sqrt{t}} \left(\frac{(|x|_\rho + |y|_\rho)^2}{t} \right)^{(d-1)/2} e^{-(|x|_\rho + |y|_\rho)^2/t} \\
&\asymp \frac{1}{t^{d/2}} e^{-(|x|_\rho + |y|_\rho)^2/t}. \tag{3.12}
\end{aligned}$$

By (2.1) and (3.9), we obtain

$$\begin{aligned} p(t, x, y) &\asymp \frac{1}{t^{d/2}} \left(1 \wedge \frac{|x|_\rho}{\sqrt{t}}\right) \left(1 \wedge \frac{|y|_\rho}{\sqrt{t}}\right) e^{-|x-y|^2/t} + \frac{e^{-(|x|_\rho+|y|_\rho)^2/t}}{\sqrt{t}(|y|_\rho+\varepsilon)^{d-1}} \\ &\lesssim \frac{1}{t^{d/2}} \left(e^{-|x-y|^2/t} + e^{-(|x|_\rho+|y|_\rho)^2/t}\right). \end{aligned} \quad (3.13)$$

If $|x|_\rho \wedge |y|_\rho \leq \sqrt{t}$, then we have $p(t, x, y) \asymp \frac{1}{t^{d/2}} e^{-\rho(x,y)^2/t}$ in the same way as Case1.

If $|x|_\rho \wedge |y|_\rho > \sqrt{t}$, then we have $p(t, x, y) \asymp \frac{1}{t^{d/2}} e^{-\rho(x,y)^2/t}$ since $\rho(x, y) = |x - y| \wedge (|x|_\rho + |y|_\rho)$.

This completes the proof. \square

Proposition 3.5 (Theorem 1.3(i)). *Fix $T \geq 1$, then for all $t \in (0, T]$, $x, y \in \mathbb{R}_\varepsilon^{d'}$, it holds that*

$$p(t, x, y) \asymp \frac{e^{-\rho(x,y)^2/t}}{\sqrt{t}} + \frac{e^{-|x-y|^2/t}}{t^{d'/2}} \left(1 \wedge \frac{|x|_\rho}{\sqrt{t}}\right) \left(1 \wedge \frac{|y|_\rho}{\sqrt{t}}\right) \text{ if } |x|_\rho \vee |y|_\rho \leq 1.$$

$$p(t, x, y) \asymp \frac{1}{t^{d'/2}} e^{-\rho(x,y)^2/t} \text{ if } |x|_\rho \vee |y|_\rho > 1.$$

Proof. The proof is the same as that of Proposition 3.4. \square

This completes the proof of Theorem 1.3.

4 Large time estimate($d' = 1$)

In this section, we prove Theorem 1.4. Let $d' = 1$. When $d = 1$, $\mathbb{R}_+ \cup \mathbb{R}_+ \cup \{a^*\}$ can be identified with \mathbb{R} . In this case, BMVD is 1-dimensional Brownian motion, so there is nothing to prove. When $d = 2$, it was proved by [5]. Hence we consider the case of $d \geq 3$. Let $\varepsilon > 0$ and $S_\varepsilon^{d-1} := \{x \in \mathbb{R}^d ; |x| = \varepsilon\}$. We will prove Theorem 1.4 by projecting $(\mathbb{R}_+ \times S_\varepsilon^{d-1}) \# \mathbb{R}^d$ to $\mathbb{R}_+ \cup \mathbb{R}_\varepsilon^d \cup \{a^*\}$.

The following theorem is a special case of [14, Corollary 6.13].

Theorem 4.1. *Let K be central part of $M := (\mathbb{R}_+ \times S_\varepsilon^{d-1}) \# \mathbb{R}^d$. Let $E_1 := (M \setminus K) \cap (\mathbb{R}_+ \times S_\varepsilon^{d-1})$, $E_2 := (M \setminus K) \cap \mathbb{R}^d$, and $E_0 \subset M$ be a precompact open set having smooth boundary and containing K . Then heat kernel $\check{p}(t, x, y)$ of standard Brownian motion \check{X} on M satisfies the following estimates for $1 \leq t$.*

(i) For $x, y \in E_1$,

$$\check{p}(t, x, y) \asymp \frac{|x|_e |y|_e}{\sqrt{t}(|x|_e + \sqrt{t})(|y|_e + \sqrt{t})} e^{-d(x,y)^2/t}.$$

(ii) For $x, y \in E_2$,

$$\check{p}(t, x, y) \asymp \frac{1}{t^{3/2}|x|_e^{d-2}|y|_e^{d-2}} e^{-(|x|_e+|y|_e)^2/t} + \frac{1}{t^{d/2}} e^{-d(x,y)^2/t}.$$

(iii) For $x \in E_0 \cup E_1, y \in E_0 \cup E_2$,

$$\check{p}(t, x, y) \asymp \left(\frac{1}{t^{d/2}} + \frac{|x|_e}{t^{3/2}|y|_e^{d-2}} \right) e^{-\rho(x,y)^2/t}.$$

Here, d is a geodesic distance, and $|x|_e := \sup_{z \in K} d(x, z) \asymp 1 + d(x, K)$.

From now on, we fix $K := (\{0\} \times \{x \in \mathbb{R}^d, |x| < 1 + \varepsilon\}) \cup ([0, 1) \times S_\varepsilon^{d-1})$. Then it holds that $M = (\mathbb{R}_+ \cup \{0\}) \times S_\varepsilon^{d-1} \cup \mathbb{R}_\varepsilon^d$. See Figure 2.

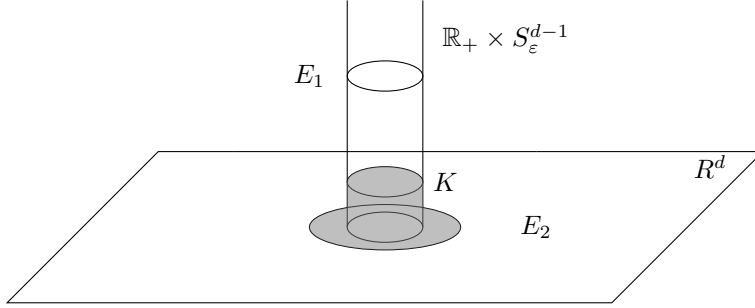


Figure 2: $M := (\mathbb{R}_+ \times S_\varepsilon^{d-1}) \# \mathbb{R}^d = (\mathbb{R}_+ \cup \{0\}) \times S_\varepsilon^{d-1} \cup \mathbb{R}_\varepsilon^d$

We define $q := p/|\partial\mathbb{R}_\varepsilon^d|$ and $\tilde{m}_q(A) := m^{(d)}(A \cap \mathbb{R}^d) + q m^{(1,d-1)}(A \cap (\mathbb{R}_+ \times S_\varepsilon^{d-1}))$ for a Borel set $A \subset M$. Here, $m^{(d)}$ and $m^{(1,d-1)}$ are the Lebesgue measures on \mathbb{R}^d and $\mathbb{R}_+ \times S_\varepsilon^{d-1}$, respectively. Then \tilde{m}_q -symmetric Brownian motion $\{\tilde{X}_t\}$ on M is a time-changed process of standard Brownian motion $\{\check{X}_t\}$ on M by a positive continuous additive functional having the Revuz measure \tilde{m}_q . To be precise, we have $\tilde{X}_t = \check{X}_{\tau_t}$, where $A_t := \int_0^t (\mathbf{1}_{\mathbb{R}^d} + q \mathbf{1}_{\mathbb{R}_+ \times S_\varepsilon^{d-1}})(\check{X}_s) ds$ and $\tau_t := \{s > 0 \mid A_s > t\}$. Let $\tilde{p}(t, x, y)$ (resp. $\check{p}(t, x, y)$) be the heat kernel of $\{\tilde{X}_t\}$ (resp. $\{\check{X}_t\}$). Since $(1 \wedge q)t \leq A_t \leq (1 \vee q)t$ and $\frac{t}{1 \vee q} \leq \tau_t \leq \frac{t}{1 \wedge q}$, we have $\tilde{p}(t, x, y) \asymp \check{p}(t, x, y)$. Thus $\tilde{p}(t, x, y)$ satisfies the same estimates as Theorem 4.1.

We define

$$v(x) := \begin{cases} -x^{(1)} : x = (x^{(1)}, x^{(2)}) \in (\mathbb{R}_+ \cup \{0\}) \times S_\varepsilon^{d-1}, \\ |x|_\rho : x \in \mathbb{R}_\varepsilon^d, \end{cases}$$

and $\tilde{Y}_t := v(\tilde{X}_t)$.

Theorem 4.2. \tilde{Y} has the same law as Y . Here, Y is the signed radial process of X defined in Section 3.

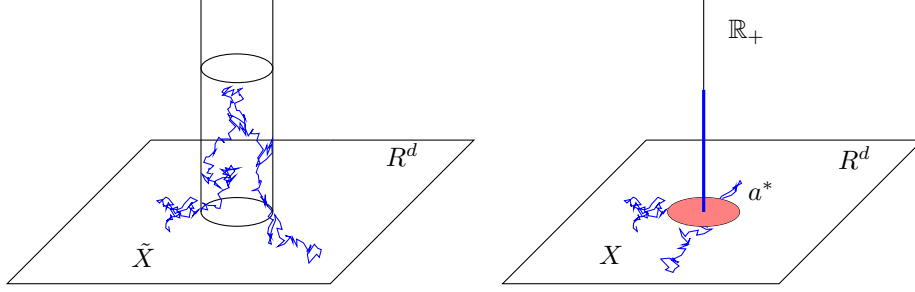


Figure 3: Projection M to $\mathbb{R}_+ \cup \mathbb{R}_\varepsilon^d \cup \{a^*\}$, and \tilde{X} and X

Proof. By Proposition 3.1, it holds that

$$dY_t = dB_t + \frac{(d-1)\mathbf{1}_{\{Y_t > 0\}}}{2(Y_t + \varepsilon)} dt + \frac{|\partial\mathbb{R}_\varepsilon^d| - p}{|\partial\mathbb{R}_\varepsilon^d| + p} d\hat{L}_t^0(Y), \quad (4.1)$$

where B is one-dimensional Brownian motion. We will prove \tilde{Y} also satisfies (4.1). Let $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ on $L^2(M; d\tilde{m}_q)$ be the Dirichlet form associated with \tilde{X} . Then we have $v \in \tilde{\mathcal{F}}^{loc}$. By the Fukushima decomposition, there exist local martingale additive functional $M^{[v]}$ and continuous additive functional locally having zero energy $N^{[v]}$ such that $\tilde{Y}_t - \tilde{Y}_0 = M_t^{[v]} + N_t^{[v]}$, \mathbb{P}_x -a.s. for q.e. $x \in M$. For any $\psi \in C_c^\infty(M)$, it holds that

$$\begin{aligned} \tilde{\mathcal{E}}(v, \psi) &= \frac{1}{2} \int_{\mathbb{R}_\varepsilon^d} \nabla|x|_\rho \cdot \nabla\psi dx + \frac{q}{2} \int_{(\mathbb{R}_+ \cup \{0\}) \times S_\varepsilon^{d-1}} \nabla(-x^{(1)}) \cdot \nabla\psi dm^{(1,d-1)} \\ &= -\frac{1}{2} \int_{\mathbb{R}_\varepsilon^d} \frac{d-1}{|x|} \psi dx + \frac{1}{2} \int_{\partial\mathbb{R}_\varepsilon^d} \psi(x) \frac{\partial|x|_\rho}{\partial\mathbf{n}} \sigma(dx) - \frac{q}{2} \int_{S_\varepsilon^{d-1}} \int_0^\infty \frac{\partial\psi}{\partial x^{(1)}} dm^{(1,d-1)} \\ &= - \int_M \frac{d-1}{2|x|} \psi \mathbf{1}_{\mathbb{R}_\varepsilon^d} dx - \int_M \frac{\psi}{2} \mathbf{1}_{\partial\mathbb{R}_\varepsilon^d} d\sigma + \int_M \frac{q\psi}{2} \mathbf{1}_{\{0\} \times S_\varepsilon^{d-1}} d\sigma = - \int_M \psi d\nu, \end{aligned}$$

where \mathbf{n} is the outward normal vector of the surface $\partial\mathbb{R}_\varepsilon^d$, σ is the surface measure on $\partial\mathbb{R}_\varepsilon^d = \{0\} \times S_\varepsilon^{d-1}$, and

$$\nu(dx) := \frac{d-1}{2|x|} \mathbf{1}_{\mathbb{R}_\varepsilon^d}(x) dx + \frac{1-q}{2} d\sigma.$$

By [7, Theorem 5.5.5], it holds that

$$dN_t^{[v]} = \frac{(d-1)\mathbf{1}_{\{\tilde{Y}_t > 0\}}}{2(\tilde{Y}_t + \varepsilon)} dt + \frac{1-q}{2} dL_t \quad (4.2)$$

Here, L is the positive continuous additive functional of \tilde{X} whose Revuz measure is σ . By the same proof as that of Proposition 3.1, it holds that $M^{[v]}$ is one-dimensional Brownian motion \hat{B} , and $\hat{L}_t^0(\tilde{Y}) = \frac{1+q}{2} dL_t$, where $\hat{L}_t^0(\tilde{Y})$ is a

symmetric semimartingale local time of \tilde{Y} at 0. Combining these with (4.2) and $q = p/|\partial\mathbb{R}_\varepsilon^d|$, \tilde{Y} satisfies (4.1). By [1, Theorem 2.1], weak solutions of (4.1) have the same law, so this completes the proof. \square

Proof of Theorem 1.4.

Step1 (the case of x or $y \in \mathbb{R}_+$) Fix large $T > 0$ and $t \geq T$. For $f \in C_c(\mathbb{R}_+ \cup \mathbb{R}_\varepsilon^d \cup \{a^*\})$ with $\text{supp}(f) \subset \mathbb{R}_+$, we define $\tilde{f} : M \rightarrow \mathbb{R}$ by

$$\tilde{f}(\tilde{y}) := \begin{cases} f(\tilde{y}^{(1)}) & : \tilde{y} = (\tilde{y}^{(1)}, \tilde{y}^{(2)}) \in \mathbb{R}_+ \times S_\varepsilon^{d-1} \\ 0 & : \text{otherwise.} \end{cases}$$

For $x \in \mathbb{R}_+ \cup \mathbb{R}_\varepsilon^d \cup \{a^*\}$ and $x_2 \in S_\varepsilon^{d-1}$, we define $\tilde{x} \in M$ by

$$\tilde{x} := \tilde{x}(x_2) := \begin{cases} (x, x_2) & : x \in \mathbb{R}_+ \cup \{a^*\}, \\ x & : x \in \mathbb{R}_\varepsilon^d. \end{cases} \quad (4.3)$$

Here, we defined $(a^*, x_2) := (0, x_2)$. Now, we take $x_2, x_2^* \in S_\varepsilon^{d-1}$ and define $\tilde{x}(x_2), \tilde{x}(x_2^*) \in M$ as in (4.3). Then, since \tilde{f} is independent of x_2 and x_2^* , it holds that $\mathbb{E}_{\tilde{x}(x_2)}(\tilde{f}(\tilde{X}_t)) = \mathbb{E}_{\tilde{x}(x_2^*)}(\tilde{f}(\tilde{X}_t))$, so we simply write $\tilde{x}(x_2)$ as \tilde{x} .

By Theorem 4.2, we have $\mathbb{E}_x(f(X_t)) = \mathbb{E}_{u(x)}(f(-Y_t)) = \mathbb{E}_{v(\tilde{x})}(f(-\tilde{Y}_t)) = \mathbb{E}_{\tilde{x}}(\tilde{f}(\tilde{X}_t))$. While, we have

$$\begin{aligned} \mathbb{E}_{\tilde{x}}(\tilde{f}(\tilde{X}_t)) &= \int_{\mathbb{R}_+ \times S_\varepsilon^{d-1}} \tilde{f}(\tilde{y}) \tilde{p}(t, \tilde{x}, \tilde{y}) \tilde{m}_q(d\tilde{y}) \\ &\asymp \int_{\mathbb{R}_+} f(y) \left(\int_{S_\varepsilon^{d-1}} \tilde{p}(t, \tilde{x}, (y, y_2)) dy_2 \right) m_p(dy). \end{aligned}$$

Thus, for $x \in \mathbb{R}_+ \cup \mathbb{R}_\varepsilon^d \cup \{a^*\}$ and $y \in \mathbb{R}_+$, we have

$$p(t, x, y) \asymp \int_{S_\varepsilon^{d-1}} \tilde{p}(t, \tilde{x}, (y, y_2)) dy_2. \quad (4.4)$$

We next consider the relation between the distance d on M and ρ on $\mathbb{R}_+ \cup \mathbb{R}_\varepsilon^d \cup \{a^*\}$.

- (i) (Figure 4, left) For $x, y \in \mathbb{R}_+$, since S_ε^{d-1} is bounded, there exists a constant $C > 0$ with $\rho(x, y) \leq d(\tilde{x}, \tilde{y}) \leq C + \rho(x, y)$. Hence, for $t \geq T$, it holds that $e^{-\rho(x, y)^2/t} \asymp e^{-d(\tilde{x}, \tilde{y})^2/t}$ and $|\tilde{x}|_e \asymp 1 + d(\tilde{x}, K) = |x| = |x|_\rho$.
- (ii) (Figure 4, right) For $x \in \mathbb{R}_\varepsilon^d \cup \{a^*\}, y \in \mathbb{R}_+$, since S_ε^{d-1} is bounded, there exists a constant $C > 0$ with $\rho(x, y) \leq d(x, \tilde{y}) \leq C + \rho(x, y)$. Then for $t \geq T$, it holds that $e^{-\rho(x, y)^2/t} \asymp e^{-d(x, \tilde{y})^2/t}$.

Thus, for x or $y \in \mathbb{R}_+$, the desired estimates follow from (4.4), Theorem 4.1 and the boundedness of S_ε^{d-1} . In particular, for all $x \in \mathbb{R}_+$ and $t \geq T$, it holds that

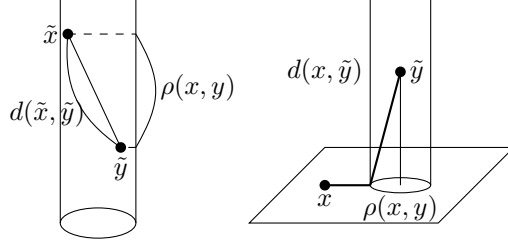


Figure 4: the relation between d and ρ

$p(t, x, x) \asymp t^{-3/2}$ and by continuity of p , we have $p(t, a^*, a^*) \asymp t^{-3/2}$ for $t \geq T$. Combining this with the small time estimates, we have

$$p(t, a^*, a^*) \asymp t^{-1/2} \wedge t^{-3/2} \quad \text{for } t > 0. \quad (4.5)$$

Step2 (the case of $x, y \in \mathbb{R}_\varepsilon^d \cup \{a^*\}$) Fix large $T \geq 2$ and $t \geq T$.

(1) For $y \in \mathbb{R}_\varepsilon^d$, by (4.5), Lemma 2.5, Lemma 2.7 and Lemma 2.8, we have

$$\begin{aligned} p(t, a^*, y) &= \int_0^t p(t-s, a^*, a^*) \mathbb{P}_y(\sigma_{a^*} \in ds) \\ &\asymp \int_0^{t/2} (t-s)^{-3/2} \mathbb{P}_y(\sigma_{a^*} \in ds) + \int_{t/2}^t (t-s)^{-1/2} \mathbb{P}_y(\sigma_{a^*} \in ds) \\ &\asymp \frac{\mathbb{P}_y(\sigma_{a^*} \leq t/2)}{t^{3/2}} + \left(\int_{t/2}^{t-1} (t-s)^{-3/2} ds + \int_{t-1}^t (t-s)^{-1/2} ds \right) \frac{|y|_\rho e^{-|y|_\rho^2/t}}{|y| t^{d/2}} \\ &\asymp \frac{1}{t^{3/2}} \frac{e^{-|y|_\rho^2/t}}{|y|^{d-2}} + \frac{|y|_\rho e^{-|y|_\rho^2/t}}{|y| t^{d/2}} \end{aligned} \quad (4.6)$$

If $1 \leq |y|_\rho$, by $\frac{1}{1+\varepsilon} \leq \frac{|y|_\rho}{|y|} \leq 1$ and (4.6), it holds that

$$p(t, a^*, y) \asymp \left(\frac{1}{t^{d/2}} + \frac{1}{t^{3/2}|y|^{d-2}} \right) e^{-|y|_\rho^2/t}.$$

If $1 > |y|_\rho$, by

$$\frac{1}{t^{3/2}|y|^{d-2}} e^{-|y|_\rho^2/t} \geq \frac{T^{(d-3)/2}}{t^{d/2}(1+\varepsilon)^{d-2}} e^{-|y|_\rho^2/t}, \quad \frac{|y|_\rho}{|y|} \leq 1$$

and (4.6), it holds that

$$p(t, a^*, y) \asymp \left(\frac{1}{t^{d/2}} + \frac{1}{t^{3/2}|y|^{d-2}} \right) e^{-|y|_\rho^2/t}.$$

(2) For $x, y \in \mathbb{R}_\varepsilon^d$, with $|x|_\rho \wedge |y|_\rho > 1$ and $t \geq T$, by (1), Lemma 2.8, Theorem 1.3, Lemma 2.5, Proposition 2.4, Lemma 2.10 and Lemma 2.7, we obtain that

$$\begin{aligned}
p(t, x, y) &= \int_0^{t/2} + \int_{t/2}^{t-1} + \int_{t-1}^t p(t-s, a^*, y) \mathbb{P}_x(\sigma_{a^*} \in ds) + p_{\mathbb{R}_\varepsilon^d}(t, x, y) \\
&\asymp \left(\frac{e^{-|y|_\rho^2/t}}{t^{d/2}} + \frac{e^{-|y|_\rho^2/t}}{t^{3/2}|y|^{d-2}} \right) \frac{e^{-|x|_\rho^2/t}}{|x|^{d-2}} \\
&\quad + \int_{t/2}^{t-1} + \int_{t-1}^t p(t-s, a^*, y) \frac{e^{-|x|_\rho^2/t}}{t^{d/2}} ds + \frac{1}{t^{d/2}} e^{-|x-y|^2/t} \\
&\lesssim \frac{1}{t^{3/2}|x|^{d-2}|y|^{d-2}} e^{-(|x|_\rho+|y|_\rho)^2/t} + \frac{1}{t^{d/2}} e^{-\rho(x,y)^2/t}
\end{aligned}$$

and

$$\begin{aligned}
p(t, x, y) &\geq \int_0^{t/2} p(t-s, a^*, y) \mathbb{P}_x(\sigma_{a^*} \in ds) + p_{\mathbb{R}_\varepsilon^d}(t, x, y) \\
&\gtrsim \frac{1}{t^{3/2}|x|^{d-2}|y|^{d-2}} e^{-(|x|_\rho+|y|_\rho)^2/t} + \frac{1}{t^{d/2}} e^{-\rho(x,y)^2/t}.
\end{aligned}$$

(3) For $x, y \in \mathbb{R}_\varepsilon^d$ with $|x|_\rho \wedge |y|_\rho \leq 1$ and $t \geq T$, we may assume $|x|_\rho \leq 1$ without loss of generality since $p(t, x, y)$ is symmetric. By (1), $|x| \asymp 1$, Lemma 2.8, Theorem 1.3, Lemma 2.5, Proposition 2.4, Lemma 2.10 and Lemma 2.7, we obtain that

$$\begin{aligned}
p(t, x, y) &= \int_0^{t/2} + \int_{t/2}^{t-1} + \int_{t-1}^t p(t-s, a^*, x) \mathbb{P}_y(\sigma_{a^*} \in ds) + p_{\mathbb{R}_\varepsilon^d}(t, x, y) \\
&\asymp \left(\frac{e^{-|x|_\rho^2/t}}{t^{d/2}} + \frac{|x| e^{-|x|_\rho^2/t}}{t^{3/2}|y|^{d-2}} \right) e^{-|y|_\rho^2/t} \\
&\quad + \int_{t/2}^{t-1} + \int_{t-1}^t p(t-s, a^*, x) \frac{e^{-|y|_\rho^2/t}}{t^{d/2}} ds + \frac{|x|_\rho(1 \wedge |y|_\rho)}{t^{d/2}} e^{-|x-y|^2/t} \\
&\lesssim \left(\frac{1}{t^{d/2}} + \frac{|x|}{t^{3/2}|y|^{d-2}} \right) e^{-\rho(x,y)^2/t}
\end{aligned}$$

and if $|y|_\rho \leq 1$ or $2 \leq |y|_\rho$, by Lemma 2.10, then we have

$$p(t, x, y) \geq \int_0^{t/2} p(t-s, a^*, x) \mathbb{P}_y(\sigma_{a^*} \in ds) \asymp \left(\frac{1}{t^{d/2}} + \frac{|x|}{t^{3/2}|y|^{d-2}} \right) e^{-\rho(x,y)^2/t}.$$

Since there exists constant $c > 0$ such that it holds that $\mathbb{P}_x(\sigma_{a^*} \leq 1) > c$ for x with $|x|_\rho \leq 1$, if $1 < |y|_\rho < 2$, then we have

$$\begin{aligned}
p(t, x, y) &\geq \int_0^1 p(t-s, a^*, y) \mathbb{P}_x(\sigma_{a^*} \in ds) \\
&\gtrsim \int_0^1 \left(\frac{1}{t^{d/2}} + \frac{|x|}{t^{3/2}|y|^{d-2}} \right) e^{-|y|_\rho^2/t-1} \mathbb{P}_x(\sigma_{a^*} \in ds) \\
&\gtrsim \left(\frac{1}{t^{d/2}} + \frac{|x|}{t^{3/2}|y|^{d-2}} \right) e^{-1/T-1} \gtrsim \left(\frac{1}{t^{d/2}} + \frac{|x|}{t^{3/2}|y|^{d-2}} \right) e^{-\rho(x,y)^2/t}.
\end{aligned}$$

This completes the proof of Theorem 1.4. \square

Remark 4.3. In [11], the heat kernel estimate for Brownian motion on $(\mathbb{R}_+ \times S_\varepsilon^1) \# \mathbb{R}^2$ is obtained. Therefore, by the same way as in this section, we can obtain the large time estimate on $\mathbb{R}_+ \cup \mathbb{R}_\varepsilon^2 \cup \{a^*\}$. By elementary computations, this estimate is the same as the one appearing in [5].

5 Large time estimate ($d' \geq 3$)

In this section, we will prove Theorem 1.7. We assume $d \geq d' \geq 3$. Moreover, we may assume $\varepsilon, \varepsilon' < 1$ without loss of generality. Unlike the case $d = 1$ in Section 4, we cannot project $(\mathbb{R}^{d'} \times S^{d-d'}) \# \mathbb{R}^d$ to get $\mathbb{R}_{\varepsilon'}^{d'} \cup \mathbb{R}_\varepsilon^d \cup \{a^*\}$ when $d' \geq 2$. Hence, we will take careful approach.

For $x, y \in \mathbb{R}_\varepsilon^d$, it holds that

$$p(t, x, y) = p_{\mathbb{R}_\varepsilon^d}(t, x, y) + \int_0^t \int_0^{t-s} p(t-s-u, a^*, a^*) \mathbb{P}_x(\sigma_{a^*} \in du) \mathbb{P}_y(\sigma_{a^*} \in ds).$$

For $x \in \mathbb{R}_\varepsilon^d, y \in \mathbb{R}_{\varepsilon'}^{d'}$, it holds that

$$p(t, x, y) = \int_0^t \int_0^{t-s} p(t-s-u, a^*, a^*) \mathbb{P}_x(\sigma_{a^*} \in du) \mathbb{P}_y(\sigma_{a^*} \in ds).$$

So, we consider the estimate of $p(t, a^*, a^*)$ in order to prove Theorem 1.7.

Proposition 5.1. *For $t > 0$, we have*

$$p(t, a^*, a^*) \lesssim \frac{1}{\sqrt{t}} \wedge \frac{1}{t^{d'/2}}.$$

Proof. For $t > 1$, it holds that $t^{-d'/2} \leq t^{-1/2}$ and $p(t, a^*, a^*) \lesssim t^{-d'/2}$ by Proposition 2.2. For $t \leq 1$, it holds that $t^{-1/2} \leq t^{-d'/2}$ and $p(t, a^*, a^*) \lesssim t^{-1/2}$ by the small time estimate (Theorem 1.3). Thus for $t > 0$, we have

$$p(t, a^*, a^*) \lesssim \frac{1}{\sqrt{t}} \wedge \frac{1}{t^{d'/2}}.$$

\square

Proposition 5.2. *For $t > 0$,*

$$p(t, a^*, a^*) \asymp \frac{1}{\sqrt{t}} \wedge \frac{1}{t^{d'/2}}. \quad (5.1)$$

Proof. Take $t \geq 2$ and $x \in \mathbb{R}_\varepsilon^{d'}$ with $\sqrt{t} \leq |x| \leq 2\sqrt{t}$. For $s > 0$ with $t-1 < s < t$, it holds that $t-s < 1$, so we can apply Theorem 1.3 to $p(t-s, a^*, a^*)$. Thus, by Theorem 1.3 and Lemma 2.5, we have

$$\begin{aligned} p(t, a^*, x) &= \int_0^t p(t-s, a^*, a^*) \mathbb{P}(\sigma_{a^*} \in ds) \geq \int_{t-1}^t p(t-s, a^*, a^*) \mathbb{P}(\sigma_{a^*} \in ds) \\ &\gtrsim \int_{t-1}^t (t-s)^{-1/2} \frac{|x|_\rho}{|x|} \frac{e^{-|x|_\rho^2/s}}{s^{d'/2} + s^{3/2}|x|^{(d'-3)/2}} ds. \end{aligned} \quad (5.2)$$

Since $t/2 \leq t-1$ and $\sqrt{2}-\varepsilon \leq |x|_\rho$, we have

$$p(t, a^*, x) \gtrsim \frac{e^{-|x|_\rho^2/t}}{t^{d'/2} + t^{3/2}|x|^{(d'-3)/2}} \geq \frac{e^{-(2\sqrt{t})^2/t}}{t^{d'/2} + t^{3/2}(2\sqrt{t})^{(d'-3)/2}} \gtrsim \frac{1}{t^{d'/2}}. \quad (5.3)$$

By the Markov property and (5.3), we have

$$\begin{aligned} p(2t, a^*, a^*) &\geq \int_{\{x \in \mathbb{R}_\varepsilon^{d'}; \sqrt{t} \leq |x| \leq 2\sqrt{t}\}} p(t, a^*, x)^2 m_p(dx) \\ &\gtrsim \int_{\{x \in \mathbb{R}_\varepsilon^{d'}; \sqrt{t} \leq |x| \leq 2\sqrt{t}\}} t^{-d'} m_p(dx) \\ &= \int_{\sqrt{t}}^{2\sqrt{t}} r^{d'-1} dr \times pt^{-d'} \asymp \frac{1}{t^{d'/2}}, \end{aligned} \quad (5.4)$$

where we used polar coordinates $r := |x|$. (5.4) and the small time estimate (Theorem 1.3) imply $p(t, a^*, a^*) \gtrsim \frac{1}{\sqrt{t}} \wedge \frac{1}{t^{d'/2}}$ for $t > 0$. Thus (5.1) follows from it and Proposition 5.1. \square

We will prove Theorem 1.7, by using the on-diagonal estimate at a^* and hitting probability.

Proposition 5.3. *Let $d \geq d' \geq 3$. Then $p(t, x, y)$ satisfies the following estimates when $1 \leq t$:*

- (i) For $x, y \in \mathbb{R}_\varepsilon^{d'}$, $p(t, x, y) \lesssim t^{-d'/2} e^{-\rho(x,y)^2/t}$.
- (ii) For $x, y \in \mathbb{R}_\varepsilon^d$ with $|x|_\rho \vee |y|_\rho \leq 1$, $p(t, x, y) \lesssim t^{-d'/2} e^{-\rho(x,y)^2/t}$.
For $x, y \in \mathbb{R}_\varepsilon^d$ with $|x|_\rho \vee |y|_\rho > 1$,

$$p(t, x, y) \lesssim \frac{1}{t^{d'/2}|x|^{d-2}|y|^{d-2}} e^{-(|x|_\rho + |y|_\rho)^2/t} + \frac{1}{t^{d/2}} e^{-\rho(x,y)^2/t}.$$

- (iii) For $x \in \mathbb{R}_\varepsilon^d \cup \{a^*\}, y \in \mathbb{R}_\varepsilon^{d'} \cup \{a^*\}$,

$$p(t, x, y) \lesssim \left(\frac{1}{t^{d'/2}|y|^{d'-2}} + \frac{1}{t^{d'/2}|x|^{d-2}} \right) e^{-\rho(x,y)^2/t}.$$

Proof. In order to avoid a long calculation, we will prove the estimates by comparing $\mathbb{R}_\varepsilon^d \cup \mathbb{R}_\varepsilon^{d'} \cup \{a^*\}$ with a manifold with ends. First, we assume $\varepsilon \leq \varepsilon'$ (See Figure 5).

Let $\tilde{p}(t, x, y)$ be the heat kernel of Brownian motion \tilde{X} on $(\mathbb{R}^{d'} \times S_\varepsilon^{d-d'}) \# \mathbb{R}^d$, where $S_\varepsilon^{d-d'} := \{x \in \mathbb{R}^{d-d'+1} : |x| = \varepsilon\}$. According to [14, Example 4.5 and Example 5.5], for $t > 1$, $\tilde{p}(t, x, y)$ has sharp estimates as the right hands side of this proposition up to the difference between distances ρ and d , where d is a geodesic distance on $(\mathbb{R}^{d'} \times S_\varepsilon^{d-d'}) \# \mathbb{R}^d$. Furthermore, let $K := (\overline{B}^{d'}(0; \varepsilon') \times S_\varepsilon^{d-d'}) \cup \overline{B}^d(0; \varepsilon)$ then, for $t > 1$ and $\tilde{x}, \tilde{y} \in K$, it holds that $\tilde{p}(t, \tilde{x}, \tilde{y}) \asymp t^{-d'/2}$.

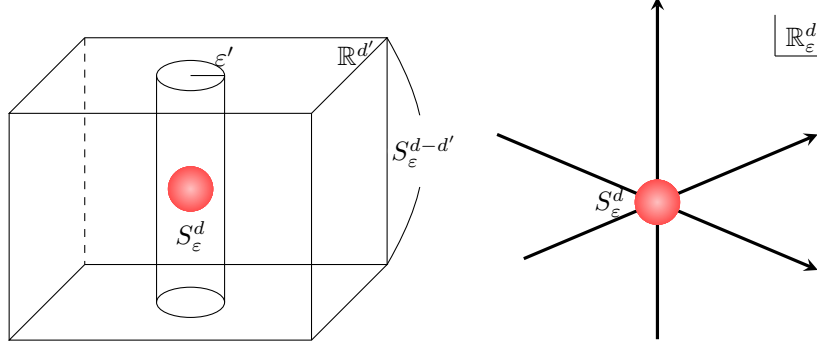


Figure 5: $(\mathbb{R}^{d'} \times S_\varepsilon^{d-d'}) \# \mathbb{R}^d$

Here \overline{B}^d is a closed ball on \mathbb{R}^d . By combining this with small time estimates ([14, Theorem 5.10]), for $t > 0$ and $\tilde{x}, \tilde{y} \in K$, we obtain $\tilde{p}(t, \tilde{x}, \tilde{y}) \asymp t^{-d/2} e^{-d(\tilde{x}, \tilde{y})/t} \vee t^{-d'/2}$.

By proposition 5.2, we have $p(t, a^*, a^*) \asymp t^{-1/2} \wedge t^{-d'/2} \leq t^{-d/2} \vee t^{-d'/2}$,

$\mathbb{P}_x(\sigma_{a^*} \in ds) = \tilde{\mathbb{P}}_x(\tilde{\sigma}_K \in ds)$, $p_{\mathbb{R}_\varepsilon^d}(t, x, y) = \tilde{p}_{\mathbb{R}^d \setminus K}(t, x, y)$ for $x, y \in \mathbb{R}_\varepsilon^d$ and

$\mathbb{P}_x(\sigma_{a^*} \in ds) = \tilde{\mathbb{P}}_{(x, x_2)}(\tilde{\sigma}_K \in ds)$ for $x \in \mathbb{R}_{\varepsilon'}^{d'}$, $x_2 \in S_\varepsilon^{d-d'}$,

where $\tilde{\mathbb{P}}$, $\tilde{\sigma}_K$ and $\tilde{p}_{\mathbb{R}^d \setminus K}$ are those for the process \tilde{X} . Moreover, for the part process on $(\mathbb{R}^{d'} \times S_\varepsilon^{d-d'}) \setminus K = \mathbb{R}_{\varepsilon'}^{d'} \times S_\varepsilon^{d-d'}$ of \tilde{X} , the projection to $\mathbb{R}_{\varepsilon'}^{d'}$ has the same law as the part process on $\mathbb{R}_{\varepsilon'}^{d'}$ of X . Thus, for $x, y \in \mathbb{R}_{\varepsilon'}^{d'}$, $x_2 \in S_\varepsilon^{d-d'}$, by the same reason as the proof of Theorem 1.4, and continuity of \tilde{p} ,

$$\begin{aligned} p_{\mathbb{R}_{\varepsilon'}^{d'}}(t, x, y) &= \int_{S_\varepsilon^{d-d'}} \tilde{p}_{(\mathbb{R}^{d'} \times S_\varepsilon^{d-d'}) \setminus K}(t, (x, x_2), (y, y_2)) dy_2 \\ &\lesssim \max_{y_0 \in S_\varepsilon^{d-d'}} \tilde{p}_{(\mathbb{R}^{d'} \times S_\varepsilon^{d-d'}) \setminus K}(t, (x, x_2), (y, y_0)). \end{aligned}$$

Hence we have

$$\begin{aligned} p(t, x, y) &= p_{\mathbb{R}_{\varepsilon'}^{d'}}(t, x, y) + \int_0^t \int_0^{t-s} p(t-s-w, a^*, a^*) \mathbb{P}_x(\sigma_{a^*} \in dw) \mathbb{P}_y(\sigma_{a^*} \in ds) \\ &= p_{\mathbb{R}_{\varepsilon'}^{d'}}(t, x, y) + \tilde{\mathbb{E}}_{\tilde{x}} \tilde{\mathbb{E}}_{\tilde{y}} \int_0^t \int_0^{t-s} p(t-s-w, a^*, a^*) \mathbb{P}_x(\sigma_{a^*} \in dw) \mathbb{P}_y(\sigma_{a^*} \in ds) \\ &\lesssim \tilde{p}_{(\mathbb{R}^{d'} \times S_\varepsilon^{d-d'}) \setminus K}(t, \tilde{x}, \tilde{y}) \\ &\quad + \tilde{\mathbb{E}}_{\tilde{x}} \tilde{\mathbb{E}}_{\tilde{y}} \int_0^t \int_0^{t-s} \tilde{p}(t-s-w, \tilde{X}_s, \tilde{X}_w) \tilde{\mathbb{P}}_{\tilde{x}}(\tilde{\sigma}_K \in dw) \tilde{\mathbb{P}}_{\tilde{y}}(\tilde{\sigma}_K \in ds) \\ &= \tilde{p}(t, \tilde{x}, \tilde{y}), \end{aligned}$$

where we denote $\tilde{x} := (x, x_2)$, $\tilde{y} := (y, y_2)$ for $x, y \in \mathbb{R}_{\varepsilon'}^{d'}$ and $x_2, y_2 \in S_\varepsilon^{d-d'}$ with

$$\max_{y_0 \in S_\varepsilon^{d-d'}} \tilde{p}_{(\mathbb{R}^{d'} \times S_\varepsilon^{d-d'}) \setminus K}(t, (x, x_2), (y, y_0)) = \tilde{p}_{(\mathbb{R}^{d'} \times S_\varepsilon^{d-d'}) \setminus K}(t, (x, x_2), (y, y_2)).$$

In the above inequalities, we used the following estimates in order to treat the effect of $e^{-d(\tilde{x}, \tilde{y})/t}$ appearing in the estimate of $\tilde{p}(t, \tilde{x}, \tilde{y})$ for $t < 1$, $\tilde{x}, \tilde{y} \in K$. For $x, y \in \mathbb{R}_\varepsilon^d$, we have

$$\begin{aligned}
& \int_{\{0 \leq t-s-w \leq 1, s \geq w\}} p(t-s-w, a^*, a^*) \mathbb{P}_y(\sigma_{a^*} \in ds) \mathbb{P}_x(\sigma_{a^*} \in dw) \\
\lesssim & \int_0^1 \int_{t-1}^{t-w} + \int_0^{t/2} \int_{(t-w-1) \vee (t/2-1)}^{t/2-1} + \int_{(t-1)/2}^{t/2} \int_{w \vee (t-w-1)}^{t/2} (t-s-w)^{-1/2} ds \frac{e^{-|y|_\rho^2/t}}{t^{d/2}} \mathbb{P}_x(\sigma_{a^*} \in dw) \\
\leq & 2 \frac{e^{-|y|_\rho^2/t}}{t^{d/2}} \mathbb{P}_x(\sigma_{a^*} \leq t) + \int_{(t-1)/2}^{t/2} \left(\frac{t}{2} - w\right)^{-1/2} \frac{e^{-|y|_\rho^2/t}}{t^{d/2}} \mathbb{P}_x(\sigma_{a^*} \in dw) \\
\lesssim & \frac{e^{-(|x|_\rho + |y|_\rho)^2/t}}{|x|^{d-2} t^{d/2}} + \frac{e^{-(|x|_\rho + |y|_\rho)^2/t}}{t^d} \lesssim \tilde{p}(t, x, y).
\end{aligned}$$

Thus, by the symmetry, we have

$$\int_{\{t-s-w \leq 1\}} p(t-s-w, a^*, a^*) \mathbb{P}_x(\sigma_{a^*} \in dw) \mathbb{P}_y(\sigma_{a^*} \in ds) \lesssim \tilde{p}(t, x, y).$$

The same inequalities hold for the cases of $x \in \mathbb{R}_\varepsilon^d, y \in \mathbb{R}_\varepsilon^{d'}$ and $x, y \in \mathbb{R}_\varepsilon^{d'}$.

By the compactness of K , we can ignore the difference between ρ and d and derive upper estimates similarly as in the proof of Theorem 1.4.

If $\varepsilon > \varepsilon'$, we can prove in the same way as above by exchanging $(\mathbb{R}^{d'} \times S_\varepsilon^{d-d'}) \# \mathbb{R}^d$ and K to $(\mathbb{R}^{d'} \times S_{\varepsilon'}^{d-d'}) \# \mathbb{R}^d$ and $(\overline{B}^{d'}(0; \varepsilon') \times S_{\varepsilon'}^{d-d'}) \cup \overline{B}^d(0; \varepsilon)$, respectively. \square

Remark 5.4. One can prove Proposition 5.3 directly by using the estimates of $p(t, a^*, a^*)$ and $\mathbb{P}_x(\sigma_{a^*} \in ds)$.

Proof of Theorem 1.7. The upper estimates is already proved in Proposition 5.3, so we consider the lower estimates. In this proof, let $T > 3$ be large, and $t \in [T, \infty)$.

Step1 (the estimate of $p(t, x, a^*)$)

(1) For $x \in \mathbb{R}_\varepsilon^d$ with $|x|_\rho \geq 1$, by the Markov property, Theorem 1.3, (5.1), Lemma 2.5, Lemma 2.7 and Lemma 2.8, we have

$$\begin{aligned}
p(t, x, a^*) & \geq \int_0^{t/2} p(t-s, a^*, a^*) \mathbb{P}_x(\sigma_{a^*} \in ds) + \int_{t-1}^t p(t-s, a^*, a^*) \mathbb{P}(\sigma_{a^*} \in ds) \\
& \asymp t^{-d'/2} \mathbb{P}\left(\sigma_{a^*} \leq \frac{t}{2}\right) + \int_{t-1}^t (t-s)^{-1/2} \frac{|x|_\rho}{|x|} \frac{e^{-|x|_\rho^2/t}}{t^{d/2} + t^{3/2} |x|^{(d-3)/2}} ds \\
& \gtrsim \left(\frac{1}{t^{d'/2} |x|^{d-2}} + \frac{1}{t^{d/2}}\right) e^{-|x|_\rho^2/t}.
\end{aligned}$$

For $x \in \mathbb{R}_\varepsilon^d$ with $|x|_\rho < 1$, by the Markov property, (5.1) and Lemma 2.8, we have

$$p(t, x, a^*) \geq \int_0^{t/2} p(t-s, a^*, a^*) \mathbb{P}_x(\sigma_{a^*} \in ds) \gtrsim \frac{1}{t^{d'/2}} e^{-|x|_\rho^2/t}.$$

(2) For $x \in \mathbb{R}_{\varepsilon'}^{d'}$, we can prove in the same way as in the case of $x \in \mathbb{R}_{\varepsilon}^d$. Since the estimate of $p(t, a^*, a^*)$ depends only on d' , we can derive $p(t, x, a^*) \gtrsim t^{-d'/2} e^{-|x|_{\rho}^2/t}$ from (1) by changing d to d' .

Step2 (Theorem 1.7 (i) and (ii))

(1) For $x, y \in \mathbb{R}_{\varepsilon}^d$, by (5.1), (2.1), Step1, Lemma 2.10 and Lemma 2.8, we have

$$\begin{aligned}
p(t, x, y) &= p_{\mathbb{R}_{\varepsilon}^d}(t, x, y) + \bar{p}_{\mathbb{R}_{\varepsilon}^d}(t, x, y) & (5.5) \\
&\geq p_{\mathbb{R}_{\varepsilon}^d}(t, x, y) + \int_0^{t/2} p(t-s, a^*, x) \mathbb{P}_y(\sigma_{a^*} \in ds) \\
&\gtrsim \frac{(1 \wedge |x|_{\rho})(1 \wedge |y|_{\rho})}{t^{d/2}} e^{-|x-y|^2/t} + p(t, a^*, x) \mathbb{P}_y\left(\sigma_{a^*} \leq \frac{t}{2}\right) \\
&\gtrsim \frac{(1 \wedge |x|_{\rho})(1 \wedge |y|_{\rho})}{t^{d/2}} e^{-\rho(x,y)^2/t} + p(t, a^*, x) \frac{e^{-|y|_{\rho}^2/t}}{|y|^{d-2}}.
\end{aligned}$$

(a) If $|x|_{\rho} \vee |y|_{\rho} \leq 1$, by (5.5) and Lemma 2.10, we have

$$p(t, x, y) \gtrsim 0 + 0 + \frac{1}{t^{d/2}} e^{-\rho(x,y)^2/t} \gtrsim \frac{1}{t^{d/2}} e^{-\rho(x,y)^2/t}.$$

(b) If $|x|_{\rho} > 1 \geq |y|_{\rho} > \frac{1}{2}$, by (5.5), we have

$$p(t, x, y) \gtrsim \frac{1}{t^{d/2}} e^{-\rho(x,y)^2/t} + \frac{e^{-(|x|_{\rho}+|y|_{\rho})^2/t}}{t^{d'/2}|x|^{d-2}|y|^{d-2}} + 0.$$

(c) If $|x|_{\rho} > 1$, $\frac{1}{2} \geq |y|_{\rho}$, by (5.5) and Lemma 2.10 (iii), we have

$$\begin{aligned}
p(t, x, y) &\gtrsim \frac{(1 \wedge |x|_{\rho})(1 \wedge |y|_{\rho})}{t^{d/2}} e^{-\rho(x,y)^2/t} + \left(\frac{1}{t^{d'/2}|x|^{d-2}} + \frac{1}{t^{d/2}} \right) \frac{e^{-(|x|_{\rho}+|y|_{\rho})^2/t}}{|y|^{d-2}} \\
&\gtrsim 0 + \left(\frac{1}{t^{d'/2}|x|^{d-2}} + \frac{1}{t^{d/2}} \right) \frac{e^{-\rho(x,y)^2/t}}{|y|^{d-2}} \\
&\asymp \frac{1}{t^{d'/2}|x|^{d-2}|y|^{d-2}} e^{-(|x|_{\rho}+|y|_{\rho})^2/t} + \frac{1}{t^{d/2}} e^{-\rho(x,y)^2/t}.
\end{aligned}$$

By the above estimates (a)-(c) and using the symmetry of $p(t, x, y)$, we obtain the estimates in Theorem 1.7 (ii).

(2) For $x, y \in \mathbb{R}_{\varepsilon'}^{d'}$, we can prove in the same way as in the case of $x, y \in \mathbb{R}_{\varepsilon}^d$. Since the estimate of $p(t, a^*, x)$ depends only on d' , and we can derive

$$p(t, x, y) \gtrsim \frac{e^{-\rho(x,y)^2/t}}{t^{d'/2}}$$

from (1) by changing d to d' .

Step3 (Theorem 1.7(iii))

For $x \in \mathbb{R}_{\varepsilon}^d, y \in \mathbb{R}_{\varepsilon'}^{d'}$, by Step1, Lemma 2.5, Lemma 2.7 and Lemma 2.8, we

obtain

$$\begin{aligned}
p(t, x, y) &\geq \int_0^{t/2} + \int_{t/2}^{t-1} p(t-s, a^*, y) \mathbb{P}_x(\sigma_{a^*} \in ds) \\
&\gtrsim \frac{e^{-|x|_\rho^2/t} \mathbb{P}_x(\sigma_{a^*} \leq \frac{t}{2})}{t^{d'/2}} + \int_{t/2}^{t-1} \frac{e^{-|y|_\rho^2/(t-s)} ds}{(t-s)^{d'/2}} \frac{|x|_\rho}{|x|} \frac{e^{-|x|_\rho^2/t}}{t^{d/2} + t^{3/2}|x|^{(d-3)/2}} \\
&\gtrsim \frac{1}{t^{d'/2}|x|^{d-2}} e^{-\rho(x,y)^2/t} + \int_{t/2}^{t-1} \frac{e^{-|y|_\rho^2/(t-s)} ds}{(t-s)^{d'/2}} \frac{|x|_\rho}{|x|} \frac{e^{-|x|_\rho^2/t}}{t^{d/2}}. \quad (5.6)
\end{aligned}$$

(a) If $|x|_\rho < 1$, by (5.6) and $|y| \geq \epsilon'$, we have

$$p(t, x, y) \gtrsim \frac{1}{t^{d'/2}|x|^{d-2}} e^{-\rho(x,y)^2/t} + 0 \gtrsim \left(\frac{1}{t^{d'/2}|x|^{d-2}} + \frac{1}{t^{d/2}|y|^{d-2}} \right) e^{-\rho(x,y)^2/t}.$$

(b) If $|x|_\rho \geq 1$, $|y|_\rho \leq 1$, by (5.6) and $3 < T \leq t$, we have

$$\begin{aligned}
p(t, x, y) &\gtrsim \frac{1}{t^{d'/2}|x|^{d-2}} e^{-\rho(x,y)^2/t} + \int_{t/2}^{t-1} \frac{e^{-1}}{(t-s)^{d'/2}} ds \frac{e^{-|x|_\rho^2/t}}{t^{d/2}} \\
&\gtrsim \frac{1}{t^{d'/2}|x|^{d-2}} e^{-\rho(x,y)^2/t} + \left(1 - \left(\frac{t}{2} \right)^{1-d'/2} \right) \frac{e^{-|x|_\rho^2/t}}{t^{d/2}} \\
&\gtrsim \left(\frac{1}{t^{d'/2}|x|^{d-2}} + \frac{1}{t^{d/2}|y|^{d'-2}} \right) e^{-\rho(x,y)^2/t}.
\end{aligned}$$

(c) If $|x|_\rho \geq 1$, $1 < |y|_\rho < |y| < \sqrt{t}/2$, by (5.6) and let $\theta := \frac{|y|_\rho^2}{t-s}$, we have

$$\begin{aligned}
p(t, x, y) &\gtrsim \frac{1}{t^{d'/2}|x|^{d-2}} e^{-\rho(x,y)^2/t} + \int_{t/2}^{t-1} \frac{e^{-|y|_\rho^2/(t-s)} ds}{(t-s)^{d'/2}} \frac{|x|_\rho}{|x|} \frac{e^{-|x|_\rho^2/t}}{t^{d/2}} \\
&\asymp \frac{1}{t^{d'/2}|x|^{d-2}} e^{-\rho(x,y)^2/t} + \int_{2|y|_\rho^2/t}^{|y|_\rho^2} e^{-\theta} \theta^{d'/2-2} d\theta \frac{e^{-|x|_\rho^2/t}}{t^{d/2}|y|^{d'-2}} \\
&\gtrsim \frac{1}{t^{d'/2}|x|^{d-2}} e^{-\rho(x,y)^2/t} + \int_{1/2}^1 e^{-\theta} \theta^{d'/2-2} d\theta \frac{e^{-|x|_\rho^2/t}}{t^{d/2}|y|^{d'-2}} \\
&\asymp \left(\frac{1}{t^{d'/2}|x|^{d-2}} + \frac{1}{t^{d/2}|y|^{d'-2}} \right) e^{-\rho(x,y)^2/t}.
\end{aligned}$$

(d) If $|x|_\rho \geq 1$, $\sqrt{t}/2 \leq |y|$, by (5.6) and $2t/3 < t-1$, we have

$$\begin{aligned}
p(t, x, y) &\gtrsim \frac{1}{t^{d'/2}|x|^{d-2}} e^{-\rho(x,y)^2/t} + \int_{t/2}^{2t/3} \frac{e^{-|y|_\rho^2/(t-s)} ds}{(t-s)^{d'/2}} \frac{1}{t^{d/2}} e^{-|x|_\rho^2/t} \\
&\asymp \frac{1}{t^{d'/2}|x|^{d-2}} e^{-\rho(x,y)^2/t} + \int_{t/2}^{2t/3} t^{-d'/2} ds \frac{1}{t^{d/2}} e^{-\rho(x,y)^2/t} \\
&\asymp \frac{1}{t^{d'/2}|x|^{d-2}} e^{-\rho(x,y)^2/t} + \frac{1}{t^{(d+d'-2)/2}} e^{-\rho(x,y)^2/t} \\
&\gtrsim \left(\frac{1}{t^{d'/2}|x|^{d-2}} + \frac{1}{t^{d/2}|y|^{d'-2}} \right) e^{-\rho(x,y)^2/t}.
\end{aligned}$$

By (a)-(d), we obtain the assertion of Theorem 1.7 (iii). \square

6 Large time estimate($d' = 2$)

In this section, we will prove Theorem 1.5 and Theorem 1.6. Let $d' = 2$, $d \geq 2$ and without loss of generality, we assume $\varepsilon, \varepsilon' < 1$.

For a same reason as in the case of $d' = 3$, we consider the estimate of $p(t, a^*, a^*)$. When $d = d' = 2$, this is easy. When $d \geq 3, d' = 2$, we will obtain the estimate by using Doob's h -transform and the relative Faber-Krahn inequality.

Proposition 6.1. *Let $d \geq d' = 2$. Then, for $t > 0$, it holds that*

$$t^{-1/2} \wedge t^{-d/2} \lesssim p(t, a^*, a^*) \lesssim t^{-1/2} \wedge t^{-1}. \quad (6.1)$$

Proof. The upper estimate follows from Proposition 2.2 and Theorem 1.3. By the Markov property and the Cauchy-Schwarz inequality, for large $M > 0$, we have

$$\begin{aligned} p(t, a^*, a^*) &= \int p\left(\frac{t}{2}, a^*, x\right)^2 m_p(dx) \geq \int_{\{|x| \leq M\sqrt{t}\}} p\left(\frac{t}{2}, a^*, x\right)^2 m_p(dx) \\ &\geq m_p\left(\{|x| \leq M\sqrt{t}\}\right)^{-1} \times \left(\int_{\{|x| \leq M\sqrt{t}\}} p\left(\frac{t}{2}, a^*, x\right) m_p(dx)\right)^2 \\ &\gtrsim \left(\frac{1}{t} \wedge \frac{1}{t^{d/2}}\right) \mathbb{P}_{a^*}\left(|X_t| \leq M\sqrt{t}\right)^2. \end{aligned} \quad (6.2)$$

By the proof of [5, Theorem 5.10], there is large $M > 0$ such that for all $t > 0$, $\mathbb{P}_{a^*}\left(|X_t| \leq M\sqrt{t}\right) \geq \frac{1}{2}$. Thus the right hand side of (6.2) is equal to $t^{-1} \wedge t^{-d/2}$ up to a constant multiple. Therefore, by Theorem 1.3 again, we have

$$p(t, a^*, a^*) \gtrsim \frac{1}{\sqrt{t}} \wedge \left(\frac{1}{t} \wedge \frac{1}{t^{d/2}}\right) = \frac{1}{\sqrt{t}} \wedge \frac{1}{t^{d/2}}. \quad (6.3)$$

\square

Corollary 6.2. *Let $d = d' = 2$. Then, for $t > 0$, $p(t, a^*, a^*) \asymp t^{-1/2} \wedge t^{-1}$.*

Proof of Theorem 1.5. Let $d = d' = 2$. We may assume $\varepsilon \geq \varepsilon'$ without loss of generality. $\tilde{p}(t, x, y)$ denotes the heat kernel for Brownian motion \tilde{X} on $\mathbb{R}^2 \# \mathbb{R}^2$. Then, by [11, Example 2.12], $\tilde{p}(t, x, y)$ has the estimates of this theorem as a sharp estimate. In particular, it holds that

$$\tilde{p}(t, x, y) \asymp t^{-1} e^{-d(x,y)/t} \text{ for } t > 0 \text{ and } x, y \in K := \overline{B}^2(0; \varepsilon) \cup \overline{B}^2(0; \varepsilon'),$$

where d is a geodesic distance on $\mathbb{R}^2 \# \mathbb{R}^2$. By Corollary 6.2, we have $p(t, a^*, a^*) \lesssim t^{-1}$. Furthermore, it holds that $\mathbb{P}_x(\sigma_{a^*} \in ds) = \tilde{\mathbb{P}}_x(\tilde{\sigma}_K \in ds)$ and heat kernels of part processes of X and \tilde{X} on \mathbb{R}_ε^2 are equivalent, where $\tilde{\mathbb{P}}$ and $\tilde{\sigma}_K$ are those

for \tilde{X} . Thus, by the same way as the proof of Proposition 5.3, it holds that $p(t, x, y) \lesssim \tilde{p}(t, x, y)$ for $x, y \in \mathbb{R}_\varepsilon^2 \cup \mathbb{R}_\varepsilon^2 \cup \{a^*\}$, so the upper estimates are proved.

Next, we prove the lower estimates. Let $T > 0$ be large and $t \in [T, \infty)$.

(1) (a) For $x \in \mathbb{R}_\varepsilon^2$ with $|x| \leq 1$, by Corollary 6.2, we have

$$p(t, x, a^*) \geq \int_0^{t/2} p(t-s, a^*, a^*) \mathbb{P}_x(\sigma_{a^*} \in ds) \gtrsim \frac{1}{t} \mathbb{P}_x(\sigma_{a^*} \leq 1) \asymp \frac{e^{-|x|_\rho^2/t}}{t}.$$

(b) For $x \in \mathbb{R}_\varepsilon^2$ with $1 < |x| \leq \sqrt{t}/2$, by Lemma 2.6, Lemma 2.9, and Corollary 6.2, we have

$$\begin{aligned} p(t, x, a^*) &\geq \int_0^{t/2} p(t-s, a^*, a^*) \mathbb{P}_x(\sigma_{a^*} \in ds) + \int_{t/2}^{t-1} p(t-s, a^*, a^*) \mathbb{P}_x(\sigma_{a^*} \in ds) \\ &\asymp \frac{1}{t} \left(1 - \frac{\log|x|}{\log\sqrt{t/2}} \right) + \int_{t/2}^{t-1} \frac{1}{t-s} ds \mathbb{P}_x(\sigma_{a^*} \in dt) \\ &\gtrsim \frac{1}{t} \left(1 - \frac{\log|x|}{\log\sqrt{t/2}} \right) e^{-|x|_\rho^2/t} + \frac{\log|x|}{t \log t} e^{-|x|_\rho^2/t} \\ &\gtrsim \frac{1}{t} e^{-|x|_\rho^2/t}. \end{aligned}$$

(c) For $x \in \mathbb{R}_\varepsilon^2$ with $\sqrt{t}/2 < |x|$, by Lemma 2.6, and Corollary 6.2, we have

$$\begin{aligned} p(t, x, a^*) &\geq \int_{t/2}^{t-1} p(t-s, a^*, a^*) \mathbb{P}_x(\sigma_{a^*} \in ds) \asymp \int_{t/2}^{t-1} (t-s)^{-1} ds \mathbb{P}_x(\sigma_{a^*} \in dt) \\ &\gtrsim \log t \frac{1 + \log|x|}{(1 + \log(1+t))(1 + \log(|x|^2 + |x|))} \frac{e^{-|x|_\rho^2/t}}{t} \asymp \frac{1}{t} e^{-|x|_\rho^2/t}. \end{aligned}$$

(2) For $x, y \in \mathbb{R}_\varepsilon^2$, by (2.1), Lemma 2.10 and Proposition 6.2, it holds that

$$\begin{aligned} p(t, x, y) &\geq p_{\mathbb{R}_\varepsilon^2}(t, x, y) + \int_0^{t/2} p(t-s, a^*, y) \mathbb{P}_x(\sigma_{a^*} \in ds) \\ &\asymp \frac{(1 \wedge |x|_\rho)(1 \wedge |y|_\rho)}{t} e^{-\rho(x, y)^2/t} + \frac{1}{t} \mathbb{P}_x(\sigma_{a^*} \leq t/2). \end{aligned} \quad (6.4)$$

(a) If $\frac{1}{2} \leq |x|_\rho$, $1 \leq |y|_\rho$, by (6.4), we have

$$p(t, x, y) \gtrsim \frac{1}{t} e^{-\rho(x, y)^2/t} + 0 = \frac{1}{t} e^{-\rho(x, y)^2/t}.$$

(b) If $|x|_\rho < \frac{1}{2}$, $1 \leq |y|_\rho$, by (6.4) and Lemma 2.10 (iii), we have

$$p(t, x, y) \gtrsim 0 + \frac{1}{t} \mathbb{P}_x(\sigma_{a^*} \leq 1) \asymp \frac{1}{t} e^{-(|x|_\rho + |y|_\rho)^2/t} \asymp \frac{1}{t} e^{-\rho(x, y)^2/t}.$$

(c) If $|x|_\rho \leq 1$, $|y|_\rho \leq 1$, by (6.4) and Lemma 2.10 (ii), we have

$$p(t, x, y) \gtrsim 0 + \frac{1}{t} \mathbb{P}_x(\sigma_{a^*} \leq 1) \asymp \frac{1}{t} e^{-(|x|_\rho + |y|_\rho)^2/t} \asymp \frac{1}{t} e^{-\rho(x, y)^2/t}.$$

(3) For $x \in \mathbb{R}_\varepsilon^2, y \in \mathbb{R}_{\varepsilon'}^2$, it holds that

$$p(t, x, y) \geq \int_0^{t/2} p(t-s, a^*, x) \mathbb{P}_y(\sigma_{a^*} \in ds). \quad (6.5)$$

(a) If $|x| \vee |y| \leq \sqrt{t}/2$, by (6.5) and Lemma 2.9, we have

$$p(t, x, y) \gtrsim \frac{1}{t} \left(1 - \frac{\log |y|}{\log \sqrt{t}/2} \right) \asymp \frac{\frac{1}{2} \log \frac{t}{2} - \log |y|}{t \log t}.$$

By the symmetry, we have

$$\begin{aligned} p(t, x, y) &\gtrsim \frac{\log \frac{t}{2} - \log |x| - \log |y|}{t \log t} \\ &\gtrsim \frac{U_t(x)}{t} \left(U_t(y) + \frac{\log |y|}{\log(t|y|)} \right) + \frac{U_t(y)}{t} \left(U_t(x) + \frac{\log |x|}{\log(t|x|)} \right) \\ &\asymp \frac{e^{-\rho(x,y)^2/t}}{t} \left(U_t(x)U_t(y) + \frac{U_t(x) \log |y|}{\log(1+t|y|)} + \frac{U_t(y) \log |x|}{\log(1+t|x|)} \right). \end{aligned}$$

(b) If $|x| \wedge |y| \geq \sqrt{t}/2$, by (6.5) and Lemma 2.9, we have

$$p(t, x, y) \gtrsim \frac{1}{t} \left(1 - \frac{\log |y|}{\log \sqrt{t}/2} \right) \asymp \frac{1}{t \log |y|} e^{-\rho(x,y)^2/t}.$$

By the symmetry, we have

$$\begin{aligned} p(t, x, y) &\gtrsim \frac{1}{t} \left(\frac{1}{\log |x|} + \frac{1}{\log |y|} \right) e^{-\rho(x,y)^2/t} \\ &\asymp \frac{e^{-\rho(x,y)^2/t}}{t} \left(U_t(x)U_t(y) + \frac{U_t(x) \log |y|}{\log(1+t|y|)} + \frac{U_t(y) \log |x|}{\log(1+t|x|)} \right). \end{aligned}$$

(c) If $|y| \leq \sqrt{t}/2 \leq |x|$, by (6.5) and Lemma 2.9, we have

$$p(t, x, y) \gtrsim \frac{1}{t} \left(1 - \frac{\log |y|}{\log \sqrt{t}/2} \right) e^{-\rho(x,y)^2/t}.$$

Moreover, it holds that

$$p(t, x, y) \geq \int_{t/2}^{2t/3} p(t-s, a^*, y) \mathbb{P}_x(\sigma_{a^*} \in ds) \gtrsim \frac{1}{t \log t} e^{-\rho(x,y)^2/t}.$$

Thus we obtain

$$\begin{aligned} p(t, x, y) &\gtrsim \frac{1 + \log \left(\frac{1}{|y|} \sqrt{\frac{t}{2}} \right)}{t \log t} e^{-\rho(x,y)^2/t} \\ &\gtrsim \frac{e^{-\rho(x,y)^2/t}}{t} \left(U_t(x)U_t(y) + \frac{U_t(x) \log |y|}{\log(1+t|y|)} + \frac{U_t(y) \log |x|}{\log(1+t|x|)} \right). \end{aligned}$$

By the symmetry, the all cases have been proved. \square

Next, we prove Theorem 1.6.

Proposition 6.3. *Let $d \geq 3$, $d' = 2$. Then, for $t > 0$, we have*

$$p(t, a^*, a^*) \gtrsim \frac{1}{\sqrt{t}} \wedge \frac{1}{(t+1)(\log(t+1))^2}.$$

Proof. For $t > 3$ and $x \in \mathbb{R}_{\varepsilon'}^2$ with $\sqrt{t} \leq |x| \leq 2\sqrt{t}$, by Theorem 1.3, Lemma 2.6 and Proposition 6.1, we have

$$\begin{aligned} p(t, a^*, x) &\geq \int_{t-1}^t p(t-s, a^*, a^*) \mathbb{P}_x(\sigma_{a^*} \in ds) \\ &\gtrsim \int_{t-1}^t (t-s)^{-1/2} ds \frac{e^{-|x|_\rho^2/t}}{t} \frac{1 + \log |x|}{(1 + \log(1+t/|x|))(1 + \log(t+|x|))} \\ &\gtrsim \frac{1}{t} \frac{1 + \log t}{(1 + \log(1 + \sqrt{t}))(1 + \log(t + \sqrt{2t}))} \\ &\gtrsim \frac{1}{(t+1)\log(t+1)}. \end{aligned}$$

Thus, by the Markov property, we have

$$\begin{aligned} p(2t, a^*, a^*) &\geq \int_{\{x \in \mathbb{R}_{\varepsilon'}^2; \sqrt{t} \leq |x| \leq 2\sqrt{t}\}} p(t, a^*, x)^2 m_p(dx) \\ &\gtrsim \int_{\{x \in \mathbb{R}_{\varepsilon'}^2; \sqrt{t} \leq |x| \leq 2\sqrt{t}\}} \frac{1}{(t+1)^2 (\log(t+1))^2} m_p(dx) \\ &\asymp \int_{\sqrt{t}}^{\sqrt{2t}} \frac{r}{(t+1)^2 (\log(t+1))^2} dr \asymp \frac{1}{(t+1)(\log(t+1))^2}, \end{aligned}$$

where we used polar coordinates. Combining this with Theorem 1.3, the desired estimate is proved. \square

Next, we prove that the estimate in Proposition 6.3 is sharp by using Doob's h -transform. First, we construct a harmonic function, which is comparable to 1 on $\mathbb{R}_{\varepsilon}^d$ and $1 + \log |x|_\rho$ on $\mathbb{R}_{\varepsilon'}^2$. In the following proposition, we use some ideas from [22, Theorem 2.6].

Proposition 6.4. *Let $d \geq 3$. Then, there exists a positive harmonic function h on $\mathbb{R}_{\varepsilon}^d \cup \mathbb{R}_{\varepsilon'}^2 \cup \{a^*\}$ such that $h \asymp 1$ on $\mathbb{R}_{\varepsilon}^d$ and $h(x) \asymp 1 + \log |x|_\rho$ for $x \in \mathbb{R}_{\varepsilon'}^2$, with $|x|_\rho$ large enough.*

Proof. Let $R > 0$ be large, $K_1 := \mathbb{R}_{\varepsilon}^d \cap \{|x|_\rho \leq R\}$ and $K_2 := \mathbb{R}_{\varepsilon'}^2 \cap \{|x|_\rho \leq R\}$. By [14, Lemma 6.1], there exists a positive harmonic function h_1 on $\mathbb{R}_{\varepsilon}^d \setminus K_1$ such that $h_1 = 0$ on $K_1 \cup \{a^*\}$ and, for large $|x|_\rho$, $h_1 \asymp 1$. By [14, Lemma 6.1] again, there exists a positive harmonic function h_2 on $\mathbb{R}_{\varepsilon'}^2 \setminus K_2$ such that $h_2 = 0$

on $K_2 \cup \{a^*\}$ and $h_2(x) \asymp \log|x|_\rho$ for large $|x|_\rho$.
Let $K := \{|x|_\rho \leq R\}$ and

$$f(x) := \begin{cases} h_1(x) & : x \in \mathbb{R}_\varepsilon^d \cup \{a^*\}, \\ h_2(x) & : x \in \mathbb{R}_{\varepsilon'}^{d'} \cup \{a^*\}. \end{cases}$$

We take $\eta \in C^\infty(\mathbb{R}_\varepsilon^d \cup \mathbb{R}_{\varepsilon'}^2 \cup \{a^*\})$ satisfying $\eta = 1$ on $\{|x|_\rho > 2R\}$ and $\eta = 0$ on K . Let

$$h(x) := (\eta f)(x) + \int_{\mathbb{R}_\varepsilon^d \cup \mathbb{R}_{\varepsilon'}^2 \cup \{a^*\}} G(x, y) \Delta(\eta f)(y) dy,$$

where $G(x, y) := \int_0^\infty p(t, x, y) dt$. Since $\mathbb{R}_\varepsilon^d \cup \mathbb{R}_{\varepsilon'}^2 \cup \{a^*\}$ is non-parabolic, we have $G(x, y) < \infty$. It holds that $\Delta(\eta f) \in C_c^\infty(\mathbb{R}_\varepsilon^d \cup \mathbb{R}_{\varepsilon'}^2 \cup \{a^*\})$ since $(\eta f)(x) = f(x)$ and $\Delta(\eta f)(x) = 0$ for x with $|x|_\rho > 2R$. Hence, for all x , we have $\Delta h(x) = 0$, so h is a harmonic function.

For x with $|x|_\rho > 4R$,

$$\begin{aligned} |f(x) - h(x)| &= \left| \int_{\{|y|_\rho \leq 2R\}} G(x, y) \Delta(\eta f)(y) dy \right| \\ &\leq \sup_{\{|y|_\rho \leq 2R\}} G(x, y) \times |\{|y|_\rho \leq 2R\}| \times \sup \Delta(\eta f) \\ &\leq C \sup_{\{|y|_\rho \leq 2R\}} G(x, y). \end{aligned}$$

By using the elliptic Harnack inequality on $\mathbb{R}_\varepsilon^d \cup \{a^*\}$ and $\mathbb{R}_{\varepsilon'}^{d'} \cup \{a^*\}$ (see for example [10, Theorem 13.10]), it holds that $|f(x) - h(x)| \leq CG(x, a^*)$ for x with $|x|_\rho > 4R$.

Let fix $x_1 \in \mathbb{R}_\varepsilon^d$ and $x_2 \in \mathbb{R}_{\varepsilon'}^2$, with $|x_1|_\rho = |x_2|_\rho = 4R$. For $x \in \mathbb{R}_\varepsilon^d$ with $|x|_\rho > 4R$, by Lemma 2.5, we have $\mathbb{P}_x(\sigma_{a^*} \in ds) \lesssim \mathbb{P}_{x_1}(\sigma_{a^*} \in ds)$. Furthermore, for $x \in \mathbb{R}_{\varepsilon'}^2$, with $|x|_\rho > 4R$, by Lemma 2.6 and for large R , we have $\mathbb{P}_x(\sigma_{a^*} \in ds) \lesssim \mathbb{P}_{x_2}(\sigma_{a^*} \in ds)$. Thus, we have

$$p(t, x, a^*) = \int_0^t p(t-s, a^*, a^*) \mathbb{P}_x(\sigma_{a^*} \in ds) \lesssim p(t, x_1, a^*) + p(t, x_2, a^*)$$

and $G(x, a^*) \lesssim G(x_1, a^*) + G(x_2, a^*) < \infty$. Then, $|f - h|$ is bounded on $\{|x|_\rho > 4R\}$ and, by the continuity of G , $|f - h|$ is bounded. \square

Let h be a positive harmonic function constructed as above. Define

$$H : L^2(\mathbb{R}_\varepsilon^d \cup \mathbb{R}_{\varepsilon'}^2 \cup \{a^*\}; h^2 m_p) \ni f \mapsto fh \in L^2(\mathbb{R}_\varepsilon^d \cup \mathbb{R}_{\varepsilon'}^2 \cup \{a^*\}; m_p),$$

$$\mathcal{E}^h(f, f) := \mathcal{E}(fh, fh) = \int |\nabla(fh)|^2 dm_p \text{ for } f \in \mathcal{F}^h := H^{-1}\mathcal{F}.$$

Then, $H^{-1} \circ P_t \circ H$ admits a transition density $p^h(t, x, y)$ with respect to $dm_p^h := h^2 dm_p$ and $p^h(t, x, y)h(x)h(y) = p(t, x, y)$ holds ([16, Lemma 5.6]). Since h is harmonic, by [16, Proposition 5.7], we have

$$\mathcal{E}^h(f, f) = \int |\nabla f|^2 dm_p^h \quad \text{for } f \in \mathcal{F}^h.$$

The next lemma follows from [12, Lemma 4.8].

Lemma 6.5. *Let $q(t, x, y)$ be the transition density function with respect to m_p on $\mathbb{R}_\varepsilon^2 \cup \{a^*\}$ and $q^h(t, x, y)$ be the h -transform of $q(t, x, y)$. Then it holds that $q^h(t, x, x) \lesssim m_p(\{y \in \mathbb{R}_\varepsilon^2 \cup \{a^*\} | \rho(x, y) \leq \sqrt{t}\})^{-1}$ for $t > 0, x \in \mathbb{R}_\varepsilon^2 \cup \{a^*\}$.*

In order to get the sharp estimate of $p(t, a^*, a^*)$, we imitate the technique of the relative Faber-Krahn inequality appearing [15].

Lemma 6.6. *For some constant $c > 0, \alpha_2 > 0$ and any ball $B := B(x_0; R)$, let*

$$\Lambda_1(B, v) := \frac{c}{R^2} \left(\frac{m_p^h(B)}{v} \right)^{2/d}, \quad \Lambda_2(B, v) := \frac{c}{R^2} \left(\frac{m_p^h(B)}{v} \right)^{\alpha_2}.$$

Then, for any ball $B_1 \subset \mathbb{R}_\varepsilon^d \cup \{a^*\}$ and a non-empty open subset $\Omega \subset B_1$, we have

$$\inf_{f \in C_c^\infty(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla f|^2 dm_p^h}{\int_\Omega |f|^2 dm_p^h} \geq \Lambda_1(B_1, m_p^h(\Omega))$$

and, for any ball $B_2 \subset \mathbb{R}_\varepsilon^2 \cup \{a^*\}$ and non-empty open subset $\Omega \subset B_2$, we have

$$\inf_{f \in C_c^\infty(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla f|^2 dm_p^h}{\int_\Omega |f|^2 dm_p^h} \geq \Lambda_2(B_2, m_p^h(\Omega)).$$

These inequalities are called the relative Faber-Krahn inequality.

Proof. From [15, Proposition 4.2], for a complete weighted manifold, the relative Faber-Krahn inequality holds if the diagonal upper estimate of heat kernel holds. $h \asymp 1$ and $q(t, x, x) \lesssim t^{-d/2}$ on $\mathbb{R}_\varepsilon^d \cup \{a^*\}$, where q is a heat kernel with respect to m_p^h on $\mathbb{R}_\varepsilon^d \cup \{a^*\}$, so the first inequality holds. The second inequality follows from Lemma 6.5. \square

Theorem 6.7. *Let $\alpha := \alpha_2 \wedge \frac{2}{d}$. For $B := B(x_0; R) \subset \mathbb{R}_\varepsilon^d \cup \mathbb{R}_{\varepsilon'}^2 \cup \{a^*\}$ and any open subset $\Omega \subset B$, it holds that*

$$\inf_{f \in C_c^\infty(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla f|^2 dm_p^h}{\int_\Omega |f|^2 dm_p^h} \geq \Lambda(B, m_p^h(\Omega)), \quad \text{where } \Lambda(B, v) := \frac{c}{R^2} \left(\frac{F(B)}{v} \right)^\alpha,$$

$$F(B) := \begin{cases} m_p^h(B) & : B \subset \mathbb{R}_{\varepsilon'}^i \text{ for } i = 2, d, \\ m_p^h(\{x \in \mathbb{R}_{\varepsilon'}^2 \cup \{a^*\} | \rho(x, y_2) \leq R\}) & : a^* \in B \text{ and large } R. \end{cases}$$

Here, $\varepsilon_2 := \varepsilon', \varepsilon_d := \varepsilon$ and $y_2 \in \mathbb{R}_{\varepsilon'}^2 \cup \{a^*\}$ with $|x_0|_\rho = |y_2|_\rho$.

Proof. When $B \subset \mathbb{R}_{\varepsilon_i}^i$ for $i = 2, d$, the estimate holds by Lemma 6.6. When $a^* \in B := B(x_0; R) \subset \mathbb{R}_{\varepsilon}^d \cup \mathbb{R}_{\varepsilon'}^2 \cup \{a^*\}$ for large $R > 0$, for any open subset $\Omega \subset B$ and $f \in C_c^\infty(\Omega) \setminus \{0\}$, it holds that $f|_{\mathbb{R}_{\varepsilon}^d \cup \{a^*\}} \in C_c^\infty(\Omega \cap (\mathbb{R}_{\varepsilon}^d \cup \{a^*\})) \setminus \{0\}$ and $f|_{\mathbb{R}_{\varepsilon'}^2 \cup \{a^*\}} \in C_c^\infty(\Omega \cap (\mathbb{R}_{\varepsilon'}^2 \cup \{a^*\})) \setminus \{0\}$. For $i = 2, d$, fix $y_i \in \mathbb{R}_{\varepsilon_i}^i \cup \{a^*\}$ satisfying $|y_i|_\rho = |x_0|_\rho$ and $B^i := \{x \in \mathbb{R}_{\varepsilon_i}^i \cup \{a^*\}; \rho(x, y_i) \leq 3R + 2\varepsilon + 2\varepsilon'\}$.
(1) For $x_0 \in \mathbb{R}_{\varepsilon'}^2$, we have (see Figure 6)

$$\begin{aligned} B \cap (\mathbb{R}_{\varepsilon'}^2 \cup \{a^*\}) &\subset B(y_2; |x_0|_\rho + 2\varepsilon' + |x_0|_\rho + R) \cap (\mathbb{R}_{\varepsilon'}^2 \cup \{a^*\}) \subset B^2, \\ B \cap (\mathbb{R}_{\varepsilon}^d \cup \{a^*\}) &\subset B(y_d; 2\varepsilon + 2R) \cap (\mathbb{R}_{\varepsilon}^d \cup \{a^*\}) \subset B^d \text{ if } R - |x_0|_\rho \leq |y_d|_\rho, \\ B \cap (\mathbb{R}_{\varepsilon}^d \cup \{a^*\}) &\subset B(y_d; 2R) \cap (\mathbb{R}_{\varepsilon}^d \cup \{a^*\}) \subset B^d \text{ if } R - |x_0|_\rho > |y_d|_\rho. \end{aligned}$$

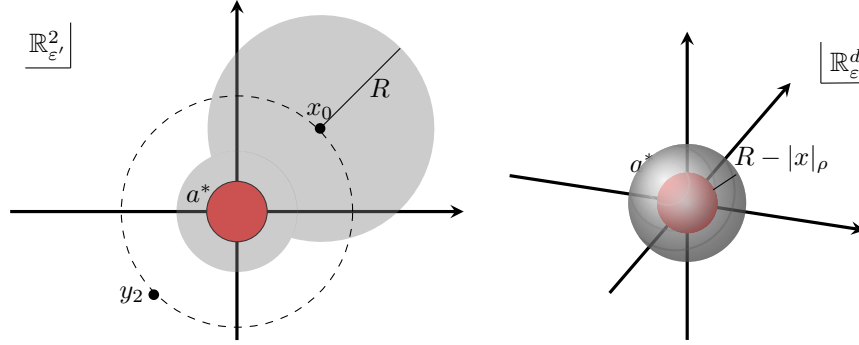


Figure 6: locations of y_2 and $B = B(x_0; R)$

(2) For $x_0 \in \mathbb{R}_{\varepsilon}^d$, by the same way as (1), it holds that $B \cap (\mathbb{R}_{\varepsilon}^d \cup \{a^*\}) \subset B^d$ and $B \cap (\mathbb{R}_{\varepsilon'}^2 \cup \{a^*\}) \subset B^2$. Hence, for $i = 2$ or d satisfying $\int_{\Omega \cap \mathbb{R}_{\varepsilon_i}^i} |f|^2 dm_p^h \geq \frac{1}{2} \int_{\Omega} |f|^2 dm_p^h$, by Lemma 6.6 and $m_p^h(B^i) \geq m_p^h(\Omega \cap \mathbb{R}_{\varepsilon_i}^i)$, we have

$$\begin{aligned} \int_{\Omega} |\nabla f|^2 dm_p^h &\geq \int_{\Omega \cap \mathbb{R}_{\varepsilon_i}^i} |\nabla f|^2 dm_p^h \\ &\geq \Lambda_i(B^i, m_p^h(\Omega \cap \mathbb{R}_{\varepsilon_i}^i)) \int_{\Omega \cap \mathbb{R}_{\varepsilon_i}^i} |f|^2 dm_p^h \\ &\geq \frac{c}{(3R + 2\varepsilon + 2\varepsilon')^2} \left(\frac{m_p^h(B^i)}{m_p^h(\Omega \cap \mathbb{R}_{\varepsilon_i}^i)} \right)^\alpha \frac{1}{2} \int_{\Omega} |f|^2 dm_p^h \\ &\gtrsim \frac{c}{R^2} \left(\frac{m_p^h(B^i)}{m_p^h(\Omega)} \right)^\alpha \int_{\Omega} |f|^2 dm_p^h. \end{aligned} \tag{6.6}$$

Hence, the proof is finished if $i = 2$. If $i = d$, we have

$$\begin{aligned}
m_p^h(B^d) &\geq m_p(B(y_d; 3R) \cap \mathbb{R}_\varepsilon^d) \geq m_p(B(y'_d; R) \cap \mathbb{R}_\varepsilon^d) \asymp R^d \\
&\geq (1 + \log(5R))^2 (8R)^2 \\
&\geq \int_{B(y_2; 4R) \cap \mathbb{R}_{\varepsilon'}^2} (1 + \log(|y_2|_\rho + 3R + 2\varepsilon + 2\varepsilon'))^2 \\
&\geq m_p^h(B^2) \geq m_p^h((B(y_2; R) \cap \mathbb{R}_{\varepsilon'}^2)), \tag{6.7}
\end{aligned}$$

where y'_d is the point with $|y'_d|_\rho = 2R$ on the line joining a^* and y_d . Therefore, by (6.6) and (6.7), the desired inequality holds. \square

Proposition 6.8. *Let $d \geq 3$ and $d' = 2$. Then, it holds that for $t > 0$,*

$$p(t, a^*, a^*) \asymp \frac{1}{\sqrt{t}} \wedge \frac{1}{(t+1)(\log(t+1))^2}.$$

Proof. The lower estimate is given in Proposition 6.3, so we prove the upper estimate.

By the same proof of [9, Theorem 5.2], for large $T > 0$, $t \in [T, \infty)$ and $x, y \in \mathbb{R}_\varepsilon^d \cup \mathbb{R}_{\varepsilon'}^2 \cup \{a^*\}$, it holds that

$$p^h(t, x, y) \lesssim (t \wedge R^2)^{-1/\alpha} \left(\frac{F(B(x; R))^\alpha}{R^2} \frac{F(B(y; R))^\alpha}{R^2} \right)^{-1/2\alpha}. \tag{6.8}$$

Indeed, in [9], Grigor'yan proved (6.8) for $t > 0$ on a smooth connected non-compact complete Riemannian manifold. In the proof, it is used that $|\nabla \hat{\rho}| \leq 1$, where $\hat{\rho}$ is a Riemannian distance. In our setting, we consider the space attached by two manifolds on which $|\nabla \rho| \leq 1$ still holds. Hence (6.8) holds by the proof of [9, Theorem 5.2].

We take $R := \sqrt{t}$ and large t , by Theorem 6.7, we have

$$\begin{aligned}
p^h(t, a^*, a^*) &\lesssim t^{-1/\alpha} \left(\frac{m_p^h(\{x \in \mathbb{R}_{\varepsilon'}^2 \cup \{a^*\}; |x|_\rho \leq \sqrt{t}\}^\alpha)}{t} \right)^{-1/\alpha} \\
&= m_p^h(\{x \in \mathbb{R}_{\varepsilon'}^2 \cup \{a^*\}; |x|_\rho \leq \sqrt{t}\})^{-1}. \tag{6.9}
\end{aligned}$$

Let $\tilde{B} := B((3\sqrt{t}/2, 0); \sqrt{t}/2) \cap \mathbb{R}_{\varepsilon'}^2$, then we obtain

$$\begin{aligned}
m_p^h(\{x \in \mathbb{R}_{\varepsilon'}^2 \cup \{a^*\}; |x|_\rho \leq \sqrt{t}\}) &\geq m_p^h(\tilde{B}) \tag{6.10} \\
&\asymp \int_{\{|x - (3\sqrt{t}/2, 0)| \leq \sqrt{t}/2\}} (1 + \log|x|_\rho)^2 dx \\
&\geq (1 + \log \sqrt{t}/2)^2 |\tilde{B}| \gtrsim (t+1)(\log(t+1))^2.
\end{aligned}$$

By (6.9), (6.10), $p^h(t, a^*, a^*) = p(t, a^*, a^*)h(a^*)^2$ and Theorem 1.3, the upper estimate holds. \square

Proof of Theorem 1.6. By comparing with the heat kernel on $\mathbb{R}^d \# (\mathbb{R}^2 \times S_\varepsilon^{d-2})$ ([14]), the upper estimates can be proved in the same way as the proof of Proposition 5.3. We prove the lower estimates. Let $T > 0$ be large and $t \in [T, \infty)$.

Step1 (the estimate of $p(t, x, a^*)$)

(1) For $x \in \mathbb{R}_\varepsilon^d \cup \{a^*\}$, by Proposition 6.8, Theorem 1.3, Lemma 2.8, Lemma 2.5, and Lemma 2.7, we have

$$\begin{aligned} p(t, x, a^*) &\geq \int_0^{t/2} + \int_{t-2}^t p(t-s, a^*, a^*) \mathbb{P}_x(\sigma_{a^*} \in ds) \\ &\asymp \frac{1}{t(\log t)^2} \frac{1}{|x|^{d-2}} e^{-|x|_\rho^2/t} + \frac{1}{t^{d/2}} e^{-|x|_\rho^2/t}. \end{aligned}$$

(2) For $y \in \mathbb{R}_\varepsilon^2 \cup \{a^*\}$ with $|y| < \sqrt{t}/2$, by Proposition 6.8, Theorem 1.3, Lemma 2.9 and Lemma 2.6, we have

$$\begin{aligned} p(t, y, a^*) &\geq \int_0^{t/2} + \int_{t-2}^t p(t-s, a^*, a^*) \mathbb{P}_y(\sigma_{a^*} \in ds) \\ &\asymp \frac{1}{t(\log t)^2} \left(1 - \frac{\log |y|}{\log \sqrt{t}/2}\right) + \frac{|y|}{t(\log t)^2} \asymp \frac{1}{t(\log t)^2} e^{-|y|_\rho^2/t}. \end{aligned}$$

(3) For $y \in \mathbb{R}_\varepsilon^2 \cup \{a^*\}$ with $|y| \geq \sqrt{t}/2$, by Theorem 1.3 and Lemma 2.9, we have

$$\begin{aligned} p(t, y, a^*) &\geq \int_{t-2}^t p(t-s, a^*, a^*) \mathbb{P}_y(\sigma_{a^*} \in ds) \\ &\asymp \frac{1 + \log |y|}{(1 + \log(1 + t/|y|))(1 + \log(t + |y|))} \frac{(|y| + t)^{1/2}}{t^{3/2}} e^{-|y|_\rho^2/t} \\ &\gtrsim \frac{1}{t(\log t)} e^{-|y|_\rho^2/t} \gtrsim \frac{1}{t(\log t)^2} e^{-|y|_\rho^2/t}. \end{aligned}$$

Since $H_t(y) \leq (\log(1 + \varepsilon')^{-2}) + (2 \log(1 + \varepsilon'))^{-1} \asymp 1$, these estimates are sharp.

Step2 (the proof of Theorem 1.6 (i))

(1) For $x, y \in \mathbb{R}_\varepsilon^d$ with $1 \leq |x|_\rho \wedge |y|_\rho$, by (2.1), Step1 and Lemma 2.8, we have

$$\begin{aligned} p(t, x, y) &\geq \int_0^{t/2} p(t-s, a^*, x) \mathbb{P}_y(\sigma_{a^*} \in ds) + p_{\mathbb{R}_\varepsilon^d}(t, x, y) \\ &\gtrsim \frac{1}{t(\log t)^2 |x|^{d-2}} e^{-|x|_\rho^2/t} \mathbb{P}_y(\sigma_{a^*} \leq t/2) + (|x|_\rho \wedge 1)(|y|_\rho \wedge 1) \frac{1}{t^{d/2}} e^{-\rho(x, y)^2/t} \\ &\asymp \frac{1}{t(\log t)^2 |x|^{d-2} |y|^{d-2}} e^{-(|x|_\rho + |y|_\rho)^2/t} + \frac{1}{t^{d/2}} e^{-\rho(x, y)^2/t}. \end{aligned}$$

(2) For $x, y \in \mathbb{R}_\varepsilon^d$ with $|x|_\rho < 1$, by Step1, Lemma 2.5, (2.1), Lemma 2.8, Lemma

2.7 and Lemma 2.10 (iii), we have

$$\begin{aligned}
p(t, x, y) &\geq \int_0^{t/2} + \int_{t-1}^t p(t-s, a^*, x) \mathbb{P}_y(\sigma_{a^*} \in ds) + p_{\mathbb{R}_\varepsilon^d}(t, x, y) \\
&\asymp \frac{\mathbb{P}_y(\sigma_{a^*} \leq t/2)}{t(\log t)^2 |x|^{d-2}} + \frac{e^{-|y|_\rho^2/t}}{t^{d/2} + t^{3/2}|y|^{(d-3)/2}} + \frac{(|x|_\rho \wedge 1)(|y|_\rho \wedge 1)}{t^{d/2}} e^{-\rho(x, y)^2/t} \\
&\gtrsim \frac{e^{-(|x|_\rho + |y|_\rho)^2/t}}{t(\log t)^2 |x|^{d-2} |y|^{d-2}} + \frac{e^{-(|x|_\rho + |y|_\rho)^2/t}}{t^{d/2}} + \frac{(|x|_\rho \wedge 1)(|y|_\rho \wedge 1)}{t^{d/2}} e^{-\rho(x, y)^2/t} \\
&\gtrsim \frac{1}{t(\log t)^2 |x|^{d-2}} e^{-(|x|_\rho + |y|_\rho)^2/t} + \frac{1}{t^{d/2}} e^{-\rho(x, y)^2/t}.
\end{aligned}$$

Step3 (the proof of Theorem 1.6 (ii))

(1) For $x, y \in \mathbb{R}_\varepsilon^2$, with $|x|_\rho \leq 1, |y|_\rho \leq \sqrt{t}/2$, by Theorem 1.3, Lemma 2.9, (2.1), and (5.3), we have

$$\begin{aligned}
p(t, x, y) &\geq \int_{t-2}^{t-1} p(t-s, x, a^*) \mathbb{P}_y(\sigma_{a^*} \in ds) + p_{\mathbb{R}_\varepsilon^2}(t, x, y) \\
&\gtrsim \int_{t-2}^{t-1} \frac{e^{-|x|_\rho^2/(t-s)} ds \log(1+|y|)}{\sqrt{t-s} t(\log t)^2} + \frac{(|x|_\rho \wedge 1)(|y|_\rho \wedge 1)}{t} e^{-\rho(x, y)^2/t} \\
&\asymp \frac{\log(1+|y|)}{t(\log t)^2} + \frac{|x|_\rho(|y|_\rho \wedge 1)}{t} e^{-\rho(x, y)^2/t} \\
&\asymp \frac{\log(1+|x|) \log(1+|y|)}{t(\log t)^2} + \frac{|x|_\rho(|y|_\rho \wedge 1)}{t} e^{-\rho(x, y)^2/t} \\
&\gtrsim \frac{\log(1+|x|) \log(1+|y|)}{t(\log t)^2} e^{-\rho(x, y)^2/t} \\
&\asymp \frac{\log(1+|x|) \log(1+|y|)}{t(\log(1+t|x|))(\log(1+t|y|))} e^{-\rho(x, y)^2/t}.
\end{aligned}$$

(2) For $x, y \in \mathbb{R}_\varepsilon^2$, with $1 \leq |x|_\rho \leq \sqrt{t}/2, |y|_\rho \leq \sqrt{t}/2$, by (2.1), we have

$$\begin{aligned}
p(t, x, y) &\geq p_{\mathbb{R}_\varepsilon^2}(t, x, y) \gtrsim \frac{1}{t} e^{-\rho(x, y)^2/t} \gtrsim \frac{\log(1+|x|) \log(1+|y|)}{t(\log t)^2} e^{-\rho(x, y)^2/t} \\
&\asymp \frac{\log(1+|x|) \log(1+|y|)}{t(\log(1+t|x|))(\log(1+t|y|))} e^{-\rho(x, y)^2/t}.
\end{aligned}$$

(3) For $x, y \in \mathbb{R}_\varepsilon^2$, with $|x|_\rho \leq 1, \sqrt{t}/2 \leq |y|_\rho$, by Theorem 1.3, Lemma 2.6 and

Lemma 2.10 (iii), we have

$$\begin{aligned}
p(t, x, y) &\geq \int_{t-2}^{t-1} p(t-s, a^*, x) \mathbb{P}_y(\sigma_{a^*} \in ds) \\
&\gtrsim \frac{1}{t \log t} e^{-(|x|_\rho + |y|_\rho)^2/t} \gtrsim \frac{\log(1+|x|)}{t \log t} \frac{|y|_\rho^2}{t} e^{-(|x|_\rho + |y|_\rho)^2/t} \\
&\gtrsim \frac{\log(1+|x|) \log(1+|y|)}{t(\log t)^2} e^{-(|x|_\rho + |y|_\rho)^2/t} \\
&\asymp \frac{\log(1+|x|) \log(1+|y|)}{t(\log(1+t|x|))(\log(1+t|y|))} e^{-\rho(x,y)^2/t}.
\end{aligned}$$

(4) For $x, y \in \mathbb{R}_\varepsilon^2$, with $1 \leq |x|_\rho \leq \sqrt{t}/2$, $\sqrt{t}/2 < |y|_\rho$, by (2.1), we have

$$\begin{aligned}
p(t, x, y) &\geq p_{\mathbb{R}_\varepsilon^2}(t, x, y) \gtrsim \frac{1}{t} e^{-\rho(x,y)^2/t} \gtrsim \frac{\log(1+|x|)}{t \log t} e^{-\rho(x,y)^2/t} \\
&\asymp \frac{\log(1+|x|) \log(1+|y|)}{t(\log(1+t|x|))(\log(1+t|y|))} e^{-\rho(x,y)^2/t}.
\end{aligned}$$

(5) For $x, y \in \mathbb{R}_\varepsilon^2$, with $\sqrt{t}/2 \leq |x|_\rho \wedge |y|_\rho$, by (2.1), we have

$$p(t, x, y) \geq p_{\mathbb{R}_\varepsilon^2}(t, x, y) \gtrsim \frac{1}{t} e^{-\rho(x,y)^2/t} \gtrsim \frac{\log(1+|x|) \log(1+|y|)}{t(\log(1+t|x|))(\log(1+t|y|))} e^{-\rho(x,y)^2/t}.$$

Step4 (the proof of Theorem 1.6 (iii))

(1) For $x \in \mathbb{R}_\varepsilon^d$, $y \in \mathbb{R}_\varepsilon^2$, with $|x|_\rho < 1$, by Theorem 1.3 and Lemma 2.6, we have

$$\begin{aligned}
p(t, x, y) &\geq \int_{t-2}^{t-1} p(t-s, a^*, x) \mathbb{P}_y(\sigma_{a^*} \in ds) \\
&\gtrsim \frac{1}{t \log t} \frac{1 + \log |y|}{1 + \log(t+|y|)} e^{-\rho(x,y)^2/t} \\
&\gtrsim \frac{1}{t(\log t)^2} e^{-\rho(x,y)^2/t} \\
&\gtrsim \left(\frac{1}{t(\log t)^2 |x|^{d-2}} + \frac{H_t(y)}{t^{d/2}} \right) e^{-\rho(x,y)^2/t}.
\end{aligned}$$

(2) For $x \in \mathbb{R}_\varepsilon^d$, $y \in \mathbb{R}_\varepsilon^2$, with $1 \leq |x|_\rho < |x| \leq \sqrt{t}/2$, $|y|_\rho \leq 1$, by Step1, Theorem 1.3, Lemma 2.8, Lemma 2.5 and $H_t(y) \lesssim 1$, we have

$$\begin{aligned}
p(t, x, y) &\geq \int_0^{t/2} + \int_{t-2}^{t-1} p(t-s, a^*, y) \mathbb{P}_x(\sigma_{a^*} \in ds) \\
&\gtrsim \frac{1}{t(\log t)^2 |x|^{d-2}} e^{-\rho(x,y)^2/t} + \int_{t-2}^{t-1} \frac{e^{-|y|_\rho^2/(t-s)}}{\sqrt{t-s}} ds \frac{e^{-|x|_\rho^2/t}}{t^{d/2}} \\
&\gtrsim \left(\frac{1}{t(\log t)^2 |x|^{d-2}} + \frac{H_t(y)}{t^{d/2}} \right) e^{-\rho(x,y)^2/t}.
\end{aligned}$$

(3) For $x \in \mathbb{R}_\varepsilon^d, y \in \mathbb{R}_\varepsilon^2$ with $1 \leq |x|_\rho < |x| \leq \sqrt{t}/2$, $1 \leq |y|_\rho < |y| \leq \sqrt{t}/2$, by Step1 and Lemma 2.8, we have

$$p(t, x, y) \geq \int_0^{t/2} p(t-s, a^*, y) \mathbb{P}_x(\sigma_{a^*} \in ds) \gtrsim \frac{1}{t(\log t)^2 |x|^{d-2}} e^{-\rho(x,y)^2/t}.$$

Furthermore, by Step1 and Lemma 2.9, we have

$$\begin{aligned} p(t, x, y) &\geq \int_0^{t/2} p(t-s, a^*, x) \mathbb{P}_y(\sigma_{a^*} \in ds) \gtrsim \frac{e^{-\rho(x,y)^2/t}}{t^{d/2}} \left(1 - \frac{\log |y|}{\log \sqrt{t/2}}\right) \\ &\gtrsim \frac{H_t(y)}{t^{d/2}} e^{-\rho(x,y)^2/t}. \end{aligned}$$

(4) For $x \in \mathbb{R}_\varepsilon^d, y \in \mathbb{R}_\varepsilon^2$ with $1 \leq |x|_\rho < |x| \leq \sqrt{t}/2$, $\sqrt{t}/2 \leq |y|$, by Step1 and Lemma 2.8, we have

$$p(t, x, y) \geq \int_0^{t/2} p(t-s, a^*, y) \mathbb{P}_x(\sigma_{a^*} \in ds) \gtrsim \frac{1}{t(\log t)^2 |x|^{d-2}} e^{-\rho(x,y)^2/t}.$$

Furthermore, by Step1, Lemma 2.9 and $H_t(y) = (\log(1+|y|))^{-2}$,

$$p(t, x, y) \geq \int_0^{t/2} p(t-s, a^*, x) \mathbb{P}_y(\sigma_{a^*} \in ds) \gtrsim \frac{e^{-\rho(x,y)^2/t}}{t^{d/2} \log |y|} \gtrsim \frac{H_t(y)}{t^{d/2}} e^{-\rho(x,y)^2/t}.$$

(5) For $x \in \mathbb{R}_\varepsilon^d, y \in \mathbb{R}_\varepsilon^2$, with $\sqrt{t}/2 \leq |x|$, by Step1, Lemma 2.8, we have

$$\begin{aligned} p(t, x, y) &\geq \int_0^{t/2} p(t-s, a^*, y) \mathbb{P}_x(\sigma_{a^*} \in ds) \asymp \frac{1}{t(\log t)^2 |x|^{d-2}} e^{-\rho(x,y)^2/t}, \\ p(t, x, y) &\geq \int_0^{t/2} p(t-s, a^*, x) \mathbb{P}_y(\sigma_{a^*} \in ds) \asymp \frac{e^{-\rho(x,y)^2/t}}{t^{d/2}} \mathbb{P}_y(\sigma_{a^*} \leq t/2). \end{aligned} \quad (6.11)$$

By the same way as in Step 4 (3),(4), the right hand side of (6.11) is larger than $H_t(y)e^{-\rho(x,y)^2/t}/t^{d/2}$ up to a constant multiple.

By the symmetry, we have proved all the cases and complete the proof of Theorem 1.6. \square

Remark 6.9. We already proved Theorem 1.4 for the case of $d' = 1, d \geq 3$ in Section 4. Since it is the mixed case of transient and recurrent, it can also be proved by the same way as in this section.

References

- [1] R. ATAR AND A. BUDHIRAJA, On the multi-dimensional skew Brownian motion, Stochastic Process. Appl. 125 (2015), 1911–1925.

- [2] T. BYCZKOWSKI, J. MAŁECKI AND M. RYZNAR, Hitting times of Bessel processes, *Potential Anal.* 38 (2013), 753–786.
- [3] E. A. CARLEN, S. KUSUOKA AND D. W. STROOCK, Upper bounds for symmetric Markov transition functions, *Ann. Inst. H. Poincaré Probab. Statist.* 23, (1987), 245–287.
- [4] Z.-Q. CHEN AND M. FUKUSHIMA, *Symmetric Markov processes, time change, and boundary theory*, Princeton University Press, Princeton, 2012.
- [5] Z.-Q. CHEN AND S. LOU, Brownian motion on some spaces with varying dimension, *Ann. Probab.* 47 (2019), 213–269.
- [6] T. DELMOTTE, Parabolic Harnack inequality and estimates of Markov chains on graphs, *Rev. Mat. Iberoamericana* 15 (1999), 181–232.
- [7] M. FUKUSHIMA, Y. OSHIMA AND M. TAKEDA, *Dirichlet forms and symmetric Markov processes*. 2nd rev. and ext. ed. de Gruyter, Berlin, 2011.
- [8] A. GRIGOR’YAN, The heat equation on noncompact Riemannian manifolds, (Russian) *Mat. Sb.* 182 (1991), 55–87, translation in *Math. USSR-Sb.* 72 (1992), 47–77.
- [9] A. GRIGOR’YAN, Heat kernel upper bounds on a complete non-compact manifold, *Rev. Mat. Iberoamericana* 10 (1994), 395–452.
- [10] A. GRIGOR’YAN, *Heat kernel and analysis on manifolds*, American Mathematical Society, Providence, International Press, Boston, 2009.
- [11] A. GRIGOR’YAN, S. ISHIWATA AND L. SALOFF-COSTE, Heat kernel estimates on connected sums of parabolic manifolds, *J. Math. Pures Appl.* 113 (2018), 155–194.
- [12] A. GRIGOR’YAN AND L. SALOFF-COSTE, Dirichlet heat kernel in the exterior of a compact set, *Comm. Pure Appl. Math.* 55 (2002), 93–133.
- [13] A. GRIGOR’YAN AND L. SALOFF-COSTE, Hitting probabilities for Brownian motion on Riemannian manifolds, *J. Math. Pures Appl.* 81 (2002), 115–142.
- [14] A. GRIGOR’YAN AND L. SALOFF-COSTE, Heat kernel on manifolds with ends, *Ann. Inst. Fourier (Grenoble)* 59 (2009), 1917–1997.
- [15] A. GRIGOR’YAN AND L. SALOFF-COSTE, Surgery of the Faber-Krahn inequality and applications to heat kernel bounds, *Nonlinear Anal* 131 (2016), 243–272.
- [16] P. GYRYA AND L. SALOFF-COSTE, Neumann and Dirichlet heat kernels in inner uniform domains, *Astérisque* No. 336, Soc. Math, France 2011.

- [17] S. LOU, Brownian motion with drift on spaces with varying dimension, *Stochastic Process. Appl.* 129 (2019) 2086–2129.
- [18] S. LOU, Explicit heat kernels of a model of distorted Brownian motion on spaces with varying dimension, [arXiv:2001.09226](https://arxiv.org/abs/2001.09226)
- [19] L. SALOFF-COSTE, A note on Poincaré, Sobolev, and Harnack inequalities. *Internat. Math. Res. Notices* no.2 (1992), 27–38.
- [20] K.-T. STURM, Analysis on local Dirichlet spaces. II. Upper Gaussian estimates for the fundamental solutions of parabolic equations, *Osaka J. Math.* 32 (1995), 275–312.
- [21] K.-T. STURM, Analysis on local Dirichlet spaces. III. The parabolic Harnack inequality, *J. Math. Pures Appl.* 75 (1996), 273–297.
- [22] C.-J. SUNG, L.-F. TAM AND J. WANG, Spaces of harmonic functions, *J. London Math. Soc.* 61 (2000), 789–806.
- [23] Q. S. ZHANG, The global behavior of heat kernels in exterior domains, *J. Funct. Anal.* 200 (2003), 160–176.