Heat kernel upper estimates for symmetric jump processes with small jumps of high intensity

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Abstract

We consider the following non-local operator

$$\mathcal{A}f(x) = \lim_{\varepsilon \to 0} \int_{\{y \in \mathbb{R}^d : \, |x-y| > \varepsilon\}} (f(y) - f(x)) n(x,y) \, dh.$$

where

$$n(x,y) \asymp \frac{1}{|x-y|^{d+2} \left(\ln \frac{2}{|x-y|} \right)^{1+\beta}} \quad \text{for} \quad |x-y| \le 1$$

and $\beta \in (0,1]$.

We prove upper estimates for the transition density of the associated symmetric Markov jump process X. Examples of Lévy processes with generator of the type above are studied.

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1 Introduction

Recently the interest in non-local opearators within the theory of partial differential equations has increased. From the probabilistic point of view, it is interesting that many non-local operators can be understood as infinitesimal generators of some discontinuous Markov processes.

A typical example is the fractional Laplacian $(-\Delta)^{\alpha/2}$ in $\mathbb{R}^d$, $\alpha \in (0,2)$. It is given by

$$(-\Delta)^{\alpha/2} f(x) = \lim_{\varepsilon \to 0+} \int_{\{y \in \mathbb{R}^d :\, |x-y| > \varepsilon\}} (f(y) - f(x)) \frac{c_{d,\alpha}}{|x-y|^{d+\alpha}}\, dy,$$

where $c_{d,\alpha} = \frac{\alpha 2^{\alpha-1}\Gamma((d+\alpha)/2)}{\pi^{d/2}\Gamma(1-\alpha/2)}$. The probabilistic counterpart of this operator is the rotationally invariant α-stable process, which is a Markov jump process with the jumping kernel

$$n(x,y) = \frac{c_{d,\alpha}}{|x-y|^{d+\alpha}}.$$

One way to associate a non-local operator on $\mathbb{R}^d$ to a Markov jump process in $\mathbb{R}^d$ is given by the theory of Dirichlet forms. To a regular and symmetric Dirichlet form we can associate a symmetric Hunt process X and a properly exceptional set $\mathcal{N} \subset \mathbb{R}^d$ such that X can start from any point in $\mathbb{R}^d \setminus \mathcal{N}$.

In this paper we take $d \geq 1$ and consider jumping kernels $n(x,y)$ in $\mathbb{R}^d$ satisfying the following assumptions:

(A1) there exist $K \geq 1$ and $\beta \in (0,1]$ such that

$$\frac{K^{-1}}{|x-y|^{d+2}\left(\ln \frac{2}{|x-y|}\right)^{1+\beta}} \leq n(x,y) \leq \frac{K}{|x-y|^{d+2}\left(\ln \frac{2}{|x-y|}\right)^{1+\beta}}$$

for all $x, y \in \mathbb{R}^d$ such that $|x-y| \leq 1$;

(A2) there exists $M \geq 0$ such that

$$\sup_{x \in \mathbb{R}^d} \int_{|x-y|>1} n(x,y)\, dy \leq M;$$

(A3) n is symmetric, i.e.

$$n(x,y) = n(y,x) \quad \text{for all} \quad x, y \in \mathbb{R}^d.$$

Define for $f \in C_c^1(\mathbb{R}^d)$

$$\mathcal{E}(f,g) = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (f(y) - f(x))(g(y) - g(x))\, n(x,y)\, dx\, dy \tag{1.1}$$

and set

$$\mathcal{F} = \overline{C_c^1(\mathbb{R}^d)}^{\mathcal{E}_1}, \tag{1.2}$$

where $\mathcal{E}_1(f,f) = \mathcal{E}(f,f) + \|f\|_2^2$.

Similarly as in [FOT94, Example 1.2.4] we can prove that $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form and thus there exists an associated Hunt process X. The singularity of n at diagonal is stronger then the corresponding singularity for stable or stable-like processes. Hence the process X is "between" stable-like processes and diffusions. On the other hand, n is not scale invariant.

Denote by $\mathcal{N}$ the properly exceptional set of the Hunt process X. Then for any $x \in \mathbb{R}^d \setminus \mathcal{N}$ and $t > 0$ we can define the transition semigroup $\{P_t\}_{t\geq 0}$ by

$$P_t f(x) = \mathbb{E}_x[f(X_t)] \quad \text{for} \quad t > 0 \quad \text{and} \quad f \geq 0. \tag{1.3}$$

By $P(t,x,dy)$ we denote the transition probabilities for $t>0$ and $x \in \mathbb{R}^d \setminus \mathcal{N}$. It can be proved (cf. Corollary 4.4) that there exists a positive symmetric kernel $p(t,x,y)$ defined on $(0,\infty) \times (\mathbb{R}^d \setminus \mathcal{N}) \times (\mathbb{R}^d \setminus \mathcal{N})$ such that

$$P(t,x,dy) = p(t,x,y)dy.$$

Now we can state our main result.

Theorem 1.1 Let $\beta \in (0,1]$. Then there exists a constant $C > 0$ and a properly exceptional set $\mathcal{N} \subset \mathbb{R}^d$ such that for all $t \leq 1$ and $x,y \in \mathbb{R}^d \setminus \mathcal{N}$, $|x-y| \leq 1$ we have

$$p(t,x,y) \leq C \min\left\{ t^{-d/2}\left(\ln\frac{2}{t}\right)^{\beta d/2}, \frac{t}{|x-y|^{d+2}\left(\ln\frac{2}{|x-y|}\right)^{\beta}} \right\}.$$

□

We remark that in order to derive upper bounds of $p(t,x,y)$ for $|x-y| \geq 1$ some additional assumptions on $n(x,y)$ are needed.

Now we explain our strategy of proof of the main result. To get on-diagonal upper estimates of $p(t,x,y)$ we use Nash inequality. Application of techniques from the proof of Theorem 3.1 in [CK08] would lead to weaker upper estimates.

To obtain an appropriate Nash inequality we have to find a Lévy process Y whose Dirichlet form is comparable to the Dirichlet form of the process X. The construction of process Y is explained in Section 3. The process Y is obtained by subordination and has the characteristic exponent

$$\Phi(\xi) = \phi(|\xi|^2),$$

where

$$\phi(\lambda) = \phi_\beta(\lambda) = \frac{\lambda}{[\ln(1+\lambda)]^\beta}$$

when $\beta \in (0,1)$ and

$$\phi(\lambda) = \phi_1(\lambda) = \frac{\lambda}{\ln(1+\lambda)} - 1$$

when $\beta = 1$.

Also, Y is an example of a jump process whose jumping kernel satisfies conditions (A1)-(A3) (cf. Proposition 3.3).

In Section 4 we obtain Nash inequality for a particular class of Lévy processes that includes process Y. Comparing Dirichlet forms (cf. Proposition 3.4) of processes X and Y, we get the following Nash inequality for the process X (cf. Proposition 4.2)

$$\|f\|_2^2 \phi\left(\frac{\|f\|_2^{4/d}}{\|f\|_1^{4/d}}\right) \le A\,\mathcal{E}(f,f) + \delta\|f\|_2^2, \quad \text{for all} \quad f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d), \tag{1.4}$$

where $\delta > 0$ and $A' > 0$ are constants. Inequality (1.4) implies the following estimate (cf. Corollary 4.4)

$$p(t,x,y) \le c \begin{cases} t^{-d/2}\left(\ln \frac{2}{t}\right)^{\beta d/2} e^{\delta t} & 0 < t < 1 \\ t^{-d/2} e^{\delta t} & t \ge 1, \end{cases} \tag{1.5}$$

for all $x, y \in \mathbb{R}^d \backslash \mathcal{N}$. In particular, for the process Y the estimate (1.5) holds with $\delta = 0$.

We note that equivalence between Nash inequalities of the form (1.4) and transition density estimates of the corresponding process is also considered in [KS10]. Some more general Nash inequalities were studied in [BCSC07, BM07, Cou96].

In Section 5 we extend Davies' method (cf. [Dav87, Dav89, CKS87]) to obtain off-diagonal estimates and apply it in the case of the process X. The idea of Davies was to define a new semigroup $(P_t^\psi)_{t\ge 0}$ by

$$P_t^\psi f = e^{-\psi} P_t(e^\psi f),$$

for some function $\psi\colon \mathbb{R}^d \to [0,\infty)$ and to try to find an estimate of the form

$$\|P_t^\psi\|_{1\to\infty} \le m_\psi(t),$$

for some function $m_\psi\colon (0,\infty) \to [0,\infty)$. As a result

$$p(t,x,y) \le e^{\psi(x)-\psi(y)} m_\psi(t).$$

If we chose ψ within some class of functions to make $e^{\psi(x)-\psi(y)} m_\psi(t)$ as small as possible, we can get satisfying upper estimates.

Originally, this method was developed to obtain Gaussian estimates. It was extended to more general Markov semigroups in [CKS87]. Then it was applied to stable and stable-like jump processes (see [BL02], [CK03], [BBCK09], [CKK08], [BGK09]). In [CKS87] the authors start from a Nash inequality of the form

$$\|f\|_2^{2+2\alpha/d} \le (A\mathcal{E}(f,f) + \delta\|f\|_2^2)\|f\|_1^{2\alpha/d} \tag{1.6}$$

for some $\alpha \in (0,2)$. Since our version of Nash inequality is not of the form (1.6), we need to extend Davies' method. As an intermediate step we use the logarithmic Sobolev inequality, results from Chapter 2 in [Dav89] and some estimates involving Dirichlet forms from [CKS87].

In Subsection 5.1 we start with the on-diagonal upper estimate of the form

$$\|P_t\|_{1\to\infty} \le C\, t^{-d/\alpha}\ell(t)e^{\delta\, t},$$

where $\ell\colon (0,\infty)\to(0,\infty)$ varies slowly at infinity (cf. Section 2) and prove that

$$\|P_t^{\psi}\|_{1\to\infty} \le C' t^{-d/\alpha}\ell(t)e^{D_\psi t+\delta\, t},$$

for a constant $D_\psi > 0$ depending on ψ (cf. Theorem 5.2). Finally, in Subsection 5.2 we apply this technique to get off-diagonal upper bounds for the process X and we also finish the proof of Theorem 1.1.

Let us fix some notation. For $x\in\mathbb{R}^d$ and $r>0$ by $B(a,r)$ we denote the open ball in $\mathbb{R}^d$ with radius r and center a. We say that $f(x)\sim g(x)$ as $x\to\infty$ if

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = 1.$$

The Lebesgue measure of the set $A\subset\mathbb{R}^d$ is denoted by $|A|$.

If $(S,\mathcal{F},\mu)$ is a measure space, then for $1\le p,q\le\infty$ and a linear operator $A\colon L^p(S,\mu)\to L^q(S,\mu)$ we set

$$\|A\|_{p\to q} = \sup\{\|A\,f\|_q\colon f\in L^p(S,\mu),\ \|f\|_p\le 1\}.$$

2 Preliminaries

We start this section with some results from the theory of regular variation. For detailed exposition of this theory the reader is referred to [BGT87].

A function $\ell\colon (0,\infty)\to(0,\infty)$ is *slowly varying* (at infinity) if for any $\lambda>0$

$$\lim_{x\to\infty}\frac{\ell(\lambda\, x)}{\ell(x)} = 1.$$

Now we state some of the results of this theory that will be used in this paper.

Theorem 2.1 Let $\ell\colon (0,\infty)\to(0,\infty)$ be a function which is slowly varying at infinity.

(i) (*Karamata's theorem*) If $\alpha<-1$, then

$$\frac{x^{\alpha+1}\ell(x)}{\int_x^\infty t^\alpha\ell(t)\,dt}\sim -\alpha-1 \quad\text{as}\quad x\to\infty.$$

(ii) (*Potter's theorem*) If ℓ is bounded on every compact subset of $(0,\infty)$ then for any $\gamma>0$ there exists a constant $B=B(\gamma)>0$ such that

$$\ell(x)/\ell(y)\le B\max\left\{(x/y)^\gamma,(y/x)^\gamma\right\}\quad\text{for}\quad x,y>0.$$

(iii) (*Asymptotic inversion*) Assume that ℓ satisfies the following condition

$$\lim_{x\to\infty} \frac{\ell\left(x\,\ell(x)\right)}{\ell(x)} = 1. \tag{2.1}$$

If $\rho > 0$, then for $f\colon (0,\infty) \to (0,\infty)$ defined by

$$f(x) = x^\rho \ell(x), \ x > 0,$$

there exists a function $g\colon (0,\infty) \to (0,\infty)$ such that

$$f(g(x)) \sim g(f(x)) \sim x \quad \text{as} \quad x \to \infty.$$

The function g has the following asymptotic behavior

$$g(x) \sim \frac{x^{1/\rho}}{\ell\left(x^{1/\rho}\right)^{1/\rho}} \quad \text{as} \quad x \to \infty.$$

Proof. (i) [BGT87, Theorem 1.5.10], (ii) [BGT87, Theorem 1.5.6 (ii)], (iii) [BGT87, Theorem 1.5.12, Proposition 1.5.15, Corollary 2.3.4] □

Let $Y = (Y_t, \mathbb{P}_x)$ be a purely discontinuous Lévy process in $\mathbb{R}^d$. In this case we have

$$\mathbb{E}_x[e^{i\xi\cdot(Y_t - Y_0)}] = e^{-t\Phi(\xi)}, \ \xi \in \mathbb{R}^d$$

where the *characteristic exponent* Φ is of the form

$$\Phi(\xi) = \int_{\mathbb{R}^d\setminus\{0\}} \left(1 - e^{i\xi\cdot y} + i\xi\cdot y 1_{\{|y|\le 1\}}\right)\nu(dy).$$

The measure ν is called the *Lévy measure*. It is a measure on $\mathbb{R}^d$ satisfying $\nu(\{0\}) = 0$ and

$$\int_{\mathbb{R}^d} (1 \wedge |y|^2)\nu(dy) < \infty.$$

We can associate a Dirichlet form $(\mathcal{Q}, \mathcal{D}(\mathcal{Q}))$ to Y in the following way (cf. [FOT94, Example 1.4.1])

$$\begin{aligned} \mathcal{Q}(f,g) &= \int_{\mathbb{R}^d} \hat{f}(\xi)\overline{\hat{g}(\xi)}\Psi(\xi)\,d\xi \\ \mathcal{D}(\mathcal{Q}) &= \{f \in L^2(\mathbb{R}^d)\colon \mathcal{Q}(f,f) < \infty\}. \end{aligned} \tag{2.2}$$

Here $\hat{f}$ denotes the Fourier transform of f, that is

$$\hat{f}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i\xi\cdot x} f(x)\,dx$$

for $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and it is then extended to $L^2(\mathbb{R}^d)$ in the usual way.

A function $\phi\colon (0,\infty) \to (0,\infty)$ is called a *Bernstein function* if it has derivatives of all orders and

$$(-1)^{n-1}\phi^{(n)}(\lambda) \ge 0 \ \text{ for all } \ n \in \mathbb{N} \ \text{ and } \ \lambda > 0.$$

We say that $\phi\colon (0,\infty) \to (0,\infty)$ is a *completely monotone function* if it has derivatives of all orders and

$$(-1)^n \phi^{(n)}(\lambda) \geq 0 \quad \text{for all} \quad n \in \mathbb{N} \cup \{0\} \quad \text{and} \quad \lambda > 0.$$

If ϕ is a Bernstein function, then it has the following representation (cf. [SSV10, Theorem 3.2])

$$\phi(\lambda) = a + b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t})\, \mu(dt), \tag{2.3}$$

where $a, b \geq 0$ and μ is a measure on $(0,\infty)$ satisfying

$$\int_{(0,\infty)} (1 \wedge t) \mu(dt) < \infty. \tag{2.4}$$

Since $1 - e^{-\lambda t} \leq 2 \min\{\lambda t, 1\}$, by the dominated convergence theorem and (2.4) we deduce that

$$\lim_{\lambda \to 0+} \phi(\lambda) = a.$$

A *subordinator* is a Lévy process $S = (S_t)_{t \geq 0}$ taking values in $[0,\infty)$. The Laplace transform of the law of S_t is given by

$$\mathbb{E} e^{-\lambda S_t} = e^{-t\phi(\lambda)}, \ \lambda > 0.$$

Here $\phi\colon (0,\infty) \to \mathbb{R}$ is called the *Laplace exponent* and it is a Bernstein function. More precisely, it has the representation (2.3) with $a = 0$ (see p. 72 in [Ber96]). In probabilistic context, the measure μ in (2.3) is called the *Lévy measure*.

Conversely, for a Bernstein function ϕ with $a = 0$ in representation (2.3) there exists a subordinator $S = (S_t)_{t \geq 0}$ whose Laplace exponent is ϕ (cf. [Ber96, Theorem I.1]).

The *potential measure* U of the subordinator S is defined by

$$U(A) = \mathbb{E} \int_0^\infty 1_{\{S_t \in A\}}\, dt = \int_0^\infty \mathbb{P}(S_t \in A)\, dt \quad \text{for a measurable} \quad A \subset [0,\infty).$$

If we define the Laplace transform of the measure U by

$$\mathcal{L}U(\lambda) = \int_0^\infty e^{-\lambda s} U(ds), \ \lambda > 0$$

then by the use of the Fubini's theorem we obtain

$$\mathcal{L}U(\lambda) = \int_0^\infty \int_{(0,\infty)} e^{-\lambda s} \mathbb{P}(S_t \in ds)\, dt = \int_0^\infty e^{-t\phi(\lambda)}\, dt = \frac{1}{\phi(\lambda)}.$$

A Bernstein function ϕ is called a *special Bernstein function* if the function

$$\phi^*(\lambda) = \frac{\lambda}{\phi(\lambda)}$$

is also a Bernstein function. We call a Bernstein function ϕ a *complete Bernstein function* if the Lévy measure μ in the representation (2.3) has a completely monotone density with respect to the Lebesgue measure.

It follows from [SSV10, Proposition 7.1] that if ϕ is a complete Bernstein, then ϕ^* is also a complete Bernstein function. In particular, ϕ is a special Bernstein function and the Lévy measure of ϕ^* has a completely monotone density.

Let $B = (B_t, \mathbb{P}_x)$ be a Brownian motion in $\mathbb{R}^d$ independent of the subordinator $S = (S_t)_{t\geq 0}$. We define the subordinate Brownian motion $Y = (Y_t, \mathbb{P}_x)$ by

$$Y_t = B_{S_t},\ t \geq 0.$$

It follows from [Sat99, Theorem 30.1] that Y is a Lévy process with the characteristic exponent

$$\Psi(\xi) = \phi(|\xi|^2).$$

The process Y is purely discontinuous Lévy process and the characteristic exponent Φ of Y is of the form

$$\Phi(\xi) = \int_{\mathbb{R}^d} (1 - e^{i\xi\cdot h} + i\xi \cdot h 1_{\{|h|\leq 1\}}(x))\, J(h)\, dh \tag{2.5}$$

with $J(h) = j(|h|)$, where

$$j(r) = (4\pi)^{-d/2} \int_{(0,\infty)} t^{-d/2} e^{-\frac{r^2}{4t}}\, \mu(dt),\ r > 0. \tag{2.6}$$

It is easy to see that j is a non-increasing function. The function J is called the *jumping kernel*.

3 Estimates of jumping kernel and comparison of Dirichlet forms

The main idea of this section is to find a Lévy process $Y = (Y_t, \mathbb{P}_x)$ whose jumping kernel J behaves like the jumping kernel n for small jumps. This is important, because, once we have such Lévy process we can use Fourier analysis to obtain the Nash inequality for this process.

Let $S = (S_t)_{t\geq 0}$ be a subordinator with the Laplace exponent

$$\phi(\lambda) = \phi_1(\lambda) = \frac{\lambda}{\ln(1+\lambda)} - 1 \tag{3.1}$$

when $\beta = 1$ and

$$\phi(\lambda) = \phi_\beta(\lambda) = \frac{\lambda}{[\ln(1+\lambda)]^\beta} \tag{3.2}$$

when $\beta \in (0, 1)$.

Remark 3.1 First we explain why ϕ_β defines a Laplace exponent of some subordinator.

Define $\ell\colon (0,\infty) \to (0,\infty)$ by $\ell(\lambda) = \ln(1+\lambda)$. It is easy to see that

$$\ell(\lambda) = \int_0^\infty (1 - e^{-\lambda t}) t^{-1} e^{-t}\, dt. \tag{3.3}$$

Since

$$t \mapsto t^{-1}e^{-t} = \int_0^\infty e^{-st} 1_{(1,\infty)}(s)\, ds,\ t > 0, \tag{3.4}$$

is a completely monotone function, we deduce that ℓ is also a special Bernstein function.

(i) Let $\beta = 1$. In this case $\ell^*(\lambda) = \lambda/\ell(\lambda)$ is a Bernstein function and it has representation given by (2.3) with $a = \lim_{\lambda\to 0+} \ell^*(\lambda) = 1$. Thus $\phi(\lambda) = \ell^*(\lambda) - 1$ is the Laplace exponent of some subordinator.

(ii) Let $\beta \in (0, 1)$. Since a constant function is a complete Bernstein function, from [SSV10, Proposition 7.10] we conclude that

$$\lambda \mapsto [\ln(1 + \lambda)]^\beta$$

is also a complete Bernstein function. Because

$$\lim_{\lambda\to 0+} \frac{\lambda}{[\ln(1 + \lambda)]^\beta} = 0$$

we conclude that ϕ is the Laplace exponent of some subordinator.

◊

Let $B = (B_t)_{t\geq 0}$ be a Brownian motion in $\mathbb{R}^d$ independent of the subordinator S and let $Y = (Y_t)_{t\geq 0}$ be the subordinate Brownian motion defined by

$$Y_t = B_{S_t},\ t \geq 0.$$

We know that Y is a Lévy process with the characteristic exponent

$$\Phi(\xi) = \phi(|\xi|^2).$$

Let $T = (T_t)_{t\geq 0}$ be the subordinator with the Laplace exponent

$$\vartheta(\lambda) = [\ln(1 + \lambda)]^\beta.$$

By Remark 3.1 the Lévy measure of T has a completely monotone density. Therefore the Lévy measure of S has also a completely monotone density, which we denote by $\mu(t)$. It follows from [SSV10, Corollary 10.7 and Corollary 10.8] that the potential measure V of T has a non-increasing density $v(t)$ and that the following is true

$$v(t) = 1 + \int_t^\infty \mu(s)\, ds,\ t > 0. \tag{3.5}$$

The following proposition has been proved in [ŠSV06] in the case $\beta = 1$.

Proposition 3.2 For any $\beta \in (0, 1]$ we have

$$v(t) \sim \frac{1}{\beta\, t\, \left(\ln \frac{1}{t}\right)^{1+\beta}},\ t \to 0+ .$$

Proof. The Laplace transform of V is

$$\mathcal{L}V(\lambda) = \frac{1}{\vartheta(\lambda)} = \frac{1}{[\ln(1+\lambda)]^{\beta}}.$$

It can be directly checked that for any $x > 0$

$$\lim_{\lambda\to 0+} \frac{\mathcal{L}V(\frac{1}{\lambda x}) - \mathcal{L}V(\frac{1}{\lambda})}{\left(\ln\frac{1}{\lambda}\right)^{-(1+\beta)}} = \frac{1}{\beta}\ln x$$

and thus by the de Haan's Tauberian theorem (cf. [BGT87, Theorem 3.9.1]) and de Haan's monotone density theorem (cf. [BGT87, Theorem 3.6.8]) we deduce that

$$v(t) \sim \frac{1}{\beta\, t\,\left(\ln\frac{1}{t}\right)^{1+\beta}},\ t\to 0+\,.$$

□

Now we can prove the asymptotic behavior of the jumping function J of Y. Recall that $J(x) = j(|x|)$, where j is given by (2.6).

The proof of the following proposition is basically the proof of [ŠSV06, Lemma 3.1]. Here we use Potter's theorem (cf. Theorem 2.1 (iii)) in the proof. The same idea was used in [KSV, Lemma 5.1].

Proposition 3.3 The following asymptotic behavior of the function j holds

$$j(r) \sim \frac{4\Gamma(\frac{d}{2}+1)}{\beta\,\pi^{d/2}}\frac{1}{r^{d+2}\left(\ln\frac{1}{r^2}\right)^{1+\beta}},\ r\to 0+\,.$$

Proof. Using Proposition 3.2 and (3.5) we get

$$\int_t^{\infty}\mu(s)\,ds \sim \frac{1}{\beta\, t\left(\ln\frac{1}{t}\right)^{1+\beta}},\ r\to 0+$$

and thus by the Karamata's monotone density theorem (cf. [BGT87, Theorem 1.7.2]) we have

$$\mu(t) \sim \frac{1}{\beta\, t^2\left(\ln\frac{1}{t}\right)^{1+\beta}},\ r\to 0+\,. \tag{3.6}$$

Changing variable in (2.6) we get

$$\begin{aligned} j(r) &= (4\pi)^{-d/2}\int_0^{\infty} t^{-d/2}e^{-\frac{r^2}{4t}}\mu(t)\,dt \\ &= 4^{-1}\pi^{-d/2}r^{-d+2}\int_0^{\infty} t^{d/2-2}e^{-t}\mu\left(\frac{r^2}{4t}\right)\,dt \\ &= 4^{-1}\pi^{-d/2}r^{-d+2}\mu(r^2)\int_0^{\infty} t^{d/2-2}e^{-t}\,\frac{\mu\left(\frac{r^2}{4t}\right)}{\mu(r^2)}\,dt. \end{aligned} \tag{3.7}$$

By Potter's theorem (cf. Theorem 2.1 (ii)) we see that there is a constant $c_1 > 0$ such that

$$\frac{\mu\left(\frac{r^2}{4t}\right)}{\mu(r^2)} \le c_1(t^{2-1/2} \vee t^{2+1/2}) \quad \text{for all } t > 0 \text{ and } r > 0.$$

Therefore we can apply the dominated convergence theorem in (3.7) to obtain

$$\lim_{r\to 0+} \frac{j(r)}{4\pi^{-d/2} r^{-d+2} \mu(r^2)} = \Gamma(d/2+1)$$

since

$$\lim_{r\to 0+} \frac{\mu\left(\frac{r^2}{4t}\right)}{\mu(r^2)} = 16t^2.$$

□

Recall that $(\mathcal{E}, \mathcal{F})$ and $(\mathcal{Q}, \mathcal{D}(\mathcal{Q}))$ are the Dirichlet forms associated with processes X and Y, respectively.

Proposition 3.4 There exists $A > 0$ such that

$$\mathcal{Q}(f,f) \le A\mathcal{E}_1(f,f) \quad \text{for all} \quad f \in \mathcal{F}.$$

Proof. Let $f \in \mathcal{F}$. By Proposition 3.3 and property (A1) we conclude that there is a constant $c_1 > 0$ such that

$$\begin{aligned}\int\int_{|x-y|\le 1} (f(y)-f(x))^2 j(|y-x|)\,dx\,dy &\le c_1 \int\int_{|x-y|\le 1} \frac{(f(y)-f(x))^2}{|x-y|^{d+2}\left(\ln\frac{2}{|x-y|}\right)^{1+\beta}}\,dx\,dy \\ &\le c_1 K \mathcal{E}(f,f). \qquad (3.8)\end{aligned}$$

Using Proposition 3.3 again we get

$$\begin{aligned}\int\int_{|x-y|>1} (f(y)-f(x))^2 j(|y-x|)\,dx\,dy &\le 2\int\int_{|x-y|>1} \left(f(y)^2+f(x)^2\right) j(|y-x|)\,dh\,dx \\ &= 4\|f\|_2^2 \sup_{x\in\mathbb{R}^d} \int_{|x-y|>1} j(|y-x|)\,dy \\ &\le 4c_2\|f\|_2^2, \qquad (3.9)\end{aligned}$$

for some constant $c_2 > 0$. By combining (3.8) and (3.9) we obtain desired inequality. □

4 Nash inequality and on-diagonal upper bounds

In this section we prove the Nash inequality for the class of Lévy processes that contains the Lévy process Y. More precisely, let $S = (S_t)_{t\geq 0}$ be a subordinator with the Laplace exponent

$$\phi(\lambda) = \frac{\lambda}{\ell(\lambda)} - c.$$

Here $\ell\colon (0,\infty) \to (0,\infty)$ is a special Bernstein function which is slowly varying at ∞ such that

$$\lim_{\lambda\to\infty} \frac{\ell(\lambda\ell(\lambda))}{\ell(\lambda)} = 1 \tag{4.1}$$

and such that

$$c = \lim_{\lambda\to 0+} \frac{\lambda}{\ell(\lambda)}$$

exists.

Let $Z = (Z_t)_{t\geq 0}$ be the corresponding subordinate Brownian motion and denote by $(\mathcal{Q}', \mathcal{D}(\mathcal{Q}'))$ the corresponding Dirichlet form. Recall that we can write

$$\mathcal{Q}'(f,g) = \int_{\mathbb{R}^d} \hat{f}(\xi)\overline{\hat{g}(\xi)}\phi\left(|\xi|^2\right)\, d\xi.$$

Remark 4.1 (i) Note that $\ell(\lambda) = [\ln(1+\lambda)]^\beta$ satisfies condition (4.1) and that $c = 0$, for $\beta \in (0,1)$ and $c = 1$ for $\beta = 1$.

(ii) ϕ is non-decreasing since it is a Bernstein function.

$\Diamond$

Proposition 4.2 (Nash inequality) There exist constants $B_1, B_2 > 0$ such that for any $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ the following inequality holds

$$B_1 \|f\|_2^2 \phi\left(B_2 \frac{\|f\|_2^{4/d}}{\|f\|_1^{4/d}}\right) \leq \mathcal{Q}'(f,f).$$

Proof. Let $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ such that $\|f\|_1 = 1$. Then

$$|\hat{f}(\xi)| \leq 1.$$

By Plancherel theorem, for $R > 0$ we obtain

$$\begin{aligned}
\|f\|_2^2 &= \int_{|\xi|\leq R} |\hat{f}(\xi)|^2\, d\xi + \int_{|\xi|>R} |\hat{f}(\xi)|^2\, d\xi \\
&\leq c_1 R^d + \phi(R^2)^{-1} \int_{|\xi|>R} |\hat{f}(\xi)|^2 \phi(|\xi|^2)\, d\xi \\
&\leq c_1 R^d + \phi(R^2)^{-1} \mathcal{Q}'(f,f),
\end{aligned} \tag{4.2}$$

since ϕ is non-decreasing.

Let $R_0 > 0$ be chosen so that

$$c_1 R_0^d = \phi(R_0^2)^{-1} \mathcal{Q}'(f,f).$$

This is possible, since $\varphi\colon (0,\infty) \to \mathbb{R}$ defined by

$$\varphi(x) = x^d \phi(x^2)$$

satisfies $\lim_{x\to 0+} \varphi(x) = 0$ and $\lim_{x\to\infty} \varphi(x) = \infty$. Moreover, since φ is strictly increasing, we have

$$R_0 = \varphi^{-1}(c_1^{-1} \mathcal{Q}'(f,f)).$$

If we set $R = R_0$ in (4.2) we obtain

$$\|f\|_2^2 \le 2c_1 \left[\varphi^{-1}(c_1^{-1}\mathcal{Q}'(f,f))\right]^d$$

and consequently

$$\varphi\left((2c_1)^{-1/d} \|f\|_2^{2/d}\right) \le 2\mathcal{Q}'(f,f). \tag{4.3}$$

If $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, we can replace f by $\|f\|_1^{-1} f$ in (4.3) to obtain desired inequality. □

Now we can obtain on-diagonal upper estimates for the process X. Recall that the transition semigorup $(P_t)_{t\ge 0}$ of the process X is defined by (1.3) and the process can start at any point which is not in the properly exceptional set $\mathcal{N}$.

For $\beta \in (0,1)$ we define $m_\beta\colon (0,\infty) \to \mathbb{R}$ by

$$m_\beta(t) = \begin{cases} t^{-d/2}\left(\ln \frac{2}{t}\right)^{\beta d/2} & t \le 1 \\ t^{-d/(2(1-\beta))} & t > 1. \end{cases} \tag{4.4}$$

If $\beta = 1$, we define $m_1\colon (0,\infty) \to \mathbb{R}$ by

$$m_1(t) = \begin{cases} t^{-d/2}\left(\ln \frac{2}{t}\right)^{d/2} & t \le 1 \\ t^{-d/2} & t > 1. \end{cases} \tag{4.5}$$

Proposition 4.3 Let $\beta \in (0,1]$. Then there exists a constant $D_1 > 0$ such that for any $t > 0$

$$\|P_t f\|_{1\to\infty} \le D_1\, m_\beta(t) e^{2At}.$$

Proof. Using Proposition 3.4 and Proposition 4.2 we conclude that the following Nash inequality holds

$$B_1 \|f\|_2^2 \phi\left(B_2 \frac{\|f\|_2^{4/d}}{\|f\|_1^{4/d}}\right) \le A\mathcal{E}(f,f) + A\|f\|_2^2,$$

where for $\beta = 1$ we have

$$\phi(x) = \frac{x}{\ln(1+x)} - 1$$

and for $\beta \in (0,1)$

$$\phi(x) = \frac{x}{[\ln(1+x)]^\beta}.$$

By [KS10, Proposition 3] we conclude that there exists a constant $c_1 > 0$ such that

$$\|P_t f\|_{1\to\infty} \leq \left[\phi^{-1}\left(\frac{1}{c_1\, t}\right)\right]^{d/2} e^{2At}.$$

Let $\beta = 1$. Since

$$\phi(x) \sim \frac{x}{2},\ x \to 0+ \qquad \text{and} \qquad \phi(x) \sim \frac{x}{\ln x},\ x \to \infty$$

we get (cf. Theorem 2.1 (iii))

$$\phi^{-1}(x) \leq c_2\, x \ \text{ for } \ x \leq 1 \qquad \text{and} \qquad \phi^{-1}(x) \leq c_3\, x \ln x \ \text{ for } \ x > 1.$$

Similarly, for $\beta \in (0,1)$ we have

$$\phi(x) \sim x^{1-\beta},\ x \to 0+ \qquad \text{and} \qquad \phi(x) \sim \frac{x}{(\ln x)^\beta},\ x \to \infty$$

and so

$$\phi^{-1}(x) \leq c_4\, x^{1/(1-\beta)} \ \text{ for } \ x \leq 1 \qquad \text{and} \qquad \phi^{-1}(x) \leq c_5\, x (\ln x)^\beta \ \text{ for } \ x > 1.$$

□

Corollary 4.4 There exists a properly exceptional set $\mathcal{N}' \subset \mathbb{R}^d$ and a positive symmetric kernel $p(t,x,y)$ defined on $(0,\infty) \times (\mathbb{R}^d \setminus \mathcal{N}') \times (\mathbb{R}^d \setminus \mathcal{N}')$ such that

$$P_t f(x) = \int_{\mathbb{R}^d} p(t,x,y) f(y)\, dy \quad \text{for all } \ x \in \mathbb{R}^d \setminus \mathcal{N}' \ \text{ and } \ f \geq 0.$$

Moreover, for any $\beta \in (0,1]$ we have the following estimate

$$p(t,x,y) \leq D_1 m_\beta(t) e^{2At} \ \text{ for all } \ x, y \in \mathbb{R}^d \setminus \mathcal{N}' \ \text{ and } \ t > 0.$$

Proof.
Follows directly from Proposition 4.3 and [BBCK09, Theorem 3.1]. □

5 Davies' method and off-diagonal upper bounds

In this section we extend Davies' method for off-diagonal upper estimates given in [CKS87]. Since this technique is of independent interest, we give it in a more general setting in Subsection 5.1. We apply this technique to our case in Subsection 5.2

5.1 Davies' method

Let S be a locally compact separable metric space and let μ be a Radon measure on S. Consider a regular symmetric Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(S,\mu)$ and assume that there exists a positive and symmetric bilinear form

$$\Gamma\colon \mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \to L^1(S,\mu),$$

such that

$$\mathcal{E}(fh,g) + \mathcal{E}(gh,f) - \mathcal{E}(h,fg) = \int_S h\Gamma(f,g)\,d\mu. \tag{5.1}$$

It follows from [BH91, Proposition 4.1.3] that such bilinear form is unique if it exists. The bilinear form Γ is sometimes called the *carré du champ*.

Using (5.1) we can easily check the following Leibniz rule

$$\mathcal{E}(fg,h) = \frac{1}{2}\int_S f\Gamma(g,h)\,d\mu + \frac{1}{2}\int_S g\Gamma(f,h)\,d\mu \tag{5.2}$$

for all $f,g,h \in \mathcal{D}(\mathcal{E})$.

Let $(P_t)_{t\geq 0}$ be the symmetric Markovian semigroup on $L^2(S,\mu)$ corresponding to the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ (cf. [FOT94, Theorem 1.4.1]) and assume that there exists a kernel $P(t,x,dy)$ such that

$$P_t f(x) = \int_S f(y)P(t,x,dy) \ \text{ for all } f \in L^2(S,\mu).$$

The symmetry of $(P_t)_{t\geq 0}$ can now be expressed in the following way

$$\int_S\int_S f(x)g(y)P(t,x,dy)\mu(dx) = \int_S\int_S f(y)g(x)P(t,x,dy)\mu(dx) \tag{5.3}$$

for all $t>0$, $x\in S$ and bounded and measurable functions $f,g\colon S \to [0,\infty)$.

Remark 5.1 Let $X=(X_t)_{t\geq 0}$ be a jump process in $\mathbb{R}^d$ with the jumping kernel $n(x,y)$.

The corresponding Dirichlet form is given by

$$\begin{aligned}\mathcal{E}(f,g) &= \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(f(y)-f(x))(g(y)-g(x))n(x,y)\,dx\,dy\\ \mathcal{D}(\mathcal{E}) &= \{f \in L^2(\mathbb{R}^d)\colon \mathcal{E}(f,f)<\infty\}.\end{aligned}$$

In this case $S=\mathbb{R}^d$ and μ is the Lebesgue measure on $\mathbb{R}^d$. Using the last display, symmetry of X and (5.1) we can check that the carré du champ exists and it is given by

$$\Gamma(f,g)(x) = \int_{\mathbb{R}^d}(f(y)-f(x))(g(y)-g(x))n(x,y)\,dx\,dy \ \text{ for } \ f,g\in\mathcal{D}(\mathcal{E}).$$

$\Diamond$

Let $\psi \in \mathcal{D}(\mathcal{E}) \cap C_b(S)$ such that

$$\Lambda(\psi) = \|e^{-2\psi}\Gamma(e^{\psi}, e^{\psi})\|_\infty^{1/2} \vee \|e^{2\psi}\Gamma(e^{-\psi}, e^{-\psi})\|_\infty^{1/2} < \infty.$$

Now we can follow the proof of Theorem 3.9 in [CKS87] and use (5.1), Leibniz rule (5.2) and the symmetry assumption (5.3) to obtain the following inequality

$$\mathcal{E}(f^{p/2}, f^{p/2}) \leq \frac{p}{2}\mathcal{E}(e^{-\psi} f^{p-1}, e^{\psi} f) + \frac{9p^2}{2}\Lambda(\psi)^2 \|f\|_p^p \tag{5.4}$$

for $p \geq 2$ and for all non-negative $f \in \mathcal{D}(\mathcal{E}) \cap C_b(S)$. Note that the expression on the left hand side of (5.4) is finite by [CKS87, Lemma 3.5].

Assume that the semigroup $(P_t)_{t\geq 0}$ satisfies

$$\|P_t f\|_{2\to\infty} \leq t^{-\alpha}\ell(t)e^{\delta t} \quad \text{for all } t > 0. \tag{5.5}$$

where $\alpha > 0$, $\delta \geq 0$ and $\ell\colon (0,\infty) \to (0,\infty)$ is a function which is slowly varying at 0. By symmetry and duality

$$\|P_t f\|_{1\to\infty} \leq t^{-2\alpha}\ell(t/2)^2 e^{2\delta t} \quad \text{for all } \ t > 0.$$

It is well known that then P_t has a kernel $p(t, x, y)$, that is

$$P_t f(x) = \int_S p(t, x, y) f(y)\, \mu(dy).$$

Davies' idea was to define a new semigroup by

$$P_t^{\psi} f(x) = e^{-\psi(x)} P_t(e^{\psi} f)(x)$$

and to obtain similar estimate for P_t^{ψ} as in (5.5). Note that

$$P_t^{\psi} f(x) = \int_S e^{-\psi(x)} p(t, x, y) e^{\psi(y)} f(y)\, \mu(dy).$$

Choosing suitable ψ it is possible to obtain off-diagonal upper heat kernel estimates. In [CKS87, Theorem 3.25)] this was done in the case when $\ell \equiv 1$ with the help of the Nash inequality. In our case we use the logarithmic Sobolev inequality.

By (5.5) and Theorem 2.2.3 in [Dav89] we deduce the following logarithmic Sobolev inequality

$$\int_S f(x)^2 \ln f(x)\, \mu(dx) \leq \varepsilon\mathcal{E}(f, f) + (\delta\varepsilon - \alpha \ln \varepsilon + \ln \ell(\varepsilon))\, \|f\|_2^2 + \|f\|_2^2 \ln \|f\|_2 \tag{5.6}$$

for all non-negative $f \in \mathcal{D}(\mathcal{E}) \cap L^1(S, \mu) \cap C_b(S)$ and $\varepsilon > 0$.

Theorem 5.2 Suppose that the logarithmic Sobolev inequality (5.6) holds. There exists a constant $D_2 > 0$ such that for any $\psi \in \mathcal{D}(\mathcal{E}) \cap L^1(S, \mu) \cap C_b(S)$ we have

$$\|P_t^{\psi} f\|_{2\to\infty} \leq D_2\, t^{-\alpha}\ell(t)\, e^{\delta t + 36\Lambda(\psi)^2 t} \quad \text{for all } t > 0.$$

Proof. First we need the logarithmic Sobolev inequality for $(P_t^\psi)_{t\geq 0}$. Let $f \in \mathcal{D}(\mathcal{E}) \cap L^1(S,\mu) \cap C_b(S)$ be a non-negative function, $p \geq 2$ and $\varepsilon > 0$. Applying (5.6) to $f^{p/2}$ we get

$$\int_S f(x)^p \ln f(x)\,\mu(dx) \leq 2p^{-1}\varepsilon\mathcal{E}(f^{p/2}, f^{p/2}) + 2p^{-1}\left(\delta\varepsilon - \alpha\ln\varepsilon + \ln\ell(\varepsilon)\right)\|f\|_p^p + \|f\|_p^p \ln\|f\|_p.$$

Using (5.4) in the last display we arrive at

$$\begin{aligned}\int_S f(x)^p \ln f(x)\,\mu(dx) &\leq \varepsilon\mathcal{E}(e^{-\psi}f^{p-1}, e^{\psi}f) + 9p\varepsilon\Lambda(\psi)^2\|f\|_p^p \\ &\quad + 2p^{-1}\left(\delta\varepsilon - \alpha\ln\varepsilon + \ln\ell(\varepsilon)\right)\|f\|_p^p + \|f\|_p^p\ln\|f\|_p. \qquad (5.7)\end{aligned}$$

Let $\mathcal{L}^\psi$ be the L^2-generator of the semigroup $(P_t^\psi)_{t\geq 0}$. Then

$$\begin{aligned}\mathcal{E}(e^{-\psi}f^{p-1}, e^{\psi}f) &= \lim_{t\to 0+}\left(e^{-\psi}f^{p-1}, \frac{P_t(e^{\psi}f) - e^{\psi}f}{t}\right)_{L^2(S,\mu)} \\ &= \lim_{t\to 0+}\left(f^{p-1}, \frac{P_t^\psi f - f}{t}\right)_{L^2(S,\mu)} = (f^{p-1}, \mathcal{L}^\psi f)_{L^2(S,\mu)}\end{aligned}$$

Therefore, for any $t > 0$, we may apply [Dav89, Theorem 2.2.7] with

$$\varepsilon(p) = \frac{8t}{p^2}.$$

We obtain

$$\begin{aligned}\ln\|P_t^\psi f\|_{2\to\infty} &\leq \left\{\int_2^\infty \left[9\Lambda(\psi)^2\varepsilon(p) + 2p^{-2}\left(\delta\varepsilon(p) - \alpha\ln\varepsilon(p) + \ln\ell(\varepsilon(p))\right)\right]dp\right\} \\ &= \left\{36\Lambda(\psi)^2 t + 2\delta t/3 + c_1 - \alpha\ln t + 2\int_2^\infty \ln\ell\left(8t/p^2\right)p^{-2}\,dp\right\}. \qquad (5.8)\end{aligned}$$

By Potter's theorem (cf. Theorem 2.1 (ii)) we deduce that there is a constant $c_2 > 0$ such that

$$\frac{\ln\ell\left(8t/p^2\right)}{\ln\ell(t)} \leq c_2 p^{1/2} \quad \text{for all } p \geq 2.$$

Thus we can estimate the integral in (5.8)

$$\begin{aligned}\int_2^\infty \ln\ell\left(8t/p^2\right)p^{-2}\,dp &= \int_2^\infty \frac{\ln\ell\left(8t/p^2\right)}{\ln\ell(t)}p^{-2}\,dp + \ln\ell(t)/2 \\ &\leq c_3 + \ln\ell(t)/2.\end{aligned}$$

From the last display and (5.8) we get

$$\|P_t^\psi\|_{2\to\infty} \leq c_4 t^{-\alpha}\ell(t)e^{\left(36\Lambda(\psi)^2+\delta\right)t}\|f\|_2 \quad \text{for all } t > 0. \qquad (5.9)$$

□

Corollary 5.3 Suppose that the logarithmic Sobolev inequality (5.6) holds. There exists a constant $D_3 > 0$ such that for any $\psi \in \mathcal{D}(\mathcal{E}) \cap L^1(S, \mu) \cap C_b(S)$ the following is true

$$p(t,x,y) \leq D_3\, t^{-2\alpha} \ell(t)^2\, e^{2\delta t + 72\Lambda(\psi)^2 t - \psi(y) + \psi(x)} \quad \text{for all } t > 0.$$

□

Proof. It can be easily checked that the adjoint operator of P_t^{ψ} is $P_t^{-\psi}$. By using duality and Theorem 5.2 we obtain the result. □

5.2 Off-diagonal upper bounds

In this subsection we use the setting from Section 4.

Theorem 5.4 For any $\beta \in (0,1]$ there exists a constant $D_4 > 0$ such that for all $x, y \in \mathbb{R}^d \setminus \mathcal{N}$ satisfying $0 < |x-y| \leq 1$ we have

$$p(t,x,y) \leq D_4 \frac{t}{|x-y|^{d+2} \left(\ln \frac{2}{|x-y|} \right)^{\beta}}$$

for all $0 < t \leq |x-y|^2 \left(\ln \frac{2}{|x-y|} \right)^{\beta}$.

Proof. Let $\beta \in (0,1]$. By Corollary 4.4 and (5.6) we obtain the following logarithmic Sobolev inequality

$$\int f(x)^2 \ln f(x)\, dx \leq \varepsilon \mathcal{E}(f,f) + \ln m_\beta(\varepsilon) \|f\|_2^2 + \|f\|_2^2 \ln \|f\|_2 \tag{5.10}$$

for all non-negative $f \in \mathcal{F} \cap L^1(\mathbb{R}^d) \cap C_b(\mathbb{R}^d)$ and $\varepsilon > 0$. Recall that m_β is given by (4.4) or (4.5).

We split the jumping kernel n according to the size of jumps. More precisely, for $R \in (0,1)$ we define

$$n_1^R(x,y) = n(x,y) 1_{\{|x-y| \leq R\}}, \quad n_2^R(x,y) = n(x,y) 1_{\{|x-y| > R\}} = n(x,y) - n_1^R(x,y).$$

The appropriate R will be chosen later in the proof.

Let $Z = (Z_t)_{t \geq 0}$ be the jump process with the Lévy density n_1^R and denote by $p_R(t,x,y)$ its transition density. Denote by $(\mathcal{E}_R, \mathcal{F}_R)$ and Γ_R the corresponding Dirichlet form and carré du champ operator. By Meyer's construction given in [Mey75] (cf. Lemma 3.1 in [BGK09]) it follows that

$$p(t,x,y) \leq p_R(t,x,y) + t \|n_2^R\|_\infty. \tag{5.11}$$

Similarly as in (3.9), for any $f \in C_c^1(\mathbb{R}^d) \subset \mathcal{F} \subset \mathcal{F}_R$ we get

$$\mathcal{E}(f,f) - \mathcal{E}_R(f,f) \le 4\|f\|_2^2 \sup_{x\in\mathbb{R}^d} \int_{|x-y|>R} n(x,y)\,dy. \tag{5.12}$$

By (A1), (A2) and Karamata's theorem (cf. Theorem 2.1 (i)) we get

$$\begin{aligned} \sup_{x\in\mathbb{R}^d} \int_{|x-y|>R} n(x,y)\,dy &\le \int_{R<|x-y|\le 1} \frac{K}{|x-y|^{d+2}\left(\ln\frac{2}{|x-y|}\right)^{1+\beta}}\,dy + \sup_{x\in\mathbb{R}^d} \int_{|x-y|>1} n(x,y)\,dy \\ &\le \frac{c_1}{R^2\left(\ln\frac{2}{R}\right)^{1+\beta}}. \end{aligned} \tag{5.13}$$

Using (5.10), (5.12) and (5.13) we obtain the following logarithmic Sobolev inequality for Z

$$\int f(x)^2 \ln f(x)\,dx \le \varepsilon\mathcal{E}_R(f,f) + \left(\ln m_\beta(\varepsilon) + \frac{c_2\varepsilon}{R^2\left(\ln\frac{2}{R}\right)^{1+\beta}}\right)\|f\|_2^2 + \|f\|_2^2 \ln\|f\|_2 \tag{5.14}$$

Let $x_0, y_0 \in \mathbb{R}^d$ be such that $0 < |x_0 - y_0| \le 1$ and set

$$R = \frac{|x_0-y_0|}{3(d+2)}. \tag{5.15}$$

We apply Davies' method, which is described in Section 5.1, to the process Z. Let $\lambda > 0$ and define

$$\psi(x) = \lambda\left(|x_0-y_0| - |x-y_0|\right)^+$$

Since

$$(e^z-1)^2 \le z^2\, e^{2|z|} \quad \text{for all} \quad z\in\mathbb{R}$$

and

$$|\psi(x)-\psi(y)| \le \lambda|x-y|,$$

we have

$$\begin{aligned} e^{-2\psi(x)}\Gamma_R(e^\psi, e^\psi)(x) &= \int_{|x-y|\le R} \left(e^{\psi(y)-\psi(x)}-1\right)^2 n(x,y)\,dy \\ &\le \lambda^2 e^{2\lambda R} \int_{|x-y|\le R} |y-x|^2\, n(x,y)\,dy \\ &\le K\,\frac{\lambda^2 e^{2\lambda R}}{\left(\ln\frac{2}{R}\right)^\beta} \le K\,\frac{e^{3\lambda R}}{R^2\left(\ln\frac{2}{R}\right)^\beta}, \end{aligned}$$

where in the last line we have used (A1) together with Karamata's theorem. Similarly we obtain the same bound for $e^{2\psi(x)}\Gamma_R(e^{-\psi}, e^{-\psi})(x)$ and thus

$$\Lambda(\psi)^2 = \|e^{-2\psi}\Gamma_R(e^\psi, e^\psi)\|_\infty \vee \|e^{2\psi}\Gamma_R(e^{-\psi}, e^{-\psi})\|_\infty \le c_3 \frac{e^{3\lambda R}}{R^2\left(\ln\frac{2}{R}\right)^\beta}. \tag{5.16}$$

Suppose

$$t < R^2 \left(\ln \frac{2}{R}\right)^{\beta}. \tag{5.17}$$

By Corollary 5.3, (5.14), (5.16) and (5.17) we obtain

$$\begin{aligned} p_R(t,x_0,y_0) &\le c_2 t^{-d/2}\left(\ln \frac{2}{t}\right)^{\beta d/2} \exp\left\{\psi(x_0)-\psi(y_0)+72\Lambda(\psi)^2 t + \frac{c_4 t}{R^2\left(\ln \frac{2}{R}\right)^{1+\beta}}\right\} \\ &\le c_2 t^{-d/2}\left(\ln \frac{2}{t}\right)^{\beta d/2} \exp\left\{-\psi(y_0)+\frac{c_3 e^{3\lambda R}\, t}{R^2\left(\ln \frac{2}{R}\right)^{\beta}} + \frac{c_5}{\ln \frac{2}{R}}\right\}. \end{aligned} \tag{5.18}$$

Let

$$\lambda = \frac{1}{6R} \ln \frac{R^2\left(\ln \frac{2}{R}\right)^{2\beta/(d+2)}\left(\ln \frac{2}{t}\right)^{\beta d/(d+2)}}{t}.$$

By Potter's theorem we deduce that there is a constant $b>0$ such that

$$\varphi(s) = s^{1/2}\left(\ln \frac{2}{s}\right)^{\beta d/(2(d+2))}$$

satisfies

$$\varphi(s_1) \le b\,\varphi(s_2) \quad \text{for all} \quad 0 < s_1 < s_2 < 1.$$

Combining this with (5.17) we get

$$\frac{e^{3\lambda R}\, t}{R^2\left(\ln \frac{2}{R}\right)^{\beta}} = \frac{\left(\ln \frac{2}{R}\right)^{\beta/(d+2)}}{R\left(\ln \frac{2}{R}\right)^{\beta}}\varphi(t) \le c_6$$

Therefore, from (5.18) we obtain

$$p_R(t,x,y) \le \frac{c_7 t}{R^{d+2}\left(\ln \frac{2}{R}\right)^{\beta}} \tag{5.19}$$

By (A1) and (5.15) we conclude

$$\|n_2^R\|_\infty \le \frac{K}{R^{d+2}\left(\ln \frac{2}{R}\right)^{1+\beta}} \le \frac{c_8}{|x_0-y_0|^{d+2}\left(\ln \frac{2}{|x_0-y_0|}\right)^{1+\beta}}$$

and thus from (5.11) and (5.19) we get

$$\begin{aligned} p(t,x,y) &\le \frac{c_9 t}{|x_0-y_0|^{d+2}\left(\ln \frac{2}{|x_0-y_0|}\right)^{\beta}} + \frac{c_{10} t}{|x_0-y_0|^{d+2}\left(\ln \frac{2}{|x_0-y_0|}\right)^{1+\beta}} \\ &\le \frac{(c_{10}+c_{11})t}{|x_0-y_0|^{d+2}\left(\ln \frac{2}{|x_0-y_0|}\right)^{\beta}}. \end{aligned}$$

□

Proof of Theorem 1.1

The theorem now follows directly from Corollary 4.4 and Theorem 5.4. □

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