HEAT TRANSFER BETWEEN SOLIDS AND GASSES UNDER NONLINEAR BOUNDARY CONDITIONS*

BY

W. ROBERT MANN

University of North Carolina

AND

FRANTISEK WOLF

University of California at Berkeley

1. Introduction. In the theory of heat transfer between solids and gasses, it is commonly assumed that the rate of heat exchange across a gas-solid interface is proportional to the difference between the temperature of the solid surface and that of the ambient gas. This assumption is known as Newton's Law of Cooling and it gives rise to a boundary condition of the following general form

$$k\left(\frac{\partial U}{\partial n}\right) = -f\Delta U, \tag{1}$$

where k is the thermal conductivity of the solid; $\partial U/\partial n$ is the thermal gradient at the surface evaluated from the interior in the direction of the outward normal; ΔU is the difference in temperature between the surface and the gas, considered positive when the solid is warmer than the gas; and f is the factor of proportionality, frequently referred to as the film transfer factor.** If f is a constant the above boundary condition is linear. At ordinary temperatures, where most of the heat transfer is due to conduction-convection, f varies but slightly with temperature and it is not a bad approximation to regard it as a constant. At higher temperatures however, most of the heat is transferred by radiation and the film transfer factor varies greatly with temperature. Neglecting conduction and convection, we find from the "fourth power law" that f_r, the film transfer factor due to radiation, is given by

$$f_r = A\epsilon \frac{T_s^4 - T_g^4}{T_s - T_g}, \tag{2}$$

where T_s is the absolute temperature of the solid surface, T_g the absolute temperature of the ambient gas, ϵ the emissivity, and A is a constant depending upon the units of measurement. Even when all the heat exchange is by conduction-convection the film transfer factor, f_c, changes somewhat with temperature but the dependence is much more complicated than equation (2) and is expressed in the form of empirically determined relations between certain dimensionless moduli.†

Henceforth we shall not be concerned with any particular form of the relationship between f and the temperature. The important point is that when the film transfer factor is a function of the temperature, the boundary condition (1) becomes nonlinear. It is our purpose in this paper to investigate a nonlinear boundary value problem under

*Received Nov. 8, 1949.

**L. M. K. Boelter, V. H. Cherry, H. A. Johnson, *Heat transfer*, University of California Press, pp. IIb-1, IIb-2.

†Boelter, Cherry, and Johnson, *op. cit.*, Chap. XII.

the most general physically significant relationship between the film transfer factor and the temperature.

2. Statement of problem. In order to keep geometrical considerations as simple as possible we shall consider only the semi-infinite solid. To simplify further the non-essential aspects of the problem, we shall assume the gas to be maintained constantly at unit temperature. This will enable us to regard the film transfer factor as a function of the surface temperature alone. Requiring that the initial temperature of the solid be zero throughout, the problem of finding the ensuing temperature-time distribution, $U(x, t)$, in the solid can be formulated as follows:

$$U_t(x, t) = U_{xx}(x, t), \qquad x > 0, t > 0, \tag{3}$$

$$U(x, 0) = 0, \tag{3a}$$

$$-U_x(0, t) = \frac{[1 - U(0, t)]f[U(0, t)]}{k} = G[U(0, t)], \tag{3b}$$

$$| U(x, t) | < M > 1 \qquad x > 0, t > 0. \tag{3c}$$

As usual, the following functions will be assumed to be continuous for the values of x and t indicated:

$$U(x, t) \qquad \text{for} \qquad x \geq 0, t \geq 0, \tag{3d}$$

$$U_x(x, t) \qquad \text{for} \qquad x \geq 0, t > 0, \tag{3e}$$

$$U_{xx}(x, t), U_t(x, t) \qquad \text{for} \qquad x > 0, t > 0. \tag{3f}$$

Equation (3) is the well-known heat flow equation, where the units of time and distance have been so chosen as to make the diffusivity 1. (3b) is the special form of boundary condition (1) corresponding to this particular problem. Observe that $-kU_x(0, t)$ is the rate of heat flow per unit area into the solid from the gas. The function $G[U(0, t)]$, which is proportional to this rate of heat exchange, occurs continually throughout the following work and will be referred to as the input function. Condition (3c) serves the double purpose of restricting the behavior of $U(x, t)$ as x tends toward infinity, and of excluding the possibility of an instantaneous heat source at the surface when $t = 0$.

To complete the statement of the problem, some hypotheses must be made concerning the input function, $G[U(0, t)]$. We know from experience that heat transfer takes place in a continuous manner; that a net exchange of heat takes place between two media only when they are at different temperatures; and that the net rate of heat transfer is a monotone increasing function of the difference in temperature between the two media. Referring to the definition of the input function above, we see that in any physically significant problem the following three hypotheses must hold:

A. $G[U]$ is continuous for all U;

B. $G[U]$ is zero when $U = 1$;

C. $G[U]$ is a monotone decreasing function of U.

In the following work we shall repeatedly invoke the above hypotheses, especially in Sec. 5.

For a brief summary of the principal results obtained, the reader is referred to Sec. 6.

3. Reduction of the problem to a nonlinear integral equation. The temperature distribution in the solid, which is the unknown quantity in the problem stated in Sec. 2, is a function of the two variables, x and t. We can easily show, however, by Duhamel's Principle for example, [1] that $U(x, t)$ is completely determined by the surface temperature, $U(0, t)$. This leads us to expect that the problem admits a more concise formulation in terms of the function of a single variable, $U(0, t)$. We shall effect such a re-formulation by means of the Laplace transform. This well-known transformation will not, of course, eliminate the essentially nonlinear character of the problem but it does restate it in terms of a nonlinear integral equation for $U(0, t)$.

In view of the conditions imposed upon $U(x, t)$ in Sec. 2, it is easily verified that the Laplace transformation with respect to t carries (3), (3b), and (3c) into the following linear ordinary boundary value problem:

$$u_{xx}(x, s) - su(x, s) = 0, \tag{3'}$$

$$u_x(0, s) = -\mathfrak{L}\{G[U(0, t)]\} = -g(s), \tag{3b'}$$

$$| u(x, s) | < \frac{M}{s} \qquad \text{for} \qquad x > 0. \tag{3c'}$$

The general solution of (3′) is

$$u(x, s) = Ae^{-xs^{1/2}} + Be^{xs^{1/2}},$$

where A and B may depend on s but not on x. (3c′) requires that $B = 0$ and from (3b′) we find that $A = g(s)s^{-1/2}$. Hence we get that

$$u(x, s) = \frac{g(s)}{s^{1/2}} e^{-xs^{1/2}} \tag{4}$$

for the solution of the transformed boundary value problem. Inverting (4) by means of the Borel formula [2] we obtain

$$U(x, t) = \int_0^t \frac{G[U(0, \tau)]}{\pi^{1/2}(t - \tau)^{1/2}} \exp \frac{-x^2}{4(t - \tau)} \, d\tau \tag{5}$$

which expresses $U(x, t)$ in terms of all the values of $U(0, \tau)$ between 0 and t. We can now prove the following theorem:

THEOREM 1. *A necessary condition that a function $y(t)$ should be a surface temperature function for the problem stated in Part II is that it satisfy the following singular, nonlinear integral equation of the Volterra type*

$$y(t) = \int_0^t \frac{G[y(\tau)]}{\pi^{1/2}(t - \tau)^{1/2}} \, d\tau \tag{6}$$

and that

$$| y(t) | \leq M \qquad t > 0. \tag{7}$$

If $| y(t) | \leq 1$, then equation (6) *is also a sufficient condition.* The proof of the theorem is facilitated by the following lemma:

LEMMA I. *Any function, $y(t)$, satisfying both* (6) *and* (7) *is continuous for $t \geq 0$ and $y(0) = \lim_{t \to 0^+} y(t) = 0$.*

The proof of Lemma I is a straightforward exercise in elementary calculus and will not be included here.

To establish the necessity condition stated in Theorem 1, we assume that $y(t)$ is a surface temperature function for the problem under consideration. This means that there exists a function $U(x, t)$ satisfying the conditions (3) through (3f) and having the property that $y(t) = \lim_{x\to 0} U(x, t) = U(0, t)$. Applying the continuity condition (3d) to equation (5) we get directly that

$$U(0, t) = \int_0^t \frac{G[U(0, \tau)]}{\pi^{1/2}(t - \tau)^{1/2}} \, d\tau,$$

and since $U(0, t) = y(t)$ we see that (6) must hold. (3c) and (3d) give us (7) immediately. This completes the proof of the necessity.

Turning now to the proof of the sufficiency condition, we assume that $y(t)$ satisfies (6) and (7) and show that the function defined as follows

$$U(x, t) = \int_0^t \frac{G[y(\tau)]}{\pi^{1/2}(t - \tau)^{1/2}} \exp \frac{-x^2}{4(t - \tau)} \, d\tau$$

is a solution of the problem stated in Sec. 2. The proof consists in verifying that $U(x, t)$ defined above satisfies the conditions (3) through (3f). Most of these verifications are trivial and will be omitted. We do include however the proof for condition (3b) which is not completely obvious, and that for (3c) which will make clear why the sufficiency is proved only for $| y(t) | \leq 1$.

Considering first (3b) we differentiate (8) to get

$$U_x(x, t) = -\frac{x}{2\pi^{1/2}} \int_0^t \frac{G[y(\tau)]}{(t - \tau)^{3/2}} \exp \frac{-x^2}{4(t - \tau)} \, d\tau.$$

Since both this integral and that on the right in (8) converge uniformly in t on any interval of the form $0 \leq t \leq T$ for any $x > 0$, the above differentiation is valid. Making the substitution $\xi^2 = x^2/4(t - \tau)$ the above becomes

$$U_x(x, t) = -\frac{2}{\pi^{1/2}} \int_{x/2t^{1/2}}^{\infty} G\left[y\left(t - \frac{x^2}{4\xi^2}\right)\right] e^{-\xi^2} \, d\xi.$$

Form now the following difference

$$\Delta = \left| \int_0^{\infty} G[y(t)] e^{-\xi^2} \, d\xi - \int_{x/2t^{1/2}}^{\infty} G\left[y\left(t - \frac{x^2}{4\xi^2}\right)\right] e^{-\xi^2} \, d\xi \right|.$$

We wish to show that for any given fixed $t > 0$ one can find a δ such that $| x | < \delta \to \Delta < \epsilon$ where ϵ is arbitrarily small. To this end we consider the following inequality

$$\Delta \leq \Delta_1 + \Delta_2 + \Delta_3 ,$$

where

$$\Delta_1 = \left| \int_{x/2t^{1/2}}^{\theta} \left\{ G\left[y\left(t - \frac{x^2}{4\xi^2}\right)\right] - G[y(t)] \right\} e^{-\xi^2} \, d\xi \right|,$$

$$\Delta_2 = \left| \int_{\theta}^{\infty} \left\{ G[y(t)] - G\left[y\left(t - \frac{x^2}{4\xi^2}\right)\right] \right\} e^{-\xi^2} \, d\xi \right|,$$

$$\Delta_3 = \left| \int_0^{x/2t^{1/2}} G[y(t)] e^{-\xi^2} \, d\xi \right|.$$

By the continuity of G and y (see Hypothesis I and Lemma I) we know that there exists a bound $\overline{G}$, such that $\Delta_1 < 2\overline{G}\theta$. Hence, when $\theta = \epsilon/6\overline{G}$

$$\Delta_1 < \epsilon/3, \qquad \text{for } x < \epsilon/3\overline{G}\, t^{1/2}.$$

Similarly, there exists a number η such that

$$\frac{x^2}{4\xi^2} < \eta \rightarrow \left| G[y(t)] - G\left[y\left(t - \frac{x^2}{4\xi^2}\right)\right] \right| < \frac{\epsilon}{3}.$$

On the interval $\theta \leq t < \infty$,

$$\frac{x^2}{4\xi^2} \leq \frac{x^2}{4\theta^2}$$

$$\frac{x^2}{4\theta^2} < \eta \Rightarrow x < 2\theta\eta^{1/2} = \frac{\epsilon}{3\overline{G}}\,\eta^{1/2}.$$

By taking $x < (\epsilon/3\overline{G})\eta^{1/2}$ we have

$$\Delta_2 < \frac{\epsilon}{3}\int_0^\infty e^{-\xi^2}\,d\xi < \frac{\epsilon}{3}.$$

Turning now to Δ_3 we can write

$$\Delta_3 < \int_0^\theta G[y(t)]e^{-\xi^2}\,d\xi < \overline{G}\int_0^{\epsilon/3\overline{G}} e^{-\xi^2}\,d\xi < \frac{\epsilon}{3}.$$

Hence, if we choose δ to be the smaller of $\epsilon/3\overline{G}\,\eta^{1/2}$ and $\epsilon/3\overline{G}\,t^{1/2}$ we get

$$x < \delta \Rightarrow \Delta < \epsilon$$

This implies that

$$\lim_{x\to 0} U_x(x, t) = -\lim_{x\to 0}\frac{2}{\pi^{1/2}}\int_{x/2t^{1/2}}^\infty G\left[y\left(t - \frac{x^2}{4\xi^2}\right)\right]e^{-\xi^2}\,d\xi$$

$$= \frac{-2}{\pi^{1/2}}\int_0^\infty G[y(t)]e^{-\xi^2}\,d\xi = \frac{-2}{\pi^{1/2}}G[y(t)]\int_0^\infty e^{-\xi^2}\,d\xi.$$

Therefore,

$$U_x(0, t) = -G[y(t)]$$

Considering now condition (3c) we recall from Hypotheses II and III that $|\,y(t)\,| \leq 1$ implies $G[y] \geq 0$. Making use of the fact that $G[y]$ does not change sign on the interval of integration, we can invoke the theorem of the mean for definite integrals [3] to get

$$|\,U(x, t)\,| = \int_0^t \frac{G[y(\tau)]}{\pi^{1/2}(t - \tau)^{1/2}}\exp\frac{-x^2}{4(t - \tau)}\,d\tau = \exp\frac{-x^2}{4(t - \tau_1)}\int_0^t \frac{G[y(\tau)]}{\pi^{1/2}(t - \tau)^{1/2}}\,d\tau,$$

where $0 < \tau_1 < t$.
By (6) this becomes

$$U(x, t) = \exp\left[\frac{-x^2}{4(t - \tau_1)}\right]y(t) < y(t) \leq 1.$$

The remaining steps in the proof of Theorem 1 are somewhat lengthy but not difficult and we shall leave them to the reader.

The result of our work so far has been to show that any solution of equation (6) which is bounded in absolute value by 1 will yield, through formula (8), a solution of the nonlinear boundary value problem stated in Part II. We shall see later that when the input function, $G[y]$, satisfies a Lipschitz condition on the unit interval, the nonlinear integral equation (6) and the differential boundary value problem are completely equivalent, i.e. every solution of one will give a solution of the other. From now on we shall be concerned exclusively with equation (6) which we shall refer to as the surface temperature equation.

4. The linear case. Before studying the surface temperature equation in its most general form, we shall consider briefly the important special case in which it is linear. This case arises when the film transfer factor is assumed to have a constant value, say f_0. The input function $G[y(t)]$, then becomes

$$\frac{[1 - y(t)]f_0}{k} = h_0[1 - y(t)]$$

and equation (6) reduces to

$$y(t) = h_0 \int_0^t \frac{1 - y(\tau)}{\pi^{1/2}(t - \tau)^{1/2}}\, d\tau. \tag{6*}$$

The above equation is easily solved by the Laplace transform and gives

$$y(t) = \frac{h_0 t^{1/2}}{\Gamma(3/2)} - h_0^2 \int_0^t e^{h^2_0 \tau} \operatorname{erfc}\,(h\tau^{1/2})\, d\tau, \tag{9}$$

$$\text{where } \operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = 1 - \frac{2}{\pi^{1/2}} \int_0^x e^{-\lambda^2}\, d\lambda = \frac{2}{\pi^{1/2}} \int_x^\infty e^{-\lambda^2}\, d\lambda.$$

In the more general case where (6) is nonlinear, the Laplace transform will no longer be useful, since the transform of the product of two functions cannot be expressed in terms of the transforms of the individual functions. For this reason, we solve (6*) by a different method which can be carried over to the nonlinear case. This second method is that of successive approximations. We begin with the function $y_0(t) \equiv 0$ as the first approximation to $y(t)$. We then construct the sequence of functions $\{y_n(t)\}$ defined by the recursive formula

$$y_{n+1}(t) = h_0 \int_0^t \frac{1 - y_n(\tau)}{\pi^{1/2}(t - \tau)^{1/2}}\, d\tau. \tag{10}$$

Making use of the formula

$$\int_0^t \frac{\tau^{n/2}}{[\pi(t - \tau)]^{1/2}}\, d\tau = \frac{\Gamma(n/2 + 1)}{\Gamma(n/2 + 3/2)}\, t^{(n+1)/2}, \tag{11}$$

we obtain

$$y_1(t) = \frac{h_0}{\Gamma(3/2)}\, t^{1/2},$$

$$y_2(t) = \frac{h_0}{\Gamma(3/2)}\, t^{1/2} - \frac{h_0^2}{\Gamma(2)}\, t,$$

$$y_3(t) = \frac{h_0}{\Gamma(3/2)}\, t^{1/2} - \frac{h_0^2}{\Gamma(2)}\, t + \frac{h_0^3}{\Gamma(5/2)}\, t^{3/2},$$

etc.

In this way we are led to the series

$$y(t) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{h_0^k}{\Gamma(k/2 + 1)} t^{k/2} \tag{12}$$

which obviously converges uniformly on any interval of the form $0 \leq t \leq T$. We can therefore verify that $y(t)$ as given by (12) satisfies (6*) by substituting it into that equation and interchanging the order of summation and integration.

We shall see in Part V that the above well-known procedure, with a slight modification, can be successfully applied to the nonlinear surface temperature equation.

5. The nonlinear integral equation. In this section we shall study the following singular, nonlinear, integral equation of the Volterra type

$$y(t) = \int_0^t \frac{G[y(\tau)]}{\pi^{1/2}(t - \tau)^{1/2}} d\tau. \tag{6}$$

We showed in Part III that it must be satisfied by the surface temperature, $U(0, t)$, of the problem stated in Part II, and we therefore refer to it as the surface temperature equation. If we can find a solution of (6) which is bounded in absolute value by 1, then by formula (8) we can construct a solution, $U(x, t)$ for the problem stated in Part II. Furthermore, if (6) can be shown, under certain conditions on G, to have a unique solution and this solution is bounded in absolute value by 1, then it will follow that under these same conditions the original heat transfer problem has a unique solution. All our results will be obtained by exploiting Hypotheses A, B, and C of Part II and it is important that the reader always keep these clearly in mind.

First we shall prove an existence theorem stating that (6) always has at least one solution which is bounded in absolute value by 1. To this end we introduce the following functional transformation

$$\mathfrak{I}[z](t) = \int_0^t \frac{G^*[z(\tau)]}{\pi^{1/2}(t - \tau)^{1/2}} d\tau, \tag{13}$$

where

$$G^*[z(t)] = \begin{cases} G[z(t)] & \text{when} \quad z(t) \leq 1, \\ 0 & \text{when} \quad z(t) > 1. \end{cases}$$

Since $G[1] = 0$, $G^*[z]$ is continuous. Now pick $T > 0$ arbitrarily large and regard it as fixed. Henceforth we shall confine our attention to the interval $0 \leq t \leq T$.

Consider the sequence of functions defined by the recursive formula

$$z_{n+1}(t) = \mathfrak{I}[z_n](t), \qquad \text{where} \qquad z_0(t) \equiv 0. \tag{14}$$

We can evaluate $z_1(t)$ explicitly since $G[z_0(t)]$ has the constant value $G[0]$ which we shall denote by G_0 . We get

$$z_1(t) = \frac{G_0}{\Gamma(3/2)} t^{1/2},$$

$z_2(t)$ cannot be evaluated without knowing the input function, $G[y]$, but the following lemma enables us to infer some important facts about the behavior of the sequence $\{z_n(t)\}$.

Lemma II. If $u(t)$ and $v(t)$ are two functions which are continuous for $t \geq 0$, which satisfy the inequality $u(t) > v(t)$ for all $t > 0$, and if $u(0) < 1$, then

$$\mathfrak{I}[u](t) < \mathfrak{I}[v](t), \qquad \text{for all} \qquad t > 0.$$

The above lemma is a very simple consequence of the monotone decreasing nature of $G[y]$ stated in Hypothesis C. Since $z_0(0) = z_1(0) = 0$ and since $z_1(t) > z_0(t)$ for all $t > 0$, it follows from Lemma II by the method of induction that the following inequalities hold for all $t > 0$.

$$z_1(t) > z_3(t) > \cdots > z_{2n+1}(t) > \cdots \tag{15}$$

$$z_0(t) < z_2(t) < \cdots < z_{2n}(t) < \cdots \tag{16}$$

and also

$$z_{2n+1}(t) > z_{2k}(t), \tag{17}$$

where n and k are independent positive integers or zero. Since $z_0(t) \equiv 0$, it follows from the above inequalities that for all $n > 0$, $z_n(t) > 0$ for all $t > 0$. The first few of these functions are sketched in the figure below.

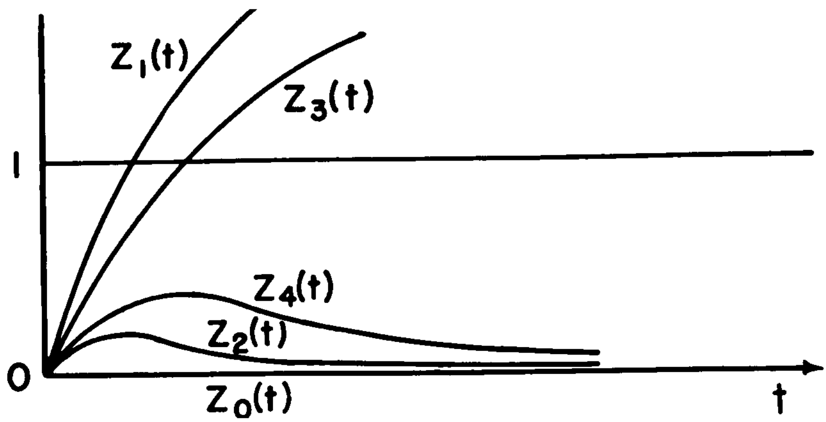

Fig. 1.

All of these functions are bounded from below by $z_0(t)$ and from above by $z_1(t)$. Therefore, on the arbitrary fixed interval $[0, T]$, they are equibounded. We shall find shortly that the $\{z_n(t)\}$ are also equicontinuous. From this it follows by Arzela's theorem that the closure of the set $\{z_n(t)\}$ in the usual topology is compact. It is easy to see, further, that the $\{z_{2n+1}(t)\}$ converge uniformly from above to some function, say $u(t)$, and the $\{z_{2n}(t)\}$ converge uniformly from below to a function, $v(t)$. If $u(t) \equiv v(t)$, then we have a fixpoint of the transformation (13). If $u(t)$ and $v(t)$ are not the same function, then each will be a fixpoint of the square of the transformation (13) but neither will be a fixpoint of the transformation itself.

In order to prove that the transformation (13) does have a fixpoint, we enlarge our attention to the collection of all functions which are continuous on the arbitrarily large fixed interval $[0, T]$. We consider these functions as elements of a Banach space, $\mathfrak{C}$, by introducing the usual topology, i.e. we denote the norm of an element f by $\|f\|$ and define it as follows

$$\|f\| = \text{l.u.b.}\ |f(t)| \qquad \text{on} \qquad [0, T].$$

By the distance between two elements f and g we mean the norm of their difference $\|f - g\|$, and by convergence in $\mathfrak{C}$ we mean uniform convergence. It is easily verified that these definitions do make $\mathfrak{C}$ a Banach space, i.e., a normed linear complete space.

In this space we consider the set Z consisting of all members of $\mathfrak{C}$ which are non-negative, bounded above by $z_1(t)$ and which have a modulus of continuity given by

$$|z(t_1) - z(t_2)| \leq \frac{4G_0}{\pi^{1/2}} (t_1 - t_2)^{1/2}.$$

It is easy to show that Z is convex and compact.

We shall now show that the transformation carries H into itself. From Lemma II it is easily seen that the transform under (13) of any member of $\mathfrak{C}$ which is contained between $z_0(t)$ and $z_1(t)$ must also lie within these bounds. Moreover, assuming $t_2 > t_1$ and denoting $\mathfrak{I}[z](t)$ by $u(t)$, we shall have

$$|u(t_2) - u(t_1)| = \left| \int_0^{t_2} \frac{G^*[z(\tau)]}{\pi^{1/2}(t - \tau)^{1/2}}\, d\tau - \int_0^{t_1} \frac{G^*[z(\tau)]}{\pi^{1/2}(t - \tau)^{1/2}}\, d\tau \right|$$

$$\leq \left| \int_0^{t_1} \frac{G^*[z(\tau)]}{\pi^{1/2}} [(t_2 - \tau)^{-1/2} - (t_1 - \tau)^{-1/2}]\, d\tau \right| + \left| \int_{t_1}^{t_2} \frac{G^*[z(\tau)]}{\pi^{1/2}(t_2 - \tau)^{1/2}}\, d\tau \right|$$

$$< \left| \frac{2G_0}{\pi^{1/2}} [t_2^{1/2} - (t_2 - t_1)^{1/2} - t_1^{1/2}] \right| + \frac{2G_0}{\pi^{1/2}} (t_2 - t_1)^{1/2}$$

since $G^*[z(t)] \leq G[0]$ for all $t > 0$, and $G[0] = G_0$. From here we find that

$$|u(t_2) - u(t_1)| < \frac{4G_0}{\pi^{1/2}} (t_2 - t_1)^{1/2}.$$

This means that $u(t)$ has the same modulus of continuity as that prescribed for the members of Z. In other words, we have shown that $\mathfrak{I}$ carries H into a subset of itself, namely Z. Moreover, it is easily seen that $\mathfrak{I}$ is continuous on H, since if u and v are any two members of H, then

$$|\mathfrak{I}[u](t) - \mathfrak{I}[v](t)| = \left| \int_0^t \frac{G^*[u(\tau)] - G^*[v(\tau)]}{\pi^{1/2}(t - \tau)^{1/2}}\, d\tau \right|$$

$$= \left| \int_0^t \frac{G[u^*(\tau)] - G[v^*(\tau)]}{\pi^{1/2}(t - \tau)^{1/2}}\, d\tau \right|$$

where

$$u^*(t) = \begin{cases} u(t) & \text{when} \quad u(t) \leq 1 \\ 1 & \text{when} \quad u(t) > 1 \end{cases} \tag{18}$$

From this and the continuity of G, it is very simple to show that $\mathfrak{I}$ is continuous on H.

Summarizing the above results, we have that $\mathfrak{I}$ is a functional transformation which carries a convex, compact set H of the Banach space $\mathfrak{C}$ into itself continuously. Using

Schauder's generalization from Euclidian space to Banach space of the Brouwer fixpoint theorem, [5] we can infer that $\mathfrak{J}$ has a fixpoint in H. This result will be stated in the following theorem.

THEOREM 2. *On any interval of the form* $[0, T]$ *there is at least one continuous function,* $y(t)$, *such that* $0 < y(t) < (G_0 t^{1/2}/\Gamma(3/2))$ *for all* $t > 0$ *and such that* $y(t) = \mathfrak{J}[y](t)$.

In order to prove the existence of a solution of equation (6), we must prove the following theorem.

THEOREM 3. *If* $y(t)$ *is a continuous fixpoint of the transformation* (13) *for* $0 \leq t \leq T$, *then* $y(t) \leq 1$ *for* $0 \leq t \leq T$.

To prove this, we shall assume that the theorem is false and arrive at a contradiction. Let S be the set of points on the open interval $0 < t < T$ for which $y(t) > 1$. Since $y(t)$ is continuous, S is open. In fact, S consists of disjoint open intervals. Now define

$$y^*(t) = \begin{cases} y(t) & \text{when} \quad t \not\in S, \\ 1 & \text{when} \quad t \in S. \end{cases}$$

The continuity of $y^*(t)$ follows from that of $y(t)$, and obviously

$$\mathfrak{J}[y](t) = \mathfrak{J}[y^*](t). \tag{19}$$

Let t be an arbitrary point of S. Then t belongs to an open interval of the form $t_1 < t < t_2$, and

$$\mathfrak{J}[y](t) = \int_0^{t_1} \frac{G^*[y^*(\tau)]}{[\pi(t-\tau)]^{1/2}}\,d\tau + \int_{t_1}^{t} \frac{G^*[y^*(\tau)]}{[\pi(t-\tau)]^{1/2}}\,d\tau.$$

But since $y^*(t) = 1$ and $G[y^*(t)] = 0$ when $t_1 \leq t \leq t_2$, and since $G^*[y^*] = G[y^*]$ is never negative we can write

$$\mathfrak{J}[y](t) = \int_0^{t_1} \frac{G[y^*(\tau)]}{\pi^{1/2}(t-\tau)^{1/2}}\,d\tau + \int_{t_1}^{t} \frac{G[y^*(\tau)]}{\pi^{1/2}(t-\tau)^{1/2}}\,d\tau = \int_0^{t_1} \frac{G[y^*(\tau)]}{\pi^{1/2}(t-\tau)^{1/2}}\,d\tau$$

$$\therefore \mathfrak{J}[y](t) < \int_0^{t_1} \frac{G[y^*]\,d\tau}{\pi^{1/2}(t_1-\tau)^{1/2}} = 1.$$

This means that $\mathfrak{J}[y^*](t) < 1$ on S. But since $\mathfrak{J}[y^*](t) = \mathfrak{J}[y](t)$, we infer that $y(t) = \mathfrak{J}[y](t) < 1$ on S. This is a contradiction unless S is empty and thus the theorem is proved.

We now notice that for any function, $f(t)$, which does not exceed 1, $G[f(t)] = G^*[f(t)]$ and for continuous f

$$\mathfrak{J}[f](t) = \int_0^t \frac{G^*[f(\tau)]}{\pi^{1/2}(t-\tau)^{1/2}}\,d\tau = \int_0^t \frac{G[f(\tau)]}{\pi^{1/2}(t-\tau)^{1/2}}\,d\tau.$$

Since we have just shown that the transformation $\mathfrak{J}$ always has at least one continuous fixpoint, $y(t)$, which is bounded between 0 and 1, $y(t)$ has the following property

$$y(t) = \mathfrak{J}[y](t) = \int_0^t \frac{G^*[y(\tau)]}{\pi^{1/2}(t-\tau)^{1/2}}\,d\tau = \int_0^t \frac{G[y(\tau)]}{\pi^{1/2}(t-\tau)^{1/2}}\,d\tau. \tag{20}$$

We have now obtained the following existence theorem.

THEOREM 4. *For any input function, $G[y]$, there is, on any interval of the form $0 \leq t \leq T$, at least one continuous surface temperature function, $y(t)$, satisfying equation* (6) *and having the property that $0 < y(t) \leq 1$.*

By virtue of Theorem 1, this means that the nonlinear differential boundary value problem stated in Part II always has a solution.

Theorem 4 expresses the best result which we have been able to obtain with the most general input function, i.e. with the input function subject only to Hypotheses A, B, and C of Part II. After strengthening Hypothesis A, we shall proceed to give a method for obtaining a solution of the surface temperature equation; to prove the uniqueness of this solution; and to find other properties of the surface temperature. This procedure of deriving increasingly better results by progressively strengthening our hypotheses is of mathematical rather than of physical interest, since the strongest hypothesis which we shall ever use in place of A, namely that G is analytic, can safely be assumed in all heat transfer problems of physical significance.

THEOREM 5. *If $G[u]$ satisfies a Lipschitz condition on the closed unit interval $0 \leq u \leq 1$, then the set of approximating functions, $\{z_n(t)\}$ defined by the recursive formula* (14) *converge to a solution of* (6) *for all $t > 0$ and the convergence is uniform on every finite interval. If we denote this solution by $y(t)$, then $0 < y(t) \leq 1$ for all $t > 0$.*

The proof is elementary, but since we shall need the same method, with slight variations, in following demonstrations, we shall go through it once here and merely refer to it later.

In view of the fact that $G^*[u(t)] = G[u^*(t)]$ where $u^*(t)$ is defined as in (18), and since $z_n^*(t)$ is between 0 and 1 for all $t \geq 0$ we can write

$$| z_2(t) - z_1(t) | = \int_0^t \frac{| G[z_1^*(\tau)] - G[z_0^*(\tau)] |}{[\pi(t - \tau)]^{1/2}} \, d\tau < L \int_0^t \frac{| z_1^*(\tau) - z_0^*(\tau) |}{[\pi(t - \tau)]^{1/2}} \, d\tau,$$

where L is the Lipschitz constant. Since $| z_1^*(t) - z_0^*(t) | < 1$ for all $t > 0$, this becomes

$$| z_2(t) - z_1(t) | < L \int_0^t \frac{d\tau}{\pi^{1/2}(t - \tau)^{1/2}} = \frac{L}{\Gamma(3/2)} \, t^{1/2}.$$

Similarly,

$$| z_3(t) - z_2(t) | = \int_0^t \frac{| G[z_2^*(\tau)] - G[z_1^*(\tau)] |}{\pi^{1/2}(t - \tau)^{1/2}} \, d\tau < L \int_0^t \frac{| z_2^*(\tau) - z_1^*(\tau) |}{\pi^{1/2}(t - \tau)^{1/2}} \, d\tau.$$

It is easily seen that

$$| z_{k+1}^*(t) - z_k^*(t) | \leq | z_{k+1}(t) - z_k(t) |.$$

Therefore,

$$| z_2^*(t) - z_1^*(t) | < \frac{L}{\Gamma(3/2)} \, t^{1/2}$$

Substituting this into the above bound for $| z_3(t) - z_2(t) |$ gives

$$| z_3(t) - z_2(t) | < \frac{L^2}{\Gamma(3/2)} \int_0^t \frac{\tau^{1/2}}{\pi^{1/2}(t - \tau)^{1/2}} \, d\tau = \frac{L^2}{\Gamma(3/2)} \frac{\Gamma(3/2)}{\Gamma(2)} \, t = \frac{L^2}{\Gamma(2)} \, t.$$

Continuing in this way with the help of formula (11) we get

$$| z_{2n+1}(t) - z_{2n}(t) | < \frac{L^{2n}}{\Gamma(n+1)} t^n.$$

This difference approaches 0 as $n \to \infty$ for any value of t. By the inequalities (15), (16), and (17) it is therefore clear that on any finite interval the functions $\{z_n(t)\}$ converge uniformly toward a limit function, $y(t)$. By virtue of the uniformity of convergence we can pass to the limit under the integral sign as $n \to \infty$ in the recursive formula (14) and in this way show that $y(t)$ must be a fixpoint of the transformation $\mathfrak{I}$. But we proved in Theorem 3 that any continuous fixpoint of $\mathfrak{I}$ is bounded above by 1. This implies that $y(t)$ satisfies (20) and is therefore a solution of (6).

Extending the Lipschitz condition to hold beyond the unit interval we prove the uniqueness of the surface temperature function.

THEOREM 6. *If $G[u]$ satisfies a Lipschitz condition on the interval* $[0, 1 + \epsilon]$ *for any* ϵ *greater than zero, then equation* (6) *has a unique bounded solution.*

We already know from Theorem 4 that (6) has at least one solution, $y(t)$ such that $0 < y(t) \leq 1$ for all $t > 0$. Let $u(t)$ be any other bounded solution of (6). We know by Lemma 1 that $u(t)$ is continuous and that $u(0) = 0$. Since the integrand of (6) is positive when $u < 1$, we also know that the origin is not a limit point of zeros of $u(t)$. Suppose that $u(t)$ has zeros to the right of the origin and let t_0 be the first one. Let t_k be the first value of t on the interval $0 < t < t_0$ for which $y(t) = 1 + \epsilon/k$ for any positive integer k. Then for $0 < t \leq t_k$ we have $0 < u(t) \leq 1 + \epsilon$ and thus the Lipschitz condition will hold for G. Applying the method of successive approximations illustrated in the proof of Theorem 5 to the difference $u(t_k) - y(t_k)$ we obtain

$$| u(t_k) - y(t_k) | < \frac{(1+\epsilon)L^n}{\Gamma(n/2+1)} t_k^{n/2} \qquad \text{for all } n. \tag{21}$$

Since this difference tends to zero as n approaches infinity, we see that $u(t_k) = y(t_k) \leq 1$ which is a contradiction. Hence there is no value of t on the interval $0 < t < t_0$ for which $u(t)$ exceeds 1. But here again is a contradiction because it is impossible that $u(t_0) = 0$ since by Hypotheses B and C there can have been no negative contribution to the integral in (6). This means that $u(t)$ remains between 0 and 1. Therefore we can use the method of successive approximations to show that (21) holds for all t and hence $u(t) \equiv y(t)$.

THEOREM 7. *If $G[u]$ is analytic on the closed unit interval* $[0, 1]$, *then equation* (6) *has a unique bounded solution, $y(t)$, and $y(t)$ is analytic for all $t > 0$.*

The uniqueness follows from the fact that since G is analytic on the closed unit interval, it can be analytically continued onto a somewhat larger interval $[0, 1 + \epsilon]$. G obviously satisfies a Lipschitz condition on this larger interval and by Theorem 6 we have the uniqueness of the solution.

The proof of the analyticity, which is somewhat lengthy, divides itself into the following three steps. First we shall show how the solution, $y(t)$, of (6) can be represented as the limit of a certain uniformly convergent sequence $\{u_n(t)\}$. Second, we shall show how $y(t)$ can be represented as the limit of a uniformly convergent sequence, $\{w_n(t)\}$, of analytic functions of the real variable, t. Third, we consider t to be a complex variable and show that a sequence, $\{W_n(t)\}$, of analytic functions of the complex variable converge uniformly to a function, $Y(t)$, which coincides with $y(t)$ along the real axis.

We begin the first step by introducing a set of functions defined as follows:

$$u_0(t) \equiv 0,$$

$$u_{n+1}(t) = \mathfrak{J}_\epsilon[u_n](t) = \int_0^t \frac{G_\epsilon[u_n(\tau)]}{[\pi(t-\tau)]^{1/2}}\,d\tau = \int_0^t \frac{G[\bar{u}_n(\tau)]}{[\pi(t-\tau)]^{1/2}}\,d\tau, \tag{22}$$

$$G_\epsilon[u_n(t)] = \begin{cases} G[u_n(t)], & \text{when} \quad u_n(t) \le 1+\epsilon, \\ G[1+\epsilon], & \text{when} \quad u_n(t) > 1+\epsilon, \end{cases}$$

and where

$$\bar{u}_n(t) = \begin{cases} u_n(t), & \text{when} \quad u_n(t) \le 1+\epsilon, \\ 1, & \text{when} \quad u_n(t) > 1+\epsilon. \end{cases}$$

We see immediately that when $\epsilon = 0$, the sequence $\{u_n(t)\}$ reduces to the sequence $\{z_n(t)\}$ already introduced by the recursive formula (14). Also, $u_1(t) = z_1(t) = G_0/\Gamma(3/2)t^{1/2}$ for any ϵ, but $u_2(t)$, unlike $z_2(t)$, will eventually fall below the t-axis for any $\epsilon > 0$ because of the negative contribution to the integral in (22) when the argument

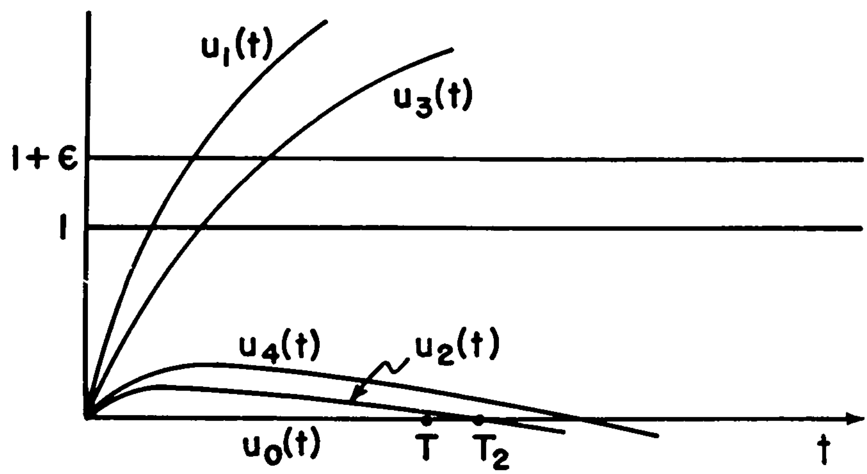

FIG. 2.

of the input function is greater than 1. Let us denote by T_2 this zero of $u(t)$. It is a simple matter to verify the intuitively obvious fact that T_2 is a continuous function of ϵ which decreases from ∞ to some positive number $T_2(0)$ as ϵ increases from 0 to ∞. Confining our attention as usual to a fixed arbitrarily large interval $[0, T]$ on the t-axis we fix ϵ at a constant value sufficiently small that $T_2(\epsilon) > T$ and that $1 + \epsilon$ lies within the interval of analyticity of G.

Recalling the monotone decreasing nature of G, and making use of the fact that $u_2(t) > u_0(t)$ for $0 < t \leq T$, we can easily verify the following inequalities for $0 < t \leq T$:

$$u_1(t) > u_3(t) > u_5(t) > \cdots > u_{2n+1}(t) > \cdots , \tag{23}$$

$$u_0(t) < u_2(t) < u_4(t) < \cdots < u_{2n}(t) < \cdots \tag{24}$$

and

$$u_{2n+1}(t) > u_{2k}(t), \tag{25}$$

where n and k are independent positive integers or zero. Using these inequalities and the method demonstrated in the proof of Theorem 5, we can prove that the sequence $\{u_n(t)\}$ converges uniformly on $[0, T]$ to a fixpoint, $y(t)$, of the transformation $\mathfrak{I}_\epsilon[u](t)$ defined in (22).

We must now show that $y(t)$ is the solution of (6) on the interval under consideration. From the above inequalities we know that $y(t) > 0$ for $0 < t \leq T$. Assume now that t_1 is the first value of t for which $y(t) = 1 + \epsilon$. Then on the closed interval $[0, t_1]$ $G_\epsilon[y(t)] = G[y(t)]$ and $y(t)$ is therefore the solution of (6) on this subinterval. But this means that $y(t) \leq 1$ and contradicts the assumption that $y(t_1) = 1 + \epsilon$. Hence $y(t)$ is the solution of (6) on the entire interval $[0, T]$.

The functions $\{u_n(t)\}$ are not analytic because of the truncating process employed in their definition. We shall now represent their limit function, $y(t)$, as the limit of a uniformly convergent sequence of analytic functions. Choose N so large that

$$n > N \rightarrow 0 < u_n(t) < 1 + \epsilon/2 \qquad \text{for} \qquad 0 \leq t \leq T \tag{26}$$

and pick k so large that $2k > N$. It is our purpose to interpose an analytic function, say $w_0(t)$, between $u_{2k}(t)$ and $u_{2k+2}(t)$ on $0 < t \leq T$. By inequalities (23), (24), and (25), and by reasoning similar to that on which Lemma 2 is based, it is easily seen that the successive transforms of $w_0(t)$ under $\mathfrak{I}_\epsilon$ will lie between those of $u_{2k}(t)$ and $u_{2k+2}(t)$. If we designate these successive transforms by $\{w_n(t)\}$ it is therefore clear that $\lim_{n\to\infty} w_n(t) = y(t)$ uniformly on $[0, T]$. Moreover, these functions will be analytic on the open interval $0 < t < T$, since none will be truncated in the recursive formula. From (22) and (26) we see that their recursive definition could be written as follows:

$$w_{n+1}(t) = \int_0^t \frac{G[w_n(\tau)]}{[\pi(t - \tau)]^{1/2}} \, d\tau. \tag{27}$$

In other words, each $w_n(t)$ is the fractional integral of order 1/2 of a function analytic for $t > 0$.

The $u_n(t)$ themselves are analytic on the open interval from the origin up to the point $t = \Gamma[(3/2)(1 + \epsilon)/G_0]^2 = T_\epsilon$ where $u_1(t)$ is truncated. On this interval $[0, T_\epsilon]$ the $\{u_n(t)\}$ have the same recursive definition as that for the $\{w_n(t)\}$ shown in (27) except that $u_0(t) \equiv 0$. In the neighborhood $0 < t < T$, $u_{2k}(t)$ and $u_{2k+2}(t)$ can be expanded in powers of $\sqrt{t}$ as follows.

$$u_{2k}(t) = a_1 t^{1/2} + a_2 t + a_3 t^{3/2} + \cdots + a_n t^{n/2} + \cdots ,$$

$$u_{2k+2}(t) = b_1 t^{1/2} + b_2 t + b_3 t^{3/2} + \cdots + b_n t^{n/2} + \cdots .$$

Suppose that $a_i = b_i$ for $i = 1, 2, 3, \cdots, m$ and that $a_{m+1} \neq b_{m+1}$.

Clearly $b_{m+1} > a_{m+1}$. Now form the following functions

$$U_{2k}(t) = \frac{u_{2k}(t) - f_m(t)}{t^{(m+1)/2}}, \qquad U_{2k+2}(t) = \frac{u_{2k+2}(t) - f_m(t)}{t^{(m+1)/2}},$$

where

$$f_m(t) = \sum_{i=1}^{m} a_i t^{i/2} = \sum_{i=1}^{m} b_i t^{i/2}.$$

This gives two functions, $U_{2k+2}(t)$ and $U_{2k}(t)$, such that $U_{2k+2}(t) > U_{2k}(t)$ for $0 \leq t \leq T$. There will be a minimum distance, say δ, between the two functions on $[0, T]$. Approximate to the mean of the two functions by closer than $\delta/2$ with a polynomial, $P(t)$. Then we can write

$$U_{2k}(t) < P(t) < U_{2k+2}(t), \qquad 0 \leq t \leq T,$$

$$t^{(m+1)/2}U_{2k}(t) < t^{(m+1)/2}P(t) < t^{(m+1)/2}U_{2k+2}(t), \qquad 0 < t \leq T,$$

$$t^{(m+1)/2}U_{2k}(t) + f_m(t) < t^{(m+1)/2}P(t) + f_m(t) < t^{(m+1)/2}U_{2k+2}(t) + f_m(t), \qquad 0 < t \leq T.$$

But

$$t^{(m+1)/2}U_{2k}(t) + f_m(t) = u_{2k}(t) \qquad \text{and} \qquad t^{(m+1)/2}U_{2k+2}(t) + f_m(t) = u_{2k+2}(t).$$

Hence if we denote $t^{(m+1)/2}P(t) + f_m(t)$ by $w_0(t)$ we have

$$u_{2k}(t) < w_0(t) < u_{2k+2}(t) \qquad \text{for} \qquad 0 < t \leq T$$

and $w_0(t)$ is analytic on this interval. Using $w_0(t)$ in the recursive formula (27) we get the desired sequence of analytic functions, $\{w_n(t)\}$ converging uniformly to $y(t)$.

Now consider t to be a complex variable, $t = u + iv$. The functions $\{w_n(t)\}$ will

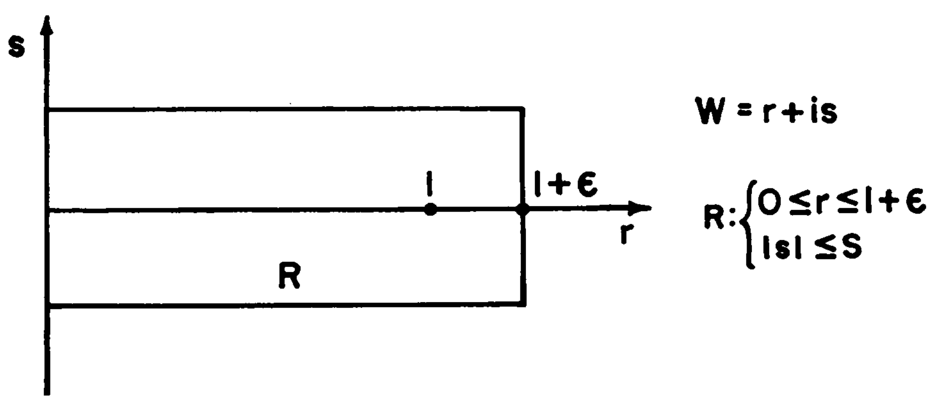

FIG. 3.

then become complex functions which we shall denote by $\{W_n(t)\}$ where $W_n = r_n + is_n$. Since G is analytic on the closed interval $[0, 1 + \epsilon]$ we know that it has an analytic continuation into a neighborhood of this interval in the complex W-plane. In particular we know that there is a closed domain, R, of the form shown below on which G is analytic.

Since G is analytic on R which is closed, we know that there exists a number, D, such that

$$W \varepsilon R \rightarrow | G[W] | < D, \tag{28}$$

and since $W_0(t)$ is a polynomial in powers of $\sqrt{t}$, we know that it is analytic in any region of the complex t-plane which does not include the origin. Consider now a region Δ in the t-plane of the form shown below

$$\Delta : \begin{cases} 0 < u < T \\ | v | < V \end{cases}$$

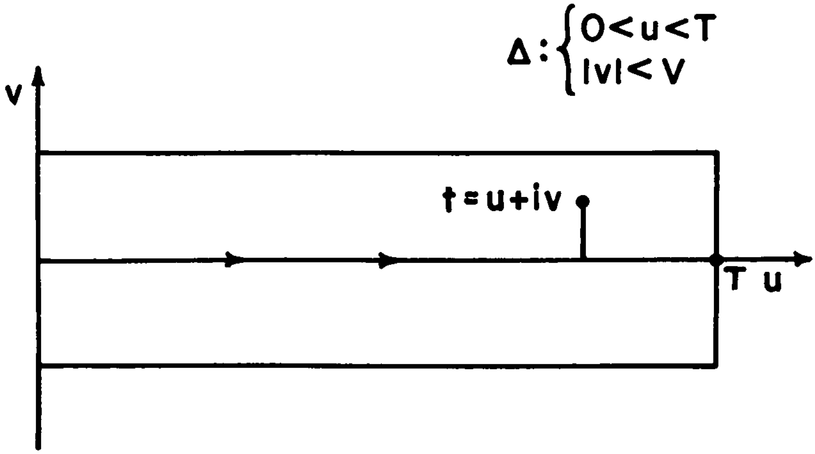

Fig. 4.

$W_0(t)$ can be regarded as a transformation which maps the region, Δ, conformally from the t-plane into the W-plane. Recalling (26) and the construction of $w_0(t)$, we see that $W_0(t)$ carries the segment $[0, T]$ into a subset of $[0, 1 + \epsilon/2]$ on the r-axis. From this and the continuity of $W_0(t)$, it follows that by taking V sufficiently small, say $V < V_1$, $W_0(t)$ will map Δ into R, i.e.

$$t\epsilon\Delta \rightarrow W_0(t)\epsilon R \rightarrow | G[W_0(t)] | < D.$$

It is now our purpose to show that by taking Δ sufficiently narrow,

$$t\epsilon\Delta \rightarrow W_n(t)\epsilon R, \qquad \text{for all } n. \tag{29}$$

Integrating along the path shown in Figure 4 we have

$$W_1(t) = \int_0^u \frac{G[W_0(\tau)]}{[\pi(t - \tau)]^{1/2}} \, d\tau + \int_u^t \frac{G[W_0(\tau)]}{[\pi(t - \tau)]^{1/2}} \, d\tau.$$

The real part of $W_1(t)$ can be estimated as follows

$$\Re\{W_1(t)\} < \int_0^u \frac{G[W_0(\tau)]}{\pi^{1/2}(u - \tau)^{1/2}} \, d\tau + \left| \int_u^t \frac{G[W_0(\tau)]}{\pi^{1/2}(t - \tau)^{1/2}} \, d\tau \right|$$

$$< 1 + \frac{\epsilon}{2} + D \int_u^{u+iv} \frac{d\tau}{\pi^{1/2}(u + iv - u - i\xi)^{1/2}}.$$

Along the vertical strip, $\tau = u + i\xi$ and $d\tau = id\xi$

$$\mathcal{R}\{W_1(t)\} < 1 + \frac{\epsilon}{2} + D \int_0^v \frac{d\xi}{\pi^{1/2}(v - \xi)^{1/2}} = 1 + \frac{\epsilon}{2} + \frac{D}{\Gamma(3/2)} v^{1/2}.$$

This shows that for all values of t belonging to Δ, we have

$$\mathcal{R}\{W_1(t)\} < 1 + \frac{\epsilon}{2} + \frac{D}{\Gamma(3/2)} V^{1/2}.$$

By requiring that

$$V < \left[\frac{\epsilon\Gamma(3/2)}{2D}\right]^2 = V_2,$$

we insure that $D/\Gamma(3/2)V^{1/2} < \epsilon/2$ and therefore $\mathcal{R}\{W_1(t)\} < 1 + \epsilon$.

Turning now to the imaginary part, we observe that along the stretch $[0, u]$,

$$\mathcal{I}\{(t - \tau)^{-1/2}\} < \frac{(2v)^{1/2}}{2[(u - \tau)^2 + v^2]^{1/2}},$$

$$\therefore \mathcal{I}\{W_1(t)\} < \frac{v^{1/2}}{(2\pi)^{1/2}} \int_0^u \frac{G[W_0(\tau)]}{[(u - \tau)^2 + v^2]^{1/2}} d\tau + \frac{D}{\Gamma(3/2)} v^{1/2}$$

$$< \frac{G_0 v^{1/2}}{(2\pi)^{1/2}} [\ln \{(u^2 + v^2)^{1/2} + u\} - \ln v] + \frac{D}{\Gamma(3/2)} v^{1/2}.$$

Thus, for any t belonging to Δ, the following inequality holds

$$\mathcal{I}\{W_1(t)\} \leq \frac{G_0 V^{1/2}}{(2\pi)^{1/2}} [\ln \{(T^2 + V^2)^{1/2} + T\} - \ln V] + \frac{D}{\Gamma(3/2)} V^{1/2}.$$

Since the first term on the right-hand side of this inequality tends to zero with V, there exists a number, V_3, such that

$$V < V_3 \rightarrow | \mathcal{I}\{W_1(t)\} | < S.$$

Therefore, by taking V to be smaller than the least of the three numbers V_1, V_2, V_3, we define a region Δ, in the complex t-plane which is mapped by the function $W_1(t)$ into the region R in the complex W-plane. By induction it is easy to show that each one of the functions $W_n(t)$ maps Δ into R.

We have now shown that $\{W_n(t)\}$ is a sequence of functions which are analytic in Δ, equibounded in Δ, and which converges uniformly on the interval $0 < t \leq T$, which is an infinite subset of Δ, to the function $y(t)$. By Vitali's theorem we can conclude that $\{W_n(t)\}$ converges uniformly throughout Δ to a function, $Y(t)$, which is the analytic continuation of $y(t)$ into the complex t-plane. $y(t)$ is therefore an analytic function of the real variable t for $0 < t < T$. This completes the proof of Theorem 7, since T was chosen arbitrarily large.

THEOREM 8. *If $G[y]$ is analytic on the unit interval $0 \leq y \leq 1$, then $y(t)$ is monotone increasing for all $t > 0$.*

By differentiating equation (6) we get

$$y'(t) = \int_0^t \frac{G'[y(\tau)]y'(\tau)}{[\pi(t - \tau)]^{1/2}} \, d\tau + \frac{G_0}{(\pi t)^{1/2}}. \tag{30}$$

Perhaps the easiest way of justifying the above result of differentiating the singular integral in the surface temperature equation with respect to the parameter t is to observe that (6) can be written $y(t) = G[y(t)]*1/(\pi t)^{1/2}$ where the star indicates the convolution or faltung operation. Then, by a well-known theorem on the differentiation of the faltung under hypotheses which are satisfied by the above functions [6], formula (30) follows immediately.

$y(t)$ is a power series in $t^{1/2}$ and is analytic for $t > 0$, but whereas by Lemma 1 $y(0) = 0$, $y'(t)$ approaches infinity like $t^{-1/2}$ as t approaches 0. If $y'(t)$ ever becomes negative, there must be a first zero, say t_0, to the right of which is an open interval, I, on which $y'(t)$ is less than 0. Let t_1 be any point belonging to I. Then we must have

$$y'(t_0) = \int_0^{t_0} \frac{G'[y(\tau)]}{\pi^{1/2}(t_0 - \tau)^{1/2}} \, y'(\tau) \, d\tau + \frac{G_0}{(\pi t_0)^{1/2}} = 0, \tag{31}$$

$$y'(t_1) = \int_0^{t_0} \frac{G'[y(\tau)]y'(\tau)}{\pi^{1/2}(t_1 - \tau)^{1/2}} \, d\tau + \int_{t_0}^{t_1} \frac{G'[y(\tau)]y'(\tau)}{\pi^{1/2}(t_1 - \tau)^{1/2}} \, d\tau + \frac{G_0}{(\pi t_1)^{1/2}}.$$

The latter equation can be written as follows

$$y'(t_1) = \int_0^{t_0} \left(\frac{t_0 - \tau}{t_1 - \tau}\right)^{1/2} \frac{G'[y(\tau)]}{\pi^{1/2}(t_0 - \tau)^{1/2}} \, y'(\tau) \, d\tau$$

$$+ \int_{t_0}^{t_1} \frac{G'[y(\tau)]}{\pi^{1/2}(t_1 - \tau)^{1/2}} \, y'(\tau) \, d\tau + \frac{G_0}{(\pi t_0)^{1/2}} \frac{t_0^{1/2}}{t_1^{1/2}}.$$

In the first integral on the right, the factor $[(t_0 - \tau)/(t_1 - \tau)]^{1/2}$ decreases monotonically from $t_0^{1/2}/t_1^{1/2}$ to zero. By the theorem of the mean we know that there exists a number τ_1 between 0 and t_0 such that

$$y'(t_1) = \left(\frac{t_0 - \tau_1}{t_1 - \tau_1}\right)^{1/2} \int_0^{t_0} \frac{G'[y(\tau)]}{\pi^{1/2}(t_0 - \tau)^{1/2}} \, y'(\tau) \, d\tau + \int_{t_0}^{t_1} \frac{G'[y(\tau)]y'(\tau)}{\pi^{1/2}(t_1 - \tau)^{1/2}} \, d\tau + \frac{t_0^{1/2}}{t_1^{1/2}} \frac{G_0}{(\pi t_0)^{1/2}}.$$

Since $G'[y]$ is negative for all y, and since $y'(t)$ has been assumed to be negative on the interval I, the second integral above must be positive. Comparing this last expression for $y'(t)$ with equation (31) and noticing that

$$\left(\frac{t_0 - \tau_1}{t_1 - \tau_1}\right)^{1/2} < \frac{t_0^{1/2}}{t_1^{1/2}},$$

it is obvious that $y'(t_1) < 0$. This contradiction shows that $y'(t)$ cannot become negative, and since $y'(t)$ is analytic for $t > 0$, we know that $y(t)$ must be monotone increasing.

THEOREM 9. *Let G_L be the space of input functions $G[u]$ which satisfy a Lipschitz condition with constant $<L$ on an interval of the form $0 \leq u \leq 1 + \epsilon$ for any $\epsilon > 0$. Let Y be the space of corresponding surface temperature functions, $y(t)$. As a metric we take the norm of the difference. The equation (6) represents a continuous implicit functional transformation from G_L to Y.*

By Theorem (6) we know that for each element G of G_L there is a unique element $y(t)$ of Y, and that $0 \leq y(t) \leq 1$ for all $t \geq 0$. Select arbitrarily any member G_1 of G_L and let $y_1(t)$ be its corresponding element in Y. Let G_2 and $y_2(t)$ be any other corresponding pair of functions. The difference between the two surface temperature can be written as follows

$$y_2(t) - y_1(t) = \int_0^t \frac{G_2[y_2(\tau)] - G_2[y_1(\tau)]}{\pi^{1/2}(t-\tau)^{1/2}}\, d\tau + \int_0^t \frac{G_2[y_1(\tau)] - G_1[y_1(\tau)]}{\pi^{1/2}(t-\tau)^{1/2}}\, d\tau.$$

If we specify that G_2 be chosen in such a way that $| G_2[u] - G_1[u] | < \delta$ for $0 \leq u \leq 1$, then a bound for the above difference is shown below:

$$| y_2(t) - y_1(t) | < L \int_0^t \frac{| y_2(\tau) - y_1(\tau) |}{\pi^{1/2}(t-\tau)^{1/2}}\, d\tau + \int_0^t \frac{\delta}{\pi^{1/2}(t-\tau)^{1/2}}\, d\tau. \tag{32}$$

In view of the fact that $| y_2(t) - y_1(t) | < 1$ for all $t > 0$, (32) becomes

$$| y_2(t) - y_1(t) | < L \int_0^t \frac{d\tau}{\pi^{1/2}(t-\tau)^{1/2}} + \delta \int_0^t \frac{d\tau}{\pi^{1/2}(t-\tau)^{1/2}} = \frac{L+\delta}{\Gamma(3/2)}\, t^{1/2}.$$

Substituting this bound back into (32) we get

$$| y_2(t) - y_1(t) | < \frac{L(L+\delta)}{\Gamma(2)}\, t + \frac{\delta}{\Gamma(3/2)}\, t^{1/2}.$$

By repeating this process of successive approximations n times we get

$$| y_2(t) - y_1(t) | < \frac{\delta t^{1/2}}{\Gamma(3/2)} + \frac{L\delta}{\Gamma(2)}\, t + \frac{L^2\delta}{\Gamma(5/2)}\, t^{3/2} + \cdots + \frac{L^n\delta}{\Gamma((n+3)/2)}\, t^{(n+1)/2}$$

$$+ \frac{L^{n+1}(L+\delta)}{\Gamma(n/2+2)}\, t^{n/2+1}.$$

Letting n tend toward infinity, the above can be written as follows

$$| y_2(t) - y_1(t) | < L\delta t\left[\frac{1}{\Gamma(2)} + \frac{L^2 t}{\Gamma(3)} + \frac{L^4 t^2}{\Gamma(4)} + \frac{L^6 t^3}{\Gamma(5)} + \cdots\right]$$

$$+ \delta t^{1/2}\left[\frac{1}{\Gamma(3/2)} + \frac{L^2 t}{\Gamma(5/2)} + \frac{L^4 t^2}{\Gamma(7/2)} + \frac{L^6 t^3}{\Gamma(7/2)} + \cdots\right].$$

Restricting ourselves to an arbitrarily large fixed interval $[0, T]$ we can now show that

$$| y_2(t) - y_1(t) | < \delta T^{1/2}[1 + LT^{1/2}]e^{L^2T}.$$

For any given L and T the above difference can be made arbitrarily small by taking δ sufficiently small. This completes the proof of Theorem 9.

Making use of the fact that any monotone continuous function, $G[u]$, which satisfies a Lipschitz condition with constant L can be approximated arbitrarily closely on any closed finite interval by a monotone analytic function which also satisfies a Lipschitz condition with constant L, we can draw the following conclusion.

THEOREM 10. *If the input function $G[u]$ satisfies a Lipschitz condition on the interval*

$0 \leq u \leq 1 + \epsilon$, *for any* ϵ *greater than zero, then the associated surface temperature function,* $y(t)$, *is non-decreasing for all* t *greater than zero.*

This theorem is a simple consequence of Theorems 8 and 9, and the proof will be left to the reader. We shall merely indicate here one way in which any input function, $G[u]$, can be approximated on $[0, 1 + \epsilon]$ by an analytic input function. Define $H[u, \lambda]$ as follows

$$H[u, \lambda] = \frac{\lambda}{\pi} \int_{-\infty}^{\infty} \frac{G_1[u + \xi]}{\xi^2 + \lambda^2} \, d\xi = \frac{\lambda}{\pi} \int_{-\infty}^{\infty} \frac{G_1[x]}{(x - u)^2 + \lambda^2} \, dx$$

$$= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} G_1[u + \lambda \tan \theta] \, d\theta,$$

$$G_1[u] = \begin{cases} G[1 + \epsilon] & \text{when} \quad u > 1 + \epsilon, \\ G[u] & \text{when} \quad 0 \leq u \leq 1 + \epsilon, \\ G[0] & \text{when} \quad u < 0. \end{cases}$$

It is easily verified that $H[u, \lambda]$ is continuous in λ, that $\lim_{\lambda \to 0} H[u, \lambda] = G_1[u]$ and hence that the convergence is uniform on the closed interval $[0, 1 + \epsilon]$. By examining the difference $H[u_2, \lambda] - H[u_1, \lambda]$ it is evident that the monotony of G_1 on $[0, 1 + \epsilon]$ implies that of H, and it is obvious that if G_1 satisfies a Lipschitz condition with constant L, the same is true of H. Furthermore, $H[u, \lambda]$ is analytic in u.

THEOREM 11. *If the input function* $G[u]$ *satisfies a Lipschitz condition on an interval* $[0, 1 + \epsilon]$ *for any* ϵ *greater than zero, and if* $y(t)$ *is the corresponding surface temperature function, then* $y(t) < 1$ *for all* $t \geq 0$, *and* $\lim_{t \to \infty} y(t) = 1$.

Assume that $y(t_1) = 1$. Then since by Theorem 10 $y(t)$ is non-decreasing, and since $0 < y(t) \leq 1$ for all $t > 0$, it follows that $y(t) \equiv 1$ for all $t \geq t_1$. Let t_2 be any value of t greater than t_1. Then

$$y(t_1) = \int_0^{t_1} \frac{G[y(\tau)]}{\pi^{1/2}(t_1 - \tau)^{1/2}} \, d\tau = 1,$$

$$y(t_2) = \int_0^{t_1} \frac{G[y(\tau)]}{\pi^{1/2}(t_2 - \tau)^{1/2}} \, d\tau + \int_{t_1}^{t_2} \frac{G[y(\tau)]}{\pi^{1/2}(t - \tau)^{1/2}} \, d\tau.$$

But since $y(t) \equiv 1$ for $t_1 \leq t \leq t_2$, it follows from Hypothesis B that $G[y(\tau)] = 0$ for $t_1 \leq \tau \leq t_2$. Therefore the last equation reduces to

$$y(t_2) = \int_0^{t_1} \frac{G[y(\tau)]}{\pi^{1/2}(t_2 - \tau)^{1/2}} \, d\tau < \int_0^{t_1} \frac{G[y(\tau)]}{\pi^{1/2}(t_1 - \tau)^{1/2}} \, d\tau = 1,$$

since both integrands are positive for $0 \leq t \leq t_1$ and $t_2 > t_1$. This last conclusion that $y(t_2) < 1$ contradicts the earlier implication that $y(t) \equiv 1$ for all $t \geq t_1$. Hence our assumption that $y(t_1) = 1$ is impossible and we have that $y(t) < 1$ for all $t \geq 0$.

In order to prove that $\lim_{t \to \infty} y(t) = 1$, assume that this is false. Then by the non-decreasing nature of $y(t)$ we infer that there exists some number $\delta > 0$ such that $y(t) <$

$1 - \delta$ for all $t > 0$. By Hypotheses A, B, and C this means that $G[y(t)] > G[1 - \delta] > 0$ for all $t > 0$. This means that

$$y(t) > \int_0^t \frac{G[1 - \delta]}{\pi^{1/2}(t - \tau)^{1/2}}\, d\tau = \frac{G[1 - \delta]}{\Gamma(3/2)}\, t^{1/2}.$$

But the right-hand side of the above inequality can be made arbitrarily large by taking t sufficiently large. This contradicts the fact that $y(t) < 1$ for all $t > 0$. Hence δ does not exist and therefore $\lim_{t\to\infty} y(t) = 1$. This completes the proof of Theorem 11.

In view of this last result it is not hard to see how the hypotheses in Theorems 6, 7, 8, 10, and 11 can be somewhat weakened. Let $G[u]$ be any input function which satisfies a Lipschitz condition on the closed unit interval $[0, 1]$. Clearly, $G[u]$ can be continued outside this interval as an input function in such a way as to satisfy a Lipchitz condition in the large. One such extension is shown below:

$$G_1[u] = \begin{cases} 1 - u & \text{when} \quad u \geq 1, \\ G[u] & \text{when} \quad 0 \leq u < 1, \\ G[0] - u & \text{when} \quad u < 0. \end{cases}$$

By Theorem 6 there is a unique surface temperature function, $y(t)$, corresponding to G_1, and $y(t)$ is independent of the behavior of G_1 outside the unit interval. Since, by Theorem 11, $y(t) < 1$ for all $t \geq 0$, the uniqueness proof in Theorem 6 can be carried thru without regard to the behavior of G_1 outside the unit interval. Hence, the Lipschitz condition assumed in Theorems 6, 10, and 11 need hold only on the unit interval. Similarly, it is sufficient in Theorems 7 and 8 to require that G be analytic on the open interval $(0, 1)$ and satisfy a Lipschitz condition on the closed interval.

4. Summary. Recalling that the input function, $G[U(0, t)]$ is simply the product of the film transfer factor, $f[U(0, t)]$ and the term $[1 - U(0, t)]/k$ we can summarize as follows the results which we have obtained:

Conclusion 1. For any film transfer factor of physical significance, the heat transfer problem stated in Sec. 2 always has at least one solution, $U(x, t)$, for all $x \geq 0$, $t \geq 0$. It can be constructed in the following way

$$U(x, t) = \int_0^t \frac{G[U(0, \tau)]}{\pi^{1/2}(t - \tau)^{1/2}} \exp \frac{-x^2}{4(t - \tau)}\, d\tau \tag{8}$$

from the surface temperature function $U(0, t)$ which must be a bounded solution of the following nonlinear integral equation

$$U(0, t) = \int_0^t \frac{G[U(0, \tau)]}{\pi^{1/2}(t - \tau)^{1/2}}\, d\tau. \tag{6}$$

Equation (6) always has at least one continuous bounded solution which satisfies the inequalities $0 < U(0, t) < 1$ for all $t > 0$, and having the property that $U(0, 0) = 0$. From these inequalities it follows by applying the mean value theorem to equation (8) that $0 < U(x, t) < 1$ for all $x \geq 0$, $t > 0$.

Conclusion 2. If $f[U(0, t)]$ satisfies a Lipschitz condition on the closed unit interval, then we can add to Conclusion 1 that

a) $U(0, t)$, and therefore $U(x, t)$, is unique.
b) $U(0, t)$ is non-decreasing for all $t > 0$.
c) $U(0, t) < 1$ for all $t \geq 0$.
d) $\lim_{t\to\infty} U(0, t) = 1$.

CONCLUSION 3. If the film transfer factor is analytic on the open unit interval and satisfies a Lipschitz condition on the closed unit interval, then we can add to Conclusion 2 that $U(0, t)$ is analytic and monotone increasing for all $t > 0$.

CONCLUSION 4. If the film transfer factor satisfies a Lipschitz condition on the closed unit interval, then $U(0, t)$ can be approximated uniformly, arbitrarily closely in the large by the method of successive approximations defined in formula (14). Since the successive approximations lie alternately above and below $U(0, t)$, an upper bound for the error in the nth approximating function is simply the difference $| U_n(0, t) - U_{n-1}(0, t) |$

As a final remark we point out that the methods developed here in the treatment of the surface temperature equation,

$$y(t) = \int_0^t \frac{G[y(\tau)]}{[\pi(t - \tau)]^{1/2}} \, d\tau \tag{6}$$

are also applicable to a much more general equation,

$$y(t) = \int_0^t K(t, \tau) G[y(\tau)] \, d\tau \tag{33}$$

in which the kernel, $K(t, \tau)$, need only be positive definite and satisfy appropriate integrability conditions. The theory for the more general equation (33) will be developed in a later paper.

BIBLIOGRAPHICAL REFERENCES

1. R. V. Churchill, *Modern operational mathematics in engineering*, McGraw-Hill Book Company, Inc., New York and London, 1944, p. 214.
2. R. V. Churchill, Loc. cit.[1], p. 37.
3. L. M. Graves, *The theory of functions of real variables*, McGraw-Hill Book Company, Inc., New York and London, 1946, p. 96.
4. S. Mazur, *Ueber die kleinste konvexe Menge die eine gegebene kompakte Menge enthaelt*, Studia Mathematica 2, 7-9 (1930).
5. J. Schauder, *Der Fixpunktsatz in Funktionalraeumen*, Studia Mathematica 2, 171-180 (1930).
6. G. Doetsch, *Theorie und Anwendung der Laplace-Transformation*, Dover Publications, New York, 1943, p. 159.