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HEAT TRANSFER BY FREE CONVECTION ACROSS A CLOSED CAVITY BETWEEN VERTICAL BOUNDARIES AT DIFFERENT TEMPERATURES*

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Summary. The two-dimensional convective motion generated by buoyancy forces on the fluid in a long rectangle, of which the two long sides are vertical boundaries held at different temperatures, is considered with a view to the determination of the rate of transfer of heat between the two vertical boundaries. The governing equations are set up; they reveal that the flow is determined uniquely by the Prandtl number σ, the Rayleigh number $A = g(T_1 - T_0)d^3/(T_0\kappa\nu)$, and the ratio of the sides of the rectangle l/d. In the case of cavities used for thermal insulation of buildings, which is kept specially in mind throughout the paper, A is usually about 1000 d^3 (where d is in centimeters), and l/d takes values between about 5 and 200.

The essence of the problem is to determine which of several different flow regimes occurs at any given values of A and l/d. It appears that with the above practical values the flow is not decisively of one single kind, and the discussion of the heat transfer for each of several ranges of values of A and l/d is necessary. Sections 4, 5 and 6 are concerned with laminar flow regimes characterized by very small values of A, large values of l/d, and large values of A, respectively. In Section 7 approximate criteria for these flows to be stable are established, and in Section 8 the expressions for the heat transfer when the flow is turbulent are considered briefly. The unified picture provided by all these different results is considered in Section 9.

A comparison of the theoretical predictions about the heat transfer with the limited experimental data, mostly obtained by Mull and Reiher (1930), is made. Theory and experiment agree in suggesting that, under practical conditions, the effect of convection is negligible for $d \leq 1$ cm, and that the heat transfer per unit area of vertical boundary decreases as d increases, provided $d < 2.5$ cm, and remains approximately constant (at a value proportional to $l^{-1/4}$) for further increase of d.

1. The background to the problem. The purpose of this paper is to determine the rate at which heat is transferred across the air space between two plane parallel vertical boundaries which are held at different temperatures. The air space is closed by horizontal boundaries distance l apart (l being large compared with the distance d between the vertical walls, in general), as sketched in Fig. 1. In the remaining direction, at right angles to the plane of the sketch, the air space is regarded as extending to infinity. All boundary conditions will be assumed uniform in this latter direction, and the convective

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motion generated by buoyancy forces can then be assumed to be two-dimensional, lying in the plane of Fig. 1. Only the heat transferred from one vertical boundary to the other by conduction and convection will be considered. The radiative transfer is not negligible, but it takes place independently of the conductive and convective transfer and does not depend significantly on the shape and size of the air space; once the nature of the surface of the boundaries and their temperature difference have been specified, the radiative transfer can be calculated with reasonable accuracy from known laws.

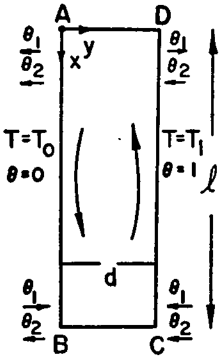

FIG. 1.

The determination of the transfer of heat across an air space of given size and shape is a problem which arises frequently in connection with the thermal insulation of buildings. This is the context in which the problem first came to my notice, and it will be kept in mind in the analysis that follows. It has long been appreciated by those concerned with the thermal insulation of buildings that a narrow gap or cavity in the interior of a wall can impede the flow of heat considerably without adding appreciably to the cost of construction of the wall. For instance the rate of transfer heat through a brick wall 9 inches thick is about 40% greater than that (with the effect of radiation included) through a composite wall consisting of two $4\frac{1}{2}''$ brick leaves with an unventilated cavity 2 inches wide between them, given the same temperature difference between the two outer brick faces in each case*. The use of cavity walls of this kind has been standard practice for some years. Double windows consisting of two panes of glass with an air space between them are also well-known in principle as a means of improving thermal insulation (although it does not seem to be so widely appreciated in Great Britian that their use may also be economically desirable, at any rate for rooms which are kept at a comfortable temperature throughout most of the year).

Now it is clear that if the air space between the two vertical boundaries is very narrow, very little convective motion can occur and the heat transfer will be due mostly to conduction. If the air gap is widened, the transfer that would result from conduction alone will decrease but the convective transfer will increase. Thus there exists the possibility that the transfer is a minimum for an air space of certain width. Inasmuch as the convective motion is mostly in the vertical direction, it is likely that the height of the air space also has an influence on the heat transfer. In the design of a building in-

*An excellent account of the principles and practice of thermal insulation of buildings (with English conditions chiefly in mind) is given by N. S. Billington, in "Thermal Properties of Buildings," Cleaver-Hume Press Ltd. 1952.

volving the use of an air space as a thermal barrier the gap d, and to a lesser extent the height l will be disposable quantities, unlike the temperature difference, so that the practical value of the investigation will lie in the information it provides about the dependence of the heat transfer on d and l. This information is of special importance for a consideration of the use of double windows, since first, the thermal resistance provided by the air space is here more than half the total resistance of the window, and second, the use of double windows will usually be economically justified by a small margin only, if at all.

Although no anlytical discussion of the determination of the heat transfer seems to have been published, a number of measurements of the heat transfer have been made; the relevant parts of the data will be described later. So far as I can gather from the literature of heating engineers, it seems to be generally accepted that an optimum size of air gap exists and that it is close to 3/4 inch, at any rate for double windows. The fact that the heat transfer may depend on the height l seems not to be well appreciated, and the possibility that the optimum value of d itself depends on l has not always been taken into account in the design of the experiments. No observations of the velocity or temperature distributions within the air space seem to have been published.

2. The equations governing the problem. Let T_0 and T_1 be the absolute temperatures of the two vertical boundaries. The main assumption underlying the equations to be used is that the temperature difference $T_1 - T_0$ is a small fraction of the absolute temperature T_0, and that the variation of temperature in the fluid can be neglected for all purposes other than the determination of the buoyancy force. The self-consistency of equations based on this assumption for problems of free convection has been explained elsewhere (Goldstein, 1938). We shall see in the next section that $(T_1 - T_0)/T_0$ is in fact small in cases of heat transfer in buildings.

Now it is clear that the speeds involved in problems of free convection of this kind are far below the speed of sound, and that the pressure differences produced by inertia forces are a minute fraction of the absolute pressure. The pressure differences produced by gravity are also very small compared with the absolute pressure (being controlled by the length scale of the space occupied by fluid), and variations of the fluid density ρ will be determined wholly by variations of the temperature T. The equation of state for gases then has the form

$$\frac{\rho - \rho_0}{\rho_0} \approx -\frac{T - T_0}{T_0}\,; \tag{2.1}$$

this and subsequent equations can be made applicable to liquids also by replacing the factor T_0^{-1} on the right hand side by a coefficient of expansion, since the magnitude of the temperature does not occur in the equations in any other connexion.

The equation expressing conservation of mass is the same, with the assumptions described, as that for an incompressible uniform fluid. Thus if coordinates x, y are chosen as in Fig. 1, and the corresponding velocity components are u, v, we have

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{2.2}$$

The only cause of change of the temperature of a moving element of fluid is heat conduction in the fluid, since neglect of temperature changes due to compression of the fluid and due to heat of viscous dissipation is consistent with the assumptions already made. Hence, on introducing

$$\theta = \frac{T - T_0}{T_1 - T_0}$$

as a more useful variable than T, we have

$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \kappa\left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2}\right), \tag{2.3}$$

where κ is the thermal diffusivity ($= k/\rho_0 c_p$ for gases, $k/\rho_0 c_v$ for liquids where k is the thermal conductivity of the fluid, and c_p, c_v are the specific heats); κ is effectively uniform in view of the smallness of $(T_1 - T_0)/T_0$.

The force equations, taking account of the effect of buoyancy, are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + g\left(\frac{\rho - \rho_0}{\rho_0}\right) + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{2.4}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{2.5}$$

where p is the pressure in the fluid. In these equations the density and kinematic viscosity ν have been taken as uniform (and equal to ρ_0 and μ_0/ρ_0 respectively, where μ_0 is the fluid viscosity at temperature T_0), again in view of the smallness of $(T_1 - T_0)/T_0$.

The boundary conditions that will be imposed on the variables u, v and θ at the vertical boundaries are

$$u = v = 0, \quad \theta = 0, \quad \text{at} \quad y = 0,$$

$$u = v = 0, \quad \theta = 1, \quad \text{at} \quad y = d,$$

the assumption being that these boundaries are made of material of high conductivity not in the form of thin sheets. On the two sides of the boundary given by $x = 0$ and $x = l$, we shall require $u = v = 0$, together with a reasonable condition on θ. In cases in which the boundary is of the same material on all four sides, the only consistent simple assumption is that $\theta = y/d$ on the horizontal boundaries. However in cases in which the vertical and horizontal boundaries are not both appreciably better conductors than the fluid (as for instance where air is enclosed by two glass panes set in a wooden frame; $k_{\text{glass}} \approx 30\, k_{\text{air}}$, and $k_{\text{wood}} \approx 6\, k_{\text{air}}$), this assumption may not be accurate, and it may be useful to consider the other extreme case of insulating horizontal boundaries, for which the boundary condition is $\partial\theta/\partial x = 0$. The length of horizontal boundary is small compared with the length of vertical boundary, and it seems unlikely that the exact form of the boundary condition at $x = 0$ and $x = l$ has much effect on the heat transferred through the vertical boundaries.

These are the general equations and boundary conditions that will govern all types of flow in the cavity. Since the time scale of changes in temperature inside or outside buildings is large compared with time scales related to the convective motion, steady motions (or steady mean motions in the case of turbulent flow) are of interest herein and we therefore put $\partial/\partial t = 0$. Further simplification is obtained by writing the equations in dimensionless form. We take d as the unit of length for the coordinates x and y

(without changing the notation), and define a dimensionless stream-function ψ, with the aid of (2.2), by the relations

$$u = \kappa d^{-1} \frac{\partial \psi}{\partial y}, \qquad v = -\kappa d^{-1} \frac{\partial \psi}{\partial x}. \tag{2.6}$$

The heat equation (2.3) can then be written as

$$\frac{\partial \theta}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial \theta}{\partial y}\frac{\partial \psi}{\partial x} = \frac{\partial(\theta, \psi)}{\partial(x, y)} = \nabla^2 \theta. \tag{2.7}$$

When the pressure p is eliminated from (2.4) and (2.5), and ρ is eliminated with the use of (2.1), we are left with the second governing equation for ψ and θ:

$$\frac{1}{\sigma}\frac{\partial(\omega, \psi)}{\partial(x, y)} = A \frac{\partial \theta}{\partial y} + \nabla^2 \omega, \tag{2.8}$$

where $\omega = -\nabla^2 \psi$ is the (dimensionless) vorticity, σ is the Prandtl number ν/κ, and

$$A = \frac{(T_1 - T_0) g d^3}{T_0 \kappa \nu} \tag{2.9}$$

is the Rayleigh number*. The boundary conditions are now

$$\psi = \frac{\partial \psi}{\partial y} = 0, \qquad \theta = 0, \qquad \text{at} \qquad y = 0,$$

$$\psi = \frac{\partial \psi}{\partial y} = 0, \qquad \theta = 1, \qquad \text{at} \qquad y = 1,$$

$$\psi = \frac{\partial \psi}{\partial x} = 0, \qquad \theta = y \quad \text{or} \quad \frac{\partial \theta}{\partial x} = 0, \qquad \text{at} \qquad x = 0 \quad \text{and} \quad x = \frac{l}{d}.$$

The form of the governing equations (2.7) and (2.8) and the above boundary conditions shows that the dimensionless parameters whose values are sufficient to determine uniquely the distributions of ψ and θ are σ, A and l/d.

The quantity to be determined from the analysis is the rate at which heat is transferred by conduction through either vertical boundary. If this rate is Q heat units per second per unit depth of boundary (in the z-direction), a suitable dimensionless quantity describing the heat transfer is the Nusselt number

$$N = \frac{Q}{k(T_1 - T_0)} = \int_0^{l/d} \left(\frac{\partial \theta}{\partial y}\right)_{y=0} dx. \tag{2.10}$$

N has the value l/d when there is no convection, and in the general case we anticipate that

$$N \equiv N(\sigma, A, l/d). \tag{2.11}$$

*The Grashof number $G = A/\sigma$ is frequently used by authors in the analysis of problems of free convection, but the number A enters directly at least as often as G. Moreover, as Professor H. B. Squire has pointed out to me, there is a much stronger case for honouring the name of Rayleigh in this field. The symbol A is used here to avoid the introduction of either R, which usually denotes Reynolds number, or the double letter Ra.

A quantity which is perhaps more useful for practical design of buildings is the heat transfer coefficient or thermal conductance of the cavity, defined by

$$C = \frac{Q}{l(T_1 - T_0)} = \frac{k}{l} N; \tag{2.12}$$

C is the rate of heat transfer through the vertical boundary, per unit area, per unit temperature difference between the boundaries.

It can be seen immediately that if the temperature in the cavity were distributed as though the air were at rest, i.e. $\theta = y$, the term representing the effect of buoyancy in Eq. (2.8) is finite, and it is not possible for both of the other two terms to vanish. Zero velocity everywhere is thus not a possible solution of the equations, and any temperature difference across the cavity, however small, will produce some steady convective motion. The problem is quite different from that in which the temperature gradient is across the two horizontal sides of a cavity, when steady convective motion can occur only if the Rayleigh number exceeds a certain critical value. A plausible inference from the existence of a convective motion at all Rayleigh numbers is that the dependence of ψ and θ on A is smooth and that it is possible to expand ψ and θ as power series in A for sufficiently small values of A. This provides a potential method of solution of the problem which will be taken up in section 4, although it will be found that the first few terms of this series provide a valid approximation only when the convective flux of heat is small compared with the transfer by conduction, and its usefulness is very limited.

3. The practical conditions. It will be useful to have some idea of the values of A appropriate to the practical problem of heat transfer through the walls of buildings. For air at 10°C and atmospheric pressure we have

$$\nu = 0.14 \text{ cm}^2 \text{sec}^{-1}, \qquad \kappa = 0.19 \text{ cm}^2 \text{sec}^{-1}, \qquad \sigma = 0.73.$$

The difference between the temperature of the air inside the building and that outside will not often exceed 20°C in Great Britain and an average value during the winter months would be about 15°C for an indoor temperature of 18°C (= 65°F)*. Of this air-to-air temperature difference, a fraction C_w/C_c (where C_w is the air-to-air conductance —including the effect of radiation—of the whole composite wall and C_c is the boundary-to-boundary conductance of the cavity) represents the difference between the temperatures of the boundaries of the cavity. For double windows C_w/C_c is roughly 0.6, and for a wall composed of two $4\frac{1}{2}''$ brick leaves with a cavity C_w/C_c is roughly 0.4. Taking the value $C_w/C_c = 0.5$ we have 7.5°C as a representative value of $T_1 - T_0$, and the corresponding value of A for $d = 1$ cm is (very conveniently)

$$A_1 = \frac{980 \times 7.5}{276 \times 0.19 \times 0.14} = 1000. \tag{3.1}$$

The values of d used in practice range from about 1 cm up to about 8 cm and the corresponding values of A for the conditions described by (3.1) (which will be referred to as the "standard" conditions) are given by

$$A = 1000d^3. \tag{3.2}$$

where d is expressed in centimetres.

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*The temperature difference appropriate to conditions in North Europe and North America might well be twice as large.

For double windows l will normally lie between 25 and 200 cm, and values of l/d ranging from about 5 to 200 should be considered in the analysis. In the case of a cavity in a brick wall, l will normally be at least 250 cm and may be much larger; standard building practice is to make the cavity 5 to 7 cm wide, giving a range of values of l/d which is included within that just mentioned.

4. The solution for very small values of A. We consider here a method of solution by means of an expansion of the quantities θ and ψ in power series in A. As already intimated the method is useful only at values of A too small for appreciable convection to occur, but it is necessary to give a few details of the method in order to be able to estimate its range of validity.

The expansions to be employed are

$$\theta(x, y) = y + A\theta_1(x, y) + A^2\theta_2(x, y) + \cdots, \tag{4.1}$$

$$\psi(x, y) = A\psi_1(x, y) + A^2\psi_2(x, y) + \cdots; \tag{4.2}$$

and the boundary conditions on the coefficients are

$$\psi_n = \frac{\partial \psi_n}{\partial y} = \theta_n = 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad y = 1,$$

$$\psi_n = \frac{\partial \psi_n}{\partial x} = 0, \quad \theta_n = 0 \quad \text{or} \quad \frac{\partial \theta_n}{\partial x} = 0, \quad \text{at} \quad x = 0 \quad \text{and} \quad x = \frac{l}{d}.$$

These series can be substituted into equations (2.7) and (2.8), whence by equating coefficients of like powers of A the following set of equations is obtained;

$$\nabla^4\psi_1 = 1, \qquad \nabla^2\theta_1 = -\frac{\partial \psi_1}{\partial x}, \tag{4.3}$$

$$\nabla^4\psi_2 = \frac{\partial \theta_1}{\partial y} + \frac{1}{\sigma}\frac{\partial(\nabla^2\psi_1, \psi_1)}{\partial(x, y)}, \qquad \nabla^2\theta_2 = -\frac{\partial \psi_2}{\partial x} + \frac{\partial(\theta_1, \psi_1)}{\partial(x, y)}, \tag{4.4}$$

. . . .

The distribution of ψ_1 is thus identical with the distribution of displacement of an elastic plane rectangular plate clamped at the edges and subjected to a (small) uniform transverse pressure. An analytic solution to this latter problem is not known, so that it is necessary to determine ψ_1 numerically. When that has been done, the determination of θ_1 requires a numerical integration (which can be carried out explicitly), and ψ_2, θ_2, etc., can then be determined by similar procedures.

It is clear from the form of Eqs. (4.3), (4.4), etc. that ψ_n and θ_n have the same symmetry properties as $(x - h/2d)^{n-1}$ $(y - 1/2)^{n-1}$ and $(x - h/2d)^n$ $(y - 1/2)^{n-1}$ respectively. Hence the horizontal flux of heat is given by

$$N = \int_0^{l/d} \left[\frac{\partial}{\partial y}\left(y + \sum_1^\infty A^n\theta_n\right)\right]_{y=0} dx$$

$$= \frac{l}{d} + \sum_{n=1}^\infty \gamma_{2n}A^{2n}\left(\gamma_{2n} = \int_0^{l/d}\left(\frac{\partial \theta_{2n}}{\partial y}\right)_0 dx\right), \tag{4.5}$$

and the first approximation to N which takes account of the convection is

$$N \approx \frac{l}{d} + \gamma_2 A^2. \tag{4.6}$$

The coefficient θ_1 does not appear in the horizontal transfer of heat, but it does determine the first approximation to the vertical flux of heat. It is easy to verify, from Eqs. (4.3), that the rate of release of potential energy that is associated with the vertical heat flux determined by the temperature and velocity distributions θ_1 and ψ_1 is equal to the rate of viscous dissipation of energy associated with the velocity distribution ψ_1 . Vertical flow of heat is the driving mechanism of the convection, and horizontal flow of heat is an accompanying effect, which is of higher order in A. The directions of the heat fluxes through different parts of the vertical boundaries produced by the temperature distributions θ_1 and θ_2 are shown in Fig. 1.

The magnitude of A for which the above power series expansion is likely to be useful can be estimated roughly in the following way without going through the labour of numerical solutions of (4.3) and (4.4). For values of l/d not too different from unity, an approximation to the solution of $\nabla^4\psi_1 = 1$ is known (Love 1927, Chap. 22) to be given by Grashof's formula, viz.

$$\psi_1 = \frac{2}{3}\left(1 + \frac{l^4}{d^4}\right)^{-1} x^2\left(\frac{l}{d} - x\right)^2 y^2(1 - y)^2. \tag{4.7}$$

θ_1 is anti-symmetrical about $x = l/2d$, so that we can regard θ_1 as being determined by (4.7) and the second of equations (4.3) together with the condition $\theta_1 = 0$ on the boundary of the rectangle $y = 0, 1, x = 0, l/2d$, (taking, for definiteness, the case in which the boundaries $x = 0$ and $x = l/d$ are perfectly conducting). As expected, the above formula shows $\partial\psi_1/\partial x$ to have the same sign over this area, with a single stationary value at the centre. For the purpose of obtaining an estimate of the magnitude of θ_1 , it will make little difference if we take the right hand side of the second of equations (4.3) as constant and equal to the average value of $-\partial\psi_1/\partial x$, with ψ_1 given by (4.7), over the region $0 \le x \le l/2d, 0 \le y \le 1$. This average value is

$$-\frac{1}{360}\frac{l^3}{d^3}\left(1 + \frac{l^4}{d^4}\right)^{-1}.$$

The equation for θ_1 has now been made identical with that describing torsion of an elastic rectangular prism, the solution of which is known (Love 1927, Chap. 14). The solution is available in series form, and the maximum value of θ_1 , which occurs at $y = 1/2$, $x = l/4d$, is

$$\frac{1}{2880}\frac{l^3/d^3}{(1 + l^4/d^4)}\left[1 - \frac{32}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3\cosh\dfrac{(2n+1)\pi l}{4d}}\right].$$

For $l/d = 2$, this maximum value is 0.97×10^{-4}, and for $l/d = 4$ it is 0.79×10^{-4}.

Thus for values of l/d not very different from unity the maximum value of θ_1 may be expected to be about 10^{-4}. When A has the standard value 1000 d^3, the maximum value of the second term $A\theta_1$ in the power series for θ is then about 1/10 d^3, whereas the value of the first term y at the same point is 1/2. It seems that even for the smallest values of d (and, consequently, of A) likely to be of interest (viz. $d = 1$ cm) the second term is not far from being as large as the first. It is fairly clear from the form of equation (4.4) that likewise θ_2 will not be a small perturbation of the temperature distribution $y + A\theta_1$ at these same values of A. It is of course still possible that the power series (4.1) and (4.2) are convergent at these same values of A, but the more important practical point is that the first few terms of the series—and only the first few could be determined numerically without excessive labour—do not by themselves provide a good approximation to the sums of the series. These remarks have been justified only for the case in which l/d is not large compared with unity, since it is only for such values that Grashof's formula (4.7) applies; however it is very unlikely that convergence of the series could vary appreciably with the value of l/d. It seems that $A = 1000$ is outside (although possibly only just outside) the range of usefulness of the expansion in powers of A.

When $l/d \gg 1$, Grashof's approximate formula for ψ_1 fails because it spreads the variation of ψ_1 over the whole of the range $0 \leq x \leq l/d$, whereas the solution of $\nabla^4\psi_1 = 1$ will be such that the distribution of ψ_1 with respect to y tends to a constant form as the distance from either end of the rectangle increases.* This asymptotic distribution of ψ_1, which will surely be a valid approximation when $\epsilon < x < l/d - \epsilon$ and ϵ is only a few multiples of unity, is

$$\psi_1 = \frac{1}{24} y^2(1 - y)^2. \tag{4.8}$$

In the part of the rectangle where (4.8) holds, $\theta_1, \psi_2, \theta_2, \cdots$ are all zero**, corresponding to the fact that $\theta = y$, $\psi = A\psi_1$, is an exact solution of the full equations. Thus at values of A such that the series expansions (5.1) and (5.2) are rapidly convergent, finite values of $\theta - y$ are confined to regions at the two ends of the rectangle which have dimensions in the x-direction only a few times the width of the rectangle. This again reveals the serious limitations of the range of usefulness of the expansion in powers of A. Anything resembling a boundary layer on the vertical walls, or having strong asymmetry about $y = 1/2$, could not be described by the power series. In short, only at values of A such that conduction has a much stronger influence on the temperature distribution than convection will the power series be useful***, whereas measurements at practical value of A show that the two are of comparable importance.

5. General value of A, and $l/d \to \infty$. The starting point for an investigation of the convection at larger values of A is supplied by the remark, at the end of the last section, that when l/d is large enough the variables θ and ψ take up their asymptotic form

$$\theta = y, \qquad \psi = \frac{A}{24} y^2(1 - y)^2, \tag{5.1}$$

at all points not near to either end of the cavity (see Fig. 2). This asymptotic state corresponds to a purely vertical flow of the fluid in which the rate of viscous dissipation of kinetic energy, per unit length (in the vertical direction) of the cavity, is just equal

*In accordance with Saint Venant's principle in elasticity theory.

**Although the difference between θ_n, $\psi_n (n > 1)$ and their zero asymptotic values probably increases as n increases, since the solutions of equations like (4.3), (4.4) ... are such that the dependent variable lags behind the right hand side when the latter tends to zero at infinity.

***It is relevant that the smallest value of the Rayleigh number at which the fluid between two horizontal boundaries at different temperatures is unstable, i.e. at which convection can overcome the damping effect of conductivity and viscosity, is also of the order of 10^3; for instance if the two boundaries are rigid perfect conductors, the critical value of A (based on distance between the planes) is known to be 1708.

to the rate at which potential energy is being released in unit length of the cavity. Since the horizontal velocity is zero the horizontal heat flow, over parts of the cavity where (5.1) holds, occurs wholly by conduction. Now at small values of A such that

$$\theta \approx y + \theta_1 A, \qquad \psi = \psi_1 A,$$

is valid everywhere, the region of the cavity in which θ and ψ do not have their asymptotic form (5.1) is confined, as already stated, to nearly-square regions at the two ends of the cavity. As A increases, and more terms in the power series become significant, the de-

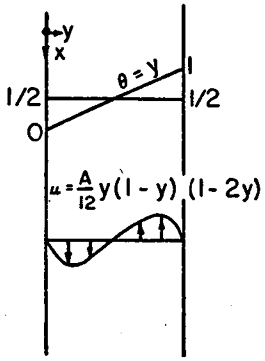

FIG. 2. The asymptotic state far from both ends of the cavity.

partures from the asymptotic state spread from each end of the cavity (especially on the side of flow *away* from the end). At large values of A at which boundary-layer analysis is appropriate, the asymptotic state will exist only at distances sufficiently far from each end for the boundary layers on the two vertical sides to have become thick enough to overlap and amalgamate.

In this section we shall assume that the value of l/d is large enough for the asymptotic state described by (5.1) to be set up over a finite range of values of x, say $X/d < x < (l - X)/d$, at the general value of A under consideration. Several useful conclusions follow from the existence of the asymptotic state at *some* values of x;

(a) The horizontal flow of heat due to convection occurs at the two ends of the cavity, i.e. within the ranges $x < X/d$, $x > l - X)/d$, and the total heat flux though a vertical boundary due to convection is independent of l; an increase in l merely increases the extent of the region in which the asymptotic state occurs and in which the transfer is by conduction alone. The general relation (2.11) thus reduces to

$$N = l/d + F(\sigma, A), \tag{5.2}$$

where the function F describes the additional heat transfer due to the existence of con vection; likewise the addition to the conductance C due to convection is inversely proportional to l.

(b) It follows from (a) that the insertion of horizontal partitions in a deep cavity can only *increase* the total horizontal heat flow; indeed, if the new cavities which are created by the insertion of horizontal partitions are themselves deep enough for the asymptotic state to be set up in each separate cavity, we shall have

$$N = l/d + nF(\sigma, A),$$

where l is the over-all height and n the number of separate cavities.

(c) The streamlines which rise on the side of the cavity adjacent to the hot boundary ultimately become the streamlines of the falling stream on the other side of the cavity. In the course of its journey from one side of the cavity, in a region of asymptotic flow, to the other, the rising stream loses heat at a rate equal to the net flux of heat across any horizontal plane in the range $X/d < x < (h - X)/d$, i.e. equal to

$$k(T_1 - T_0) \int_0^1 [u\theta]_{X/d<x<(l-X)/d}\, dy = k(T_1 - T_0) \int_0^{1/2} (1 - 2y) \frac{d}{dy}\left[\frac{A}{24} y^2(1 - y)^2\right] dy$$

$$= k(T_1 - T_0) \frac{A}{720}\,. \tag{5.3}$$

This addition to the heat content of the air in the upper end of the cavity (i.e. in the range $0 < x < X/d$) is that due wholly to the existence of convection, and must be lost by conduction through the two vertical walls* in this region, some of it—a fraction α, say,—through the cold boundary, and the remaining fraction $1 - \alpha$ through the hot boundary. Hence the contribution to N due to the existence of convection is

$$F(\sigma, A) = (2\alpha - 1) \frac{A}{720}\,. \tag{5.4}$$

α is itself a function of A, about which a little information is available. When A is very small we have already seen that the heat conveyed by the rising stream is given to the two vertical boundaries in nearly equal parts (because the effect of conduction is so powerful), the value of α being found from (4.6) and (5.4) as

$$\alpha = \tfrac{1}{2} + 360\gamma_2 A. \tag{5.5}$$

As A increases, the distribution of loss of the heat of the rising stream becomes more asymmetrical and α increases, presumably monotonically. α can never be as large as unity**, so that as $A \to \infty$ it must asymptote to a constant value, β say, which cannot be far from unity.

We thus have the result that, provided l/d is always large enough for the asymptotic flow to be set up for some values of x,

$$N \sim \frac{l}{d} + \left(\frac{2\beta - 1}{720}\right) A \tag{5.6}$$

for large A, where β is a constant. There is the interesting consequence that for a given value of l, N (and also the conductance C) has a minimum at a value of d given by

$$\frac{l}{d} = \left(\frac{2\beta - 1}{240}\right) A \tag{5.7}$$

(provided that *this* value of l/d is large enough for a region of asymptotic flow to be set up). With the standard conditions, and the approximation $\beta = 1$, this gives $d \approx (l/4)^{1/4}$ as the optimum spacing between the cavity boundaries, where l and d are

*Assuming that the flow of heat through the horizontal boundary is negligible, either because its length is small or its conductivity is small.

**For as the rising streamlines approach the upper end of the cavity, those near the boundary necessarily diverge and reduce the value of $(\partial\theta/\partial y)_{y=1}$, so that it is less than unity, and the flux of heat through this part of the hot boundary into the cavity is less than that which occurs in absence of any convection.

expressed in centimetres. At this optimum value of d, convection is responsible for 25% of the total transfer, the proportion increasing very rapidly with increase of d.

Before the significance of the results of this section can be assessed we must estimate the smallest value of l/d for which the asymptotic state described by (5.1) is set up. It is difficult to do this exactly, but a number of consistent estimates can be obtained by different methods. There is first of all the simple argument that when the rising stream turns around at the top of the cavity, a new temperature distribution across this stream has to be set up by a process of conduction across the flow and the time needed for this process is of order $d^2/4\kappa$. During this time a particle moving with a velocity equal to the mean downward velocity in the asymptotic region (i.e. equal to the average value of $A\kappa/12d\, y\,(1 - y)\,(1 - 2y)$ over the range $0 < y < 1/2$, which is found to be $A\kappa/192d$) moves through a distance

$$\frac{A}{768}\, d, \tag{5.8}$$

which will be of the same order as X.

Another estimate of X/d can be obtained by assuming that the rate at which the influence of the cold boundary spreads into the stream, after it has turned around at the upper end of the cavity, is the same as the rate at which the thermal boundary layer grows on an isolated cold plate by free convection. On defining the edge of the thermal boundary layer to be where the temperature (relative to that of the ambient fluid at infinity) is 20% of the temperature of the boundary—assumed to be $T_1 - T_0$—we find from Schmidt and Beckman's experiments (Goldstein, Ed., 1938) that the thermal boundary layer due to free convection from a vertical plate has a thickness $d/2$ when the length X of the plate is given by

$$\frac{d}{2} = 2X^{1/4}\left[\frac{g(T_1 - T_0)}{4\nu^2 T_0}\right]^{-1/4},$$

$$\text{i.e.}\quad \frac{X}{d} = \frac{A}{1024\sigma}\,. \tag{5.9}$$

Finally, we may analyse the way in which the thermal influence of the cold boundary spreads into the stream by means of the Graetz-Nusselt procedure (Goldstein, Ed., 1938), which was developed for the case of forced convection due to Poiseuille flow in a circular tube with a sudden change in temperature of the wall at a certain section. If the falling stream on the cold side of the cavity is regarded as a forced flow, with parabolic velocity distribution (more accurate representation of the real velocity distribution is not warranted) and the same mass flow (for $0 < y < 1/2$ only) as that given by (5.1), the approximate equation satisfied by θ in the falling stream is

$$\frac{A}{8}\, y(\tfrac{1}{2} - y)\, \frac{\partial \theta}{\partial x} = \frac{\partial^2 \theta}{\partial y^2}\,. \tag{5.10}$$

A particular solution is

$$\theta = y + e^{-8\lambda x/A}\, f(y),$$

provided $f(y)$ is such that

$$f'' + \lambda y(\tfrac{1}{2} - y) f = 0.$$

If $f(y)$ is now assumed to be an even power series in $(y - 1/4)$ (the odd powers represent a temperature distribution which dies out, as x increases, more rapidly than that described by the even powers), the condition that $f(y) = 0$ at $y = 0$ and $y = 1/2$ gives an equation for λ whose degree is higher as higher powers are included in the series for f. In this way the first three successive approximations to the smallest root for λ are found to be 615, 735 and 705, and the series corresponding to this last approximation to the root is

$$f(y) = F[1 - 22.1(y - \tfrac{1}{4})^2 + 139(y - \tfrac{1}{4})^4 - 840(y - \tfrac{1}{4})^6 + \cdots].$$

The general solution is then

$$\theta = y + \sum_n e^{-8\lambda_n x/A} f_n(y), \tag{5.11}$$

where $\lambda_1, \lambda_2, \cdots$ are the possible values of λ, and the constants F_n are chosen to give a correct representation of the temperature distribution at $x = 0$. Whatever the shape of the temperature disturbance, the rate at which its magnitude diminishes as x increases will ultimately be dominated by the term in the series (5.11) corresponding to the smallest value of λ. The distance over which the magnitude of this term falls to a tenth of its initial value is

$$\frac{A}{2450}, \tag{5.12}$$

which can be regarded as another estimate of X/d.

These three estimates of X/d, (5.8), (5.9) and (5.12), are consistent (some variation is to be expected, since X is not a definite length), despite the fact that two of them use the data of forced convection and one uses free convection (although the methods are not essentially different, since the assumed velocity of the forced convection is obtained from a consideration of the velocity that would ultimately be attained in free convection). As a rough average of the estimates, the value of x can be taken to be

$$\frac{X}{d} = \frac{A}{1000};$$

this gives the distance from either end of the cavity at which the asymptotic state described by (5.1) is attained approximately*. Hence for the asymptotic state to be set up in a cavity we must have

$$\frac{l}{d} > \frac{A}{500}. \tag{5.13}$$

With the standard conditions, and l and d in centimetres, this is equivalent to

$$l > 2d^4, \tag{5.14}$$

showing that results obtained in this section will be relevant to many cases of cavities used in buildings, especially those in double windows in view of the common choice of the value $d \approx 2$ cm.

*Note that it is only when A is less than about 1000 that conduction can carry the thermal effect of the boundaries across the flow in such a short time that X is of the same order as d; this is consistent with the conclusion, obtained in section 4, that $A = 1000$ is near the end of the range of values of A at which conduction is sufficiently dominant for the expansions in powers of A to be useful.

6. General value of l/d, and $A \to \infty$. Since it will also be true, according to the criterion (5.14), that the value of l/d in *some* practical cases is not large enough for the asymptotic flow to be set up, it will be useful to supplement the foregoing theory with a consideration of the extreme case in which the rising and falling streams at the vertical boundaries are far from being contiguous. The thickness of boundary layers produced by free convection always decreases as the Rayleigh number increases, so that this extreme case is achieved by holding l/d constant and allowing A to approach infinity. Ultimately, as $A \to \infty$, the thickness of the region of thermal influence of each portion of the boundary becomes small compared with both l and d, and a single continuous boundary layer surrounds the cavity. The boundary layer thickness remains finite, in spite of the fact that the path length of any fluid particle in the boundary layer increases indefinitely, because gravity produces a force on the particle in the direction of flow (for part of the time) and counteracts the loss of momentum due to viscous forces. In the core of the cavity, the flow is presumably such that the influence of the diffusivities ν and κ is slight.

This is a type of flow problem which has considerable intrinsic interest, as Pillow (1952) has pointed out in a discussion of the analogous situation in which heat is being transferred, by means of a two-dimensional convection call, between two horizontal plane boundaries. The special interest—and the difficulties—are associated mostly with the determination of conditions in the core of the cavity. Inasmuch as when $A \to \infty$ (i.e. $\nu \to 0$, $\kappa \to 0$) the terms containing second-order differentials in the governing equations (2.7) and (2.8) become negligible everywhere, except possibly in the neighbourhood of the boundaries, the flow in the core of the cavity conforms to

$$\frac{\partial(\theta, \psi)}{\partial(x, y)} = 0, \qquad \frac{1}{\sigma}\frac{\partial(\omega, \psi)}{\partial(x, y)} = A\,\frac{\partial \theta}{\partial y}\,. \tag{6.1}$$

Now in cases in which the flow outside a boundary layer is known to be free from vorticity (usually by reason of the fact that all streamlines originate at infinity where the vorticity is zero), this same flow is determined uniquely (by the condition that the normal velocity at rigid boundaries is zero). In the above case of cavity flow we have no reason to expect irrotational flow in the core, and a unique determination from the condition of zero normal velocity at the boundary is not to be expected without the introduction of information concerning the flow *outside* the core of the cavity. Pillow shows that the solution of equations (6.1) is

$$\theta = f(\psi), \qquad \omega = \sigma A x f'(\psi) + g(\psi), \tag{6.2}$$

where f and g are arbitrary functions. The condition of symmetry of the equi-temperature lines and streamlines in the core about $x = l/2d$ and about $y = 1/2$ necessitates that $f(\psi)$ is constant, and by symmetry again the value of the constant is $1/2$. This leaves the vorticity distribution arbitrary, apart from being symmetrical, and to determine the function $g(\psi)$ it would seem to be necessary to make use of the condition of compatibility of the flows in the core and in the boundary layer. So far as the form of $g(\psi)$ is concerned, we can employ the argument that if the vorticity in the cavity core were not uniform, the existence of a finite, although small, viscosity would make it uniform in a sufficiently long time (unless of course there is some mechanism generating vorticity in the core; inasmuch as the temperature is uniform in the core this possibility can be rejected).

Hence the conclusion is that

$$\theta = \tfrac{1}{2}, \qquad \omega = \omega_0(\text{const.}), \tag{6.3}$$

in the core of the cavity.

The flow in the cavity thus consists of an isothermal core in which the velocity distribution is given by

$$\nabla^2 \psi = -\omega_0 , \tag{6.4}$$

with ψ constant on a rectangular boundary, surrounded by a continuous boundary layer in which the temperature and velocity make rapid transitions to their assigned values at the boundary. If s denotes (non-dimensional) distance along the cavity boundary from the point A towards B in Fig. 1, the velocity outside the boundary layer can be written as $\omega_0\, U(s)$, where the function U is periodic in s with period $2(l/d + 1)$ and is determined by (6.4). (For the long narrow cavities with which we are concerned, $U(s)$ will have the value 1/2 at points not near the two ends of the cavity, since the velocity will vary linearly across the cavity near such points). Except in the immediate neighbourhood of the corners, the boundary layer flow will be the same as that produced on a plane wall, with the (spatially) periodic velocity $U(s)$ at the wall in the absence of the boundary layer, periodic temperature conditions at the wall, and periodic variation of the buoyancy force. If u and v now denote velocity components parallel and at right angles to the boundary, whatever its direction, the (non-dimensional) equations describing the flow in the boundary layer will be, after the usual approximations are made,

$$u\frac{\partial u}{\partial s} + v\frac{\partial u}{\partial n} = \omega_0^2 U\frac{dU}{ds} - \sigma G A(\theta - \tfrac{1}{2}) + \sigma\frac{\partial^2 u}{\partial n^2}, \tag{6.5}$$

$$u\frac{\partial \theta}{\partial s} + v\frac{\partial \theta}{\partial n} = \frac{\partial^2 \theta}{\partial n^2}, \tag{6.6}$$

where n denotes the normal distance from the boundary and G is a periodic factor which takes account of the change in relative direction of the buoyancy force and has the values shown in Fig. 3. The conditions at the wall are as before, while when n becomes large the flow in the cavity must be recovered, ie.

$$\text{as } n \to \infty, \qquad \theta \to \tfrac{1}{2} \qquad \text{and} \qquad u \to \omega_0 U(s). \tag{6.7}$$

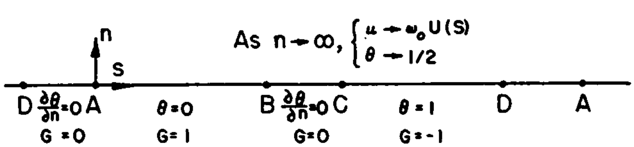

FIG. 3. (The points $ABCD$ correspond to the points $ABCD$ in Fig. 1).

The value of ω_0 is presumably to be obtained mathematically from the condition that equations (6.5) and (6.6), with the boundary conditions specified, do in fact permit a periodic solution, although I have not been able to establish this in detail. Physically, one may say that a finite positive value of ω_0 is produced by the torque exerted on the cavity core by the boundary layer surrounding it. The velocity in the boundary layer is everywhere in the direction of s increasing (which is obviously true immediately after fluid at temperature $\theta = 1/2$ throughout the cavity is released from rest, while in the steady state a negative value of $u(s)$ would imply the existence of a stagnation point in the flow which would be contrary to the whole notion of the existence of a continuous boundary layer) and so a core at rest would be subject to an anticlockwise frictional couple at all points on its perimeter. Probably the value of ω_0 is such as to make the velocity at the edge of the core comparable with the maximum velocity in the boundary layer produced by free convection on a plate of length l and temperature difference $\frac{1}{2}(T_1 - T_0)$; that is, using the measurements of Schmidt and Beckman (Goldstein, Ed., 1938),

$$\frac{1}{2}\frac{\omega_0 \kappa}{d} \approx 0.54 l^{1/2}\left[\frac{g(T_1 - T_0)}{2T_0}\right]^{1/2},$$

giving

$$\omega_0 \approx 0.76\left(\sigma A \frac{l}{d}\right)^{1/2}. \tag{6.8}$$

I have not found any reasonably simple method of solving the boundary layer equations (6.5) and (6.6). Methods which rely on a polynomial representation of θ and u are frustrated by the fact that θ and u make several oscillations as n increases and polynomials of small degree in n are quite inadequate to represent them. Linearization of the equations in the Oseen manner was also tried, but the simplification is evidently too drastic, since a solution periodic in s did not appear to be consistent with any value of ω_0.

Even though the problem remains unsolved in detail, it is possible to estimate the rate of heat transfer. The flow near each vertical wall bears a partial resemblance to forced convection due to a stream of speed $\omega_0\kappa/(2d)$ past a plate of length l with temperature difference $T_1 - T_0$ (for the boundary layer near each end of the cavity, and consequently near the leading edge of each vertical wall, can be assumed to have the temperature of the vertical wall it has just left*), on the one hand, and to free convection past a plate of length l with temperature difference $T_1 - T_0$ on the other. In the former case the heat transfer would be (Goldstein, 1938), for air, and laminar flow,

$$N = 0.59\left(\frac{1}{2}\omega_0 \frac{\kappa l}{d\nu}\right)^{1/2},$$

$$\approx 0.38\left(\frac{A}{\sigma}\frac{l^3}{d^3}\right)^{1/4} \tag{6.9}$$

in view of (6.8), whereas in the latter case the heat transfer would be (Goldstein, Ed., 1938)

$$N = 0.48\left(\frac{A}{\sigma}\frac{l^3}{d^3}\right)^{1/4}. \tag{6.10}$$

The two expressions have the same functional form (necessarily, in view of their essentially similar bases), and the closeness of the numerical coefficients makes it un-

*It seems to be appropriate to use $\frac{1}{2}(T_1 - T_0)$ as the temperature difference of the equivalent isolated plate in an estimate of the maximum velocity in the boundary layer, as has already been done, since the maximum velocity occurs near the trailing edge of the plate and the temperature of the cavity core will here be important; but to use $(T_1 - T_0)$ as the temperature difference in an estimate of the heat transfer, because this is greatest near the leading edge where the ambient fluid is the oncoming boundary layer from the other vertical boundary.

necessary to consider which of the two cases represents the real situation more closely. N is now independent of d, provided l/d is sufficiently large for the boundary layer to be dominated by its travel over the vertical walls. With the standard conditions, and with l in centimetres, the mean of (6.9) and (6.10) reduces to

$$N = 2.0l^{3/4}. \tag{6.11}$$

These expressions for N are asymptotically valid, as $A \to \infty$, and supplement the expression (5.6) which is valid when l/d is large enough for the boundary layers to have amalgamated completely near the centre of the cavity at least.

7. The criterion for the flow in the cavity to be laminar. The discussion of the conditions under which the flow may be expected to be either laminar or turbulent has been delayed until now, since it is necessary first to know what form the laminar flow regime would take before considering its stability. Since the parameters A and l/d (leaving aside σ) are sufficient to determine the flow uniquely, the criterion for the flow to be on the borderline of the laminar state can be expressed as a relation between A and l/d. The kinds of laminar flow which exist when either l/d or A is very large have been described (in sections 5 and 6), and the criteria for these two flows to break down will be considered separately.

Taking first the case in which A is very large and a continuous boundary layer surrounds the cavity, we can make use again of the general resemblance which this boundary layer bears to the boundary layer produced by free convection on an isolated flat plate of length l and temperature difference $T_1 - T_0$. Experiments with air (Goldstein, 1938) suggests that this latter flow is laminar provided the Rayleigh number based on the length of the plate is less than about 10^9. We may expect, therefore, that an approximate criterion for the flow (of air, and perhaps of other fluids also) in the cavity to be laminar at large values of A is

$$A \frac{l}{d}^3 < 10^9, \tag{7.1}$$

and the corresponding relation between A and l/d expressing the borderline state is shown in Fig. 4. With the standard conditions, (7.1) states that the flow in the cavity will be laminar provided $l < 100$ cm (and note that the initial assumption that A is large here means that d must be sufficiently large).

In the other extreme case in which l/d is so large that the asymptotic distribution of temperature and velocity is set up at positions not near the ends of the cavity, it seems likely that breakdown of the laminar flow will occur first in the region far from the cavity ends* since this is where the greatest velocity gradients exist. We thus require to know the critical Reynolds number of the flow described in Fig. 2. The boundary layer produced by free convection on an isolated plate has a velocity distribution with similar general characteristics and again we may use it as a guide. The critical Reynolds number, formed from the maximum velocity and the boundary layer thickness, in this latter case

*There is also the possibility of breakdown of the laminar flow near the ends of the cavity where the streamlines are curved and the circulation decreases with increase of distance from the centre of curvature. A comparison with the case of flow between concentric cylinders, with the outer cylinder stationary, suggests that the cavity flow is near the limit of stability for some of the relevant values of d and l, but since there is so little time for a disturbance to be amplified while turning the corner this region of the flow is unlikely to become a source of turbulence.

is about 300 (Goldstein, 1938). Hence an approximate critical relation between A and l/d for the cavity flow when l/d is large, obtained by taking d as equivalent to the boundary layer thickness and by taking the maximum velocity *difference* in the asymptotic cavity flow as equivalent to the maximum velocity in the boundary layer, is

$$d \frac{2A\kappa}{72\sqrt{3}\,d} \Big/ \nu = 300,$$

i.e.

$$A = 18700\sigma, \qquad = 13700 \text{ for air.} \tag{7.2}$$

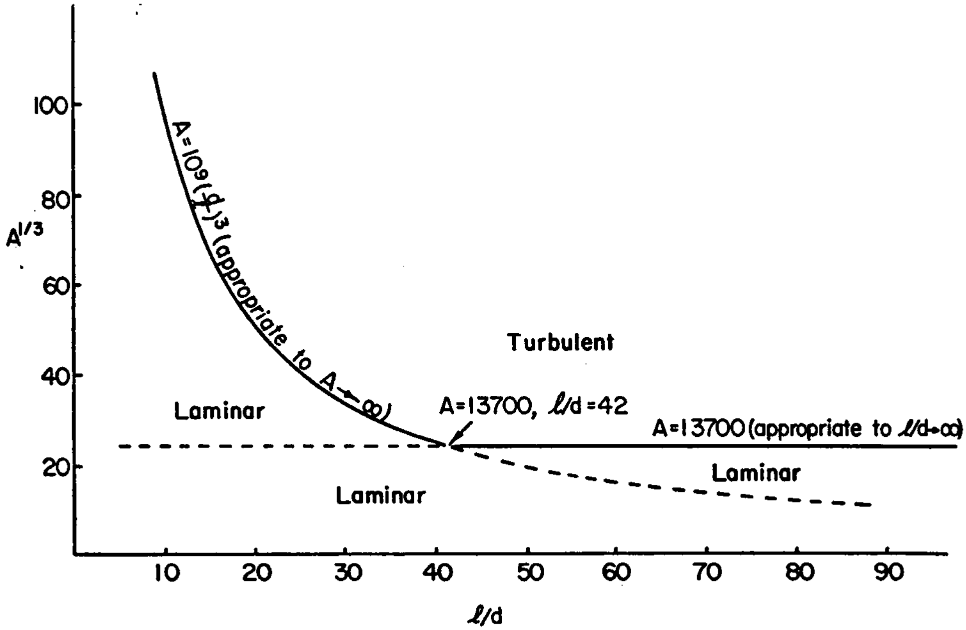

FIG. 4. The critical relation governing the existence of laminar flow.

With the standard conditions, (7.2) states that when l/d is sufficiently large, $d = (13.7)^{1/3} = 2.4$ cm is about the largest value of d for which the flow is laminar.

The two critical relations (7.1) and (7.2) cross at $l/d = 42$**, as shown in Fig. 4; presumably in the neighbourhood of the point of intersection neither of the approximations on which (7.1) and (7.2) are based is valid. The true critical relation between A and l/d will have the solid portions of the two curves as asymptotes and will lie a little above both of them in order to have a smooth transition from one to the other. For practical purposes it will be sufficiently accurate to regard the flow in the cavity as laminar if the point $(l/d, A)$ describing the flow lies below *either* of the curves (7.1) or (7.2). If the point lies only a little above both curves, the flow in the cavity may be turbulent, but it will probably be necessary for the point to lie well above the curves

**This value of l/d is implicitly an estimate of the minimum value of l/d, at $A = 13700$, for which the asymptotic velocity and temperature distributions are set up and as such is consistent with the estimate which is given by (5.13) (viz. 27); likewise the value $A = 13700$ is implicitly an estimate of the minimum value of A, at $l/d = 42$, for which the notion of a continuous boundary layer surrounding a core in which temperature and vorticity are uniform is valid.

before the turbulent region is large enough to dominate expressions for the heat transfer; for instance, the variation of N with Rayleigh number for the free convection produced by an isolated vertical plate does not settle down to the form appropriate to wholly turbulent flow until the Rayleigh number (based on length of the plate) is above about 10^{10}. As a consequence, it will be possible to use the results appropriate to laminar flow, as a reasonable approximation, even when the values of l/d and A are not quite such as to satisfy the above criteria for laminar flow.

8. Cases of turbulent flow. It seems probable, from the considerations of the previous section, that in some cases of cavities used in buildings the convective motion is turbulent. The difficulties of the problem are already great when the flow is laminar, and it would be unwise to speculate about the case of turbulent flow without some guidance from experiments. All that will be done here is to indicate how the expressions for heat transfer obtained earlier, for the cases in which either l/d or A are very large, are altered if the flow is assumed to be turbulent instead of laminar.

Consider first the case described in section 5, in which l/d is so large that the two boundary layers on the vertical walls have amalgamated and do not vary with x over the range $X/d < x < (l - X)/d$, and assume now that the flow in this latter range is turbulent. The streamlines of the mean flow are vertical, as before, but there is now a fluctuating velocity in the direction perpendicular to the vertical boundaries and the convective flux of heat is not zero in this region. The magnitude of the velocity fluctuations is readily found by returning to Eqs. (2.3), (2.4) and (2.5) and taking the mean value of both sides. If u, v, w now represent (dimensional) velocity fluctuations and U is the mean velocity, these equations show that in the region of asymptotic mean flow

$$\langle uv \rangle = \nu \frac{dU}{dy} - \frac{g(T_1 - T_0)d}{T_0} \int_0^y (\langle \theta \rangle - \langle \theta \rangle_{y=1/2})\, dy, \tag{8.1}$$

where $\langle \cdots \rangle$ indicates the mean value of the enclosed quantity. Except in the immediate neighbourhood of the walls, the first term on the right hand side will be small, as in all turbulent flows. The general magnitudes of u and v will be equal, and roughly constant, over the central part of the turbulent where the influence of neither wall is dominant, and (8.1) thus shows that

$$\langle u^2 \rangle, \qquad \langle v^2 \rangle, \qquad | \langle uv \rangle |, \qquad \frac{g(T_1 - T_0)d}{T_0},$$

are all comparable in magnitude.

Now the rate of heat transfer across the mean flow is $\rho c_p(T_1 - T_0) \langle v\theta \rangle$, per unit area, and the magnitude of $\langle v\theta \rangle$ will be the same as that of $\langle v^2 \rangle^{1/2}$ since the range of variation of θ is unity. The value of the Nusselt number N for a cavity of height l across which the heat is transferred at the above rate over the *whole* of either vertical boundary is then

$$N = \frac{Q}{k(T_1 - T_0)} = \frac{l\langle v\theta \rangle}{\kappa} \sim \frac{l}{d} (\sigma A)^{1/2}. \tag{8.2}$$

This is not the whole of the heat transfer, for some heat is convected upwards by the rising mean flow until it turns around at the upper end of the cavity and loses heat to the cold boundary. The rate at which heat is convected upwards across a horizontal line in the region of asymptotic mean flow is

$$\rho c_p(T_1 - T_0)d \int_0^{1/2} U\langle\theta\rangle \, dy,$$

and if we make the rough (but reliable) assumption that $\langle u^2\rangle^{1/2}$ and U are of the same order of magnitude, this quantity has the magnitude

$$\rho c_p\left[\frac{g(T_1 - T_0)^3 d^3}{T_0}\right]^{1/2}. \tag{8.3}$$

A certain fraction of this—not less than 1/2, not more than unity—is conducted through the cold boundary when the rising stream turns around at the top of the cavity, so that the complete expression for N is

$$N = \lambda_1 \frac{l}{d} (\sigma A)^{1/2} + \lambda_2(\sigma A)^{1/2}, \tag{8.4}$$

where λ_1 and λ_2 are constants of order unity. This expression shows what might have been expected, that in view of the ability of turbulent fluctuations to transfer heat across the cavity even when the mean flow is vertical, the upward convection of heat by the mean flow is comparatively unimportant. (It is to be expected that λ_2 will be a little larger than λ_1 from the nature of the above approximations, but l/d is large compared with unity and this will make the first term in (8.4) dominant.)

In the other limiting case, considered in section 6, in which A is so large that the flow consists of an isothermal core of uniform vorticity, surrounded by a continuous boundary layer, it may likewise be possible to make modifications to allow for the existence of turbulence. If it can be shown from experiments that the flow in the isothermal core remains laminar, still with uniform vorticity, while that in the boundary layer is turbulent, it will be possible to make use of the general resemblance between this boundary layer and that produced by free convection on an isolated plate of length l and temperature difference $T_1 - T_0$ or 1/2 $(T_1 - T_0)$ (according to the quantity under discussion). The empirical expression for the heat transfer, due to turbulent free convection, from an isolated plate with temperature difference $T_1 - T_0$ in air is (Goldstein, 1938)

$$N = 0.13A^{1/3} \frac{l}{d}, \tag{8.5}$$

which may be identified with the heat transfer across the cavity as in the case of laminar flow.

9. Summary of the results and comparison with experiment. A rather confusing picture of different types of flow occuring at different values of A and l/d has been built up in the preceding sections, and it will be useful now to recall the principal results and to see how they fit together. A comparison with the available measurements (leaving aside those for which all the relevant data, such as the value of l/d, is not specified) will be made, although these are so few in number as to leave the comparison indecisive. The measurements to be used are those described in the book by Jakob (1949), most of them having been made by Mull and Reiher (1930).

The final aim of the analysis is to determine the rate of heat transfer, and the significance of the various types of flow can be considered best with the aid of a diagram showing the variation of N over a wide range of values of A. We shall use Fig. 5 for this

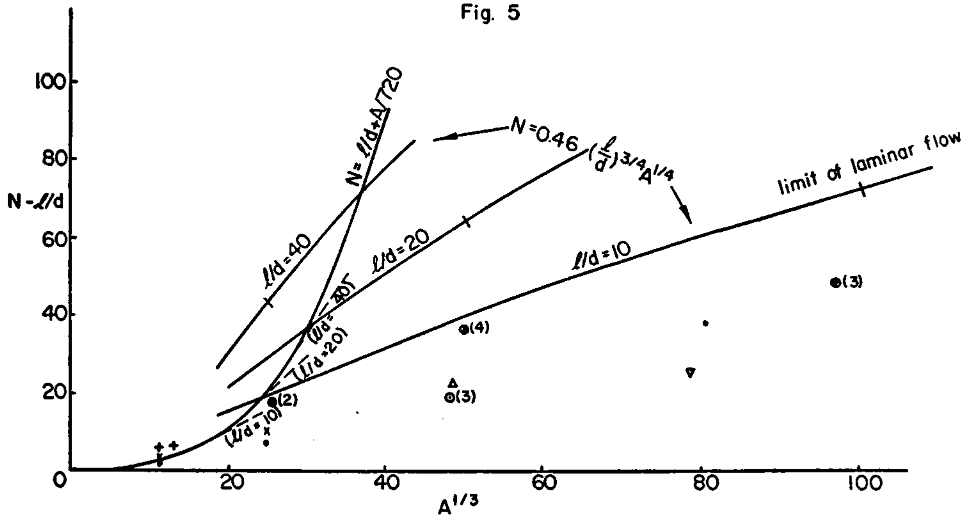

FIG. 5. Variation of heat transfer with Rayleigh number

$\left(N = \dfrac{Q}{k(T_1 - T_0)}, A = \dfrac{g(T_1 - T_0)d^3}{T_0 \kappa \nu}\right.$; note that $A^{1/3} = 10d$ under standard conditions and with d in centimetres$\Big)$

Experimental points:

▽	l/d =	6.25	Schmidt
•		10.6	Mull and Reiher
△		11.5	Nusselt
×		21.1	Mull and Reiher
+		42.2	Mull and Reiher
○	denotes multiple point		

purpose although the representation is designed to show clearly only the variation of N over the range of values of A likely to be of practical interest in problems of thermal insulation of buildings. The ordinate, $N - l/d$, is the addition to the Nusselt number (see (2.10)) due to the existence of convection, giving the curves for different values of l/d a common value at $A = 0$. (The contribution due to existence of convection is not dominant at the values of A used in the figure, so that mental allowance for the conduction term l/d must be made.) The abscissa has been chosen as $A^{1/3}$, corresponding to a linear variation of the important parameter d when $T_1 - T_0$ is kept constant; with the standard conditions described in section 3, and with d in centimetres, we have

$$A^{1/3} = 10d. \tag{9.1}$$

Considering first values of A less than about 1000, the appropriate prediction is that $N - l/d$ varies as A^2 (see (4.6)), with a constant of proportionality which very roughly is of the order of 10^{-8}. We saw that at these low values of A at which a power series expansion of θ and ψ is useful convection is much less important than conduction. Indeed we shall see that the values of $N - l/d$ at even larger values of A, at which another type of flow occurs, are too small for their exact magnitude to be of much interest, so that for practical purposes we can pass over the predictions about N described in section 4.

The first curve shown in Fig. 5, at low values of A, therefore, is not the quadratic law (4.6) but the linear law (5.6), which was derived on the assumption that l/d is large enough for a region of flow parallel to the walls to be set up near the centre of the cavity. A rough criterion for this to be a valid assumption was found to be $l/d > A/500$ (see (5.13)), so that the linear law (5.6) will be valid up to values of A which will be increasing with l/d. The linear law (5.6) is not definite without some assumption about the value of β, which was seen to lie between 1/2 and 1 and to asymptote to a value near unity as $A \to \infty$. As a tentative approximation we may assume $\beta = 1$, and the first curve shown in Fig. 5 is

$$N - \frac{l}{d} = \frac{A}{720} \, . \tag{9.2}$$

When A is about 1000, the contribution to N due to convection will be small for all values of l/d large compared with unity, as remarked above.

When A is in the neighbourhood of $(30)^3$, $= 2.7 \times 10^4$, corresponding to $d = 3$ cm with the standard conditions, the requirement for the law (9.2) to be valid is that $l/d > 54$ (equivalent to $l > 162$ cm with the standard conditions again), which is already very restrictive. At values of A too large for the criterion (5.13) to be satisfied, the flow does not attain the asymptotic form near the centre of the cavity and there is a tendency for the gradients of the temperature and velocity to be largest in the neighbourhood of the walls. At values of A *large* compared with that for which the criterion (5.13) just fails, the assumption of a continuous boundary layer surrounding a core of uniform temperature and vorticity is valid and we can use the predictions of Section 6. Two alternative expressions for N, (6.9) and (6.10), were obtained (the methods being such as to ensure the correctness of the functional form of N although not necessarily of the multiplicative constant), and the mean of these expressions, viz.

$$N = 0.43\left(\frac{A}{\sigma}\frac{l^3}{d^3}\right)^{1/4} = 0.48A^{1/4}\left(\frac{l}{d}\right)^{3/4}, \tag{9.3}$$

is plotted in Fig. 5 for various values of l/d. There is presumably a curve of transition from the law (9.2) to the appropriate member of the family (9.3), and it will be noted that the criterion $A/500 < l/d$, giving the upper limit of the range of validity of (9.2), is consistent with the position of the curve to which the transition must be made. Short dotted tangents are shown on the curve (9.2) in Fig. 5 at the place where the curve ceases to be valid, according to the criterion (5.13), for the value of l/d shown in brackets.

When A reaches a certain value, given by the criteria (7.1) and (7.2), steady laminar flow ceases to be possible. For the values of l/d employed in Fig. 5, viz. 10, 20, 40, the criterion (7.1) is appropriate, and the corresponding limiting value of $A^{1/3}$ is shown on the curves in Fig. 5 as a short cross stroke. (Note that the upper limit of A for laminar flow does not decrease indefinitely as l/d increases; when $l/d = 42$ the criterion (7.2) takes over, and laminar flow is possible for $A^{1/3} < 24$ however large l/d may be). When $l/d = 40$, the largest value of $A^{1/3}$ for which laminar flow is possible is smaller than the values of $A^{1/3}$ at which transition from the law (9.2) to the law (9.3) occurs, so that the curve corresponding to (9.3) with $l/d = 40$ in Fig. 5 has significance only as an approximation to the turbulent flow that will occur in practice, as explained at the end of Section 7. It would be useful to know where the curves, that describe the variation of N in turbulent flow for these larger values of l/d, occur in Fig. 5, but speculation about this should perhaps await some guidance from experiments.

Turning now to the experimental data about N, all the measurements described in Jakob's book (1949), except three referring to small values of l/d and very large values of A, are reproduced* in Fig. 5. Where several of the measurements were nearly identical, they have been amalgamated into a single point, the number of separate measurements being indicated in brackets. At small values of A, less than about 10^4, there are too few measurements to permit any conclusions about the validity of (9.2) and all that can be said is that the theory and the experiments are not inconsistent. (These smaller values of A are not unimportant practically, so that there is a real need for further experiments in this range). At larger values of A the experimental points seem to be consistent with a set of curves like (9.3) in form but having a numerical coefficient smaller by a factor of about 0.6. In other words, if (9.3) were replaced by

$$N = 0.3A^{1/4}\left(\frac{l}{d}\right)^{3/4}, \tag{9.4}$$

the two laws (9.2) and (9.4), with appropriate transition curves, would fit the above data adequately. That a change in the numerical coefficient of (9.3) should be necessary for agreement with observation is not impossible in view of the numerical uncertainties involved in its derivation. Moreover it will be recalled that (9.2) was based on the notion of independent boundary layers on the two vertical boundaries, which will be a valid picture only when the boundary layer thicknesses are small compared with d. Now if l/d and $T_1 - T_0$ are kept constant, the boundary layer thickness, as a fraction of d, is proportional to $d^{-3/4}$, which is a fairly slow rate of decrease, so far as the range of values covered in Fig. 5 is concerned; in fact it is readily found that at $A^{1/3} = 100$, the boundary layer on each vertical boundary near the centre has a thickness of about 0.15 d for l/d = 10, and varies as $(l/d)^{1/4}$, which is scarcely small enough for the asymptotic picture of completely separate boundary layers to be valid. It is possible, therefore, that the experimental points are approaching the curves (9.3) (or a set of curves with a numerical factor a little different from 0.48), and that values of A below 10^6 are still in the transition range.

The above interpretation of the measurements is quite different from that proposed by Jakob (1949). Jakob plots log (Nd/l) against log A, and represents the measurements at values of A above 10^5 by means of curves of the form

$$\frac{d}{l}N = 0.065\left(\frac{A}{\sigma}\right)^{1/3}\left(\frac{d}{l}\right)^{1/9}, \tag{9.5}$$

on the supposition that the flow is then turbulent (compare (8.6)) and the measurements in the range $10^4 < A < 10^5$ by curves of the form

$$\frac{d}{l}N = 0.18\left(\frac{A}{\sigma}\right)^{1/4}\left(\frac{d}{l}\right)^{1/9}. \tag{9.6}$$

The measurements at values of A less than 10^4 are represented by a curve common to all values of l/d. This is an empirical representation, and will serve as well as any other means of describing the data. However it should be observed that there is little evidence for Jakob's assumption of turbulent flow at values of A above 10^5. The criterion for

*I have been obliged to take the data from Fig. 25-7 of Jakob's book since Mull and Reiher's original paper is not available in Cambridge.

laminar flow to be possible clearly depends on l/d as well as on A, and the considerations of section 7 suggest that all the measurements reproduced in Fig. 5 (and also the three measurements not shown) correspond to laminar flow. The empirical conclusion that $\log dN/l$ is independent of l/d at values of A below about 10^4 is also at variance with the theory given herein, since this is the range in which we expect a region of parallel, vertical, flow to be set up near the centre of the cavity, with (9.2) as the corresponding expression for N.

For many practical purposes, the description of the theoretical results that is given in Fig. 5 is not the most revealing. The quantity of greatest interest is the heat transfer per unit area of a vertical boundary, which is proportional to N/l. Moreover, the width d can be chosen much more freely than the height l when a cavity is being used in a building, so that curves showing the heat transfer as a function of d for various values of l, rather than for various values of l/d, are needed. Both of these needs are met by Fig. 6,

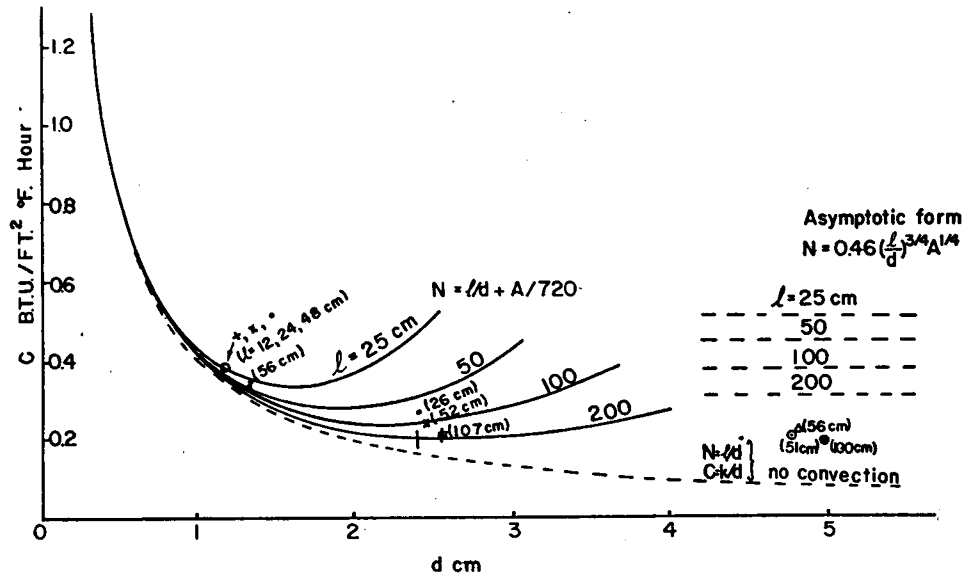

FIG. 6. Variation of conductance C with width of cavity ($C = Q/l(T_1 - T_0) = kN/l$, i.e., C BTU/ft² rF hour $= 0.394/l$ cm N. Note that d cm $= 0.1A^{1/3}$ under standard conditions).

which shows the thermal conductance C (defined by (2.12)) as a function of d (i.e. of $A^{1/3}$, when $T_1 - T_0$ is given) for various values of l, with the standard conditions. The range of values of d is here restricted to those of greatest practical interest. The units for C are those in common use by heating engineers.

At values of d less than about 1 cm, the conductance has approximately the value obtained by ignoring the effect of convection, but at larger values of d the curves obtained from (9.2) with different values of l diverge as shown in the figure. (There is the useful conclusion that if it is possible to subdivide a given rectangular cavity by the insertion of vertical partitions of small thickness, a maximum width of 1 cm for any of the new cavities so formed will ensure that the overall conductance is close to the optimum value obtained in the absence of convection.) The curves for different values of l all show the potentially important feature of a minimum value of C. However the position of the curves corresponding to (9.3), towards which N asymptotes as $d \to \infty$, is such that the value of N rises only a little, if at all, as d increases above the value at which the minimum of (9.2) occurs. The limiting curves given by (9.3) are shown on the right side of Fig. 6, but no transition curves to join the two theoretical predictions have been drawn in view of the uncertainty about the rate at which the limiting curves are approached.

The measurements described by Jakob have been plotted in Fig. 6 by the device of calculating what the values of d and l would have been if $g(T_1 - T_0)/T_0\kappa\nu$ had the standard value 1000 cm^{-3} and if A and l/d had the values given by Jakob; these equivalent values of l are shown in brackets by each point. As already seen, they are consistent with the theory only if we change the factor 0.48 in (9.3) to 0.3, or if we suppose that the law (9.3) becomes asymptotically valid at values of d well above those used in Fig. 6. Neither the theory no the measurements support the existence of a definite minimum of C as a function of d but both suggest that no further significant decrease in C occurs for values of d above about 2.5 cm (not even when l is greater than 200 cm, because for such cavities laminar flow becomes impossible when $d > 2.4$ cm, and this will prevent any decrease in C).

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