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HEAT TRANSFER IN A TURBULENT LIQUID OR GAS STREAM

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The theory of heat transfer from a solid body to a liquid stream could be presented previously** only with limiting assumptions about the movement of the fluid (potential flow, laminar frictional flow). (See references 1, 2, and 3.)

For turbulent flow, the most important practical case, the previous theoretical considerations did not go beyond dimensionless formulas and certain conclusions as to the analogy between the friction factor and the unit thermal conductance. (See references 4, 5, 6, and 7.) In order to obtain numerical results, an experimental treatment of the problem was resorted to, which gave rise to numerous investigations because of the importance of this problem in many branches of technology. However, the results of these investigations frequently deviate from one another. The experimental results are especially dependent upon the overall dimensions and the specific proportions of the equipment.

In the present work, the attempt will be made to develop systematically the theory of the heat transfer and of the dependence of the unit thermal conductance upon shape and dimensions, using as a basis the velocity distribution for turbulent flow set up by Prandtl and Von Kármán.

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*"Der Wärmeübergang an einen turbulenten Flüssigkeits- oder Gasstrom," (Abstract of a Dissertation presented to the Phil. Faculty of the Univ. of Vienna.) Z.f.a.M.M., vol. 1, no. 4, Aug. 1921, pp. 268-290.

**As long as the velocities remain much below the velocity of sound, compressible fluids (gases) and incompressible fluids follow, as is known, approximately the same laws of flow; therefore, in the following, the expression for the flow of fluids will be used for actual liquids as well as for gases.

NOTE: Translation received from Univ. of California, Berkeley 4, Calif.

1. HYDRODYNAMIC PRINCIPLES

First of all, the results of the Prandtl-Kármán theory (reference 8) which will be used constantly, will be set forth.

For the distribution of the shear stress in the immediate vicinity of the wall, dimensional considerations (see note 1 in the appendix) yield the expression:

$$\tau = \frac{7}{B} \, \tau_0^{3/7} \, \rho^{3/7} \, \mu^{1/7} \, y^{6/7} \, \frac{\partial u}{\partial y} \tag{1}$$

where the symbols are:

u   velocity in the direction of flow

τ_0 shear stress at the wall

μ   absolute viscosity

$\nu = \mu/\rho$, kinematic viscosity

B   constant (note 2), (8.82)*

y   distance from wall

ρ   mass density

In the same region, when the shear stress at the wall is assumed as known, the velocity follows from the equation (note 3):

$$u = B \left(\frac{\tau_0}{\rho}\right)^{4/7} \left(\frac{y}{\nu}\right)^{1/7} \tag{2}$$

There are two methods (note 4) of obtaining the distribution of velocity and shear stress for the entire region of the fluid. Either one starts from equation (2) and sets

$$u(y) = y^{1/7} \left(A_0 + A_1 y + A_2 y^2 + \ldots \right) \tag{2a}$$

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*The value of the constant corresponds to the equation for the velocity distribution, which is used below. Compare equation (8).

in which the constant A_0 is determined from the require-
ment that equation (2a) is transformed into equation (2) at
small values of y, or the basic equation (1) for the shear
stress transmitted between the individual layers can be ex-
tended and equation (1a) can be written

$$\tau = K \left[Y(y) \right]^{6/7} \frac{\partial u}{\partial y} \qquad (1a)$$

in which Y must change to y in the vicinity of the wall.
With this basic equation the velocity field for turbulent
flow can be calculated as long as no separation from the
boundary walls takes place.

For the special case of flow in a right circular cylin-
der it was shown by Von Kármán that the experimental results
on the velocity distribution can be reproduced with suffi-
cient accuracy if the function Y (y) is made, (called the
influence function).

$$Y = \frac{r^2 - \overline{y}^2}{2r}$$

where $\overline{y} = r - y$.

For time and volume invariant, the velocity distribu-
tion is then:

$$u = u_{max} \left[1 - \left(\frac{\overline{y}}{r} \right)^2 \right]^{1/7} \qquad (3)$$

and finally, the relation of the maximum velocity at the
axis of the tube u_{max} to the average velocity v in the
cross section is

$$v = \frac{7}{8} u_{max} \qquad (4)$$

2. TURBULENT THERMAL CONVECTIVITY

In the following, the transmission of heat by matter
only is considered, consequently limiting the study to a
temperature region in which the amount of heat carried off

by radiation is negligible in comparison with that carried off by particles of matter. Furthermore, the velocity distribution shall be affected only by external conditions; that is, the influence of the temperature field upon the velocity field is disregarded. At relatively great velocities of flow where the motion is turbulent, the resulting error need not be taken into account as long as the differences in density in the cross section, caused by temperature changes, are not too great.

Corresponding to the ideas taken from those on the conduction of heat in solid bodies, a distinction is likely to be made in the case of heat transfer in fluids, in general between the thermal conductivity, which describes the heat transported by molecular movement, and the so-called thermal convection - that is, transfer of heat by movements of the mass. The order of magnitude of the carrier of heat is thus used as the basis for distinction. A somewhat different mode of consideration, which pushes into the foreground the nature of the motion of the carrier of heat, seems, however, to be more advantageous both for mathematical treatment and for comprehension of the process. Accordingly, by "thermal conduction" in liquids is understood the transmission of heat by the random motion of the molecules, as has been represented by the concepts of the kinetic theory of gases. It then will be regarded as characteristic of the molecular movement of heat that it is a pure function of temperature at a fixed pressure and a fixed density of the fluid and especially that it is not dependent upon the state of motion of the fluid. Thermal convection, on the other hand, shall signify the transfer of heat which results when the motion of the particles is directed. In many textbooks of physics free convection is considered as the origination of a natural flow produced by differences of density under the influence of the force of gravity. This concept, then, is contained in the preceding definition.

In the case of laminar flow, all the heat transfer can be accounted for by the foregoing concepts. For turbulent flow, however, one manner of heat transfer is still unmentioned. As is known, steady-state turbulent flow is represented as having at each point a certain average velocity vector upon which is superposed another velocity vector, varying in direction and magnitude, having an average value over a sufficient span of time equal to zero. According to Von Kármán this kinematic picture can be described more exactly by the representation that vortex filaments with a random motion float in the bulk of the fluid, which moves

along with a fixed time-average velocity distribution. The
movements of the vortex filaments, as well as those of the
molecules, obey laws of statistics. The fluctuating veloc-
ity vector (time average is zero) is then defined at a point
of the fluid by the circulation and by the relative position
of all the vortex filaments.

     This concept leads to introduction, apart from the usu-
al thermal conductivity, which appears as an expression for
the statistical law of molecular motion, of a conductivity
of turbulent motion which expresses the statistical influ-
ence of vortex motion upon the transfer of heat. It then
will depend primarily upon the state of motion of the fluid,
which is especially influenced by the nature of the boundary
surfaces.

     The method of accounting for this phenomenon by intro-
duction of an increased conductivity for turbulent motion is
known. Several authors have proposed different basic equa-
tions in which the increased conductivity is regarded as an
empirical function of the velocity. Recognition of the true
circumstances was partially clarified by the considerations
of Reynolds and Prandtl. Both began with the idea that tur-
bulent friction and turbulent heat transfer are analogous
processes, and that the same mechanism which in the first
case causes a "momentum transport" leads to transfer of heat
in the second case. Reynolds (reference 4) in an intuitive
manner, according to this consideration, went directly from
the friction factor to the unit thermal conductance in cir-
cular tubes and compared, as it were, the integral processes.
On the other hand, Prandtl (reference 7) sets up the exact
conditions under which a directly analogous conclusion is
permissible; he shows that in certain cases the temperature
field is an exact image of the velocity field, so that
knowledge of the motion permits direct conclusions about the
thermal field. However, he shows that this is clearly not
the case for right circular tubes, so that conclusions can
only be drawn as to the form of the relation between the
different parameters, since numerical results cannot be ob-
tained. Recent advances (sec. 1) in the mathematical repre-
sentation of turbulent flow and the velocity distribution
corresponding to it now make possible a more exact expres-
sion of the "elementary law" for turbulent heat exchange, so
that the following statements start out from Prandtl's re-
sults in two directions, by first of all furnishing numeri-
cal results and then allowing a mathematical consideration
of the different arrangements in which there exists no spa-
tial constancy of the velocity and temperature fields. This

makes possible a detailed discussion of experimental results
to explain individual deviations.

3. THE FUNDAMENTAL LAW OF TURBULENT HEAT EXCHANGE

First of all, the processes of thought applied to the
kinetic theory of gases, which lead to the differential law
of internal friction and heat transfer by random motion of
molecules, will be applied to the case of turbulent exchange.

Consider a layer at a distance y from the wall; there
the average velocity u prevails in the direction of flow
and let; $|w|$ be the average absolute amount of the velocity
perpendicular to direction of flow. Then the average velo-
city of flow in two layers at a distance $\pm \dfrac{x}{2}$ from the layer
y under consideration (where x is a kind of "mean path")
is:

$$u \pm \frac{\partial u}{\partial y} \frac{x}{2}$$

The momentum transport per unit of surface perpendicu-
lar to the average flow is given, introducing a proportion-
ality factor β which depends upon the nature of the coher-
ent parts of the fluid and the formation of the mean value
with respect to time, by the expression:

$$\beta \rho w \frac{\partial u}{\partial y} x = \tau \tag{5}$$

and is equal to the shear stress τ at y.

If C is the heat capacity of a unit volume, then, on
the other hand, the heat transport q per unit of surface,
likewise perpendicular to the average flow, is given by:

$$q = \beta C w \frac{\partial \vartheta}{\partial y} x \tag{6}$$

This states that the same fluid particles which produce
the shear stress τ by their transmission of momentum, also

transmit the heat. The proportion of heat transmitted can
be calculated by a kind of counting of these particles.

One such "counting" is given, as is easily seen, by the
product βwx (called the coefficient of turbulence). The
coefficient of turbulence, together with the constant C,
represents an expression for the statistical law of heat
transfer in the case of turbulent flow just as does the con-
ductivity λ in the case of no flow.

The coefficient of turbulence can be calculated from
previously obtained knowledge of the state of flow.

From equations (1) and (1a)

$$\tau = \rho \varphi(y) \frac{\partial u}{\partial y}$$

from which (note 5)

$$\beta wx = \varphi(y) = \frac{7}{B} \frac{\tau_0^{3/7}}{\rho^{4/7}} \mu^{1/7} Y^{6/7} \qquad (7)$$

The basic equation (1) expresses the total effect of
the molecular conduction of momentum (internal friction) and
of the momentum transport by eddy convection. Correspond-
ingly, the basic equation itself, as well as the velocity
distribution originating from it, is to be regarded only as
an expression which becomes asymptotic at very great
Reynolds numbers, where the effect of the molecular conduc-
tion of momentum is small in comparison with the second part
of the friction mechanism – that is, eddy convection. How-
ever, it has been found that the proportionality between τ
and $v^{7/4}$ is a very good approximation even at values of the
Reynolds number which correspond to about five times the
critical velocity. From this, it is concluded that the sta-
tistical laws for the molecular and eddy transport of momen-
tum can be represented to a good approximation, even at
moderate Reynolds numbers, by the general expression (1).

By referring to equation (1) for calculation of βwx,
it is assumed that all the heat transfer can be expressed
also by a general statistical law which summarizes molecular
and eddy processes. It has been assumed, therefore, that
there exists in the molecular processes the same proportion-
ality between momentum and energy transfer as exists in the

eddy processes; that is, it is assumed that the ratio between λ and μ is the same as that between C and ρ.

The error committed is negligible for gases, as is shown by the following consideration. On the one hand, the portion of heat carried over by pure turbulent convection is several times that transferred by molecular condition, as is shown by a comparison of unit thermal conductance for laminar and turbulent flow; on the other hand, the ratio $\dfrac{\lambda\rho}{C\mu}$

lies between 1.25 and 0.97 according to the number of atoms in the gas; that is, the molecular mechanism of condition of momentum (internal friction) and that of conduction of heat are essentially similar. Hence, for gases and superheated steam, practical and quantitatively correct results can be expected from the calculation. The following derivations

are to be understood in this sense. The case where $\dfrac{\lambda\rho}{C\mu}$

differs greatly from unity will be referred to once again at the conclusion of the work.

By consideration of equations (6) and (7), there is obtained for the total amount of heat q transferred through a unit surface of a layer at a distance y:

$$q = \frac{7}{B}\left(\frac{\tau_0}{\rho}\right)^{3/7} v^{1/7}\ C\ Y^{6/7}\ \frac{\partial\vartheta}{\partial y} \tag{8}$$

In practice, the limiting value of q_0 for $y = 0$ — that is, the amount of heat going out of the wall per unit of surface — will be calculated as follows: The velocity u is represented by:

$$u(y) = y^{1/7}\left\{A_0 + A_1 y + A_2 y^2 + \ldots\right\}$$

The shear stress τ has a fixed limiting value for $y = 0$, and is a regular function of y in the vicinity of $y = 0$. Therefore, τ can be developed as a power series in y.

$$\tau = \beta w \varkappa \rho\ \frac{\partial u}{\partial y} = \tau_0 + \tau_1 y + \tau_2 y^2 + \ldots$$

Inserting $\frac{\partial u}{\partial y}$,

$$\beta w x = \frac{1}{\rho} \frac{\tau_0 + \tau_1 y + \tau_2 y^2 + \ldots}{\frac{1}{7} A_0 + \frac{8}{7} A_1 y + \ldots} y^{6/7}$$

then developing this fractional expression according to powers of y in the region $y = 0$:

$$\beta w x = \frac{7}{\rho} \frac{\tau_0}{A_0} y^{6/7} \left\{ 1 + \alpha y + \ldots \right\}$$

finally, considering equation (6), yields:

$$q_0 = \frac{7}{\rho} \frac{\tau_0}{A_0} \; C \lim_{y=0} \left[\frac{\partial \vartheta}{\partial y} y^{6/7} \right] \tag{9}$$

4. HEAT EXCHANGE IN TUBES

When there is steady-state flow through a tube, two regions can be distinguished:

1. Fully developed flow state - that is, one in which the velocity profile remains similar along the direction of flow

2. The hydrodynamic calming length at the entrance to the tube

Assume, for example, that the fluid flows into the tube through a smooth passage from a large reservoir; then at the inlet cross section the streamlines will have approximately equal velocity. On progressing further, the layers near the wall will be retarded by friction until the constant (with length) velocity profile, which corresponds to the steady state, has been developed. This part of the tube is often called the entrance section.

In the following sections the temperature field and the heat transfer in the tube are calculated for the case where a temperature distribution for the entrance section is given beforehand and the wall temperature is kept constant along

the direction of flow. Separate solutions are set up for
the two regions mentioned, but by a continuous transition
from the first solution to the second, they can satisfy the
general function through summation of the partial solutions.

5. HEAT TRANSFER FOR THE CASE IN WHICH THE VELOCITY

DISTRIBUTION HAS BEEN ESTABLISHED AT THE

ENTRANCE TO THE THERMAL SECTION

In order to set up the differential equation for the
temperature field, an element of volume, bounded on the
sides by two concentric cylindrical surfaces, parallel to
the walls of the tube, and bounded on the ends by the cross
sections perpendicular to them, is considered. In order to
complete the representation it is assumed that a warm fluid
flows through a colder tube; that is, the flow of heat shall
be from the fluid to the wall. Furthermore, the constant
temperature of the tube is set equal to 0, so that the fluid
temperature is the excess temperature above that of the wall.
However, since no assumption is made which distinguishes one
direction of heat flow from the other, all relations are
valid when ϑ changes its sign.

If   z = coordinate of the direction of flow

$\overline{y}$ = distance from the axis (note 6)

C = heat capacity per unit of volume

then the heat balance for the steady state gives:

$$\frac{\partial}{\partial y}\left\{2\pi\overline{y}q\right\} = Cu\,\frac{\partial\vartheta}{\partial z}\,2\pi\overline{y} \tag{10}$$

From equations (3) and (4), u is replaced by (note 7)

$$u = \frac{8}{7}\,v\,\left\{1 - \left(\frac{\overline{y}}{r}\right)^{2}\right\}^{1/7}$$

while q follows from equation (8); then, also considering
(1a), (3), and (4)

$$q = 0.199 \frac{v^{3/4} \nu^{1/4} C}{(2r)^{3/2}\varepsilon} \left(\frac{r^2 - \overline{y}^2}{2r}\right)^{6/7} \frac{\partial \vartheta}{\partial \overline{y}}$$

If this equation (note 8) is introduced into equation (10), there is finally obtained as the differential equation for heat transfer when the conditions of flow are hydrodynamically complete:

$$\frac{\partial}{\partial \overline{y}} \left\{ \overline{y} \left(\frac{r^2 - \overline{y}^2}{2r}\right)^{6/7} \frac{\partial \vartheta}{\partial \overline{y}} \right\} = K\overline{y} \left\{ 1 - \left(\frac{\overline{y}}{r}\right)^2 \right\}^{1/7} \frac{\partial \vartheta}{\partial z} \tag{11}$$

where

$$K = \frac{8}{7} \frac{v^{1/4} (2r)^{3/2}\varepsilon}{0.199\, \nu^{1/4}} \tag{11a}$$

The boundary conditions are:

I. $\vartheta = 0$ for $\overline{y} = r$.

II. $\frac{\partial \vartheta}{\partial \overline{y}} = 0$ for $\overline{y} = 0$, because of the universal symmetry.

III. The radial temperature distribution must be given for $z = 0$.

Since the fluid temperature approaches asymptotically the temperature of the wall, as the tube length increases, then the solution is of the form

$$\vartheta = g(\overline{y}) e^{-kz}$$

If this expression is inserted into the equation, then there is obtained for the function q the ordinary differential equation

$$\frac{d}{d\overline{y}} \left\{ \overline{y} \left(\frac{r^2 - \overline{y}^2}{2r}\right)^{6/7} \frac{dg}{d\overline{y}} \right\} = -kK\overline{y}\, g(\overline{y}) \left\{ 1 - \left(\frac{\overline{y}}{r}\right)^2 \right\}^{1/7}$$

which, after elimination of the fractional exponents by the transformation

$$\left\{1 - \left(\frac{\overline{y}}{r}\right)^2\right\}^{1/7} = x$$

becomes (note 9)

$$\frac{d}{dx}\left\{(1 - x^7)\frac{dg}{dx}\right\} = -\mathbf{w}x^7 g \tag{12}$$

where (note 10)

$$w = 49kK\left(\frac{r}{2}\right)^{8/7} \tag{12a}$$

The boundary conditions are now:

I. $g = 0$ for $x = 0$

II. $\frac{dg}{dx}$ is finite for $x = 1$

An approximate solution is obtained by means of the Ritz method (note 11), when the problem is changed to one in the Calculus of Variations (note 12); that is, as can be verified easily (note 13):

$$\int_0^1 \left\{(1 - x^7)\left(\frac{dg}{dx}\right)^2 - wx^7 g^2\right\} dx = \text{minimum} \tag{13}$$

with the boundary conditions I and II as supplementary conditions. Here the problem is one of finding the "characteristic values," since equation (13) will have solutions which also satisfy the boundary conditions, only for fixed values of w. Substituting for g:

$$g(x) = g_1 P_1(x) + g_2 P_3(x) + g_3 P_5(x) + \ldots \tag{14}$$

in which g_1, g_2, g_3 are undetermined coefficients and P_1, P_3, and so forth are the Legendre spherical functions (note 14) of the first kind, and taking only three terms first, results, for w at the minimal conditions, in an equation of the third degree the roots of which are

$$w_1 = 8.712, \; w_2 = 164.36, \; w_3 = 1700.40 \tag{14a}$$

The characteristic functions are normalized, in contrast to the customary procedure, so that

$$g_1 + g_2 + g_3 = 1 \qquad (15)$$

in order that the temperature at the axis of the tube will become unity; $g(y)$ is, therefore, the ratio of the temperature at the point under consideration to the temperature at the axis of the tube. Thus the first characteristic functions are:

$$
\begin{aligned}
g_I &= 0.9703\ P_1 + 0.0212\ P_3 + 0.0085\ P_5 \\
g_{II} &= -0.7312\ P_1 + 0.9665\ P_3 + 0.7647\ P_5 \qquad (16) \\
g_{III} &= 2.6552\ P_1 - 6.1589\ P_3 + 4.5037\ P_5
\end{aligned}
$$

The choice of development according to spherical functions must be justified. Since it is a minimal problem, the exact characteristic values can only be less than the approximate values. The magnitudes of the characteristic values which are obtained according to the choice of the basic series equations for the function to be varied, form a suitable criterion for the validity of the approximation. Now, it can be seen that, compared to a simple power equation in x, as well as several Fourier developments, the basic equation in spherical functions leads to the least characteristic values. As to their behavior on making further approximations, the first three approximations for the first characteristic functions furnish successively, for example, the values 8.75, 8.67, 8.71; consequently the convergence of the procedure ought to be satisfactory.

In order to obtain equally good results for the other characteristic values, further approximations must naturally be made; the third characteristic value, in particular, will agree only in magnitude in the case of a three-member basic equation. As will be seen, however, this has only a slight influence upon the results.

The particular merit of the spherical functions for the problem in hand also can be demonstrated by the following simple consideration, which can at the same time dispel doubt caused by the increase of the second coefficients in the second characteristic function. Considering figure 1, it is seen that the characteristic values themselves show great similarity to spherical functions. If a form, like that represented by equation (16), is now set up according to functions which are identical with the characteristic

values, then the coefficient the index of which is equal to the ordinal number of the characteristic value becomes equal to unity, all others being equal to zero. The characteristic value concerned is already represented exactly by one member of the development. If developed according to functions which are not identical with the characteristic values, but which have, however, a certain similarity to them, then, in the development the coefficient having an index equal to the ordinal number will be slightly greater. Clearly, such a development will closely approximate, with relatively few members, the function to be represented.

If the values of w from (14a) are inserted into equations (11a) and (12a), then for the coefficients of the exponents

$$k_1 = 0.1510 \, \frac{1}{d} \left(\frac{\upsilon}{\upsilon d}\right)^{1/4} ; \quad k_2 = 2.844 \, \frac{1}{d} \left(\frac{\upsilon}{\upsilon d}\right)^{1/4} ; \quad k_3 = 29.42 \, \frac{1}{d} \left(\frac{\upsilon}{\upsilon d}\right)^{1/4} \quad (17)$$

The complete solution of the partial differential equation (12) can be written as a development according to characteristic functions

$$\vartheta = a_1 g_I e^{-k_1 z} + a_2 g_{II} e^{-k_2 z} + a_3 g_{III} e^{-k_3 z} \quad (18)$$

where the coefficients are to be determined so that the prescribed temperature distribution is fulfilled for $z = 0$. The calculation will be carried out first for the case of uniform temperature distribution at the initial cross section. Therefore $a_1 \ldots a_3$ first must be determined so that $\vartheta(y)$ must be as close to $\vartheta = 1$ as possible. Every other temperature then follows with the aid of a multiplicative constant. The least square error yields the values:

$$a_1 = 1.129, \qquad a_2 = -0.180, \qquad a_3 = 0.048$$

Thus the final equation of the temperature field for turbulent flow in a hydrodynamically complete state in tubes for the case where the uniform temperature ϑ_0 prevails in the initial cross section, is

$$\vartheta = \vartheta_0 \left\{ 1.129\ e^{-0.151\left(\frac{v}{vd}\right)^{1/4}\frac{z}{d}} [0.9544\ x - 0.0212\ x^3 + 0.0668\ x^5] \right.$$
$$-0.180\ e^{-2.844\left(\frac{v}{vd}\right)^{1/4}\frac{z}{d}} [-0.7472\ x - 4.275\ x^3 + 6.022\ x^5]$$
$$\left. +0.048\ e^{-29.42\left(\frac{v}{vd}\right)^{1/4}\frac{z}{d}} [20.34\ x - 54.80\ x^3 + 35.47\ x^5] \right\} \tag{19}$$

in which the similar powers of x are collected from the P (Legendre spherical functions).

It is also recognized now that an error in the third characteristic value and in the third characteristic function is of slight significance; even if the third exponent should be still somewhat smaller, the third characteristic function dies out several centimeters from the beginning of the thermal effect, the error having no influence upon the remaining part of the tube. A fourth approximation always can be calculated.

6. DISCUSSION OF RESULTS AND AGREEMENT WITH EXPERIMENT

By reference to figure 1, the temperature distribution over the cross section (of the fluid stream) can be discussed. For $z = 0$, a square distribution was assumed; that is, the fluid enters with a uniform temperature over the whole cross section.

In the interval between 0 and 0.8 for $\frac{\overline{y}}{r}$, the uniform temperature is represented to a maximum error of $\pm 2\frac{1}{2}$ (per 1000) by equation (19).

For $\frac{\overline{y}}{r}$ between 0.9 and 1 there is a sharp temperature decrease, since only three terms were considered. Similar situations also exist in reality, since the layers near the wall will undergo a change in temperature, due to radiation, before making direct contact with the wall. On moving farther along the tube, the temperature gradient at the wall levels out more and more; the so-called final temperature

distribution, which is represented by the first characteristic function only, is reached when the second characteristic function has died away. From then on, all temperature profiles remain similar since all temperatures decrease in the z-direction according to the same exponential function; the expression "final temperature distribution" is to be understood in the above sense. As shown by equation (16), the first characteristic function and, consequently, the final temperature distribution, differ but little from the velocity distribution in the hydrodynamically complete state.

With the help of the known temperature field the point now is reached where all the questions about the heat transfer can be answered. For example, to calculate the unit thermal convective conductance α, the ratio is set up of the amount of heat transferred per unit of wall surface for the mixed-mean temperature difference at the cross section; that is,

$$\alpha = \frac{q_o}{\vartheta_m} \tag{20}$$

According to equation (9), q_o is given by

$$q_o = \frac{0.176 \, v^{3/4} \, C v^{1/4}}{r^{3/2} s} \lim_{y=o} \left[\frac{\partial \vartheta}{\partial y} \, y^{6/7} \right]$$

and the average temperature ϑ_m is defined by the equation:

$$\vartheta_m = \frac{1}{\pi r^2} \int_o^r \vartheta(\overline{y}) \, 2\pi \overline{y} d\overline{y}$$

Hence, the expression (note 15) for α is:

$$\alpha = 0.0346 \, vC \left(\frac{v}{vd} \right)^{1/4} \frac{1.078 \, e^{-k_1 z} + 0.134 \, e^{-k_2 z} + 0.980 \, e^{-k_3 z}}{0.970 \, e^{-k_1 z} + 0.024 \, e^{-k_2 z} + 0.006 \, e^{-k_3 z}} \tag{21}$$

The analogous result for laminar flow was calculated by Nusselt. (See reference 6.)

Figure 2, which is calculated for the special value

$$\frac{1}{d} \left(\frac{v}{vd}\right)^{1/4} = 0.037,$$ shows the variation of the unit thermal convective conductance with distance into the tube. At $z = 0$, α is infinite, then, corresponding to the decrease in the temperature difference at the wall; it decreases, though considerably faster than it does when the flow is laminar, finally approaching a minimum value α_{min}. Although this least value is independent of the velocity in the case of laminar flow, for turbulent flow equation (21) is changed into the form:

$$\alpha_{min} = 0.0384 \ vC \left(\frac{v}{vd}\right)^{1/4} \tag{21a}$$

Equation (21a) is analogous to the equation developed by Reynolds. Since, as mentioned already, the eddy heat transfer in turbulent flow exceeds the molecular one by a multiple, it seems justifiable that only those magnitudes which are determinative for the condition of flow and also for the eddy transport of heat should appear in the formula for the unit thermal convective conductance. These are magnitudes v, d, and v or C. The variation with temperature depends upon the values of the kinematic viscosity v. If the relation for gases $\frac{\lambda \rho}{C\mu} \cong 1$ is considered, it is observed that equation (21a) likewise agrees in form with the dimensionless formulas of Nusselt and Prandtl. (See references 2 and 7, respectively.)

The existing experimental material is not sufficient, unfortunately, for an exact test of these results, since average unit thermal convective conductances were always measured and the "entrance sections" were not chosen long enough so that in the measuring length a hydrodynamically complete state with a temperature profile which remained similar could have been attained with certainty. In most cases the point at which the thermal effect began cannot even be determined. Obviously this is an indication that the experimenters possibly did not have a clear concept of the influence of the arrangement upon the results of the measurement.

Nusselt, in a short series of experiments*, connected a

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*Research work published in reference 7, table 6.

piece of tubing 2 meters long in front of the actual experi-
mental section. According to the calculations, which are
discussed in the next section, the state of flow was cer-
tainly complete. Besides, since this entrance section and
the heated experimental tube both were made of brass and
were joined firmly to one another metallically, the added
length of tubing likewise was heated, at least in the part
directly connecting with the experimental tube. The first
point of temperature measurement was, on the average, about
15 centimeters downstream from the beginning of the test
section, so that it can safely be assumed from the results
of equation (21) that the unit thermal convective conduct-
ance had reached its minimum value. These experiments were
checked, using the equation for α_{min}. The results are pre-
sented in the following table (p, air pressure, γ, unit
weight density of air):

Experiment number	ϑ_m	l	p_m	γ_m	v	α_{meas}	α_{calc}	Difference (percent)
95	39.0	0.6133	1.161	1.273	4.24	19.29	20.09	4.14
96	37.8	.6245	1.167	1.285	5.75	24.95	25.36	1.64
97	34.2	.6368	1.164	1.255	8.29	32.75	32.57	-.5
98	31.5	.6438	1.163	1.307	13.06	46.8	47.44	1.3
99	35.6	1.0590	1.164	1.291	21.06	65.3	67.51	3.35
100	32.1	1.1300	1.167	1.309	24.05	73.0	75.25	3.0

Experiment 95, at a Reynolds number of 6100 (about three to
four times the critical velocity), was near the limit of the
region of the validity of the above-developed theory. Con-
sidering the limits of the accuracy of measurements of this
kind, the agreement seems to be absolutely satisfactory.

In the following sections, the heat transfer in the
calming length of a tube (that is, the heat transfer for the
hydrodynamically incomplete state) will be investigated.

Inasmuch as a solution for the velocity field in the calming length of a tube has not been given previously, it must first of all be determined.

7. THE VELOCITY FIELD IN THE "CALMING LENGTH"

In order to obtain an approximate expression for the velocity field in the calming length of tubes under conditions of turbulent flow (see reference 9, for the case of laminar flow), the momentum consideration introduced by Von Kármán (reference 8, pp. 235 and 256) will be used.

Consider a longitudinal section through the beginning of the tube. At A, fluid from a large reservoir flows into it with a uniform velocity distribution. The layers near the wall will be retarded under the influence of the viscosity, and the thickness of the layer, in which the shear stress is transferred (shown by the shaded lines in the figure) will increase until the two boundary layers meet. From then on, with the insertion of a short transition region, the velocity distribution over the cross section will remain constant.

Hence, it is assumed that there is at the beginning of the tube, a region in the interior of the flowing fluid where viscosity can be neglected. For this region, the validity of the Euler equation, formulated for frictionless flow, is assumed.

If the steady state is assumed, a balance on an element of the boundary layer is considered, which has a ring-shape structure; a b c d in figure 3 represents a cross section of this element.

Let

Q   = the volume flowing through the cross section in the boundary layer per second

J   = the transport of momentum per second in the direction of flow through the cross-sectional surface

U   = the velocity of free stream

u   = velocity in the boundary layer

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*L. Schiller, who most recently studied experimentally – a theoretical explanation also was published – the problem of the entrance section, treats only the laminar case.

v = average velocity over the cross section

δ = thickness of the boundary layer

p = pressure

y = distance from the wall of the tube

τ_0 = shear stress at the wall

z = coordinate in the direction of flow

The equilibrium condition for the element of the boundary layer can then be written as follows (note 16):

$$\frac{dJ}{dz} - \rho U \frac{dQ}{dz} = -\frac{dp}{dz} (2r\delta - \delta^2) \pi - 2r\pi\tau_0 \qquad (22)$$

In addition, the following condition appears: Because of continuity, the same amount of fluid must pass through all cross sections of the tube. If, therefore, the layers at the wall are retarded, then the velocity of the undisturbed fluid (undisturbed always in the sense that no shear stress is transmitted) must increase. Then, according to the equation of motion for ideal fluids, this increase in velocity must be accompanied by a decrease in pressure. This furnishes additional boundary equations:

$$Q + U(r - \delta)^2 \pi = vr^2 \pi \qquad (23)$$

$$\frac{dp}{dz} + \rho U \frac{dU}{dz} = 0 \qquad (24)$$

From equation (2a), u is set into the form:

$$u = U \left(\frac{y}{\delta}\right)^{1/7} \left\{ \alpha + \beta \left(\frac{y}{\delta}\right) \right\} \qquad (25)$$

and the coefficients α and β are determined from the requirements:

$$u = 0 \quad \text{for} \quad y = 0$$

$$u = U, \quad \frac{\partial u}{\partial y} = 0 \quad \text{for} \quad y = \delta$$

Then there results:

$$u = U \left(\frac{y}{\delta}\right)^{1/7} \left\{\frac{8}{7} - \frac{1}{7}\left(\frac{y}{\delta}\right)\right\} \qquad (25a)$$

Results of τ_0 from the condition at (25a) must change to equation (2) for small values of y:

$$\tau_0 = \rho \left(\frac{8}{7B}\right)^{7/4} U^{7/4} \left(\frac{v}{\delta}\right)^{1/4} \qquad (26)$$

Consider the relation, which follows from equation (23), between the velocity of the free stream U and the average velocity v:

$$U = \frac{165v}{4\xi^2 - 22\xi + 165} \qquad (27)$$

in which $\frac{\delta}{r} = \xi$; then the steady-state condition yields an ordinary differential equation in ξ, with variables separated. There is obtained then (note (17):

$$\int_0^1 \frac{\left\{\frac{64}{345}\xi^3 - \frac{100}{69}\xi^2 - \frac{206}{207}\xi + \frac{616}{69}\right\} \xi^{1/4}}{[4\xi^2 - 22\xi + 165]^{5/4}} d\xi = \frac{1}{(165)^{1/4}} \left(\frac{8}{7B}\right)^{7/4} \frac{v^{1/4}}{v^{1/4} r^{5/4}} z_0 \qquad (28)$$

Instead of determining the quadrature numerically, the following method of calculation is applied.

For small values of ξ, the higher powers of ξ can be neglected and there is obtained

$$\xi = F\left(z^{4/5}\right)$$

If $z^{4/5}$ is set equal to t, then ξ can be represented as a power series in t. Therefore:

$$\xi = At + Bt^2 + Ct^3 + \ldots \ldots$$

Write equation (28) in the form:

$$\frac{(1 + \alpha\xi + \beta\xi^2 + \gamma\xi^3)\xi^{1/4}}{[1 + p\xi + q\xi^2]^{5/4}} \, d\xi = K \, d z$$

and introduce the above expression; then the fractional powers of t drop out and, by developing in powers of t and comparing the coefficients on left and right revealing the history of the boundary layer at the beginning of the tube, there is obtained:

$$\xi = \frac{\delta}{r} = 1.41 \left(\frac{\nu}{vd}\right)^{1/5} \left(\frac{z}{d}\right)^{4/5} - 0.048 \left(\frac{\nu}{vd}\right)^{2/5} \left(\frac{z}{d}\right)^{8/5} + 0.168 \left(\frac{\nu}{vd}\right)^{3/5} \left(\frac{z}{d}\right)^{12/5} \quad (29)$$

The series is ended at the third term.

In figure 4 (note 18) ξ is presented as a function of

$$\chi = \frac{\nu^{1/5}}{v^{1/5} d} z^{4/5}; \quad \xi = 1 \quad \text{for} \quad \chi = 0.686; \quad \text{thus the length of the}$$

tube up to the section where the boundary layer fills the tube is (note 19):

$$z_0 = 0.625 \, d \left(\frac{vd}{\nu}\right)^{1/4} \quad (30)$$

With equation (29) the field of the average velocities in the entrance section is determined and all questions, for which knowledge of its variation is sufficient, can be answered. Thus, for example, one obtains the resistance of the tube between two cross sections at z_1 and z_2 of the initial length; that is, the integral

$$W = -2r\pi \int_{z_1}^{z_2} \tau_0(z)\,dz$$

by calculating the difference of momentum transport through the cross sections z_1 and z_2 and calculating the pressure difference times the cross-sectional area and then adding the two results.

8. HEAT TRANSFER IN THE ENTRANCE SECTION

For the calculation of the temperature field a similar consideration is employed by setting up a heat balance for one element of the boundary layer. Again, let a warm fluid flow into a tube with constant wall temperature $\vartheta_w = 0$. The only simplifying assumption made is that at a place where shear stress is not transmitted, heat transfer will not occur. In so doing the small amount of heat which is continually carried away at the inner limit of the boundary layer by molecular condition is neglected. However, at that place the temperature gradient is so small, since the calculation is carried out for velocities in excess of the critical, that the error committed can be taken directly into consideration. Therefore it is assumed that in the region where the undisturbed fluid flows with velocity U, the temperature always should be equal to the entrance temperature ϑ_0. The heat balance for the element considered is then:

$$\frac{d}{dz} \int_0^{\delta} uC\vartheta \, 2\pi \, (r - y) \, dy = C\vartheta_0 \frac{dQ}{dz} - 2r\pi q_0 \qquad (31)$$

Inasmuch as it already has been seen for hydrodynamically complete flow that in the case of turbulence the temperature distribution is very similar to the velocity distribution, ϑ is given as:

$$\vartheta = \vartheta_0 \left(\frac{y}{\delta}\right)^{1/7} \left\{\alpha + \beta \frac{y}{\delta} + \gamma \left(\frac{y}{\delta}\right)^2\right\} \qquad (32)$$

Corresponding to the above assumptions, the boundary conditions are:

$$\vartheta = 0 \quad \text{for} \quad y = 0$$

$$\vartheta = \vartheta_0, \ \frac{\partial \vartheta}{\partial y} = 0 \quad \text{for} \quad y = \delta$$

which permit reduction of the three coefficients α, β, γ to a single one. Thus is obtained

$$\vartheta = \vartheta_0 \left(\frac{y}{\delta}\right)^{1/7} \left\{\left(\frac{8}{7} - \frac{1}{7}\frac{y}{\delta}\right) + \gamma\left(1 - \frac{y}{\delta}\right)^2\right\} \tag{32b}$$

The first term is identical with the equation for the velocity distribution and the second can be interpreted as a kind of correction term to the equation of the velocity distribution. This can be easily understood in a physical sense. At $y = 0$, temperature and velocity curves begin with the same power of y. At $y = \delta$, both have horizontal tangents. The curves must therefore have a similar character in the intermediate region.

For q_0 it is found that:

$$q_0 = \frac{1.340 \ v^{3/4} v^{1/4} \ C\vartheta_0 \left\{8/7 + \gamma\right\}}{[4\xi^2 - 22\xi + 165]^{3/4} \ \xi^{1/4} d^{1/4}} \tag{32a}$$

Equation (31) will now furnish, since the variation of the boundary-layer thickness δ is known, an ordinary linear differential equation of the first order for determination of γ. This differential equation can be brought into the form:

$$\frac{d\gamma}{dz} + A(z)\gamma = B(z)$$

The functions A and B are very unwieldy, however, so that the general integral of this first order differential equation:

$$\gamma = Ke^{-\int Adz} + e^{-\int Adz} \int Be^{\int Adz} dz$$

would require a very troublesome numerical calculation. A graphical method is chosen, for it is still possible to obtain far greater accuracy than corresponds to the physical assumptions of the problem. If $\delta/r = \xi$ is introduced as a new independent variable, the equation is essentially simplified, the fractional powers of ξ drop out, and after some calculation there is obtained:

$$\gamma = \frac{-0.1855\ \xi^3 + 1.477\ \xi^2 - 2.658\ \xi}{0.1623\ \xi^3 - 0.701\ \xi^2 - 23.05\ \xi + 45.4}$$

$$- \frac{-0.269\ \xi^4 + 2.390\ \xi^3 - 16.10\ \xi^2 + 37.59\ \xi}{0.1623\ \xi^3 - 0.701\ \xi^2 - 23.05\ \xi + 45.4}\ \gamma' \qquad (33)$$

in which $\gamma' = \dfrac{d\gamma}{d\xi}$

In this form equation (33) is directly suitable for calculation of the directional field of the differential equation which is given in figure 5a. The point $\xi = 0$, $\gamma = 0$ is the point of origin; all solution curves come from plus or minus ∞ up to a unique curve which leads to the point 0,0. Since γ must likewise be finite for $\xi = 0$, then for the initial condition there is obtained

$$\gamma = 0 \quad \text{for} \quad \xi = 0$$

This is likewise readily understandable from a physical viewpoint. As long as $\delta \ll r$, the immediate beginning of the tube can differ, either in hydrodynamical or thermal respect, from the behavior of a plate in a free stream. It will be seen later that the velocity and temperature fields are the same in the case of the plate. Accordingly, at the beginning of the tube (z = 0), the temperature distribution will coincide with the velocity distribution.

If equation (33) is solved in terms of γ' and the limiting value toward which γ' tends is considered, for $\xi = 0$, then there is found:

$$\lim_{\xi=0} \gamma' = -0.032$$

All isoclines begin at the point 0,0; the isocline for $\gamma' = 0.032$ with the slope assigned to it, runs into this point; however, it is rather weakly concave toward the abscissa. The isoclines of greater slope lie entirely above it; those with a lesser slope lie below it. From this behavior it follows that the solution curve sought must lie for its entire length in the narrow strip between the isocline $\gamma' = -0.032$ and its tangent at the zero point. The line $\gamma = -0.032\,\xi$ will, therefore, represent a first approximation with a maximum error of $12\frac{1}{2}$ percent.

In order to obtain a second approximation, γ is set equal to $-0.032\,\xi + h(\xi)$ and this expression is introduced into equation (33), which then changes into a differential equation for h, the family of isoclines of which is shown in figure 5b. In order to increase the accuracy, a thousand-fold scale of ordinates is chosen.

The solution naturally begins with $h = 0$ for $\xi = 0$.

If e is set equal to $1000\,h$, then this magnitude, as is easily proved by plotting on a logarithmic scale, is given by the formula $e = 1.48\,\xi^{1.865}$, so that the following expression for γ is finally obtained:

$$\gamma = -0.032\,\xi + 0.00148\,\xi^{1.865} \tag{34}$$

This function is given in figure 5a by the deep, solid line.

In this manner the temperature field for the region of the simultaneous hydrodynamic and thermal calming length is obtained. Then for ϑ may be written:

$$\vartheta = \vartheta_0\left(\frac{y}{\delta}\right)^{1/7}\left\{\frac{8}{7} - \frac{1}{7}\frac{y}{\delta} - \left(1 - \frac{y}{\delta}\right)^2\left[0.00148\,\xi^{1.865} - 0.032\,\xi\right]\right\} \tag{35}$$

The values of δ and ξ are taken from equation (29) and figure 4, respectively. For $\xi = 1$, ϑ becomes

$$\vartheta = \vartheta_0\left(\frac{y}{r}\right)^{1/7}\left\{1.112 - 0.0819\frac{y}{r} - 0.0305\left(\frac{y}{r}\right)^2\right\} \tag{35a}$$

that is, the final temperature distribution for the hydrodynamical steady state is not attained (fig. 6), which was, perhaps, to be expected

In order to get further agreement, every solution of the differential equation for the temperature field beyond the hydrodynamic calming section, which also satisfies the initial condition (equation (35a)), is determined. To this end the function represented by equation (35a) must be developed according to the characteristic functions. (See equation (16).) Since the temperature distribution of equation (35a) does not differ very much from the first characteristic function (equation (16)), the development is carried out with only two members.

The development naturally cannot represent the function (35a) quite exactly, because at one time the velocity distribution was established with the "influence factor," the other time with a power series development. However, in order to obtain the best possible transition from the one solution to the other, it can be arranged that the two temperature curves agree completely in important properties.

The heat transfer is limited by the processes at the wall; accordingly, it will be stipulated that: (1) both curves begin with the same term of the development at the wall of the tube, and (2) the flow of energy through the whole cross section be equal.

This furnishes two equations of condition for determining the two coefficients of the development. Hence there is obtained for the temperature field in the hydrodynamically complete region

$$\vartheta = \vartheta_0 \left\{ 1.016 \ e^{-k_1 z} \ [0.9544 \ x - 0.0212 \ x^3 + 0.0668 \ x^5] \right.$$
$$\left. - 0.051 \ e^{-k_2 z} \ [- 0.7472 \ x - 4.275 \ x^3 + 6.022 \ x^5] \right\} \qquad (36)$$

Knowledge of the temperature field first shall be used for calculation of the unit thermal convective conductance. The calming length first will be considered.

Equation (36) already has given an expression for q_0. The average temperature of the cross section is then:

$$\vartheta_m = \frac{1}{r^2 \pi} \left\{ \int_0^\delta \vartheta \, 2\pi (r - y) \ dy + \vartheta_0 \int_\delta^r 2\pi (r - y) \ dy \right\}$$

The expression for α_u ($\alpha_u = \alpha$ in the hydrodynamical calming region) is then:

$$\alpha_u = 1.340 \ vC \left(\frac{\nu}{vd}\right)^{\frac{1}{4}}$$

$$\frac{1.143 - 0.032\xi + 0.00148\xi^{1.865}}{(4\xi^2 - 22\xi + 165)^{\frac{3}{4}}\,\xi^{\frac{1}{4}}\left\{1 - 0.133\xi + 0.024\xi^2 + (0.00148\xi^{1.865} - 0.032\xi)(0.520\xi - 0.1434\xi^2)\right\}} \quad (37a)$$

The continuation in the second region $(\alpha_a = \alpha$ in the hydrodynamical steady-state region) is:

$$\alpha_a = 0.03461 \ vC \left(\frac{\nu}{vd}\right)^{1/4} \frac{0.969 \ e^{-k_1 z} + 0.038 \ e^{-k_2 z}}{0.873 \ e^{-k_1 z} + 0.0068 \ e^{-k_2 z}} \quad (37b)$$

The variation of the unit thermal convective conductance with location in the calming length of the tube is shown in figure 7a (note 20). Write equation (37a) in the form

$$\alpha_u = KvC \left(\frac{\nu}{vd}\right)^{1/4} \quad (kcal/hr \ m^2 \ {}^{\circ}C)$$

then the factor K, which is a pure function of ξ, is plotted as ordinate, with ξ as abscissa. Figure 7A (note 21), in combination with figure 4, from which the particular values of ξ can be taken, covers all possible cases. In figure 7B, with z as the abscissa scale, the variation of α for a certain case is shown in comparison with the same case for the completely developed hydrodynamic flow. It is seen that the decrease of α takes place less quickly in the first case.

9. SUMMARY AND COMPARISON WITH OBSERVATIONS

Now comes the point where the heat transfer in a tube can be surveyed in all particulars. The results will be summarized briefly:

With respect to heat transfer, the following cases in which the heat transfer obeys different laws are to be distinguished according to the structure of the velocity and temperature fields.

1. Fully developed hydrodynamic and thermal fields.- This condition is attained when the fluid has passed through a considerable portion of the tube length. The unit thermal convective conductance is constant and is given by equation (21a).

2. Fully developed hydrodynamic flow field, temperature uniform at entrance.- Realized by a connected entrance section which is maintained at the original fluid temperature by suitable heating. The unit thermal convective conductance is dependent upon the location in the tube, falls very quickly from its maximum value, and asymptotically approaches a constant minimum value. (See equation (21).)

3. Uniform velocity and temperature distributions across the section at entrance.- The unit thermal convective conductance is likewise dependent upon the location in the tube, but falls to a minimum value more slowly. The point unit thermal convective conductance is given by equations (37a and 37b).

4. The application of heat begins at a section somewhere in the middle of the calming length.- For this last case a good approximation for the unit thermal convective conductance α is obtained by drawing the curves which represent the variation of α with the location in the tube for cases 2 and 3, in the same system of coordinates but with the zero point of the abscissa scale for α_a (a fully developed hydrodynamic flow) displaced by the distance l between the beginning of heating and the inlet section. Since the curve for α_a has a much steeper slope, it will cut the curve for α_u; the envelope (note 22) represents (to a first approximation) the α distribution for this special case.

The great differences in the results of the individual experimental works are now understandable. Whereas Nusselt ascribed this, at the conclusion of his work on heat transfer in laminar flow, exclusively to the conditions mentioned under point 2, now the possibility of a series of factors which influence the process by interchangeable combinations may be seen.

The two most careful investigations known, those of
Nusselt and Jordan (references 6 and 10, respectively), used
a right-angle gas approach to the measuring section. At the
beginning of this section the flow was not completely devel-
oped. Obviously case 4 is to be considered here. No fur-
ther experimental data, however, have appeared to date. The
beginning of the thermal action, as well as the exact posi-
tion of the first temperature-measuring station, cannot be
ascertained accurately.* An exact evaluation of the exper-
imental results on the basis of the above-mentioned theory
is not possible for these experiments. Nevertheless, a
series of experiments by Nusselt were investigated to deter-
mine the magnitude of the measured unit thermal conductances
with respect to the minimum α. The results are compiled in
table 2. Considering the dimensions of Nusselt's apparatus,
it is seen that these figures are affected by them, which is
to be expected according to the above-mentioned derivations.

TABLE 2

Exper- iment num- ber	l	p_m	ϑ_m	γ_m	v	α_{meas}	α_{min}	Differ- ence (percent)
7	0.5906	1.153	43.7	1.245	8.52	35.7	33.43	6.4
10	.5983	1.152	35.3	1.278	18.33	65.5	60.4	8.4
13	.5880	1.171	31.6	1.315	27.2	91.7	82.6	11.0
19	.7870	2.050	69.0	2.050	8.94	57.3	50.9	12.5
24	.6022	1.883	30.4	2.122	29.94	146.6	127.2	15.3
30	.6178	3.96	31.9	4.45	10.87	124.8	103.8	20.0
41	.5988	6.97	29.2	7.91	9.27	162.9	141.6	15.1
54	.5863	9.98	26.0	11.41	10.04	233.0	198.0	17.7

If experiment 10, for example, were evaluated with the
assumption that case 3 is to be considered here, which comes
close to the experimental conditions in any case, then a

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*These results are based on a letter sent to Latzko by
Prof. Nusselt.

value of α is obtained, which is 10.2 percent above α_{min}. In reality the hydrodynamical condition of uniform distribution of velocity at the first temperature-measuring station was not completely fulfilled, so that the value of α must fall somewhat lower; the measured value is 8.4 percent above α_{min}.

According to equation (29), the length of the calming section is proportional to $\sqrt[4]{v\rho}$; accordingly the ratio of the calming length to the total length of the measuring station must increase with increasing velocity, and density and therefore the values of α also must increase somewhat. Experiments 7, 10, and 13, in which the velocity is varied, the other parameters being maintained reasonably constant, show this clearly, as do experiments 7, 19, 41, and 54, in which the values of ρ are changed up to a ratio of 1:10.

10. PRACTICAL COMPUTATION OF THE AMOUNT

OF HEAT TRANSFERRED

Finally the question of the practical computation of the amount of heat transferred in tubes must be discussed. Because of the fact that two different solutions for the temperature field have been obtained, depending upon whether or not the hydrodynamic field is fully developed, it is necessary also, in the computation of the amount of heat transferred in a certain section of the tube, to determine in which region the flow takes place. However, several general remarks, which hold for both regions, must be presented first.

The amount of heat transferred between two cross sections at an interval $l = z_1 - z_2$ can always be found in two ways.

1. The calculation is referred to the volume of fluid passing through the cross section per unit time. Since the velocity and temperature distributions are known, then the flow of heat which passes through the cross-sectional area per unit of time can be obtained by integration. The amount of heat transferred at a strip of wall of length l between z_1 and z_2 thus is given by the difference of two integrals:

$$Q = C \int_f u(y,z_1)\vartheta(y,z_1)df - C \int_f u(y,z_2)\vartheta(y,z_2)df \qquad (38)$$

where   df   is an element of area in the cross section the
form of which depends upon the choice of   y.

2. Only the processes at the wall are considered. For
this purpose the equations for q_0 have been set up. With
the foregoing notation, the amount of heat transferred is
then:

$$Q = 2r\pi \int_{z_1}^{z_2} q_0(z)dz \tag{39}$$

the term ϑ_0 in q_0 is replaced by the difference at the
cross section z_1 between the wall temperature and the
temperature at the axis of the tube.

The expressions for q_0 are, however, rather unwieldy,
so that equation (38) is usually preferable. As an example,
equation (38) will be applied to the most important cases.

1a. For the case where a uniform temperature distribu-
tion and a uniform velocity distribution prevail in the ini-
tial cross section, the heat transfer in the entire entrance
section is:

$$Q = 0.115 \ r^2\pi v C\vartheta_0 \tag{40}$$

The value is constant since the temperature and velocity
distributions coincide at the initial and the final cross
sections. The special cases differ in the length of the
tube section from the inlet opening to the fully developed
hydrodynamic state.

1b. Considering a section of the tube from the inlet to
a cross section at a distance   L   and letting l be the
length of the entrance section, then, (using equation (36)),
the total heat transferred in the length   L   is:

$$Q = r^2\pi v C\vartheta_0 \left\{ 1 - 0.886 \ e^{-k_1 l_1} - 0.0037 \ e^{-k_2 l_1} \right\} \tag{40a}$$

where $l_1 = L - l.$

2. In the fully developed hydrodynamic state the amount
of heat transferred from the beginning of the heating ($z = 0$)
to a cross section   z   at a distance l from it, comes to:

$$Q = \pi r^2 v C \vartheta_0 \left\{ 1 - 0.985 \ e^{-k_1 l} - 0.013 \ e^{-k_2 l} - 0.0022 \ e^{-k_3 l} \right\} \qquad (41)$$

11. ESTIMATION OF THE INTERNAL HEATING

OF THE FLUID BY FRICTION

In the previous derivations the generation of heat in the interior of fluids by friction was entirely disregarded. An exact consideration of this is not possible as long as the pulsating velocities are not known individually, knowledge of the distribution of the (time) average velocity being especially insufficient for this, since the pulsation velocities contain the dissipation function in quadrature terms. However, an attempt will be made to determine, by an approximate consideration, in which velocity region the above-mentioned neglect of this term is allowable. In doing this, only processes in the fully developed hydrodynamic and thermal states are considered; that is, the unit thermal conductance must be independent of the location in the tube.

It is assumed, for example, that a cold fluid flows through a heated tube, an element of volume bounded by the tube and two cross sections at a distance dz is considered. Let ϑ_w be the constant wall temperature; let $\vartheta(z)$ be the average temperature of the cross section at the position z; and let the difference between wall and average temperature in the initial cross section be designated by ϑ_0. Then the heat balance on the element reads:

$$r^2 \pi v C d\vartheta = \alpha(\vartheta_w - \vartheta) 2r\pi dz + T\zeta v^3 2r\pi dz \qquad (42)$$

where T is the thermal equivalent of work and ζ is the ratio of the frictional resistance to the square of the velocity. Equation (42) then is written in the form:

$$\frac{d\vartheta}{dz} + \frac{2\alpha}{rvC} \vartheta = \frac{3\alpha}{rvC} \left(\vartheta_w + T \frac{\zeta}{\alpha} v^3 \right)$$

The term $T \frac{\zeta}{\alpha} v^3$ has dimensions of temperature; $T \frac{\zeta}{\alpha} v^3$ will be called the friction temperature and will be designated by ϑ_R.

With consideration of the initial conditions at $z = 0$, the solution of the differential equation (42) is:

$$\vartheta_z = (\vartheta_w + \vartheta_R) - e^{-\frac{2\alpha}{rvC} z} [\vartheta_0 + \vartheta_R] \tag{43}$$

Hence, the temperature of the fluid is proportional to the wall temperature increased by the friction temperature.

If this relation is plotted in a system of coordinates with ϑ as ordinate and z as abscissa, then there is obtained the clear results of figure 8, which are drawn for the case of air flowing with

$v = 200$ m/sec $r = 0.025$ m $p = 1$ atm

$C = 0.282$ kcal/m^3 $v = 0.175$ cm^2/sec

The fluid temperature asymptotically approaches a limit which is equal to the wall temperature plus the friction temperature. Since the amount of heat transferred is proportional to the areas (which are crosshatched in fig. 8) between the straight line $\vartheta_w =$ constant and the curve of the temperature of the fluid, it is seen that there exists such a relation between the length of the tube section and the velocity of the stream that, for a given length, there exists a certain velocity and, for a given velocity, there exists a certain length of tube for which the maximum heat is transferred.

The factor $\frac{\zeta}{\alpha}$ is then:

$$\frac{\zeta}{\alpha} = 0.103 \frac{1}{vc}$$

and, therefore, if the slight variation of the heat capacity with temperature is disregarded, it is a function of velocity alone. For $c = 0.238$ kilocalorie per kilogram $^\circ$C, ϑ_R for air has been calculated for several velocities and compiled in the following table:

$v =$	10	25	50	100	150	200 m/sec
$\vartheta_R =$	0.102°	0.634°	2.54°	10.15°	22.80°	40.6° C

It can be seen that neglecting ϑ_R is directly permissible in most practical cases.

The heat carried away is obtained by substitution of (43) into the equation:

$$dQ = \alpha 2r\pi dz \, (\vartheta_w - \vartheta_z)$$

and integration between $z_1 = 0$ and $z_2 = l$ to

$$Q = \vartheta_o r^2 \pi v C \left(1 - e^{-\frac{2\alpha l}{rvC}}\right) - \frac{1}{2} \vartheta_R r^2 \pi v C \left(\frac{2\alpha l}{rvC}\right)^2 \qquad (44)$$

in which higher powers of $\dfrac{2\alpha l}{rvC}$ are neglected.

12. HEAT TRANSFER ON A FLAT PLATE

As a second geometric configuration for which the heat transfer will be calculated, the flat plate parallel to the direction of flow is chosen. The plate shall be so thin that the influence of the forward edge can be neglected. Prandtl, in the previously mentioned work, has already shown that for the case of an infinite thin plate, which is moved parallel to itself through a fluid, the velocity field and temperature field agree, if the heat from the internal friction is neglected.

If u denotes the velocity vector and p denotes the pressure, then, for the time change of the momentum vector of an incompressible fluid referred to a unit of volume, the result is, neglecting gravity effects (note 23):

$$\left.\begin{aligned}
\rho \, \frac{Du}{dt} &= -\,\text{grad } p + \mu \Delta u \\[2mm]
\frac{D}{dt} &= \frac{\partial}{\partial t} + u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z}
\end{aligned}\right\} \qquad (45)$$

On the other hand, the Kirchhoff differential equation for the temperature field is:

$$\frac{D\vartheta}{dt} = \frac{\lambda}{c\rho} \Delta\vartheta \tag{46}$$

For the plate, p is a constant, in case it is infinitely thin; or, in case the plate thickness is finite, the average value of p over a certain region is still a constant; grad p is, therefore, equal to 0 and it may be seen that the two equations (45) and (46) agree in form. When $\frac{\lambda}{c\mu} = 1$, a solution of (45) also will be a solution of (46).

If use is made of this condition when the solution for the velocity field, as given in the previous work by Von Kármán, is accepted, there results:

$$u = U \left(\frac{y}{\delta}\right)^{1/7}, \quad \delta = 0.366 \left(\frac{v}{Uz}\right)^{1/5} z \tag{47}$$

and correspondingly:

$$\vartheta = \vartheta_0 \left(\frac{y}{\delta}\right)^{1/7}$$

The condition for thermal equilibrium in an element of the boundary layer is:

$$\frac{d}{dz} \int_0^\delta U \left(\frac{y}{\delta}\right)^{1/7} C\vartheta_0 \left(\frac{y}{\delta}\right)^{1/7} dy - C\vartheta_0 \frac{d}{dz} \int_0^\delta U \left(\frac{y}{\delta}\right)^{1/7} dy + q_0 = 0$$

Introducing equation (47), solve for q_0 and obtain:

$$q_0 = 0.0285 \; \vartheta_0 \; CU \left(\frac{v}{Uz}\right)^{1/5} \tag{48}$$

The total amount of heat leaving a plate strip of unit width is then (note 24):

$$Q = \int_0^l q_0 dz = 0.0356 \; CU\vartheta_0 l \left(\frac{v}{Ul}\right)^{1/5} \tag{49}$$

(l = length in the direction of flow).

13. TRANSMISSION IN LIQUIDS

In the preceding section it has been seen that the laws for the heat transfer in a turbulent fluid stream, derived for the two most important basic geometric forms, lead, in the case of flowing gases (and superheated steam), to results which also agree well quantitatively with experiments. Here the statistical basis of the kinetic theory of gases can be used to arrive at a uniform concept of the molecular processes as well as the eddy processes in friction, on the one hand, and in heat transfer on the other hand.

Naturally, one cannot transfer the foregoing simple considerations directly to liquids, where the effect of the molecular forces of cohesion may no longer be neglected. Whereas the heat and momentum convection through the eddy system also represent here processes which are similar in character, this is no longer true of the molecular conduction of heat and momentum, which finds its expression in that the ratio $\dfrac{\lambda \rho}{C \mu}$ is very different from 1. For water, the magnitude of this ratio, which is quite dependent upon temperature, is about 0.1.

The mutual law for the molecular and eddy phenomena of internal friction, which is represented by the coefficient of turbulence, will be applicable to the propagation of heat only in that region in which eddy processes dominate. However, this is the case for the entire mass of fluid up to a very thin layer* at the wall. Accordingly, the differential equations derived in the foregoing will maintain their validity everywhere except in this thin layer.

To attain a suitable description of the heat transfer in fluids, it will be necessary to seek a transition from

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*Closer investigation shows that this layer is much smaller than the boundary layer itself, which was defined as the region in which shear stresses are transmitted. For the right circular tube, the thickness of this layer in the hydrodynamically perfect state is given by the expression

$\delta = 5.51 \dfrac{d}{R^{7/8}}$ where d is the diameter and R is the

Reynolds number.

the statistical law for the interior of the mass of fluid
(coefficient of turbulence) to the molecular law in the very
neighborhood of the wall (thermal conductivity). Mathemati-
cally, this transition can be made by a modification of the
boundary conditions. This extension of the theory shortly
will be discussed more closely. Still, it is to be noted
that even the previous experimental results of research,
which relate exclusively to heat transfer in water, show re-
sults deviating so much from one another that only with dif-
ficulty can a picture of the process be made clear to some
degree.

At considerable expenditure, Soennecken (1911) under-
took experiments on the heat transfer of water in tubes.
These experiments are frequently cited in modern literature
as authoritative. His results are summarized in two formu-
las for the unit thermal conductance α:

1. Smooth surfaces:

$$\alpha = 2020 \frac{v^{0.9}}{d^{0.1}} (1 + 0.014 \, T_i) \frac{kcal}{hr \, m^2 \, {}^\circ C}$$

2. Rough surfaces:

$$\alpha = 735 \frac{v^{0.7}}{d^{0.3}} (1 + 0.014 \, T_i) \frac{kcal}{hr \, m^2 \, {}^\circ C}$$

where

v water velocity, meters per second

d tube diameter, meters

T_i internal tube-wall temperature, degrees centigrade

By "smooth surface" is understood a seamless drawn-brass
tube; whereas the experiments with "rough surfaces" were
performed with iron tubes. These formulas directly contra-
dict the fundamental ideas on the nature of the heat transfer
presented here. It is out of the question that the unit
thermal conductance can be smaller in the case of rough sur-
faces than it is for smooth surfaces. Because of the in-
creased eddy formation, the eddy transport is increased and
consequently the unit thermal conductance is increased. The
smaller values which were measured in the case of the iron

tubes are most probably ascribable to an incorrect determination of the actual wall temperature due to the presence of layers of boiler scale and rust.

Extensive experiments on condensers were performed by Josse in Charlottenburg. Josse measured the over-all unit thermal conductance k of condensing steam to water, which is defined by the formula (reference 11):

$$k = \frac{1}{\frac{1}{\alpha_1} + \frac{\lambda}{\delta} + \frac{1}{\alpha_2}}$$

where

α_1, α_2 unit thermal conductances of the fluids

λ thermal conductivity of the partition

δ thickness of the partition

Since Josse substitutes for λ and δ the known values and uses the figures of Nichol (about 4500 kcal/hr m^2 $^\circ$C), which represent in magnitude a mean between the figures of Soennecken for smooth and for rough surfaces, for the heat transfer in the water, he obtains unusually high values for the heat transfer from condensing steam to metals. According- ing to these experiments the unit thermal conductance for condensing steam would be about seven times as large as that for flowing water.

These results are likewise not understandable. Accord- ing to Nusselt (reference 12), the process of condensation on a cold perpendicular wall produces on the cold side of the surface a film of water in which occurs all the drop from steam to wall temperature. The film of water clings to the wall, the remaining layers flow downward under the ef- fect of gravity. Since the thickness of the film of water is very small in any case - fractions of a millimeter accord- ing to Nusselt's calculation - the flow obviously must be laminar. It is impossible to understand why the unit thermal conductance from water to the metallic wall should be seven times greater on the one side of the wall (where the state of flow is laminar) than on the other side (where there is turbulent flow).

The experimental results of Josse accordingly ought to have some other explanation so that k, the over-all unit thermal conductance, is represented by two terms, approximately equal in magnitude, $\frac{1}{\alpha_1}$ and $\frac{1}{\alpha_2}$ where α_1 and α_2 are average values.

It seems quite probable that, for this reason, the unit thermal conductance for water has higher values than were frequently assumed previously. An extensive experimental investigation of the heat transfer to fluids seems to be urgently needed in order to be able to test the accuracy of the theoretical calculations. The difficulties which are encountered due to formation of rust and scale when water is used, suggest the use of other fluids, such as oil, for example.

Translation by L. M. K. Boelter,
G. Young, and A. G. Guibert.

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EXPLANATORY NOTES BY THE TRANSLATORS

NOTE 1

The equation,

$$\tau = \frac{7}{B} \, (\tau_0)^{3/7} (\rho)^{3/7} (\mu)^{1/7} (y)^{6/7} \, \frac{\partial u}{\partial y}$$

can be obtained in the following manner. Postulate that the 1/7-power equation for the velocity distribution holds near the wall and that the shear stress at, and in the vicinity of, the wall is a function of y, the distance from the wall, and of $\frac{\partial u}{\partial y}$, the velocity gradient at that point. Then,

$$\frac{\tau}{\rho} = F\left(y, \, \frac{\partial u}{\partial y}\right)$$

divide by the particular velocity gradient, which gives

$$\frac{\dfrac{\tau}{\rho}}{\dfrac{\partial u}{\partial y}} = F^0\left(y, \, \frac{\partial u}{\partial y}\right)$$

From the 1/7-power equation (see note 3),

$$u = B \left(\frac{\tau_0}{\rho}\right)^{4/7} \left(\frac{y}{\nu}\right)^{1/7}$$

there is obtained:

$$\frac{\partial u}{\partial y} = \frac{B}{7} \left(\frac{\tau_0}{\rho}\right)^{4/7} \left(\frac{1}{\nu}\right)^{1/7} (y)^{-6/7}$$

or

$$\frac{\dfrac{\tau}{\rho}}{\dfrac{\partial u}{\partial y}} = \frac{\dfrac{\tau}{\rho}}{\dfrac{B}{7}} \left(\frac{\tau_0}{\rho}\right)^{4/7} \left(\frac{1}{\nu}\right)^{1/7} (y)^{-6/7}$$

In the vicinity near the wall, let

$$F^{o}\left(y, \frac{\partial u}{\partial y}\right) = F^{o}_{y \to o}\left(y, \frac{\partial u}{\partial y}\right)$$

therefore

$$\frac{\frac{\tau}{\rho}}{\frac{\partial u}{\partial y}} = \frac{\frac{\tau_{o}}{\rho}}{\left[\frac{B}{7}\left(\frac{\tau_{o}}{\rho}\right)^{4/7}\left(\frac{1}{\nu}\right)^{1/7} y^{-6/7}\right]_{y \to o}}$$

or

$$\tau = \frac{7}{B}\tau_{o}^{3/7}\rho^{3/7}\mu^{1/7} y^{6/7}\frac{\partial u}{\partial y}$$

Latzko dropped the subscript $(y \to o)$ and stipulated that the equation is valid only in the vicinity of the wall.

NOTE 2

Sydney Goldstein (Modern Developments in Fluid Dynamics, Clarendon Press (Oxford), 1938, pp. 339-340) recommends a value of $B = 8.7$.

NOTE 3

For fluid flow near the wall in smooth pipes the velocity u is determined by the following variables: τ_{o}, the shear stress at the wall; y, the distance from the wall; ρ, the density of the fluid; and ν, the kinematic viscosity of the fluid. By dimensional analysis there is obtained

$$\frac{u}{\sqrt{\frac{\tau_{o}}{\rho}}} = F\left(\frac{\sqrt{\frac{\tau_{o}}{\rho}}\, y}{\nu}\right) = K\left(\frac{\sqrt{\frac{\tau_{o}}{\rho}}\, y}{\nu}\right)^{n}$$

A comparison of this equation with the Blasius resistance

formula (empirical) for flow in smooth pipes (see Goldstein, Modern Development in Fluid Dynamics, pp. 339-340) allows the determination of the magnitude of the exponent   n   and K;   so there results the 1/7-power equation for velocity distribution,

$$u = \bar{B} \left(\frac{\tau_o}{\rho}\right)^{4/7} \left(\frac{y}{v}\right)^{1/7}$$

NOTE 4

As an attempt to correlate the hydrodynamic principles presented by Latzko with the more recent knowledge of velocity and shear distributions, the Prandtl mixing length was calculated from equation (7) and compared with those derived from Prandtl's and Kármán's logarithmic formulas for velocity distribution and also with that obtained from Nikuradse's data.* Although there are some inconsistencies in Latzko's equation (inconsistencies also appear in Prandtl's and Kármán's equations), the variation of the mixing length does have the same trend as that calculated from Nikuradse's experimental data and differs from it only by a constant ratio

of approximately 1.25 $\left(\frac{l \ (\text{Nikuradse})}{l \ (\text{Latzko})} = 1.25\right)$. The method of

calculation and the results are shown in table A-1 and figure A-1.

The most important inconsistency appears in the determination of the velocity distribution near the wall. Recent developments indicate that the velocity is a linear function $\left(u = \frac{\tau_o}{\rho v} y\right)$ of the distance from the wall in the laminar

sublayer. Latzko's expression for the velocity distribution, however, approaches the 1/7-power equation near the wall and would yield an infinite instead of a finite velocity gradient at the wall $\left(\left|\frac{\partial u}{\partial y}\right|_{y=0} = \infty\right)$. Latzko may be justified in

using such an expression by the fact that his expression for

*See Bakhmetev, Boris A.: The Mechanics of Turbulent Flow. Princeton Univ. Press, 1941, p. 73.

shear stress is $\tau = K (y)^{6/7} \dfrac{\partial u}{\partial y}$, which could yield a finite value for τ_0, however, because the product of zero and infinity is indeterminate.

NOTE 5

The term βwx, called here the coefficient of turbulence, is identical with the modern term "eddy diffusivity," ϵ in

$$\tau = \rho \epsilon \frac{\partial u}{\partial y}$$

NOTE 6

From this point on, Latzko uses y for the distance from the pipe axis. Up to this point, however, he used y as the distance from the wall and $\overline{y}$ for the distance from the axis. To avoid confusion, this translation continues to use y for the distance from the wall and $\overline{y}$ for the distance from the axis only. The equations that follow have been altered (from the original) to conform with these original definitions.

NOTE 7

The original article gave the equation,

$$u = \frac{7}{8} v \left\{ 1 - \left(\frac{\overline{y}}{r} \right)^2 \right\}^{1/7}$$

It should be

$$u = \frac{8}{7} v \left\{ 1 - \left(\frac{\overline{y}}{r} \right)^2 \right\}^{1/7}$$

NOTE 8

The derivation of

$$q = 0.199 \frac{v^{3/4} \, v^{1/4} \, C}{(2r)^{3/28}} \left(\frac{r^2 - \overline{y}^2}{2r} \right)^{6/7} \frac{\partial \vartheta}{\partial \overline{y}}$$

from equations (8), (2), (3), and (4).

Equation (8):

$$q = \frac{7}{B} \left(\frac{\tau_0}{\rho} \right)^{3/7} C v^{1/7} Y^{6/7} \frac{\partial \vartheta}{\partial y}$$

Equation (2):

$$u = B \left(\frac{\tau_0}{\rho} \right)^{4/7} \left(\frac{y}{v} \right)^{1/7}$$

Equation (3):

$$u = u_{max} \left[1 - \left(\frac{\overline{y}}{r} \right)^2 \right]^{1/7}$$

Equation (4):

$$v = \frac{7}{8} u_{max}$$

From equation (2) there is obtained

$$\left(\frac{\tau_0}{\rho} \right)^{3/7} = \left[\frac{u}{B \left(\frac{y}{v} \right)^{1/7}} \right]^{3/4} = \frac{u^{3/4}}{B^{3/4} \left(\frac{y}{v} \right)^{3/28}}$$

But by equations (3) and (4),

$$u = \frac{8}{7} v \left\{ 1 - \left(\frac{\overline{y}}{r} \right)^2 \right\}^{1/7}$$

or

$$u^{3/4} = \left(\frac{8}{7} \right)^{3/4} v^{3/4} \left\{ 1 - \left(\frac{\overline{y}}{r} \right)^2 \right\}^{3/28}$$

therefore

$$\left(\frac{\tau_0}{\rho}\right)^{3/7} = \frac{\left(\frac{8}{7}\right)^{3/4} v^{3/4} \left\{1 - \left(\frac{\overline{y}}{r}\right)^2\right\}^{3/2\varepsilon}}{B^{3/4} \left(\frac{y}{v}\right)^{3/2\varepsilon}}$$

Substituting into equation (8),

$$q = \frac{7}{B} \frac{\left(\frac{8}{7}\right)^{3/4} v^{3/4} \left\{1 - \left(\frac{\overline{y}}{r}\right)^2\right\}^{3/2\varepsilon}}{B^{3/4} \left(\frac{y}{v}\right)^{3/2\varepsilon}} v^{1/7} C Y^{6/7} \frac{\partial \vartheta}{\partial y}$$

$$= \frac{\left(7 \times 8^3\right)^{1/4}}{B^{7/4}} v^{3/4} v^{1/4} C \left(\frac{r^2 - \overline{y}^2}{2r}\right)^{6/7} \frac{\partial \vartheta}{\partial y} \left\{\frac{1 - \left(\frac{\overline{y}}{r}\right)^2}{y}\right\}^{3/2\varepsilon}$$

and since

$$\frac{1 - \left(\frac{\overline{y}}{r}\right)^2}{y} = \frac{r^2 - r^2 + 2ry - y^2}{r^2 y} = \frac{4}{2r}\left(1 - \frac{y}{2r}\right)$$

thus

$$q = \frac{\left(7 \times 8^3\right)^{1/4}}{B^{7/4}} \frac{4^{3/2\varepsilon} v^{3/4} v^{1/4} C}{(2r)^{3/2\varepsilon}} \left(\frac{r^2 - \overline{y}^2}{2r}\right)^{6/7} \frac{\partial \vartheta}{\partial y} \left(1 - \frac{y}{2r}\right)^{3/2\varepsilon}$$

and taking $\left(1 - \frac{y}{2r}\right)^{3/2\varepsilon} \cong 1$, which is approximately true in

the vicinity of the wall (y small), yields

$$q = \frac{0.199 \, v^{3/4} v^{1/4} C}{(2r)^{3/2\varepsilon}} \left(\frac{r^2 - \overline{y}^2}{2r}\right)^{6/7} \frac{\partial \vartheta}{\partial y}$$

NOTE 9

Transformation of equation (11) to equation (12).

Equation (11):

$$\frac{\partial}{\partial \overline{y}} \left\{ \overline{y} \left(\frac{r^2 - \overline{y}^2}{2r} \right)^{6/7} \frac{\partial \vartheta}{\partial \overline{y}} \right\} = K\overline{y} \left\{ 1 - \left(\frac{\overline{y}}{r} \right)^2 \right\}^{1/7} \frac{\partial \vartheta}{\partial z}$$

where

$$K = \frac{8 \; v^{1/4} \; (2r)^{3/2 e}}{7 \times 0.199 \; v^{1/4}}$$

Let

$$\vartheta = g \, (\overline{y}) \, e^{-kz}$$

then

$$\frac{\partial \vartheta}{\partial \overline{y}} = e^{kz} \frac{dg}{d\overline{y}}$$

and

$$\frac{\partial \vartheta}{\partial z} = - \, ke^{-kz} g(\overline{y})$$

Substituting in equation (11) yields

$$\frac{\partial}{\partial \overline{y}} \left\{ \overline{y} \left(\frac{r^2 - \overline{y}^2}{2r} \right)^{6/7} \frac{\partial g}{\partial \overline{y}} \right\} = - \, Kk\overline{y}g(\overline{y}) \left\{ 1 - \left(\frac{\overline{y}}{r} \right)^2 \right\}^{1/7}$$

To simplify, let $\left\{ 1 - \left(\frac{\overline{y}}{r} \right)^2 \right\}^{1/7} = x$

or

$$\overline{y}^2 = (1 - x^7) \, r^2$$

then

$$2\overline{y}d\overline{y} = - \, r^2 \; 7 \; x^6 dx$$

$$d\overline{y} = \frac{r^2 \; 7 \; x^6 dx}{2\overline{y}}$$

By substituting, there is obtained

$$\frac{d}{dx} \left\{ (1 - x^7) \, \frac{\partial g}{\partial x} \right\} = - \, 49kK \left(\frac{r}{2} \right)^{8/7} x^7 g = - \, wx^7 g$$

where

$$w = 49kK \left(\frac{r}{2}\right)^{8/7}$$

NOTE 10

The original article gave

$$w = 4gkK \left(\frac{r}{2}\right)^{8/7}$$

It should read

$$w = 49kK \left(\frac{r}{2}\right)^{8/7}$$

NOTE 11

(a)   Some references to the Ritz method of solution of
       differential equations.

   Ritz, Walter: Über eine neue Methode zur Lösung
       gewisser Variationsprobleme der mathematischen
       Physik. J. f. reine u. angew. Math., vol. 135,
       1908, pp. 1-61.

   Über eine neue Methode zur Lösung gewisser Rand-
       wertaufgaben. Göttinger Nachrichten, 1908, pp.
       236-248.

   Both in "Oeuvres de Walther Ritz," Paris, Gauthier-
       Villars, 1911.

   Timoshenko, S.: Vibration Problems in Engineering,
       D. Van Nostrand Co., New York, 1928, p. 259.

(b)   The manipulations involved in the Ritz method as ap-
       plied to this problem.

   1. Insert the Legendre spherical functions of the
       first kind into equation (14). Then,

$$g(x) = \left(g_1 - \frac{3}{2}g_2 + \frac{15}{8}g_3\right)x + \left(\frac{5}{2}g_2 - \frac{70}{8}g_3\right)x^3 + \frac{63}{8}g_3x^5$$

2. Now differentiate equation (14) with respect to x

to obtain $\dfrac{\partial g}{\partial x}$

$$\frac{\partial g}{\partial x} = \left(g_1 - \frac{3}{2}g_2 + \frac{15}{8}g_3\right) + 3\left(\frac{5}{2}g_2 - \frac{70}{8}g_3\right)x^2 + \frac{5 \cdot 63}{8}g_3x^4$$

3. Substitute into equation (13), the value of g and
$\dfrac{\partial g}{\partial x}$ obtained in steps (1) and (2) and integrate.
Then,

$$\int_0^1 \left\{(1-x^7)\left(\frac{dg}{dx}\right)^2 - wx^7g^2\right\}dx = \left(\frac{7}{8} - \frac{w}{10}\right)\left(g_1{}^2 - 3g_1g_2 + \frac{15}{4}g_1g_3 + \frac{9}{4}g_2{}^2 - \frac{45}{8}g_2g_3 + \frac{(15)^2}{(8)^2}g_3{}^2\right)$$

$$+ \left(\frac{14}{10} - \frac{w}{6}\right)\left(\frac{5}{2}g_1g_2 - \frac{15}{4}g_2{}^2 + \frac{5 \cdot 15}{16}g_2g_3 - \frac{70}{8}g_1g_3 + \frac{210}{16}g_2g_3 - \frac{70 \cdot 15}{64}g_3{}^2\right)$$

$$+ \left(\frac{7}{6} - \frac{w}{7}\right)\left(\frac{63}{8}g_1g_3 - \frac{3 \cdot 63}{16}g_2g_3 + \frac{15 \cdot 63}{64}g_3{}^2\right) + \left(\frac{63}{60} - \frac{w}{14}\right)\left(\frac{25}{4}g_2{}^2 - \frac{350}{8}g_2g_3 + \frac{4900}{64}g_3{}^2\right)$$

$$+ \left(\frac{30}{14} - \frac{w}{8}\right)\left(\frac{5 \cdot 63}{16}g_2g_3 - \frac{63 \cdot 70}{64}g_3{}^2\right) + \left(\frac{7 \cdot 25}{144} - \frac{w}{18}\right)\left(\frac{63}{8}\right)^2 g_3{}^2$$

4. Then differentiate the equation obtained in step (3) with
respect to g_1, g_2, and g_3, yielding the three equations
for $\dfrac{\partial J}{\partial g_1}$, $\dfrac{\partial J}{\partial g_2}$, and $\dfrac{\partial J}{\partial g_3}$, where

$$J = \int_0^1 \left\{(1 - x^7)\left(\frac{\partial g}{\partial x}\right)^2 - wx^7g^2\right\}dx$$

and set them equal to zero.

$$\frac{\partial J}{\partial g_1} = 0 = (1.75 - 0.2\,w)g_1 + (0.875 - 0.1167\,w)g_2 + (1.22 - 0.04\,w)g_3$$

$$\frac{\partial J}{\partial g_2} = 0 = (0.875 - 0.1167 \ w) \ g_1 + (6.54 - 0.092 \ w) \ g_2 + (3.43 - 0.06 \ w) \ g_3$$

$$\frac{\partial J}{\partial g_3} = 0 = (0.22 - 0.042 \ w) \ g_1 + (2.41 - 0.05 \ w) \ g_2 + (11.45 - 0.053 \ w) \ g_3$$

5. Set $\dfrac{\partial J}{\partial g_1} = \dfrac{\partial J}{\partial g_2} = \dfrac{\partial J}{\partial g_3} = 0$; by setting the determi-
    nants of the three equations to zero, a third
    degree algebraic equation involving w is ob-
    tained:

    $$109.3 - 13.3 \ w + 0.088 \ w^2 - 0.000027 \ w^3 = 0$$

6. Solve this third-degree equation in w, to obtain
    the three characteristic values, $w_1 = 8.712$,
    $w_2 = 164.36$, and $w_3 = 1700.40$ as given by
    Latzko.

7. Substitute these characteristic values into the
    three equations in step (4) to obtain three sets
    of g_1, g_2, and g_3; and insert these into
    equation (14) to obtain equation (16) and thus
    equation (18).

8. The values of a_1, a_2, and a_3 in equation (18)
    are obtained by the method of least squares,
    thus yielding equation (19).

NOTE 12

References to Calculus of Variation:

1. Woods, Frederick S.: Advanced Calculus, Ginn & Co.,
        1926.

2. Bliss, Gilbert A.: Calculus of Variations, 1925.

NOTE 13

The function

$$f\left(x_1 g_1 g'\right) = \left(1 - x^7\right)\left(\frac{dg}{dx}\right)^2 - wx^7 g^2$$

satisfies Euler's differential equation, which is the necessary though not sufficient condition for a minimum,

$$\frac{\partial f}{\partial g} - \frac{d}{dx}\left(\frac{\partial f}{\partial g'}\right) = 0$$

for

$$\frac{\partial f}{\partial g} = -2wx^7 g$$

and

$$\frac{\partial f}{\partial g'} = 2\left(1 - x^7\right)\left(\frac{\partial g}{\partial x}\right)$$

$$\frac{d}{dx}\left(\frac{\partial f}{\partial g'}\right) = 2\,\frac{d}{dx}\left\{\left(1 - x^7\right)\left(\frac{dg}{dx}\right)\right\}$$

therefore

$$\frac{\partial f}{\partial g} - \frac{d}{dx}\left(\frac{\partial f}{\partial g'}\right) = -2wx^7 g - 2\,\frac{d}{dx}\left\{\left(1 - x^7\right)\left(\frac{dg}{dx}\right)\right\}$$

and since

$$\frac{d}{dx}\left\{\left(1 - x^7\right)\frac{dg}{dx}\right\} = -wx^7 g \tag{12}$$

thus

$$\frac{\partial f}{\partial g} - \frac{d}{dx}\left(\frac{\partial f}{\partial g'}\right) = 0$$

NOTE 14

References to Legendre's Polynomials.

1. Jahnke, E., and Emde, F.: Funktionentafeln mit formeln und kurven. Dover Publications, 1943.

2. Woods, F. S.: Advanced Calculus. Ginn & Co., 1926.

NOTE 15

M. ten Bosch in "Die Wärmeübertragung" (Julius Springer (Berlin), 1936) rearranged equation (21) to yield:

$$\alpha = \frac{0.0384 \ vC}{Re^{0.25}} \left\{ 1 + 0.1 \ e^{\frac{-2.7}{Re^{0.25}} \frac{z}{d}} + \cdots \right\}$$

which when more terms are added becomes

$$\alpha = \frac{0.0384 \ vC}{Re^{0.25}} \left\{ 1 + 0.1 \ e^{\frac{-2.7}{Re^{0.25}} \frac{z}{d}} + 0.9 \ e^{\frac{-29.27}{Re^{0.25}} \frac{z}{d}} - 0.023 \ e^{\frac{-31.96}{Re^{0.25}} \frac{z}{d}} \right\}$$

which is a more convenient form of equation (21). Notice, however, that these equations do not yield an infinite unit thermal conductance at the entrance where z = 0, which is in contradiction to Latzko's statement in the paragraph following equation (21).

NOTE 16

The momentum equation of the boundary layer can be written

$$\frac{dJ}{dz} - \rho U \frac{dQ}{dz} = -\frac{dp}{dz} (2r\delta - \delta^2) \pi - 2r\pi\tau_0 \qquad (22)$$

Referring to figure 3, it can be determined that:

$$\frac{dJ}{dz} \ dz = \left(\frac{\partial}{\partial z} \int_0^\delta \rho u^2 2\pi \ (r - y) \ dy \right) dz$$

the flux of momentum across $\overline{cd}$ that exceeds that across $\overline{ab}$

$$\frac{dJ}{dz} = \frac{2}{7} \pi r^2 \rho \left[U^2 \left\{ \frac{1274}{207} - \frac{2303}{345} \xi \right\} \right.$$

$$\left. - \frac{165v \ (8\xi - 22)}{(4\xi^2 - 22\xi + 165)^2} \ 2U \left\{ \frac{1274}{207} \xi - \frac{2303}{690} \xi^2 \right\} \right] \frac{d\xi}{dz} - \rho U \frac{\partial Q}{\partial z} \ dz$$

the inward flux of momentum across $\overline{bd}$

$$- \rho U \frac{\partial Q}{\partial z} = + 2\rho U \pi r^2 \left[U \ (\xi - 1) \right.$$

$$\left. - \ \frac{165v \ (8\xi - 22)(1 - 2\xi + \xi^2)}{(4\xi^2 - 22\xi + 165)^2} \right] \frac{d\xi}{dz} - \frac{dp}{dz} \ (2r\delta - \delta^2) \ dz$$

difference in pressure between $\overline{ab}$ and $\overline{cd}$

$$- \frac{dp}{dz} \ (2r\delta - \delta^2) = - \ \frac{2\pi r^2 \rho U (2\xi - \xi^2) 165v(4\xi - 11)}{(4\xi^2 - 22\xi + 165)^2} \ \frac{d\xi}{dz} - 2r\pi\tau_0 \ dz$$

retarding force at the wall

$$- 2r\pi\tau_0 = - 2r\pi\rho \left(\frac{8}{7B}\right)^{7/4} U^{7/4} \left(\frac{v}{\xi r}\right)^{1/4}$$

where

$$Q = \int_0^\delta 2\pi \ (r - y) \ u \ dy$$

By substituting these values into equation (22), there is obtained equation (28).

<div align="center">NOTE 17</div>

Though the original article gave z in equation (28), it is obviously z_0 after considering the limits of the integration.

<div align="center">NOTE 18</div>

The original article indicates that the curve in figure

4 is a graph of equation (29). However, on replotting the expression, a discrepancy was found. It appears (see fig. A-II) that the curve given by Latzko in figure 4 is in error.

NOTE 19

From figure A-II, it is seen that for $\xi = 1$, $\chi = 0.686$

$$\chi = \left(\frac{v}{vd}\right)^{1/5} \left(\frac{z_0}{d}\right)^{4/5} = 0.686$$

or

$$z_0 = (0.686)^{5/4} d \left(\frac{vd}{v}\right)^{1/4}$$

$$= 0.625 \, d \left(\frac{vd}{v}\right)^{1/4}$$

Latzko derived from figure 4, that

$$z_0 = 0.693 \, d \left(\frac{vd}{v}\right)^{1/4}$$

NOTE 20

This plot is given in the original article under figure 7A with ordinate misrepresented as K.

NOTE 21

A plot of K against ξ, which was to have been given by figure 7A, is missing in the original article. It is presented here in figure A-III.

NOTE 22

An example of case 4.

Legend on figure:

$\alpha_a = \alpha$ for case 2

$\alpha_u = \alpha$ for case 3

$\alpha_i = \alpha$ for case 4
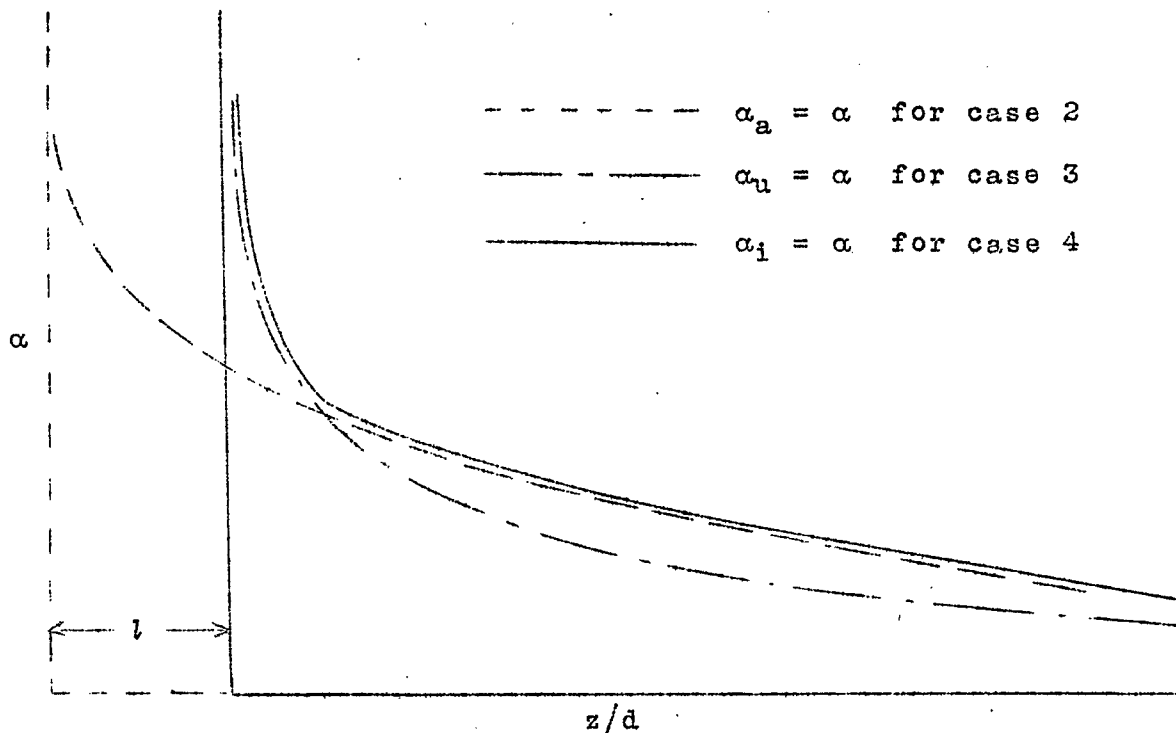
NOTE 23

The original article gave the equation,

$$\rho \frac{Du}{ds} = - \operatorname{grad} p + \mu \Delta u$$

It should be

$$\rho \frac{Du}{dt} = - \operatorname{grad} p + \mu \Delta u$$

NOTE 24

This equation is derived for a fluid the Prandtl number of which is unity. Colburn* gives an expression for heat transfer at plane surfaces which is valid for Pr other than unity.

Colburn's equation is

$$\frac{\alpha}{CU}\left(\frac{c\mu}{k}\right)^{2/3} = 0.036\left(\frac{\nu}{lU}\right)^{1/5}$$

which is practically the same as Latzko's equation for $Pr = 1$

$$\frac{\dfrac{Q}{\vartheta_0 l}}{CU} = \frac{\alpha}{CU} = 0.0356\left(\frac{\nu}{lU}\right)^{1/5}$$

---

*Colburn, Allan P.: A Method of Correlating Forced Convection Heat Transfer Data and a Comparison with Fluid Friction. Trans. Am. Inst. Chem. Eng., vol. XXIX, 1933, p. 199, equation (22).

Velocity distribution	Latzko	1/7th-power law	Logarithmic
u	$B\left(\dfrac{\tau_0}{\rho}\right)^{4/7}\left(\dfrac{y}{\nu}\right)^{1/7}$	$B\left(\dfrac{\tau_0}{\rho}\right)^{4/7}\left(\dfrac{y}{\nu}\right)^{1/7}$	$\dfrac{1}{K}\sqrt{\dfrac{\tau_0}{\rho}}\,\ln y + C$
$\dfrac{du}{dy}$	$\dfrac{B}{7}\left(\dfrac{\tau_0}{\rho}\right)^{4/7}\nu^{-1/7}\,Y^{-6/7}$ *	$\dfrac{B}{7}\left(\dfrac{\tau_0}{\rho}\right)^{4/7}\nu^{-1/7}\,y^{-6/7}$	$\dfrac{1}{Ky}\sqrt{\dfrac{\tau_0}{\rho}}$
ϵ	$\dfrac{\tau_0/\rho}{du/dy}$	$\dfrac{\tau_0/\rho}{du/dy}\left(1-\dfrac{y}{r}\right)$	$K^2 y^2\left(1-\dfrac{y}{r}\right)\dfrac{du}{dy}$
ϵ	$0.794\left(\dfrac{\tau_0}{\rho}\right)^{3/7}\nu^{1/7}y^{6/7}\left(1-\dfrac{y}{2r}\right)^{6/7}$	$0.794\left(\dfrac{\tau_0}{\rho}\right)^{3/7}\nu^{1/7}y^{6/7}\left(1-\dfrac{y}{r}\right)$	$Ky\left(1-\dfrac{y}{r}\right)\sqrt{\dfrac{\tau_0}{\rho}}$
$l^2 = \dfrac{\epsilon}{\frac{du}{dy}}$	$(0.794)^2\left(\dfrac{\tau_0}{\rho}\right)^{-1/7}\nu^{2/7}y^{12/7}\left(1-\dfrac{y}{2r}\right)^{12/7}$	$(0.794)^2\left(\dfrac{\tau_0}{\rho}\right)^{-1/7}\nu^{2/7}\left(y^{6/7}\right)^2\left(1-\dfrac{y}{r}\right)$	$K^2 y^2\left(1-\dfrac{y}{r}\right)$
l	$0.794\left(\dfrac{\tau_0}{\rho}\right)^{-1/14}\nu^{1/7}y^{6/7}\left(1-\dfrac{y}{2r}\right)^{6/7}$	$(0.794)\left(\dfrac{\tau_0}{\rho}\right)^{-1/14}\nu^{1/7}y^{6/7}\sqrt{1-\dfrac{y}{r}}$	$Ky\sqrt{1-\dfrac{y}{r}}$
$\dfrac{l}{Kr}$	$\dfrac{0.794}{Kr}\left(\dfrac{\tau_0}{\rho}\right)^{-1/14}\nu^{1/7}y^{6/7}\left(1-\dfrac{y}{2r}\right)^{6/7}$	$\dfrac{0.794}{Kr}\left(\dfrac{\tau_0}{\rho}\right)^{-1/14}\nu^{1/7}y^{6/7}\sqrt{1-\dfrac{y}{r}}$	$\dfrac{y}{r}\sqrt{1-\dfrac{y}{r}}$
$\dfrac{l}{Kr}$ ***	$0.538\,\dfrac{y^{6/7}}{r}\left(1-\dfrac{y}{2r}\right)^{6/7}$ **	$0.538\,\dfrac{y^{6/7}}{r}\sqrt{1-\dfrac{y}{r}}$ **	$\dfrac{y}{r}\sqrt{1-\dfrac{y}{r}}$

*$Y = y\left(1-\dfrac{y}{2r}\right)$.

**For $Re \sim 60{,}000$, $\nu = 0.175 \times 10^{-3}\ \dfrac{ft^2}{sec}$, $K = 0.4$.

***See fig. A-1 for plot of $\dfrac{l}{Kr}$ against $\dfrac{y}{r}$.
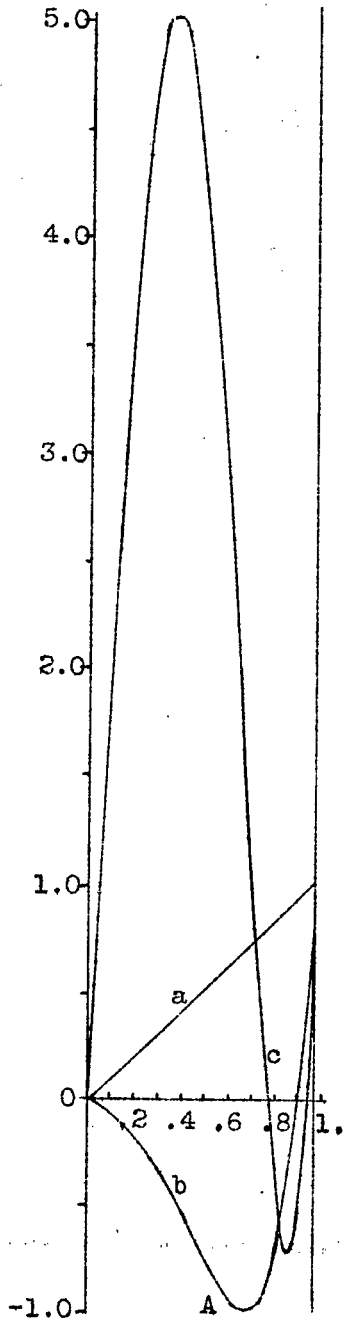
Figure 1A.- The three character- istic functions a, b, c plotted against x as abscissa.
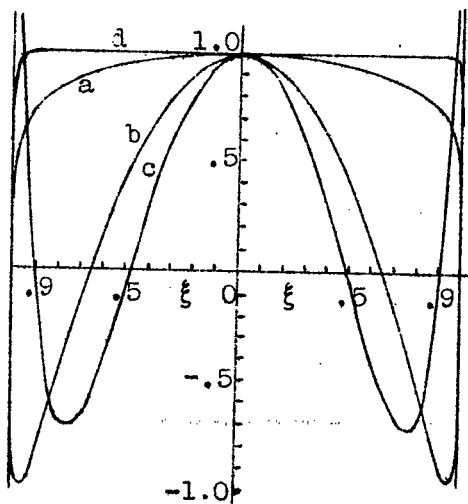
Figure 1B.- The same plotted against y as abscissa, d = initial distribu- tion for z = 0.

Figure 1A.

Figure 1B.

$$\frac{Cal}{m^2\ hr\ {}^oC}$$
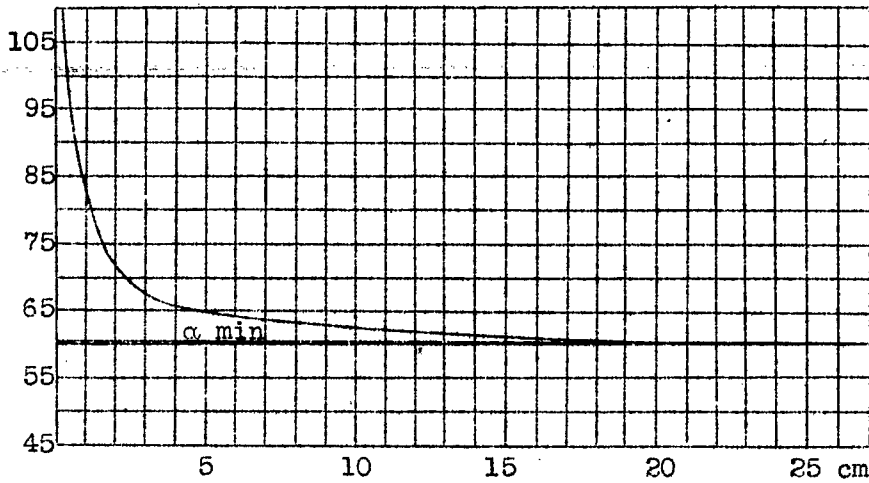

Figure 2.- Variation of heat transfer coefficient along pipe length for hydrodynamically fully developed state ($c = 0.304$ Cal/m^3, $v = 18.3$ m/sec, $\nu = 0.175$ cm^2/sec, $d = 2.2$ cm).

Figure 3.

Figure 8.

$$\xi = \frac{\delta}{r}$$

$$0.746\ \chi = \left(\frac{v}{\nu d}\right)^{\frac{1}{5}}\left(\frac{z}{d}\right)^{\frac{4}{5}}$$
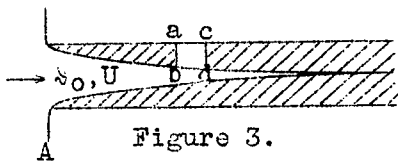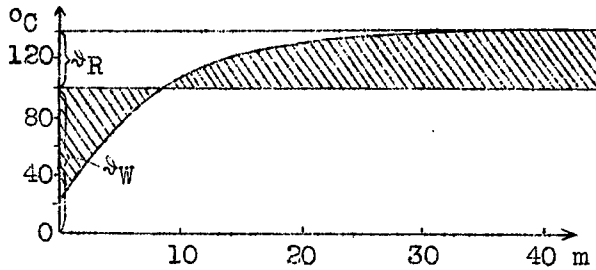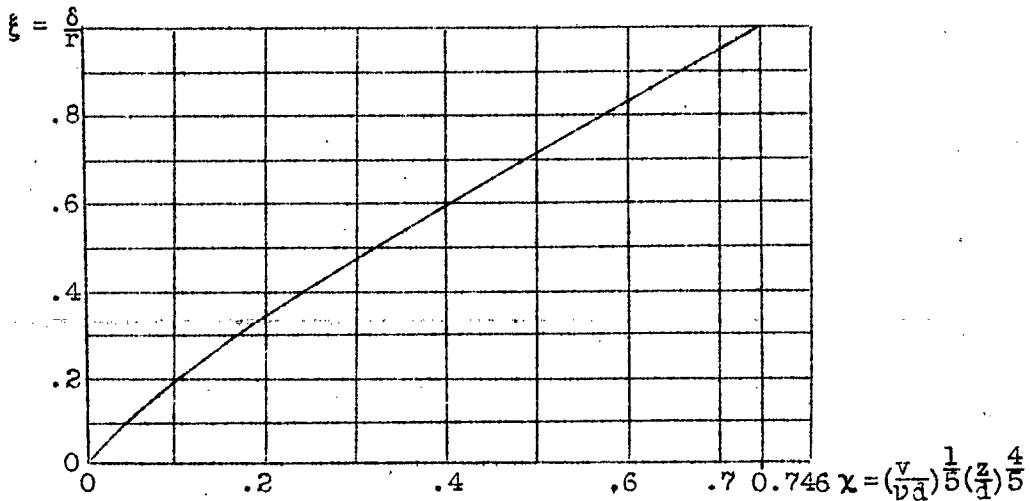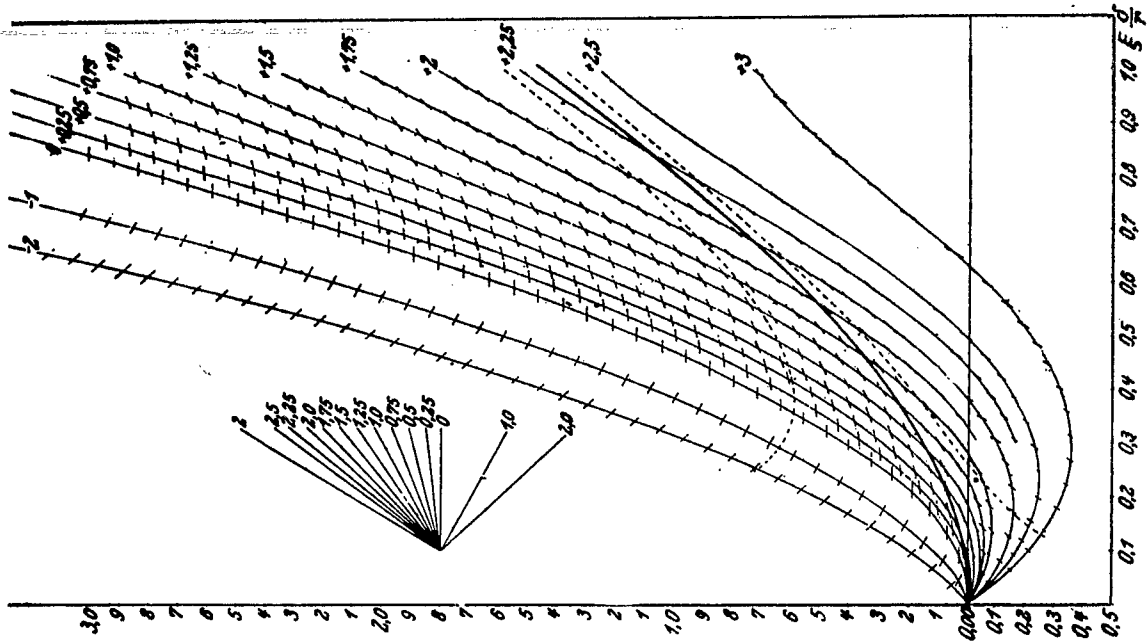
Figure 4.

Fig. 5 B
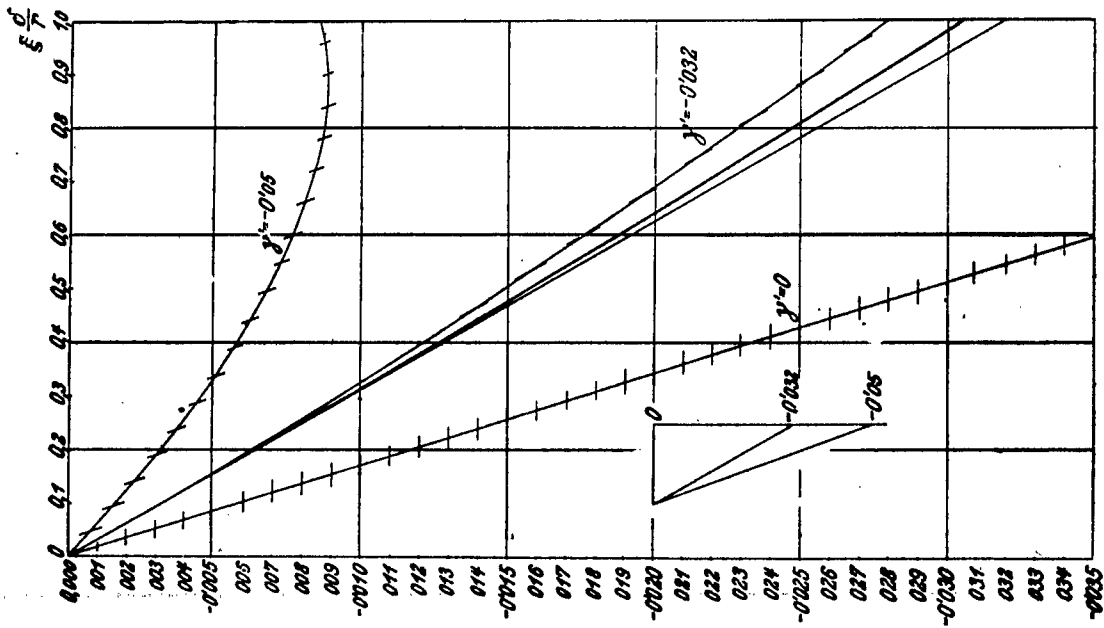
Fig. 5 A
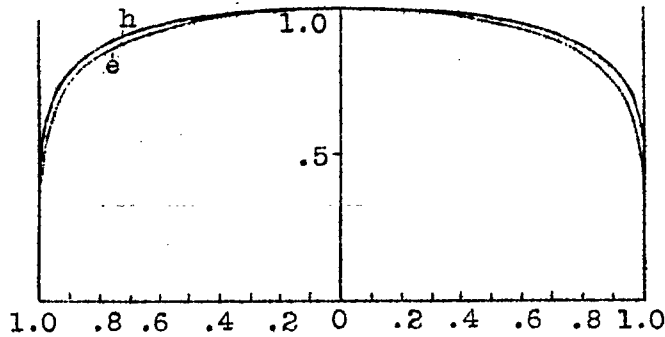
Figure 6.- End temperature distribution for hydrodynamically developed
          state (e) and distribution at the end cross-section of the
entrance run (h).
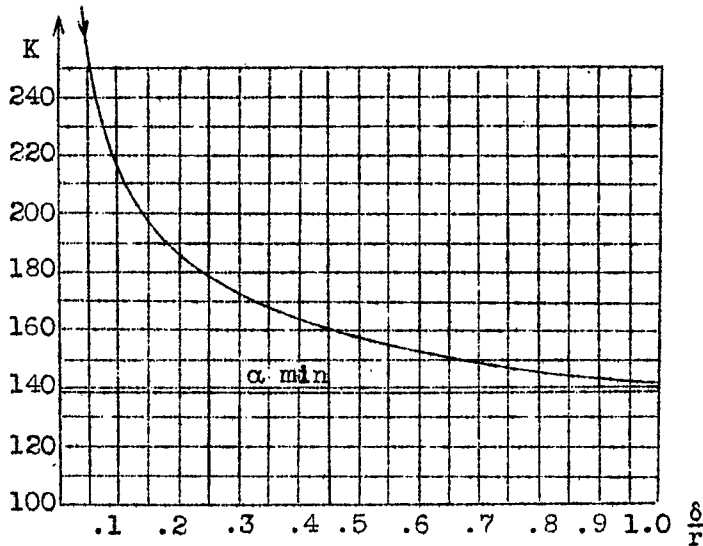
Figure 7A.
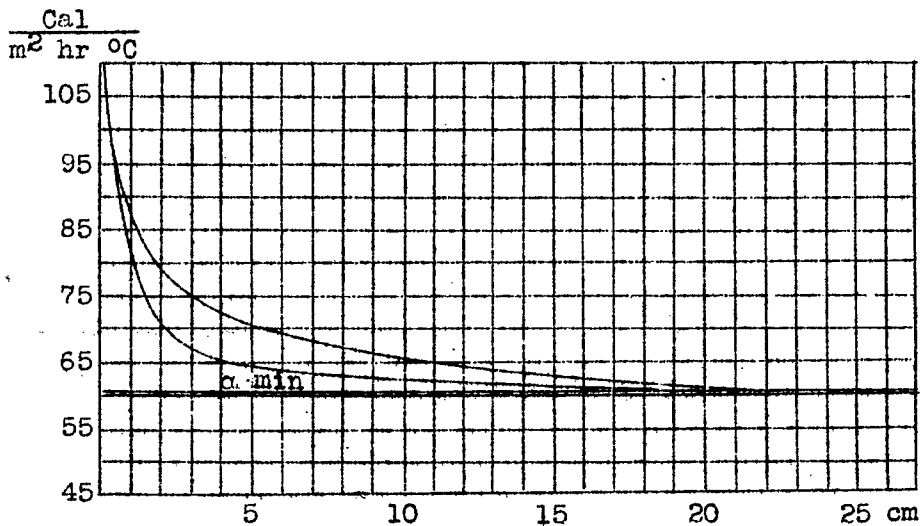
Figure 7B.
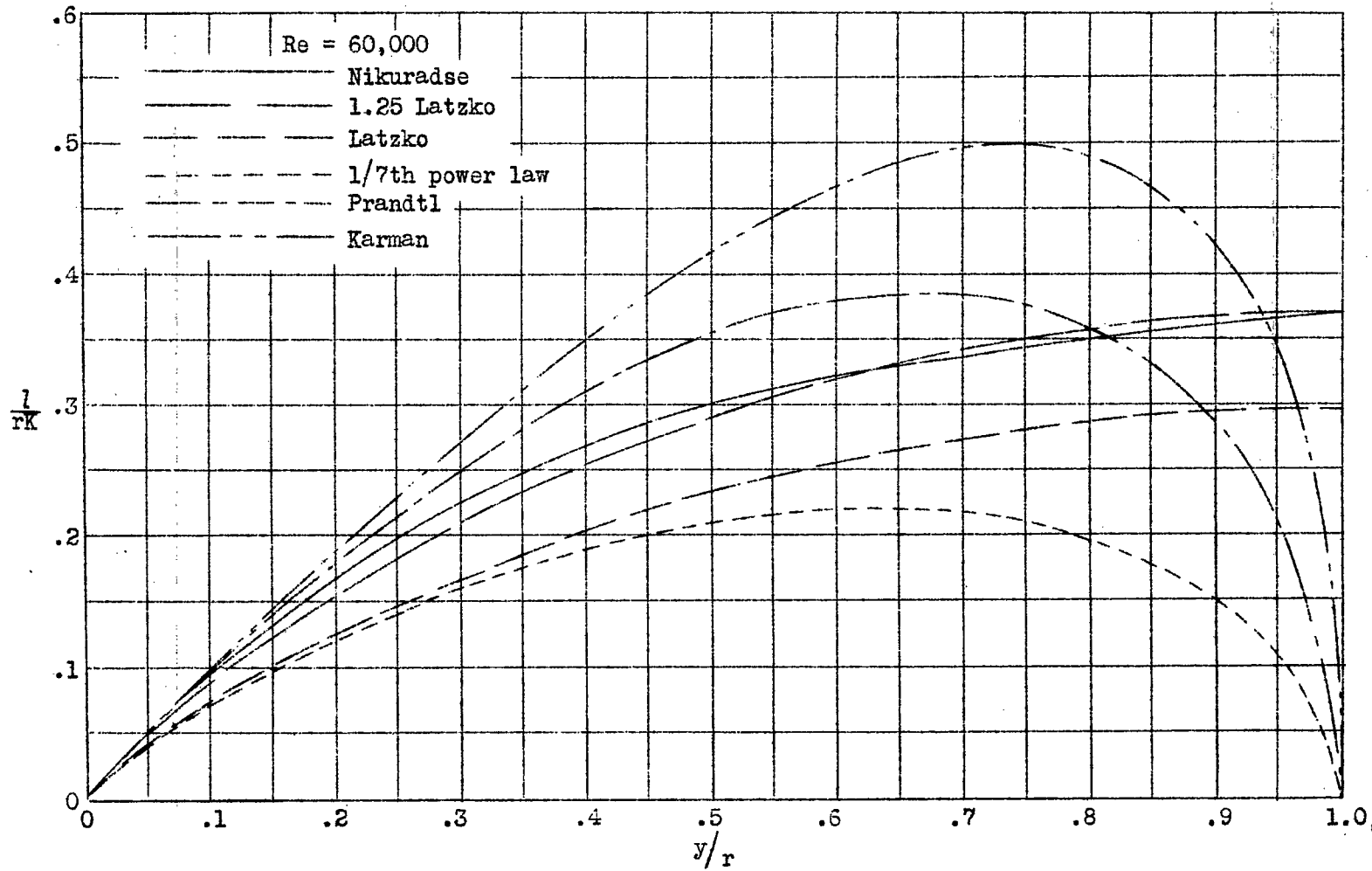
Figure A-I.— Mixing length derived from different formulae.

Figure A-II.

Figure A-III.