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Heavy-quark axial charges to nonleading order

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We combine Witten's renormalization group with the matching conditions of Bernreuther and Wetzel to calculate at next-to-leading order the complete heavy-quark contribution to the neutral-current axial-charge measurable in neutrino-proton elastic scattering. Our results are manifestly renormalization group invariant.

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This paper announces results for the next-to-leading-order (NLO) heavy-quark corrections to the axial charge $g_A^{(Z)}$ for protons to couple to the weak neutral current

$$J_{\mu 5}^Z = \frac{1}{2} \left\{ \sum_{q=u,c,t} - \sum_{q=d,s,b} \right\} \bar{q} \gamma_\mu \gamma_5 q. \quad (1)$$

The calculation is performed by decoupling heavy quarks $h = t, b, c$ sequentially, i.e. one at a time. An extension to simultaneous decoupling of t, b, c quarks is foreshadowed in our concluding remarks.

The charge $g_A^{(Z)}$ receives contributions from both light u, d, s and heavy c, b, t quarks,

$$2g_A^{(Z)} = (\Delta u - \Delta d - \Delta s) + (\Delta c - \Delta b + \Delta t) \quad (2)$$

where Δq refers to expectation value $\langle p, s | \bar{q} \gamma_\mu \gamma_5 q | p, s \rangle = 2m_p s_\mu \Delta q$ for a proton of spin s_μ and mass m_p . It governs parity-violating effects due to Z^0 exchange at low energies in elastic νp and $\bar{\nu} p$ scattering [1,2] or in light atoms [3,4]. A definitive measurement of νp elastic scattering may be possible using the MiniBooNE setup at Fermilab [5].

Once heavy-quark corrections [2,6,7] have been taken into account, $g_A^{(Z)}$ is related (modulo the issue of δ -function terms at $x=0$ [8]) to the flavor-singlet axial charge, defined scale invariantly and extracted from polarized deep inelastic scattering:

$$g_A^{(0)}|_{\text{inv}} = 0.2 - 0.35. \quad (3)$$

The small value of this quantity has inspired vast experimental and theoretical activity to understand the spin structure of the proton [9]. As a result, new experiments are being planned to map out the spin-flavor structure of the proton. These include polarized proton-proton collisions at the BNL Relativistic Heavy Ion Collider (RHIC) [10], semi-inclusive polarized deep inelastic scattering, and polarized ep collider studies [11]. Full NLO analyses are essential for a consistent interpretation of these experiments.

Many techniques for decoupling a single heavy quark are available. We rely on Witten's method [12], where the renormalization scheme is mass independent and improved

Callan-Symanzik equations [13] can be exploited. In such schemes, the decoupling of heavy particles required by the Appelquist-Carrazzone theorem [14] is not manifest. However, correct decoupling is ensured by applying the matching conditions of Bernreuther and Wetzel [15]; these relate coupling constant, mass and operator normalizations before and after the decoupling of a heavy quark. The advantages of this approach are its rigor and the fact that the final results are expressed in terms of renormalization group (RG) invariants. These invariants are Witten-style running couplings $\tilde{\alpha}_h$, one for each heavy quark $h = t, b, c$, and axial charges for nucleons in the residual theory with three light flavors.

We find that, when first t , then b , and finally c are decoupled from Eq. (2), the full NLO result is

$$2g_A^{(Z)} = (\Delta u - \Delta d - \Delta s)_{\text{inv}} + \mathcal{P}(\Delta u + \Delta d + \Delta s)_{\text{inv}} + O(m_{t,b,c}^{-1}) \quad (4)$$

where \mathcal{P} is a polynomial in the running couplings $\tilde{\alpha}_h$,

$$\mathcal{P} = \frac{6}{23\pi} (\tilde{\alpha}_b - \tilde{\alpha}_t) \left\{ 1 + \frac{125663}{82800\pi} \tilde{\alpha}_b + \frac{6167}{3312\pi} \tilde{\alpha}_t - \frac{22}{75\pi} \tilde{\alpha}_c \right\} - \frac{6}{27\pi} \tilde{\alpha}_c - \frac{181}{648\pi^2} \tilde{\alpha}_c^2 + O(\tilde{\alpha}_{t,b,c}^3) \quad (5)$$

and $(\Delta q)_{\text{inv}}$ denotes the scale-invariant version of Δq defined in the following way.

Let $\alpha_f = g_f^2/4\pi$ and $\beta_f(\alpha_f)$ be the gluon coupling and beta function for MS renormalized quantum chromodynamics (QCD) with f flavors and $N_c = 3$ colors, and let $\gamma_f(\alpha_f)$ be the gamma function for the singlet current

$$(\bar{u} \gamma_\mu \gamma_5 u + \bar{d} \gamma_\mu \gamma_5 d + \dots)_f = \sum_{k=1}^f (\bar{q}_k \gamma_\mu \gamma_5 q_k)_f. \quad (6)$$

A scale-invariant current $(S_{\mu 5})_f$ is obtained when Eq. (6) is multiplied by

$$E_f(\alpha_f) = \exp \int_0^{\alpha_f} dx \frac{\gamma_f(x)}{\beta_f(x)}. \quad (7)$$

Up to $O(m_h^{-1})$ corrections, the invariant singlet charge (3) is given by

$$g_A^{(0)}|_{\text{inv}} = E_3(\alpha_3)(\Delta u + \Delta d + \Delta s)_3 \\ = (\Delta u + \Delta d + \Delta s)_{\text{inv}}. \quad (8)$$

Flavor-dependent, scale-invariant axial charges $\Delta q|_{\text{inv}}$ such as

$$\Delta s|_{\text{inv}} = \frac{1}{3}(g_A^{(0)}|_{\text{inv}} - g_A^{(8)}) \quad (9)$$

can then be obtained from linear combinations of Eq. (8) and

$$g_A^{(3)} = \Delta u - \Delta d = (\Delta u - \Delta d)_{\text{inv}} \\ g_A^{(8)} = \Delta u + \Delta d - 2\Delta s = (\Delta u + \Delta d - 2\Delta s)_{\text{inv}}. \quad (10)$$

Here $g_A^{(3)} = 1.267 \pm 0.004$ is the isotriplet axial charge measured in neutron beta-decay, and $g_A^{(8)} = 0.58 \pm 0.03$ is the octet charge measured independently in hyperon beta decay. Taking $\tilde{\alpha}_t = 0.1$, $\tilde{\alpha}_b = 0.2$ and $\tilde{\alpha}_c = 0.35$ in Eq. (5), we find a small heavy-quark correction factor $\mathcal{P} = -0.02$, with LO terms dominant.

Our results extend and make more precise the well known work of Collins, Wilczek and Zee [6] and Kaplan and Manohar [2], where heavy-quark effective theory was used to estimate $g_A^{(Z)}$ in leading order (LO) for sequential decoupling of t, b and t, b, c respectively. Our analysis is also influenced by a discussion of [6] by Chetyrkin and Kühn [16], who considered some aspects of NLO decoupling of the t quark from the neutral current and in particular, the requirement that the result be scale invariant. Related work has been done on heavy-quark production in polarized deep inelastic scattering using the QCD parton model [17] and in high-energy polarized γp and pp at NLO [18].

The plan of this paper is as follows. First is a brief review of Witten's application of improved Callan-Symanzik equations [13] to the decoupling of a heavy quark in mass-independent renormalization schemes. Next, we combine it with matching conditions [15] to deal with next-to-leading-order (NLO) calculations involving axial-vector currents. Following is then a direct derivation of Eq. (5) from Eq. (1) for the neutral current. Our concluding remarks indicate the result of extending Eq. (5) to simultaneous decoupling of t, b, c —done not only for numerical reasons, but also to check that the t, b contributions cancel for $m_t = m_b$.

We begin by considering mass-independent schemes, such as the modified minimal subtraction scheme (MS), where renormalized masses behave like coupling constants. This key property is exploited in Witten's method.

Let μ be the scale used to define dimensional regularization and renormalization. Then the MS scale is

$$\bar{\mu} = \mu \sqrt{4\pi} e^{-\gamma/2}, \quad \gamma = 0.5772 \dots \quad (11)$$

We choose the same scale $\bar{\mu}$ irrespective of the number of flavors f being considered, and so hold $\bar{\mu}$ fixed as the heavy quarks (masses m_h) decouple:

$$F \rightarrow f \text{ flavors, } m_h \rightarrow \infty.$$

Also held fixed in this limit are the coupling α_f and light-quark masses m_{lf} of the residual f -flavor theory, and all momenta \mathbf{p} . Feynman diagrams for amplitudes

$$\mathcal{A}_F = \mathcal{A}_F(\mathbf{p}, \bar{\mu}, \alpha_F, m_{lF}, m_h) \quad (12)$$

give rise to power series in m_h^{-1} modified by polynomials in $\ln(m_h/\bar{\mu})$. We consider just the leading power $\tilde{\mathcal{A}}_F$:

$$\mathcal{A}_F = \tilde{\mathcal{A}}_F \{1 + O(1/m_h)\}. \quad (13)$$

As m_h tends to infinity, logarithms in $\tilde{\mathcal{A}}_F$ can be produced by any 1PI (one-particle irreducible) subgraph which contains at least one heavy-quark propagator and whose divergence by power counting is at least logarithmic. The effect is equivalent to shrinking all contributing 1PI parts of each diagram to a point. This means [14] that the F -flavor amplitudes $\tilde{\mathcal{A}}_F$ are the same as amplitudes \mathcal{A}_f in the residual f -flavor theory, apart from m_h -dependent renormalizations of the coupling constant, light masses, and amplitudes:

$$\tilde{\mathcal{A}}_F(\mathbf{p}, \bar{\mu}, \alpha_F, m_{lF}, m_h) \\ = \sum_{A'} \mathcal{Z}_{AA'}(\alpha_F, m_h/\bar{\mu}) \mathcal{A}'_f(\mathbf{p}, \bar{\mu}, \alpha_f, m_{lf}) \quad (14)$$

$$\alpha_f = \alpha_f(\alpha_F, m_h/\bar{\mu}), \quad m_{lf} = m_{lF} D(\alpha_F, m_h/\bar{\mu}). \quad (15)$$

Eventually, we will have to invert Eq. (15), i.e. use α_f and m_{lf} as dependent variables instead of α_F and m_{lF} , because we hold α_f and m_{lf} fixed as $m_h \rightarrow \infty$.

For any number of flavors f (including F), let

$$\mathcal{D}_f = \bar{\mu} \frac{\partial}{\partial \bar{\mu}} + \beta_f(\alpha_f) \frac{\partial}{\partial \alpha_f} + \delta_f(\alpha_f) \sum_{k=1}^f m_{kf} \frac{\partial}{\partial m_{kf}} \quad (16)$$

be the corresponding Callan-Symanzik operator. Then the amplitude \mathcal{A}_F and hence its leading power $\tilde{\mathcal{A}}_F$ both satisfy an F -flavor improved Callan-Symanzik equation:

$$\{\mathcal{D}_F + \gamma_F(\alpha_F)\} \tilde{\mathcal{A}}_F = 0. \quad (17)$$

In general, both γ_F and $\mathcal{Z} = (\mathcal{Z}_{AA'})$ are matrices.

If we substitute Eq. (14) in Eq. (17) and change variables,

$$\mathcal{D}_F = \bar{\mu} \frac{\partial}{\partial \bar{\mu}} + (\mathcal{D}_F \alpha_f) \frac{\partial}{\partial \alpha_f} + \sum_{k=1}^f (\mathcal{D}_F m_{kf}) \frac{\partial}{\partial m_{kf}} \quad (18)$$

the result is an improved Callan-Symanzik equation for each residual amplitude,

$$\{\mathcal{D}_f + \gamma_f(\alpha_f)\} \mathcal{A}_f = 0 \quad (19)$$

where the functions [12,15]

$$\beta_f(\alpha_f) = \mathcal{D}_F \alpha_f \quad (20)$$

$$\delta_f(\alpha_f) = \mathcal{D}_F \ln m_l \quad (21)$$

$$\gamma_f(\alpha_f) = \mathcal{Z}^{-1} [\gamma_F(\alpha_F) + \mathcal{D}_F] \mathcal{Z} \quad (22)$$

depend on α_f alone. The lack of m_l dependence of the renormalization factors in Eqs. (14) and (15) ensures mass-independent renormalization for the residual theory.

Although these equations hold for any $f < F$, their practical application is straightforward only when heavy quarks are decoupled one at a time. So we set $F = f + 1$, where just one quark h is heavy. Then it is convenient to introduce a running coupling [12]

$$\tilde{\alpha}_h = \tilde{\alpha}_h(\alpha_F, \ln(m_h/\bar{\mu})) \quad (23)$$

associated with the $\overline{\text{MS}}_F$ renormalized mass m_h :

$$\ln(m_h/\bar{\mu}) = \int_{\alpha_F}^{\tilde{\alpha}_h} dx [1 - \delta_F(x)] / \beta_F(x). \quad (24)$$

It satisfies the constraints

$$\tilde{\alpha}_h(\alpha_F, 0) = \alpha_F, \quad \tilde{\alpha}_h(\alpha_F, \infty) = 0 \quad (25)$$

the latter being a consequence of the asymptotic freedom of the F -flavor theory ($F \leq 16$). Also, Eqs. (16), (20) and (24) imply that $\tilde{\alpha}_h$ is renormalization group (RG) invariant:

$$\mathcal{D}_F \tilde{\alpha}_h = 0. \quad (26)$$

Witten's solution of Eq. (22) for the matrix \mathcal{Z} is

$$\begin{aligned} \mathcal{Z}(\alpha_F, m_h/\bar{\mu}) &= \exp \left\{ \int_{\alpha_F}^{\tilde{\alpha}_h} dx \frac{\gamma_F(x)}{\beta_F(x)} \right\}_{\text{ord}} \mathcal{Z}(\tilde{\alpha}_h, 1) \\ &\times \exp \left\{ \int_{\alpha_f}^{\alpha_f(\tilde{\alpha}_h, 1)} dx \frac{\gamma_f(x)}{\beta_f(x)} \right\}_{\text{ord}} \end{aligned} \quad (27)$$

where ‘‘ord’’ indicates x -ordering of matrix integrands in the exponentials. Note that it is the *relative* scaling between the initial and residual theories which matters.

For our NLO calculation, we need the formulas

$$\begin{aligned} \beta_f(x) &= -\frac{x^2}{3\pi} \left(\frac{33}{2} - f \right) - \frac{x^3}{12\pi^2} (153 - 19f) + O(x^4) \\ \gamma_f(x) &= \frac{x^2}{\pi^2} f + \frac{x^3}{36\pi^3} (177 - 2f)f + O(x^4) \\ \delta_f(x) &= -\frac{2x}{\pi} + O(x^2) \end{aligned} \quad (28)$$

where γ_f refers to the f -flavor singlet current (6) and includes the three-loop term found by Larin [19] and Chetyrkin and Kühn [16].

Our matching procedure amounts to evaluating to NLO accuracy the quantities $\tilde{\alpha}_h$, $\alpha_f(\tilde{\alpha}_h, 1)$ and $\mathcal{Z}(\tilde{\alpha}_h, 1)$ in Eq. (27), such that the answers depend on α_f and not α_F .

Bernreuther and Wetzel [15] applied the Appelquist-Carrasone decoupling theorem [14] to the gluon coupling constant α_Q^{MO} renormalized at space-like momentum Q ,

$$\alpha_Q^{\text{MO}}|_{\text{with } h} = \alpha_Q^{\text{MO}}|_{\text{no } h} + O(m_h^{-1}) \quad (29)$$

and compared calculations of α_Q^{MO} in the $F = f + 1$ and f -flavor $\overline{\text{MS}}$ theories. This reduces to a determination of the leading power of the one- h -loop $\overline{\text{MS}}_F$ gluon self-energy. The result is a matching condition

$$\alpha_F^{-1} - \alpha_f^{-1} = C_{\text{LO}} \ln(m_h/\bar{\mu}) + C_{\text{NLO}} + O(\alpha_f, m_h^{-1}) \quad (30)$$

with α_f -independent LO and NLO coefficients given by

$$C_{\text{LO}} = \frac{1}{3\pi}, \quad C_{\text{NLO}} = 0. \quad (31)$$

As a result, we find

$$\alpha_f(\tilde{\alpha}_h, 0) = \tilde{\alpha}_h + O(\tilde{\alpha}_h^3)_{\text{NLO}} = \tilde{\alpha}_h. \quad (32)$$

Bernreuther and Wetzel showed that it is possible to deduce *all* LO and NLO terms in Eq. (30) from Eq. (31) and β_f and δ_f in Eq. (28). We have done the calculation explicitly:

$$\begin{aligned} \alpha_{f+1}^{-1} &= \alpha_f^{-1} + \frac{1}{3\pi} \ln \frac{m_h}{\bar{\mu}} + c_f \ln \left[1 + \frac{\alpha_f}{3\pi} \ln \frac{m_h}{\bar{\mu}} \right] \\ &+ d_f \ln \left[1 + \frac{\alpha_f}{3\pi} \left(\frac{33}{2} - f \right) \ln \frac{m_h}{\bar{\mu}} \right] \\ c_f &= \frac{142 - 19f}{2\pi(31 - 2f)}, \quad d_f = \frac{57 + 16f}{2\pi(33 - 2f)(31 - 2f)}. \end{aligned} \quad (33)$$

From Eq. (24), we have also found $\tilde{\alpha}_h$ in NLO,

$$\begin{aligned} \tilde{\alpha}_h^{-1} &= \alpha_f^{-1} + \frac{1}{3\pi} \left(\frac{33}{2} - f \right) \ln \frac{\bar{m}_h}{\bar{\mu}} + \frac{153 - 19f}{2\pi(33 - 2f)} \\ &\times \ln \left[1 + \frac{\alpha_f}{3\pi} \left(\frac{33}{2} - f \right) \ln \frac{m_h}{\bar{\mu}} \right] \end{aligned} \quad (34)$$

where \bar{m}_h is Witten's RG invariant mass:

$$\bar{m}_h = m_h \exp \int_{\alpha_F}^{\tilde{\alpha}_h} dx \delta_F(x) / \beta_F(x). \quad (35)$$


If desired, $\ln(\bar{m}_h/\bar{\mu})$ can be eliminated by substituting

$$\ln \frac{\bar{m}_h}{\bar{\mu}} = \ln \frac{m_h}{\bar{\mu}} - \frac{12}{31 - 2f} \ln \left[1 + \frac{\alpha_f}{3\pi} \left(\frac{31}{2} - f \right) \ln \frac{m_h}{\bar{\mu}} \right]. \quad (36)$$

Therefore the asymptotic formula for $\tilde{\alpha}_h$ as $m_h \rightarrow \infty$ is

$$\tilde{\alpha}_h \sim 3\pi \left/ \left\{ \left(\frac{33}{2} - f \right) \ln \frac{m_h}{\bar{\mu}} + k_f \ln \ln \frac{m_h}{\bar{\mu}} + O(1) \right\} \right.$$

$$k_f = \frac{3(153-19f)}{2(33-2f)} - \frac{6(33-2f)}{31-2f}. \quad (37)$$

To find the matrix $\mathcal{Z}(\tilde{\alpha}_h, 1)$ in NLO, we need a matching condition for the $\overline{\text{MS}}$ amplitude $\Gamma_{\mu 5}$ for $\bar{h} \gamma_\mu \gamma_5 h$ to couple to a light quark l . We have calculated the leading power due to the two-loop diagram :

$$\Gamma_{\mu 5} = \left(\frac{\alpha_F}{\pi} \right)^2 \gamma_\mu \gamma_5 \left(\ln \frac{m_h}{\bar{\mu}} + \frac{1}{8} \right) + O(\alpha_F^3, m_h^{-1}). \quad (38)$$

Consequently, there is a NLO term $\tilde{\alpha}_h^2/8\pi^2$ in $\mathcal{Z}(\tilde{\alpha}_h, 1)$ for $\bar{h} \gamma_\mu \gamma_5 h$ to produce $\bar{l} \gamma_\mu \gamma_5 l$ as $m_h \rightarrow \infty$.

Now we consider the special case where heavy quarks are decoupled from the weak neutral axial current. Let us adopt the shorthand notation q_f for $\overline{\text{MS}}$ currents $(\bar{q} \gamma_\mu \gamma_5 q)_f$ in the f -flavor theory, e.g. the neutral current $J_{\mu 5}^{(Z)}$ and the scale-invariant singlet current $(S_{\mu 5})_f$:

$$J^Z = \frac{1}{2} (t - b + c - s + u - d)_6 \quad (39)$$

$$S_f = E_f(\alpha_f)(u + d + s + \dots)_f. \quad (40)$$

We begin by decoupling the t quark. Because of

$$(c - s + u - d)_6 = (c - s + u - d)_5 + O(1/m_t) \quad (41)$$

we see that Eq. (27) is nontrivial only for

$$(t - b)_6 = \mathcal{Z}_{6 \rightarrow 5}(u + d + s + c + b)_5 + \frac{1}{5} (u + d + s + c - 4b)_5 + O(1/m_t). \quad (42)$$

Since $(t - b)_6$ is scale invariant, we have $\gamma_F = 0$ in Eq. (27):

$$\mathcal{Z}_{6 \rightarrow 5}(\alpha_6, m_t/\bar{\mu}) = \mathcal{Z}_{6 \rightarrow 5}(\tilde{\alpha}_t, 1) \exp - \int_{\alpha_5}^{\tilde{\alpha}_t} dx \frac{\gamma_5(x)}{\beta_5(x)}. \quad (43)$$

The operator matching condition (38) corresponds to

$$t_6 = \frac{\alpha_6^2}{\pi^2} \left(\ln \frac{m_t}{\bar{\mu}} + \frac{1}{8} \right) (u + d + s + c + b)_5 + O(\alpha_6^3, m_t^{-1}) \quad (44)$$

and so we conclude:

$$\mathcal{Z}_{6 \rightarrow 5}(\tilde{\alpha}_t, 1) = -\frac{1}{5} + (8\pi^2)^{-1} \tilde{\alpha}_t^2 + O(\tilde{\alpha}_t^3). \quad (45)$$

Equation (43) is to be expanded about $\tilde{\alpha}_t \sim 0$ with α_5 held fixed. In that limit, the exponential tends to the constant factor $E_5(\alpha_5)$ of Eq. (7). This factor combines with the singlet current in Eq. (42) to form the scale-invariant operator S_5 , as required by $\text{RG}_{f=5}$ invariance. The full NLO result is then obtained by writing

$$(t - b)_6 = \mathcal{Z}_{6 \rightarrow 5}(\tilde{\alpha}_t, 1) \exp \left\{ - \int_0^{\tilde{\alpha}_t} dx \frac{\gamma_5(x)}{\beta_5(x)} \right\} S_5 + \frac{1}{5} (u + d + s + c - 4b)_5 \quad (46)$$

and expanding in $\tilde{\alpha}_t$, keeping all quadratic terms:

$$(t - b)_6 = \left\{ -\frac{1}{5} - \frac{6}{23} \frac{\tilde{\alpha}_t}{\pi} \left(1 + \frac{6167}{3312} \frac{\tilde{\alpha}_t}{\pi} \right) + O(\tilde{\alpha}_t^3) \right\} S_5 + \frac{1}{5} (u + d + s + c - 4b)_5 + O(1/m_t). \quad (47)$$

Next we decouple the b quark. Here, it is natural to define five-flavor quantities $\tilde{\alpha}_{b_5}$ and \bar{m}_{b_5} analogous to the six-flavor running coupling $\tilde{\alpha}_t$ and mass \bar{m}_t for the top quark:

$$\ln \frac{m_{b_5}}{\bar{\mu}} = \int_{\alpha_5}^{\tilde{\alpha}_{b_5}} dx \frac{1 - \delta_5(x)}{\beta_5(x)},$$

$$\ln \frac{\bar{m}_{b_5}}{m_{b_5}} = \int_{\alpha_5}^{\tilde{\alpha}_{b_5}} dx \frac{\delta_5(x)}{\beta_5(x)}. \quad (48)$$

Equations (20) and (21) imply that $\tilde{\alpha}_{b_5}$ and \bar{m}_{b_5} are both $\text{RG}_{f=5}$ and $\text{RG}_{f=6}$ invariant

$$\mathcal{D}_5 \tilde{\alpha}_{b_5} = 0 = \mathcal{D}_6 \tilde{\alpha}_{b_5}, \quad \mathcal{D}_5 \bar{m}_{b_5} = 0 = \mathcal{D}_6 \bar{m}_{b_5} \quad (49)$$

and hence physically significant in the original six-flavor theory. So we write $\tilde{\alpha}_b$ and \bar{m}_b for $\tilde{\alpha}_{b_5}$ and \bar{m}_{b_5} .

Consider decoupling the b quark from Eq. (47). The NLO matching condition (38) becomes

$$b_5 = \frac{\alpha_5^2}{\pi^2} \left(\ln \frac{\bar{m}_{b_5}}{\bar{\mu}} + \frac{1}{8} \right) (u + d + s + c)_4 + O(\alpha_5^3, m_{b_5}^{-1}) \quad (50)$$

so the nonsinglet current in Eq. (47) can be written

$$(u + d + s + c - 4b)_5 = \{ 1 - (\tilde{\alpha}_b^2/2\pi^2) \} E_4^{-1}(\tilde{\alpha}_b) S_4 + O(\tilde{\alpha}_b^3, m_{b_5}^{-1}). \quad (51)$$

For the singlet current S_5 in Eq. (47), we find

$$S_5 = E_5(\tilde{\alpha}_b) \left\{ 1 + \frac{\tilde{\alpha}_b^2}{8\pi^2} \right\} E_4^{-1}(\tilde{\alpha}_b) S_4 + O(\tilde{\alpha}_b^3, m_{b_5}^{-1}) \quad (52)$$

taking into account the definitions (7) and (40). Then we expand Eqs. (51) and (52) in $\tilde{\alpha}_b$, keeping quadratic terms:

$$(t - b)_6 = \frac{6}{23\pi} (\tilde{\alpha}_b - \tilde{\alpha}_t) \left\{ 1 + \frac{125663}{82800\pi} \tilde{\alpha}_b + \frac{6167}{3312\pi} \tilde{\alpha}_t \right\} S_4 + O(\tilde{\alpha}_{t,b}^3, m_{t,b}^{-1}). \quad (53)$$

The same technique can be applied to decouple the c quark from S_4 in Eq. (53) and $(c-s+u-d)_4$ [the result of decoupling b from Eq. (41)]. That yields the final results (4) and (5) given in the introduction.

Notice that our results depend on two key features:

(i) Like previous workers in this area, we decouple heavy quarks sequentially, i.e. one at a time.

(ii) Our running couplings $\tilde{\alpha}_t$, $\tilde{\alpha}_b$ and $\tilde{\alpha}_c$, which correspond to Witten's prescription [12], are all renormalization group invariant.

The restriction to sequential decoupling is numerically reasonable for the t quark, but dubious for the b and c quarks, because it amounts to an assumption that $\ln(m_c/\bar{\mu})$ is negligible compared with $\ln(m_b/\bar{\mu})$. This inhibits detailed comparison of NLO results with data, which ought to be carried out with NLO accuracy [20].

There is also a theoretical issue here: one would like to check that, in the limit $m_t=m_b$, the t and b contributions cancel. However, that is outside the region of validity $\ln(m_t/\bar{\mu}) \gg \ln(m_b/\bar{\mu})$ for sequential decoupling.

For these reasons, we have extended our analysis to the case of simultaneous decoupling, where the mass logarithms are allowed to grow large together: $\ln(m_c/\bar{\mu}) \sim \ln(m_b/\bar{\mu}) \sim \ln(m_t/\bar{\mu}) \rightarrow \text{large}$. This requires a considerable theoretical development of matching conditions and the renormalization group, which we will present separately. It involves the construction of running couplings α_t , α_b , α_c with the following properties: (i) They are renormalization group invariant; (ii) they are defined for $m_t \geq m_b \geq m_c$, and can have a nontrivial

dependence on more than one heavy-quark mass; (iii) in the special case of sequential decoupling, they agree with $\tilde{\alpha}_t$, $\tilde{\alpha}_b$ and $\tilde{\alpha}_c$ to NLO; and (iv) for the case of equal masses, they coincide, e.g.

$$\alpha_t = \alpha_b \quad \text{for} \quad m_t = m_b. \quad (54)$$

Then we find that the result for the simultaneous decoupling of the t, b, c quarks from the neutral current is of the same form (4) as the sequential answer, but with the sequential running couplings in Eq. (5) replaced by our simultaneous couplings α_t , α_b , and α_c :

$$\mathcal{P} = \frac{6}{23\pi} (\alpha_b - \alpha_t) \left\{ 1 + \frac{125663}{82800\pi} \alpha_b + \frac{6167}{3312\pi} \alpha_t - \frac{22}{75\pi} \alpha_c \right\} - \frac{6}{27\pi} \alpha_c - \frac{181}{648\pi^2} \alpha_c^2 + O(\alpha_{t,b,c}^3). \quad (55)$$

Notice the factorization of the terms depending on α_t and α_b . Given Eq. (54), the factor $\alpha_b - \alpha_t$ ensures that all contributions from b and t quarks cancel (as they should) for $m_t = m_b$.

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