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On the other hand, letting $\lambda \to 0$ we get

$$1 = \frac{1}{\mu}\frac{\mu_{\text{on}}}{r} + \eta_0,$$

and hence we conclude that

$$1 - \eta_0 = \frac{\mu_{\text{on}}}{\mu r}. \tag{5.12}$$

Inverting (5.11) we see that

$$V(x) = W * U(x) + \eta_0, \quad x > 0,$$

where U is a measure with Laplace transform

$$\hat{U}(\lambda) = \frac{1 - \hat{F}_{\text{on}}(\lambda(1-r))}{\mu\lambda(1-r)r}$$

and

$$1 - V(x) = 1 - \eta_0 - W * U(x) = (1-\eta_0)\left(1 - \frac{W * U(x)}{1-\eta_0}\right).$$

The $G = (1-\eta_0)^{-1}U$ has transform

$$\frac{1 - \hat{F}_{\text{on}}(\lambda(1-r))}{\mu\lambda(1-r)r(\mu_{\text{on}}/\mu r)} = \frac{1 - \hat{F}_{\text{on}}(\lambda(1-r))}{\lambda(1-r)\mu_{\text{on}}}$$

and since $\hat{G}(0) = 1$ we conclude that G is a probability measure. Note that another way to express the Laplace transform of G is

$$\int_0^\infty e^{-\lambda(1-r)x}\frac{1 - F_{\text{on}}(x)}{\mu_{\text{on}}}dx = \int_0^\infty e^{-\lambda y}\frac{1 - F_{\text{on}}(y/(1-r))}{\mu_{\text{on}}(1-r)}dy.$$

Thus we see that the density G' of G can be identified as

$$G'(y) = \frac{1 - F_{\text{on}}(y/(1-r)}{\mu_{\text{on}}(1-r)}.$$

Now if (5.6) holds (and hence also (5.7) holds) we have

$$1 - G(x) = \int_x^\infty G'(y)dy \sim \frac{x^{-(\alpha-1)}L(x)}{(1-r)^{1-\alpha}(\alpha-1)\mu_{\text{on}}}.$$

Thus, from Feller, 1971, page 278 we get as $x \to \infty$

$$\begin{aligned} 1 - V(x) &\sim \frac{\mu_{\text{on}}}{\mu r}\left\{b + \frac{1}{(1-r)^{1-\alpha}(\alpha-1)\mu_{\text{on}}}\right\} x^{-(\alpha-1)}L(x) \\ &= \left(b + \frac{(1-r)^{\alpha-1}}{\mu(\alpha-1)}\right) x^{-(\alpha-1)}L(x), \end{aligned}$$

which shows that the tail of V is heavier than the tail of W.

where the notation E_W reminds us that expectation should be computed under the assumption that $X(0)$ has distribution $W(x)$. In fact, it is easy to see that (5.10) is true for any bounded, non-negative measurable g (cf. Resnick, 1992, page 268) and so taking g to be a negative exponential function we get for $\lambda > 0$

$$\begin{aligned} Ee^{-\lambda X(\infty)} =& \frac{1}{\mu} E_W \int_0^{S_1} e^{-\lambda X(s)} ds = \frac{1}{\mu} E_W \left(\int_0^{X_1} + \int_{X_1}^{S_1} \right) \\ =& \frac{1}{\mu} \left(E_W \int_0^{X_1} e^{-\lambda(X(0)+(1-r)t)} dt + E_W \int_{X_1}^{S_1} e^{-\lambda(X(0)+(1-r)X_1 - r(t-X_1))^+} dt \right) \\ =& \frac{1}{\mu} \left(E_W e^{-\lambda X(0)} \left(\frac{1-e^{-\lambda(1-r)X_1}}{\lambda(1-r)} \right) + E_W \int_0^{Y_1} e^{-\lambda(X(0)+(1-r)X_1 - rs)^+} ds \right) \\ =& I + II. \end{aligned}$$

Write the Laplace transform of $W(x)$ as $\hat{W}(\lambda)$ and we have for I

$$I = \frac{1}{\mu} \hat{W}(\lambda) \left(\frac{1 - \hat{F}_{\text{on}}(\lambda(1-r))}{\lambda(1-r)} \right).$$

For II we decompose

$$\begin{aligned} \mu II =& E_W \int_0^\infty e^{-\lambda(X(0)+(1-r)X_1 - rs)^+} 1_{[s \le Y_1, rs \le X(0)+(1-r)X_1]} ds + E_W \int_0^\infty 1_{[s\le Y_1, rs > X(0)+(1-r)X_1]} ds \\ =& E_W e^{-\lambda(X(0)+(1-r)X_1)} \frac{\exp\{\lambda r \left(Y_1 \wedge \left(\frac{X(0)+(1-r)X_1}{r} \right)\right)\} - 1}{\lambda r} + E_W \left(Y_1 - \frac{X(0)+(1-r)X_1}{r} \right)^+ \\ =& \mu IIa + \mu IIb. \end{aligned}$$

Note that μIIb is independent of λ. For μIIa we have

$$\mu IIa = E_W \left(e^{-\lambda((X(0)+(1-r)X_1 - rY_1)\vee 0)} - e^{-\lambda(X(0)+(1-r)X_1)} \right) / \lambda r$$

and because $(X(0) + (1-r)X_1 - rY_1)^+ = X(S_1)$ also has distribution W, we get

$$\mu IIa = E_W \left(e^{-\lambda X(0)} - e^{-\lambda(X(0)+(1-r)X_1)} \right) / r\lambda.$$

Set $IIb = \eta_0$ and we have

$$\begin{aligned} \hat{V}(\lambda) =& E^{-\lambda X(\infty)} = \frac{1}{\mu} \hat{W}(\lambda) \left(\frac{1-\hat{F}_{\text{on}}(\lambda(1-r))}{\lambda(1-r)} \right) + \frac{1}{\mu} \left(\frac{\hat{W}(\lambda) - \hat{W}(\lambda)\hat{F}_{\text{on}}((1-r)\lambda)}{\lambda r} \right) + \eta_0 \\ =& \frac{1}{\mu} \frac{\hat{W}(\lambda)}{r} \frac{1-\hat{F}_{\text{on}}((1-r)\lambda)}{\lambda(1-r)} + \eta_0. \end{aligned} \tag{5.11}$$

We have

$$\lim_{\lambda\to\infty} \frac{1-\hat{F}_{\text{on}}(\lambda)}{\lambda} = 0$$

so that

$$V(0) = P[X(\infty) = 0] = \lim_{\lambda\to\infty} \hat{V}(\lambda) = \eta_0.$$

so that the tail of W is strikingly heavy.

We now consider the limit distribution for $\{X(t)\}$. Since the Markov process $\{X(S_n)\}$ has a stationary distribution W, if we use W as the initial distribution, then $\{X(S_n)\}$ is strictly stationary. Define

$$D_n(x) = \int_{S_n}^{S_{n+1}} 1_{[X(t)>x]}dt$$

for the amount of time the process X spends above level x in the nth *on/off* interval. Then $\{D_n(x)\}$ is a stationary sequence. To see this note that on $[S_n, S_{n+1})$

$$X(t) = \begin{cases} X(S_n) + (1-r)(t-S_n), & \text{if } S_n \leq t < S_n + X_{n+1}, \\ (X(S_n) + (1-r)X_{n+1} - r(t-(S_n+X_{n+1})))^+, & \text{if } S_n + X_{n+1} \leq t < S_{n+1}, \end{cases}$$

and using this we may express $D_n(x)$ as

$$\begin{aligned} D_n(x) &= \int_{S_n}^{S_{n+1}} 1_{(x,\infty)}(X(t))dt = \int_{S_n}^{S_n+X_{n+1}} + \int_{S_n+X_{n+1}}^{S_{n+1}} \\ &= \int_0^{X_{n+1}} 1_{(x,\infty)}(X(S_n)+(1-r)u)du + \int_0^{Y_{n+1}} 1_{(x,\infty)}((X(S_n)+(1-r)X_{n+1}-ru)^+)du \\ &= g(X(S_n), X_{n+1}, Y_{n+1}), \end{aligned}$$

which expresses $D_n(x)$ as an instantaneous function of the stationary sequence $\{X(S_n), X_{n+1}, Y_{n+1}\}$.

Let $N(t) = \sum_{j=0}^{\infty} 1_{[S_j \leq x]}$ be the counting renewal function so that on $[N(t) \geq 1]$

$$S_{N(t)-1} \leq t < S_{N(t)}.$$

Now observe that for any $T > 0$ such that $N(T) \geq 1$,

$$\frac{1}{T}\int_0^{S_{N(T)-1}} 1_{(x,\infty)}(X(t))dt \leq \frac{1}{T}\int_0^T 1_{(x,\infty)}(X(t))dt \leq \frac{1}{T}\int_0^{S_{N(T)}} 1_{(x,\infty)}(X(t))dt$$

and so

$$\frac{1}{T}\sum_{n=0}^{N(T)-1} D_n(x) \leq \frac{1}{T}\int_0^T 1_{(x,\infty)}(X(t))dt \leq \frac{1}{T}\sum_{n=0}^{N(T)} D_n(x).$$

Since $N(T)/T \to 1/\mu$ as $T \to \infty$ we conclude

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T 1_{(x,\infty)}(X(t))dt = E\left(D_n(x)|\mathcal{I}\right)/\mu \tag{5.8}$$

where $\mathcal{I}$ is the invariant σ-field. However, from the regenerative structure of $\{X(t)\}$ we know the limit must be a constant. We therefore conclude

$$1 - V(x) = \lim_{T\to\infty} P[X(T) > x] = ED_1(x)/\mu \tag{5.9}$$

where remember that in calculating the expectation on the right of (5.9), we have assumed that $\{X(S_n)\}$ is stationary with marginal distribution W. It remains to evaluate this expectation.

Suppose the random variable $X(\infty)$ has distribution V. If $g(s) = 1_{(x,\infty)}(s)$ we may rewrite (5.9) as

$$Eg(X(\infty)) = E_W \int_0^{S_1} g(X(t))dt/\mu \tag{5.10}$$

off periods are heavy tailed, these limit distributions will have much heavier tails than would be the case if they were light tailed with the same long term input rate and release rule.

The limit distributions can be easily expressed using Smith's Theorem. Since the limit distributions are unaffected by initial conditions, we assume as a convenience that $S_0 = 0$ so that $\{0, S_n = \sum_{i=1}^n (X_i + Y_i), n \geq 1\}$ is a pure renewal process. Define $S_n^{(X)} = \sum_{i=1}^n X_i$ and $S_n^{(Y)} = \sum_{i=1}^n Y_i$ and the stopping time

$$\bar{N} = \inf\{n : (1-r)S_n^{(X)} - rS_n^{(Y)} \leq 0\}$$

so that

$$[\bar{N} = n] = [(1-r)S_j^{(X)} - rS_j^{(Y)} > 0, j = 1, \ldots, n-1, (1-r)S_n^{(X)} - rS_n^{(Y)} \leq 0] \in \mathcal{B}(X_i, Y_i, i = 1, \ldots, n)$$

which shows that $\bar{N}$ is a stopping time. Smith's Theorem gives for $x > 0$

$$P[X(S_n) > x] \to \frac{E(\sum_{j=1}^{\bar{N}} 1_{[X(S_j) > x]})}{E(\bar{N})} =: 1 - W(x), \tag{5.3}$$

$$P[X(t) > x] \to \frac{E\left(\int_0^{C_1} 1_{[X(s) > x]} ds\right)}{E(S_{\bar{N}})} =: 1 - V(x). \tag{5.4}$$

Note by Wald's equation that $E(S_{\bar{N}}) = \mu E(\bar{N})$ and that (5.3) and (5.4) hold provided $E\bar{N} < \infty$, a fact verified in the next paragraph.

We begin by considering $\{X(S_n), n \geq 0\}$. Comparing $X(S_n)$ with $X(S_{n+1})$ we get

$$\begin{aligned} X(S_{n+1}) &= (X(S_n) + (1-r)X_{n+1} - rY_{n+1})^+ \\ &= (X(S_n) + \xi_{n+1})^+, \end{aligned} \tag{5.5}$$

where $\{\xi_n = (1-r)X_{n+1} - rY_{n+1}\}$ is iid. This equation expresses that the change of contents over a renewal interval is the input during the *on* period and the loss during the *off* period. Of course (5.3) is Lindley's equation (Resnick, 1992, page 270; Assmussen, 1987; Feller, 1971) and since (5.1) implies

$$E\xi_1 = (1-r)\mu_{\text{on}} - r\mu_{\text{off}} = \mu_{\text{on}} - r\mu < 0,$$

we know from standard theory that

$$X(S_n) \Rightarrow \bigvee_{n=0}^{\infty} \sum_{i=1}^{n} \xi_i < \infty.$$

In particular this shows $E\bar{N} < \infty$ and $EC_1 = ES_{\bar{N}} = \mu E\bar{N} < \infty$.

If we suppose that

$$1 - F_{\text{on}}(x) = x^{-\alpha} L(x), \quad 1 < \alpha < 2, \ x \to \infty, \tag{5.6}$$

where L is a slowly varying function, then setting

$$\rho = \frac{\mu_{\text{on}}}{\mu_{\text{off}}} \frac{1-r}{r} < 1,$$

we have from known results (cf. Cohen, 1972; Pakes, 1975; Bingham, Goldie and Teugels, 1987, page 387) that

$$1 - W(x) \sim \frac{\rho}{1-\rho} \frac{(1-r)^{\alpha-1}}{(\alpha-1)\mu_{\text{on}}} x^{-(\alpha-1)} L(x) =: bx^{-(\alpha-1)} L(x), \quad x \to \infty, \tag{5.7}$$

Theorem 4.3. *Assume that there is an $n \geq 1$ such that $(F_{\text{on}} * F_{\text{off}})^{n*}$ is nonsingular. If (4.7) and (4.8) hold then*

$$\gamma(t) \sim \frac{\mu_{\text{off}}^2}{(\alpha-1)\mu^3} t^{-(\alpha-1)} L(t), \ t \to \infty. \tag{4.11}$$

Proof. By (4.6) and (4.10) in Lemma 4.2 we see that the function z in (4.2) is virtually nonincreasing. Moreover, it follows from (4.9) that we are in the situation of part (ii) of Theorem 3.1, with $k = \mu_{\text{off}}$. Since

$$m = \int_0^\infty z(x)\, dx = \mu_{\text{on}} \mu_{\text{off}},$$

we immediately conclude from (3.6) that

$$\gamma(t) \sim \frac{1}{(\alpha-1)\mu^2} \left(\mu_{\text{off}} - \frac{\mu_{\text{on}}\mu_{\text{off}}}{\mu} \right) t^{-(\alpha-1)} L(t)$$

as $t \to \infty$, and (4.11) follows. □

5. Effects of heavy tails and long range dependence in a fluid model.

In order to illustrate the effects of long range dependent inputs generated by an *on/off* model, we consider the following simple fluid or storage model. Our stationary alternating renewal process feeds a reservoir. The renewal process consists of alternating *on* and *off* cycles and during the *on* periods, liquid flows in at rate 1. Using the notation of Section 2, we represent the renewal sequence as $\{S_n, n \geq 0\}$ with $S_n = S_0 + \sum_{i=1}^n (X_i + Y_i)$, $n \geq 1$ and $\{X_i\}$ representing durations of *on* periods and $\{Y_i\}$ durations of *off* periods. If $\{Z_t, t \geq 0\}$ is the indicator process which is 1 during an *on* period and 0 otherwise, then the cumulative input to the system up to time t is

$$A(t) := \int_0^t Z_v dv.$$

Since $A(t) \sim t\mu_{\text{on}}/\mu$ the long term input rate is μ_{on}/μ. Assume the release rate from the system when contents are at level x is

$$r(x) := \begin{cases} r, & \text{if } x > 0, \\ 0, & \text{if } x = 0. \end{cases}$$

We assure that the input does not overwhelm the release rate, by assuming

$$1 > r > \frac{\mu_{\text{on}}}{\mu}. \tag{5.1}$$

The contents process $\{X(t), t \geq 0\}$ satisfies the relation

$$dX(t) = dA(t) - r(X(t))dt. \tag{5.2}$$

Thus, during an *on* period, liquid enters at net rate $1 - r$ and during an *off* period liquid is released at uniform rate r.

The process $\{X(t), t \geq 0\}$ is regenerative (cf. Resnick, 1992; Feller, 1971; Asmussen, 1987). One set of regeneration times is

$$\{C_n\} := \{S_n : X(S_n-) = 0\},$$

which are the times when a dry period ends and input commences to fill the store. Thus by Smith's Theorem (Resnick, 1992, page 265), both $\{X(S_n), n \geq 0\}$ and $\{X(t), t \geq 0\}$ have limit distributions. If the *on* and

(ii) The bound (4.5) follows from (4.3) and the obvious relation

$$\overline{F_{\text{on}} * F_{\text{off}}}(t) \geq P[X_{\text{on}} > t \ \text{ or } Y_{\text{off}} > t] = \overline{F_{\text{on}}}(t) + \overline{F_{\text{off}}}(t) - \overline{F_{\text{on}}}(t)\overline{F_{\text{off}}}(t).$$

Furthermore, by (4.4) and (4.5)

$$\begin{aligned} z^*(t) &= \int_t^\infty \Big(-z'(x)\Big)\mathbf{1}\Big(z'(x) \leq 0\Big)\,dx + \int_t^\infty z'(x)\mathbf{1}\Big(z'(x) > 0\Big)\,dx \\ &= \int_t^\infty \Big(-z'(x)\Big)\,dx + 2\int_t^\infty z'(x)\mathbf{1}\Big(z'(x) > 0\Big)\,dx \\ &\leq z(t) + \delta(t), \end{aligned}$$

completing the proof of the lemma. □

The next lemma gives the order of magnitude of z and z^* in the specific situation we are considering in this paper.

Lemma 4.2. *Assume that*

$$\overline{F_{\text{on}}}(t) = t^{-\alpha}L(t),\ t \to \infty, \tag{4.7}$$

$1 < \alpha < 2$, *where L is slowly varying at infinity. Assume, moreover, that*

$$\overline{F_{\text{off}}}(t) = o\Big(\overline{F_{\text{on}}}(t)\Big),\ t \to \infty. \tag{4.8}$$

Then

$$z(t) \sim \mu_{\text{off}} t^{-\alpha} L(t),\ t \to \infty, \tag{4.9}$$

while

$$\delta(t) = o\Big(z(t)\Big),\ t \to \infty. \tag{4.10}$$

Proof. Choose an $0 < \epsilon < 1$ and write using (4.2)

$$z(t) = \int_0^{\epsilon t} \overline{F_{\text{off}}}(x)\overline{F_{\text{on}}}(t-x)\,dx + \int_{\epsilon t}^t \overline{F_{\text{off}}}(x)\overline{F_{\text{on}}}(t-x)\,dx := z_1(t) + z_2(t).$$

Observe that

$$z_2(t) \leq \overline{F_{\text{off}}}(\epsilon t)\int_{\epsilon t}^t \overline{F_{\text{on}}}(t-x)\,dx \leq \mu_{\text{on}}\overline{F_{\text{off}}}(\epsilon t) = o\Big(t^{-\alpha}L(t)\Big)$$

as $t \to \infty$ by (4.7) and (4.8). On the other hand,

$$\overline{F_{\text{on}}}(t)\int_0^{\epsilon t} \overline{F_{\text{off}}}(x)\,dx \leq z_1(t) \leq \overline{F_{\text{on}}}(t(1-\epsilon))\int_0^{\epsilon t} \overline{F_{\text{off}}}(x)\,dx.$$

Therefore,

$$\mu_{\text{off}} \leq \liminf_{t\to\infty} \frac{z_1(t)}{\overline{F_{\text{on}}}(t)} \leq \limsup_{t\to\infty} \frac{z_1(t)}{\overline{F_{\text{on}}}(t)} \leq (1-\epsilon)^{-\alpha}\mu_{\text{off}},$$

and since ϵ can be taken as close to 0 as we wish, (4.9) follows from (4.7). Finally, (4.10) follows from (4.7) and the obvious bound

$$\delta(t) \leq 2\overline{F_{\text{on}}}(t)\int_t^\infty \overline{F_{\text{off}}}(x)\,dx. \quad \square$$

We are now in a position to establish the asymptotic behavior of the covariance function γ.

4. Asymptotic behavior of the covariance function in the heavy tailed case.
We have proved in Theorem 2.3 that the covariance function γ of the *on/off* process is given by

$$\gamma(s) = \frac{\mu_{\text{on}}\mu_{\text{off}}}{\mu^2} - \frac{1}{\mu}\int_0^s z(s-w)\,U(dw),\ s \geq 0, \tag{4.1}$$

where recall $\mu = \mu_{\text{on}} + \mu_{\text{off}}$ and for $t \geq 0$ we set

$$\begin{aligned} z(t) &= \int_0^t \overline{F_{\text{off}}}(x)\overline{F_{\text{on}}}(t-x)\,dx \\ &= \mu_{\text{on}} F_{\text{on}}^{(0)} * (1 - F_{\text{off}})(t) = \mu_{\text{off}} F_{\text{off}}^{(0)} * (1 - F_{\text{on}})(t) \end{aligned} \tag{4.2}$$

and $U = \sum_{n=0}^{\infty} (F_{\text{on}} * F_{\text{off}})^{n*}$ is the renewal function. The following lemma summarizes some of the properties of the function z that we will need in the sequel.

Lemma 4.1.
(i) The function z is absolutely continuous, with density

$$z'(t) = \overline{F_{\text{on}}}(t) + \overline{F_{\text{off}}}(t) - \overline{F_{\text{on}} * F_{\text{off}}}(t) \tag{4.3}$$

Moreover,

$$z(t) = \int_t^\infty \Big(\overline{F_{\text{on}} * F_{\text{off}}}(x) - \overline{F_{\text{on}}}(x) - \overline{F_{\text{off}}}(x)\Big)\,dx,\ t > 0. \tag{4.4}$$

(ii)

$$z'(t) \leq \overline{F_{\text{on}}}(t)\overline{F_{\text{off}}}(t). \tag{4.5}$$

Moreover, the function $z^*(t) = \int_t^\infty |z'(x)|\,dx,\ t > 0$ satisfies

$$z(t) \leq z^*(t) \leq z(t) + \delta(t),\ t > 0, \tag{4.6}$$

where

$$\delta(t) = 2\int_t^\infty \overline{F_{\text{on}}}(x)\overline{F_{\text{off}}}(x)\,dx,\ t > 0.$$

Proof. (i) We have

$$\begin{aligned} z(t) &= \int_0^t F_{\text{on}}(dy)\int_{t-y}^t \overline{F_{\text{off}}}(x)\,dx + \int_t^\infty F_{\text{on}}(dy)\int_0^t \overline{F_{\text{off}}}(x)\,dx \\ &= \int_0^t \overline{F_{\text{off}}}(x)\,dx - \int_0^t F_{\text{on}}(dy)\int_0^{t-y} \overline{F_{\text{off}}}(x)\,dx \\ &= \int_0^t \Big(\overline{F_{\text{off}}}(x) - \int_0^x \overline{F_{\text{off}}}(x-y)F_{\text{on}}(dy)\Big)\,dx \\ &= \int_0^t \Big(\overline{F_{\text{off}}}(x) + \overline{F_{\text{on}}}(x) - \overline{F_{\text{on}} * F_{\text{off}}}(x)\Big)\,dx, \end{aligned}$$

which proves absolute continuity of z together with (4.3). Since z' is absolutely integrable on $(0,\infty)$ and $\lim_{t\to\infty} z(t) = 0$, we immediately obtain (4.4) as well.

$$= t^{-(\alpha+\beta-2-2\theta)} \int_0^{1-\epsilon} (1-y)^{-(\beta-\theta)} y^{-(\alpha-1-\theta)}\, dy = o\Big(t^{-(\beta-1)} L_1(t)\Big),\ t\to\infty \tag{3.27}$$

by the choice of θ. Similarly, if $z(t) = o(\overline{F}(t))$, then

$$K_1(t) = o\Big(t^{-(\alpha-1)} L(t)\Big),\ t\to\infty. \tag{3.28}$$

Finally, by (3.18) we have

$$\begin{aligned} K_2(t) \le & \frac{1}{\mu}\overline{F}_2\Big(t(1-\epsilon)\Big) \int_{t(1-\epsilon)}^t z(t-x)\, dx \\ \sim & \frac{1}{(\alpha-1)\mu^2}\Big(t(1-\epsilon)\Big)^{-(\alpha-1)} L(t) \int_{t(1-\epsilon)}^t z(t-x)\, dx \\ \sim & (1-\epsilon)^{-(\alpha-1)} \frac{m}{(\alpha-1)\mu^2} t^{-(\alpha-1)} L(t),\ t\to\infty, \end{aligned} \tag{3.29}$$

and similarly

$$\begin{aligned} K_2(t) \ge & \frac{1}{\mu}\overline{F}_2(t) \int_{t(1-\epsilon)}^t z(t-x)\, dx \\ \sim & \frac{1}{(\alpha-1)\mu^2} t^{-(\alpha-1)} L(t) \int_{t(1-\epsilon)}^t z(t-x)\, dx \\ \sim & \frac{m}{(\alpha-1)\mu^2} t^{-(\alpha-1)} L(t),\ t\to\infty. \end{aligned} \tag{3.30}$$

We are now ready to put all the pieces together. Suppose first that we are under the conditions of part (i). We have then by (3.15), (3.25), (17.27), (3.29), (3.30), (3.9), (3.11), (3.20), (3.21), (3.23) and (3.24) that

$$\lim_{t\to\infty} \frac{h(t)}{t^{-(\beta-1)} L_1(t)} = \frac{1}{(\beta-1)\mu},$$

thus proving (3.5). Under the assumptions of part (ii) of the theorem, we obtain in the same way that, for any $0<\epsilon<1$,

$$\frac{1}{(\alpha-1)\mu}\Big(k - (1-\epsilon)^{-(\alpha-1)}\frac{m}{\mu}\Big) \le \liminf_{t\to\infty} \frac{h(t)}{t^{-(\alpha-1)}L(t)} \le \limsup_{t\to\infty} \frac{h(t)}{t^{-(\alpha-1)}L(t)} \le \frac{1}{(\alpha-1)\mu}\Big(k - \frac{m}{\mu}\Big).$$

Since ϵ can be taken arbitrarily close to 0, this proves (3.6). Finally, under the assumptions of part (iii) of the theorem we have by (3.15), (3.26), (3.28), (3.29), (3.30), (3.9), (3.11), (3.20), (3.23) and (3.24) that for any $0<\epsilon<1$,

$$-(1-\epsilon)^{-(\alpha-1)} \frac{m}{(\alpha-1)\mu^2} \le \liminf_{t\to\infty} \frac{h(t)}{t^{-(\alpha-1)}L(t)} \le \limsup_{t\to\infty} \frac{h(t)}{t^{-(\alpha-1)}L(t)} \le -\frac{m}{(\alpha-1)\mu^2}.$$

Once again, since ϵ can be taken arbitrarily close to 0, this proves (3.7). The proof of the theorem is, therefore, complete. □

Remark It is very easy to see from the proof of Theorem 3.1 that one can relax the assumption that the function z is virtually nonincreasing. For example, this assumption can be replaced by

$$z^*(t) = 0\Big(t^{-(\alpha-1)} L(t)\Big),\ t\to\infty, \tag{3.31}$$

and z is directly Riemann integrable.

We conclude by (3.1) and (3.19) that

$$\lim_{t\to\infty} \frac{Q_{11}(t)}{t^{-(\alpha-1)}L(t)} = 0. \tag{3.20}$$

Further,

$$Q_{12}(t) \le C \sup_{x\ge 0} \varphi(x)\Big(z^*((1-\epsilon)n) - z^*(n)\Big). \tag{3.21}$$

We treat $Q_2(t)$ in a similar way. Write

$$Q_2(t) = C \sum_{\epsilon n\le i\le n-N} + C \sum_{1\le i<\epsilon n} := Q_{21}(t) + Q_{22}(t). \tag{3.22}$$

We have

$$Q_{21}(t) \le C \sup_{x\ge 0} \varphi(x) \sum_{\epsilon n\le i\le n} z(i). \tag{3.23}$$

Similarly,

$$Q_{22}(t) \le C\varphi(t-1)\frac{\overline{F}(t-\epsilon n)}{\overline{F}(t-1)} \sum_{i\le \epsilon n} z(i) \le C \sum_{i=1}^{\infty} z(i)\varphi(t-1)\frac{\overline{F}(t(1-\epsilon))}{\overline{F}(t)},$$

and we conclude as in (3.20) that in all cases of Theorem 3.1,

$$\lim_{t\to\infty} \frac{Q_{22}(t)}{t^{-(\alpha-1)}L(t)} = 0. \tag{3.24}$$

Suppose now that (3.4) holds. Then

$$\frac{1}{\mu}\int_{n-N}^{\infty} z(x)\,dx \sim \frac{1}{(\beta-1)\mu} n^{-(\beta-1)}L_1(n) \sim \frac{1}{(\beta-1)\mu} t^{-(\beta-1)}L_1(t),\ t\to\infty. \tag{3.25}$$

Further, if $z(t) = o(\overline{F}(t))$ as $t\to\infty$, then

$$\frac{1}{\mu}\int_{n-N}^{\infty} z(x)\,dx = o\Big(t^{-(\alpha-1)}L(t)\Big),\ t\to\infty. \tag{3.26}$$

Consider now

$$K(t) = \frac{1}{\mu}\int_{t-n+N}^{t} z(t-x)\overline{F}_2(x)\,dx.$$

Fix once again an $0<\epsilon<1$, and write for all t large enough

$$K(t) = \frac{1}{\mu}\int_{t-n+N}^{t(1-\epsilon)} + \frac{1}{\mu}\int_{t(1-\epsilon)}^{t} := K_1(t) + K_2(t).$$

If (3.4) holds with $\alpha\ge\beta>1$, take a $0<\theta<(\alpha-1)/2$. It follows from (3.4) and (3.18) that there is a finite positive constant $c = c(N,\theta)$ such that

$$K_1(t) \le c\int_{t-n+N}^{t(1-\epsilon)} (t-x)^{-(\beta-\theta)}x^{-(\alpha-1-\theta)}\,dx \le c\int_0^{t(1-\epsilon)} (t-x)^{-(\beta-\theta)}x^{-(\alpha-1-\theta)}\,dx$$

Moreover,

$$\sum_{i=1}^{n-N}\Big(\frac{1}{\mu}\int_{i-1}^{i}\Big(\int_{t-y}^{t-i+1}\overline{F}_2(x)\,dx\Big)\gamma(dy)+\frac{1}{\mu}\int_{t-i}^{t-i+1}\overline{F}_2(x)\,dx\,z(i)\Big)$$
$$=\frac{1}{\mu}\sum_{i=1}^{n-N}\Big(\int_{t-i}^{t-i+1}\Big(z(t-x)-z(i)\Big)\overline{F}_2(x)\,dx+z(i)\int_{t-i}^{t-i+1}\overline{F}_2(x)\,dx\Big)$$
$$=\frac{1}{\mu}\int_{t-n+N}^{t}z(t-x)\overline{F}_2(x)\,dx.$$

Consequently,

$$M(t)=\frac{1}{\mu}\int_0^{n-N}z(x)\,dx+\frac{1}{\mu}\int_{t-n+N}^{t}z(t-x)\overline{F}_2(x)\,dx. \tag{3.14}$$

We have, therefore,

$$\frac{1}{\mu}\int_{n-N}^{\infty}z(x)\,dx-\frac{1}{\mu}\int_{t-n+N}^{t}z(t-x)\overline{F}_2(x)\,dx-I_1(t)-I_{22}(t)-Q(t)\le h(t)$$
$$\le\frac{1}{\mu}\int_{n-N}^{\infty}z(x)\,dx-\frac{1}{\mu}\int_{t-n+N}^{t}z(t-x)\overline{F}_2(x)\,dx-I_1(t)-I_{22}(t)+Q(t). \tag{3.15}$$

We now estimate the "error" term. Write

$$Q(t)=C\int_0^{n-N}\Big(\overline{F}(t-y)\int_0^{t-y}\overline{F}_2(x)\,dx\Big)|\gamma|(dy)+C\sum_{i=1}^{n-N}\Big(\overline{F}(t-i)\int_0^{t-i}\overline{F}_2(x)\,dx\Big)z(i):=Q_1(t)+Q_2(t). \tag{3.16}$$

Fix an $0<\epsilon<1$. For all n large enough we have

$$Q_1(t)=C\int_0^{n-\epsilon n}+C\int_{n-\epsilon n}^{n-N}:=Q_{11}(t)+Q_{12}(t). \tag{3.17}$$

Let

$$\varphi(t)=\overline{F}(t)\int_0^t\overline{F}_2(x)\,dx,\ t\ge 0.$$

It follows from (3.1) that

$$\overline{F}_2(t)\sim\frac{1}{\mu(\alpha-1)}t^{-(\alpha-1)}L(t),\ t\to\infty, \tag{3.18}$$

and so

$$\varphi(t)\sim\frac{1}{\mu(2-\alpha)(\alpha-1)}\Big(t^{-(\alpha-1)}L(t)\Big)^2,\ t\to\infty. \tag{3.19}$$

In particular, the function φ is bounded. We have

$$Q_{11}(t)\le C\varphi(t)\frac{\overline{F}(t-n+\epsilon n)}{\overline{F}(t)}\int_0^{n-N}|\gamma|(dy)\le C|\gamma|(R_+)\varphi(t)\frac{\overline{F}(\epsilon t)}{\overline{F}(t)}.$$

In the same way as before we see that

$$I_{22}(t) = \int_{t-n}^{t-n+N} z(t-w)\,U(dw) \le \int_{t-n}^{t-n+N} z^*(t-w)\,U(dw) \le z^*(n-N)U(N+1) \le U(N+1)z^*(t/2). \tag{3.11}$$

To estimate $I_{21}(t)$ we use the second order refinement of Blackwell's renewal theorem given in part (a) of Theorem 3.1.26 of Frenk (1987) which says that for every $H > 0$ there is a $T > 0$ and a $C \in (0,\infty)$ such that for all $t > T$

$$\begin{aligned} \frac{p}{\mu} + \frac{1}{\mu}\int_{t-p}^{t} \overline{F}_2(x)\,dx - C\overline{F}(t-p)\int_0^{t-p} \overline{F}_2(x)\,dx &\le U(t) - U(t-p) \\ &\le \frac{p}{\mu} + \frac{1}{\mu}\int_{t-p}^{t} \overline{F}_2(x)\,dx + C\overline{F}(t-p)\int_0^{t-p} \overline{F}_2(x)\,dx \end{aligned} \tag{3.12}$$

for all $0 < p \le H$, where

$$F_2(x) = \frac{1}{\mu}\int_0^x \overline{F}(y)\,dy,\ \ x \ge 0.$$

Since

$$I_{21}(t) = \sum_{i=1}^{n-N}\left(\int_{i-1}^{i}\Big(U(t-i+1) - U(t-y)\Big)\gamma(dy) + \Big(U(t-i+1) - U(t-i)\Big)z(i)\right),$$

we conclude that there is a $C \in (0,\infty)$ such that for N large enough, for all t large enough,

$$\begin{aligned} &\sum_{i=1}^{n-N}\left[\int_{i-1}^{i}\left(\frac{y-i+1}{\mu} + \frac{1}{\mu}\int_{t-y}^{t-i+1}\overline{F}_2(x)\,dx\right)\gamma(dy) + \left(\frac{1}{\mu} + \frac{1}{\mu}\int_{t-i}^{t-i+1}\overline{F}_2(x)\,dx\right)z(i)\right] \\ &-C\sum_{i=1}^{n-N}\left[\int_{i-1}^{i}\left(\overline{F}(t-y)\int_0^{t-y}\overline{F}_2(x)\,dx\right)|\gamma|(dy) + \left(\overline{F}(t-i)\int_0^{t-i}\overline{F}_2(x)\,dx\right)z(i)\right] \\ &\le I_{21}(t) \le \sum_{i=1}^{n-N}\left[\int_{i-1}^{i}\left(\frac{y-i+1}{\mu} + \frac{1}{\mu}\int_{t-y}^{t-i+1}\overline{F}_2(x)\,dx\right)\gamma(dy) + \left(\frac{1}{\mu} + \frac{1}{\mu}\int_{t-i}^{t-i+1}\overline{F}_2(x)\,dx\right)z(i)\right] \\ &+C\sum_{i=1}^{n-N}\left[\int_{i-1}^{i}\left(\overline{F}(t-y)\int_0^{t-y}\overline{F}_2(x)\,dx\right)|\gamma|(dy) + \left(\overline{F}(t-i)\int_0^{t-i}\overline{F}_2(x)\,dx\right)z(i)\right]. \end{aligned} \tag{3.13}$$

We denote by $M(t)$ the common part of the left and right hand sides of (3.13), and by $Q(t)$ the part that comes with different signs. That is,

$$M(t) - Q(t) \le I_{21}(t) \le M(t) + Q(t).$$

We obviously think of $M(t)$ as of the main part of $I_{21}(t)$, and $Q(t)$ as the "error" part. We see that

$$\sum_{i=1}^{n-N}\left(\int_{i-1}^{i}\frac{y-i+1}{\mu}\gamma(dy) + \frac{1}{\mu}z(i)\right) = \frac{1}{\mu}\sum_{i=1}^{n-N}\left(\int_{i-1}^{i}\Big(z(x)-z(i)\Big)\,dx + z(i)\right) = \frac{1}{\mu}\int_0^{n-N} z(x)\,dx.$$

Let $m=\int_0^\infty z(x)\,dx<\infty$. Denote

$$h(t)=\frac{m}{\mu}-\int_0^t z(t-w)\,U(dw),\ t>0. \tag{3.4}$$

We say that the function z is *virtually nonincreasing* if

$$\lim_{t\to\infty}\frac{z(t)}{z^*(t)}=1.$$

Our assumptions on z imply that it is directly Riemann integrable (see e.g. Resnick (1992), Remark 3.10.5), and so the key renewal theorem says that $h(t)\to 0$ as $t\to\infty$. We would like to find out how fast is this convergence to 0.

The following is the main theorem of this section.

Theorem 3.1. *Let F be a heavy tailed renewal distribution function satisfying (3.1) such that F^{n*} is nonsingular for some $n\geq 1$.*

(i) Let z be a virtually nonincreasing nonnegative function satisfying

$$z(t)=t^{-\beta}L_1(t),\ t\to\infty,\ \beta>1 \tag{3.4}$$

where L_1 is slowly varying at infinity. If $\alpha>\beta>1$, or $\beta=\alpha$, and $L(t)=o(L_1(t))$ as $t\to\infty$, then

$$h(t)\sim\frac{1}{(\beta-1)\mu}t^{-(\beta-1)}L_1(t),\ t\to\infty. \tag{3.5}$$

(ii) Assume that z satisfies (3.4) with $\beta=\alpha$, and $L_1(t)\sim kL(t)$, $t\to\infty$, for some $0<k<\infty$. Then

$$h(t)\sim\frac{1}{(\alpha-1)\mu}\Big(k-\frac{m}{\mu}\Big)t^{-(\alpha-1)}L(t),\ t\to\infty. \tag{3.6}$$

(iii) Assume that $z(t)=o(\overline{F}(t))$ as $t\to\infty$. Then

$$h(t)\sim-\frac{m}{(\alpha-1)\mu^2}t^{-(\alpha-1)}L(t),\ t\to\infty. \tag{3.7}$$

Proof. Let $n\leq t<n+1,\ n=1,2,\ldots$. We write

$$\int_0^t z(t-w)\,U(dw)=\int_0^{t-n}z(t-w)\,U(dw)+\int_{t-n}^t z(t-w)\,U(dw):=I_1(t)+I_2(t). \tag{3.8}$$

Observe that for all t large enough,

$$I_1(t)\leq U(1)\max_{y\geq n}z(y)\leq U(1)z^*(n)\leq U(1)z^*(t/2). \tag{3.9}$$

Further, let N be a (large) positive integer. We have, for all t large enough,

$$\begin{aligned} I_2(t)&=\sum_{i=1}^n\int_{t-i}^{t-i+1}z(t-w)\,U(dw)\\ &=\sum_{i=1}^{n-N}\int_{t-i}^{t-i+1}z(t-w)\,U(dw)+\sum_{i=n-N+1}^{n}\int_{t-i}^{t-i+1}z(t-w)\,U(dw):=I_{21}(t)+I_{22}(t). \end{aligned} \tag{3.10}$$

and because U satisfies the renewal equation $U = \delta_0 + U * (F_{\text{on}} * F_{\text{off}})$, where δ_0 is the probability measure concentrating all mass at 0, we get the previous expression for the integral equal to

$$F_{\text{on}}^{(0)} * F_{\text{off}} * U(s) - F_{\text{on}}^{(0)} * (U - \delta_0)(s)$$

and thus

$$\begin{aligned} EZ_t Z_{t+s} =& \frac{\mu_{\text{on}}}{\mu} \left\{ 1 - F_{\text{on}}^{(0)}(s) + F_{\text{on}}^{(0)}(s) + F_{\text{on}}^{(0)} * F_{\text{off}} * U(s) - F_{\text{on}}^{(0)} * U(s) \right\} \\ =& \frac{\mu_{\text{on}}}{\mu} \left\{ 1 - \int_0^s (1 - F_{\text{off}}(s-u)) F_{\text{on}}^{(0)} * U(du) \right\}. \end{aligned}$$

The expression for $\gamma(s)$ now follows by writing

$$\gamma(s) = EZ_t Z_{t+s} - \left(\frac{\mu_{\text{on}}}{\mu} \right)^2. \quad \square$$

Remark: Another way to summarize the conclusion of Theorem 2.3 is to say that the covariance of the *on/off* process is

$$\gamma(s) = \frac{\mu_{\text{on}} \mu_{\text{off}}}{\mu^2} - \frac{1}{\mu} \int_0^s z(s-w)\, U(dw), \ s \ge 0, \tag{2.7}$$

where $\mu = \mu_{\text{on}} + \mu_{\text{off}}$ and for $t \ge 0$ we set

$$\begin{aligned} z(t) =& \int_0^t \overline{F_{\text{off}}}(x) \overline{F_{\text{on}}}(t-x)\, dx \\ =& \mu_{\text{on}} F_{\text{on}}^{(0)} * (1 - F_{\text{off}})(t) = \mu_{\text{off}} F_{\text{off}}^{(0)} * (1 - F_{\text{on}})(t) \end{aligned} \tag{2.8}$$

and $U = \sum_{n=0}^{\infty} (F_{\text{on}} * F_{\text{off}})^{n*}$ is the renewal function. Substituting (2.8) into (2.7) clearly shows the symmetry of the *on* and *off* characteristics in the covariance function.

3. Rate of convergence in the key renewal theorem in the heavy tailed case.

In this section we establish the rate of convergence in the key renewal theorem for heavy tailed renewal distributions. This theorem will be needed for computing the asymptotics of the covariance function in the *on/off* model and is of interest of its own right.

Let F be a distribution function concentrated on $[0, \infty)$ such that

$$1 - F(t) = \overline{F}(t) = t^{-\alpha} L(t), \ t \to \infty, \tag{3.1}$$

where $1 < \alpha < 2$, and L is a slowly varying function at infinity. F is the renewal distribution function. Let $\mu = \int_0^\infty \overline{F}(x)\, dx$ be the expected renewal time. We denote by U the renewal function associated with F so that $U = \sum_{n=0}^{\infty} F^{n*}$.

Further, let z be a right continuous nonnegative function of bounded variation on $[0, \infty)$, such that $z(t) \to 0$ as $t \to \infty$. That is,

$$z(t) = \int_t^\infty \gamma(dy), \ t \ge 0, \tag{3.2}$$

where γ is a finite signed measure on $[0, \infty)$. Denote by z^* the total variation function of γ. That is,

$$z^*(t) = \int_t^\infty |\gamma|(dy), \ t \ge 0. \tag{3.3}$$

Adding $I+II+III$ we get

$$
\begin{aligned}
Q(t) =&\frac{\theta\mu_{\text{on}}}{\mu} P[X_{\text{on}}^{(0)} > t+x_0] + \frac{\theta}{\mu}\int_{x_0}^{t+x_0}(1-F_{\text{on}}(s))ds \\
=&\frac{\theta}{\mu}\left[\int_{t+x_0}^{\infty}(1-F_{\text{on}}(s))ds + \int_{x_0}^{t+x_0}(1-F_{\text{on}}(s))ds\right] \\
=&\frac{\theta}{\mu}\int_{x_0}^{\infty}(1-F_{\text{on}}(s))ds, \qquad (2.5)
\end{aligned}
$$

which is independent of t as desired □

Corollary 2.2. *In the notation of Proposition 2.1 we have*

$$
\begin{aligned}
&P[B_{\text{on}}^{(t)} > x_0, Y_{\text{off}_0}^{(t)} > y_0, X_{\text{on}_1}^{(t)} > x_1, \dots, Y_{\text{off}_k}^{(t)} > y_k | Z_t = 1] \\
&\qquad = P[X_{\text{on}}^{(0)} > x_0]P[Y_{\text{off}} > y_0]P[X_1 > x_1]P[Y_1 > y_1]\dots P[Y_k > y_k]. \qquad (2.6)
\end{aligned}
$$

So conditional on $Z_t = 1$, the subsequent sequence of on/off periods is the same as seen from time 0 in the stationary process with $B = 1$.

Proof. Divide by $P[B=1] = P[Z_1 = 1]$ in (2.5) and use the definition of θ. □

Since long range dependence and long memory are frequently conveniently expressed in terms of slow decay of covariance functions, we now consider the second order properties of the stationary process $\{Z_t\}$. In preparation for the detailed analysis of long range dependence in subsequent sections, we compute the covariance function of $\{Z_t\}$.

Theorem 2.3. *The covariance function $\gamma(s) = Cov(Z_t, Z_{t+s})$ of the stationary process $\{Z(t), t \geq 0\}$ is*

$$
\begin{aligned}
\gamma(s) =&\frac{\mu_{\text{on}}}{\mu}\left[\frac{\mu_{\text{off}}}{\mu} - \int_0^s (1-F_{\text{off}}(s-u))F_{\text{on}}^{(0)} * U(du)\right] \\
=&\frac{\mu_{\text{on}}}{\mu}\left[\frac{\mu_{\text{off}}}{\mu} - F_{\text{on}}^{(0)} * U * (1-F_{\text{off}})(s)\right].
\end{aligned}
$$

Proof. We first compute the product moment

$$
EZ_tZ_{t+s} = P[Z_t = 1]P[Z_{t+s} = 1 | Z_t = 1]
$$

and applying Corollary 2.2, the right side is equal to

$$
=P[Z_t=1]E\left(1_{[0,X_{\text{on}}^{(0)})}(s) + \sum_{n=0}^{\infty} 1_{[X_{\text{on}}^{(0)}+Y_{\text{off}}+\sum_{i=1}^n(X_i+Y_i), X_{\text{on}}^{(0)}+Y_{\text{off}}+\sum_{i=1}^n(X_i+Y_i)+X_{n+1})}(s)\right).
$$

Applying the method used to compute $P[Z_t = 1]$ in Proposition 2.1 we get

$$
=\frac{\mu_{\text{on}}}{\mu}\left(P[X_{\text{on}}^{(0)} > s] + \int_0^s (1-F_{\text{on}}(s-w))F_{\text{on}}^{(0)} * F_{\text{off}} * U(dw)\right).
$$

Observe that the integral is

$$
F_{\text{on}}^{(0)} * F_{\text{off}} * U(s) - F_{\text{on}} * F_{\text{on}}^{(0)} * F_{\text{off}} * U(s)
$$

Let the renewal function of the stationary sequence $\{S_n, n \geq 0\}$ be

$$V(t) = \sum_{n=0}^{\infty} P[S_n \leq t],$$

and since the process is stationary we have

$$V(t) = \frac{t}{\mu}.$$

Similarly, for later use, we define U to be the renewal function of the pure renewal sequence

$$U(t) = \sum_{n=0}^{\infty} (F_{\text{on}} * F_{\text{off}})^{n*}(t).$$

We may evaluate the infinite sum in (2.3) as

$$\begin{aligned}\sum_{n=0}^{\infty} \int_0^t (1 - F_{\text{on}}(t-u)) P[S_n \in du] &= \int_0^t (1 - F_{\text{on}}(t-u)) V(du) \\ &= \int_0^t (1 - F_{\text{on}}(t-u)) \frac{du}{\mu} \\ &= \left(\int_0^t \frac{1 - F_{\text{on}}(u)}{\mu_{\text{on}}} du \right) \frac{\mu_{\text{on}}}{\mu}.\end{aligned}$$

Thus

$$EZ_t = \frac{\mu_{\text{on}}}{\mu} P[X_{\text{on}}^{(0)} > t] + P[X_{\text{on}}^{(0)} \leq t] \frac{\mu_{\text{on}}}{\mu} = \frac{\mu_{\text{on}}}{\mu},$$

as claimed.

Now we verify that $\{Z_t, t \geq 0\}$ is stationary. For this, it suffices to show that

$$Q(t) := P[Z_t = 1, B_{\text{on}}^{(t)} > x_0, {Y_{\text{off}}}_0^{(t)} > y_0, {X_{\text{on}}}_1^{(t)} > x_1, \dots, {Y_{\text{off}}}_k^{(t)} > y_k] = P(A_t) \tag{2.4}$$

is independent of t where $B_{\text{on}}^{(t)}$ is the residual life of the *on* period at time t, ${Y_{\text{off}}}_0^{(t)}$ is the length of the first subsequent *off* period, ${X_{\text{on}}}_1^{(t)}$ is the length of the next *on* period and so on.

We decompose A_t to yield

$$\begin{aligned} Q(t) =& P[A_t, B = 1, D^{(0)} > t] + P[A_t, B = 0, D^{(0)} > t] + P[A_t, D^{(0)} \leq t] \\ =& I + II + III. \end{aligned}$$

For I we note that if $B = 1$, then $D^{(0)} = X_{\text{on}}^{(0)} + Y_{\text{off}}$ so

$$I = \frac{\mu_{\text{on}}}{\mu} P[X_{\text{on}}^{(0)} > t + x_0] \theta$$

where

$$\theta = P[Y_{\text{off}} > y_0, X_1 > x_1, Y_1 > y_1, \dots, Y_k > y_k].$$

Note that $II = 0$ since $B = 0$ implies $D^{(0)} = Y_{\text{off}}^{(0)} > t$ which is incompatible with $Z_t = 1$. For III note that $D^{(0)} \leq t$ implies there is a last renewal before time t. Since the renewal function has density du/μ we have (cf. Resnick, 1992, page 204)

$$\begin{aligned} II =& \int_0^t (1 - F_{\text{on}}(t - u + x_0) \theta \frac{du}{\mu} \\ =& \theta \int_{x_0}^{t + x_0} \frac{(1 - F_{\text{on}}(s))}{\mu} ds. \end{aligned}$$

and a delayed renewal sequence by

$$\{S_n, n \geq 0\} := \{D^{(0)}, D^{(0)} + \sum_{i=1}^{n}(X_i + Y_i), n \geq 1\}$$

and we claim this delayed renewal sequence is stationary. To verify this, we merely have to show that $D^{(0)} \stackrel{d}{=} D$ where the distribution of D is given in (2.1). The probability density of D is $(1 - F_{\text{on}} * F_{\text{off}})/\mu$ and so we must show this is also the density of $D^{(0)}$. The distribution function of $D^{(0)}$ is $(x > 0)$

$$\begin{aligned} P[D^{(0)} \leq x] =& P[B(X_{\text{on}}^{(0)} + Y_{\text{off}}) + (1 - B)Y_{\text{off}}^{(0)} \leq x] \\ =& P[X_{\text{on}}^{(0)} + Y_{\text{off}} \leq x]\frac{\mu_{\text{on}}}{\mu} + P[Y_{\text{off}}^{(0)} \leq x]\frac{\mu_{\text{off}}}{\mu}. \end{aligned}$$

Since the density of $X_{\text{on}}^{(0)} + Y_{\text{off}}$ is

$$\int_0^x \frac{1 - F_{\text{on}}(x - s)}{\mu_{\text{on}}} F_{\text{off}}(ds) = \frac{F_{\text{off}}(x)}{\mu_{\text{on}}} - \frac{F_{\text{on}} * F_{\text{off}}(x)}{\mu_{\text{on}}},$$

the density of $D^{(0)}$ is

$$\left(\frac{F_{\text{off}}(x)}{\mu_{\text{on}}} - \frac{F_{\text{on}} * F_{\text{off}}(x)}{\mu_{\text{on}}}\right) \frac{\mu_{\text{on}}}{\mu} + \left(\frac{1 - F_{\text{off}}(x)}{\mu_{\text{off}}}\right) \left(\frac{\mu_{\text{off}}}{\mu}\right) = \frac{1 - F_{\text{on}} * F_{\text{off}}(x)}{\mu},$$

which is the density of D.

We now define Z_t to be 1 if t falls in an *on* period and $Z_t = 0$ if t is in an *off* period. More precisely, the process $\{Z_t, t \geq 0\}$ is defined in terms of $\{S_n, n \geq 0\}$ as follows:

$$Z_t = B 1_{[0, X_{\text{on}}^{(0)})}(t) + \sum_{n=0}^{\infty} 1_{[S_n \leq t < S_n + X_{n+1}]}. \tag{2.2}$$

Thus, if $t \geq D^{(0)}$ we have

$$Z_t = \begin{cases} 1, & \text{if } S_n \leq t < S_n + X_{n+1}, \text{ for some } n \\ 0, & \text{if } S_n + X_{n+1} \leq t < S_{n+1}, \text{ for some } n \end{cases}$$

while if $0 \leq t < D^{(0)}$ we define

$$Z_t = \begin{cases} 1, & \text{if } B = 1 \text{ and } 0 \leq t < X_{\text{on}}^{(0)}, \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 2.1. *The process $\{Z_t, t \geq 0\}$ is strictly stationary and*

$$P[Z_t = 1] = \frac{\mu_{\text{on}}}{\mu}.$$

Proof. Consider first the derivation of the marginal distribution of Z_t. We have from (2.2)

$$P[Z_t = 1] = EZ_t = \frac{\mu_{\text{on}}}{\mu} P[X_{\text{on}}^{(0)} > t] + \sum_{n=0}^{\infty} P[S_n \leq t < S_n + X_{n+1}]. \tag{2.3}$$

the covariance function of the generated traffic. To study the precise asymptotic behavior of this covariance function we first establish a rate of convergence in the key renewal theorem in the case of a heavy tailed renewal distribution. This is a second order version of Blackwell's Theorem and relies on work of H. Frenk (1987). We show that for heavy-tailed *on* and/or *off* distributions, this covariance function decays very slowly with a precise rate of decay, and that the long memory induced by the heavy tailed *on* and *off* distributions lead to strong long–range dependence.

A key issue in acceptance of non–standard traffic models is understanding the impact the non–standard features have on system performance. There is increasing evidence that long range dependence in inputs can have a dramatic effect on system performance (Resnick and Samorodnitsky (1995); Erramilli et al (1995); Livney et al (1993)) but there is not universal agreement (Heyman and Lakshman (1995)). Thus, in Section 5, we analyze the behavior of a simple fluid flow queue for which fluid from an *on/off* source arrives at unit rate during *on* periods and leaves (if the queue is non-empty) at a fixed rate. We show the existence of a limit distribution for the contents process and that heavy tailed *on* and *off* distributions lead to heavy tailed limit distributions. The important conclusion is that for sources producing traffic with long range dependence, the limiting queue length process can be heavy tailed. This clearly has important implications for the design of data networks.

Related work is given in Jelenković and Lazar (1995) who consider Markov-modulated sources with emphasis on understanding asymptotics of queue length distributions when Cramer type conditions fail. Bong and Ryu (1995) is another relevant paper which considers explanations for self-similarity in network traffic.

2. The stationary on–off process and its covariance function.

Let $\{X_{\text{on}}, X_n, n \geq 1\}$ be iid non-negative random variables representing *on* periods and similarly let $\{Y_{\text{off}}, Y_n, n \geq 1\}$ be iid non-negative random variables representing *off* periods. The X's are assumed independent of the Y's and the common distribution of *on* periods is F_{on} and the distribution of *off* periods is F_{off}. We assume both F_{on} and F_{off} have finite means μ_{on} and μ_{off} and we set $\mu = \mu_{\text{on}} + \mu_{\text{off}}$.

Consider the pure renewal sequence initiated by an *on* period $\{0, \sum_{i=1}^n (X_i + Y_i), n \geq 1\}$. The interarrival distribution is $F_{\text{on}} * F_{\text{off}}$ and the mean interarrival time is μ. This pure renewal process has a stationary version (cf. Resnick, 1992, page 224ff) $\{D, D + \sum_{i=1}^n (X_i + Y_i), n \geq 1\}$ where D is a delay random variable independent of $\{X_n, Y_n\}$ with distribution

$$P[D > x] = \int_x^\infty \frac{P[X_{\text{on}} + Y_{\text{off}} > s]}{\mu} ds = \int_x^\infty \frac{1 - F_{\text{on}} * F_{\text{off}}(s)}{\mu} ds. \tag{2.1}$$

However, defining the initial delay interval of length D in this way has the disadvantage that the interval does not decompose into an *on* and an *off* period the way subsequent interarrival intervals do and so we turn to an alternative construction of the stationary renewal process.

Define three independent random variables $B, X_{\text{on}}^{(0)}, Y_{\text{off}}^{(0)}$ which are independent of $\{Y_{\text{off}}, X_n, Y_n, n \geq 1\}$ as follows: B is a Bernoulli random variable with values $\{0, 1\}$ and mass function

$$P[B = 1] = \frac{\mu_{\text{on}}}{\mu} = 1 - P[B = 0]$$

and $(x > 0)$

$$P[X_{\text{on}}^{(0)} > x] = \int_x^\infty \frac{1 - F_{\text{on}}(s)}{\mu_{\text{on}}} ds =: 1 - F_{\text{on}}^{(0)}(x), \quad P[Y_{\text{off}}^{(0)} > x] = \int_x^\infty \frac{1 - F_{\text{off}}(s)}{\mu_{\text{off}}} ds =: 1 - F_{\text{off}}^{(0)}(x).$$

Define a delay random variable $D^{(0)}$ by

$$D^{(0)} = B(X_{\text{on}}^{(0)} + Y_{\text{off}}) + (1 - B)Y_{\text{off}}^{(0)}$$

HEAVY TAILS AND LONG RANGE DEPENDENCE IN ON/OFF PROCESSES AND ASSOCIATED FLUID MODELS

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Abstract. *On/off* models are common inputs for a variety of communication network models as well as storage and inventory models. A stationary renewal process alternating periods of activity (active transmission, fluid buildup, etc) with periods of inactivity (silence, no transmission, inputs off, etc) induces a stationary indicator process which indicates if the system is active or not. Heavy tails for the *on* periods induces a covariance function for the indicator process which decreases slowly at a rate characteristic of long range dependence. This has dramatic consequences for fluid models where fluid flows in at constant rate and there is a constant rate of release.

1. Introduction.

Traffic on data networks, e.g. Ethernet LANs, has characteristics substantially different from those of traditional voice traffic. An important feature of data traffic lies in its dependence structure; traditional models are based on assumptions of short-range dependence (like Poisson arrivals and exponential call lengths), while recent measurement and analysis of data traffic has produced strong indications of long-range dependence and self–similarity. Several empirical studies have established statistical evidence for existence of these non–standard dependence structures. See for example Leland et al (1993, 1994); Willinger, Taqqu, Leland, Wilson, (1995); Crovella and Bestavros (1995); Cunha, Bestavros and Crovella (1995).

Seeking an explanation for the observed long range dependence and self–similarity, Willinger et al (1995) have modelled traffic between a single source and destination as an *on/off* or *packet train* process. In their model, an idealized source alternates between an *on* state, in which it produces data at a constant rate, and an *off* state in which it produces no data. The durations of the *on* and *off* periods are independent; *on* times are identically distributed, and so are *off* times. The data they present indicates that both *on* and *off* times are reasonably well modelled by heavy tailed distributions with shape parameter governing heaviness represented by the parameter α. In one example, $\alpha = 1.7$ and 1.2 respectively for the *on* and *off* periods. A similar conclusion was drawn by Crovella and Bestavros (1995) who in their study of www use found evidence of heavy tails in such things as file lengths, transfer times, and operator idle periods.

To study the total traffic for all source-destination network pairs, Willinger et al (1995) first study the asymptotic behavior of the integral of the covariance function for one such pair. Their methods (Laplace transform and Tauberian theorem) yield the interesting result that the superposition of a large number of suitably scaled source-destination pairs is approximately a fractional Brownian Motion. However, they do not produce an explicit result for the decay of the covariance function. Furthermore, their model assumes that the α for both the *on* and *off* periods is the same.

We study their simple *on/off* model. Since data analysis given by Willinger et al (1995) and Crovella and Bestavros (1995) suggest means exist, we assume *on* and *off* distributions have finite means and are able to give an explicit construction of a stationary version of the source process and obtain a closed form for

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