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HECKE ALGEBRAS AND CHARACTERS OF PARABOLIC TYPE OF FINITE GROUPS WITH (B, N)-PAIRS

by C. W. CURTIS, N. IWAHORI, *and* R. KILMOYER [1]

INTRODUCTION

The purpose of this paper is to discuss the structure of the Hecke algebras (or centralizer rings) H(G, P) of a finite group G with a (B, N)-pair, with respect to an arbitrary parabolic subgroup P of G, and to discuss the irreducible complex characters of G corresponding to one-dimensional representations of H(G, P), in the sense of Curtis-Fossum [6]. These characters are constituents of the permutation character 1_B^G, where B is a Borel subgroup of G, and in some cases all irreducible constituents of 1_B^G are of this type. Such characters, which are precisely those which appear with multiplicity one in some permutation character 1_P^G, where P is a parabolic subgroup of G, are called characters of parabolic type.

Here is a survey of the contents of the paper. The first section contains the known results on the properties of a basis of H(G, B) corresponding to the double cosets relative to B, and the introduction of the generic ring of Tits corresponding to the Coxeter system (W, R) of the (B, N)-pair of G. In § 2, the Hecke algebras $H(G, G_J)$ corresponding to arbitrary parabolic subgroups G_J, with $J \subset R$, are studied by means of suitable generic rings, and it is proved that for all $J \subset R$, $H_{\mathbf{C}}(G, G_J) \cong H_{\mathbf{C}}(W, W_J)$, where W is the Weyl group, and W_J the parabolic subgroup of W defined by the subset J of the set of distinguished generators R, and **C** the complex field. A criterion for commutativity of $H(G, G_J)$ in terms of the distinguished double coset representatives of W_J in W is obtained in § 3. These results were announced by Iwahori [12].

In § 4, characters of parabolic type are introduced, and formulas for their degrees and primitive idempotents in the group algebra **C**G affording them are given, following

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Curtis-Fossum [6]. Further discussion of these characters depends on the concept of a system $\{G(q)\}$ of finite groups with (B, N)-pairs, all with the same Coxeter system, and parametrized by some infinite set of prime powers $\{q\}$. All the Chevalley groups and twisted types belong to such systems. Generic rings corresponding to parabolic subgroups $\{G_J(q), J \subset R\}$ of groups $G(q)$ in the system are defined, which are algebras over the polynomial ring $\mathbf{Q}[u]$ of polynomials in one variable with rational coefficients. Generic idempotents are constructed in these generic rings, which specialize to primitive idempotents affording characters of parabolic type, of groups $\{G(q)\}$ in the system. For each linear representation φ of a generic ring, for some $J \subset R$, in an algebraic closure $\overline{K}$ of $K = \mathbf{Q}(u)$, a generic degree d_φ, belonging to $\overline{K}$, is defined, with the property that all degrees of characters of parabolic type, corresponding to the representation φ, of groups $\{G(q)\}$ in the system, are obtained by specializing d_φ. In particular, if φ is a rational character, taking values in $\mathbf{Q}[u]$ on the basis elements of the generic ring, then the generic degree $d_\varphi \in \mathbf{Q}[u]$. When this occurs, the degree of the corresponding character of $G(q)$, for each q, is a polynomial in q with rational coefficients.

In §§ 6 and 7, further remarks on rationality of characters of generic rings are given, along with methods to determine which characters of groups in the system $\{G(q)\}$ are of parabolic type in terms of the Weyl group.

In § 8, the irreducible representations of a generic ring corresponding to a Coxeter system of dihedral type are constructed explicitly. It is then shown how to construct, for every finite irreducible Coxeter system (W, R), irreducible representations of the generic ring of (W, R) corresponding to the reflection representation of W, and its exterior powers (Kilmoyer [13]). If (G, B, N, R) is a finite group with a (B, N)-pair of type (W, R), then the characters of G corresponding to the reflection representation of W and its exterior powers are shown to be distinct, and of parabolic type. Explicit formulas for the generic degrees of the irreducible characters corresponding to the reflection representation of all known systems of groups with (B, N)-pairs are given in § 9.

In § 10 the one-dimensional representations of the Hecke algebras H(G, B) are discussed, and explicit formulas for the corresponding generic degrees are given for all known systems of groups with (B, N)-pairs.

1. Finite groups with (B, N)-pairs and Hecke algebras corresponding to Borel Subgroups.

This section is a summary of known results, originally proved by Iwahori, Matsumoto, and Tits. All are either proved or appear as exercises in Bourbaki [3], and we shall not attempt to give the original source of each result.

We shall be concerned with *finite groups with* (B, N)*-pairs* (or *Tits systems*) (G, B, N, R). Then $H = B \cap N$ is a normal subgroup of N, and the *Weyl group* $W = N/H$ of G admits a *Coxeter system* (W, R), with set of distinguished generators R. This means that W has a presentation with a set of generators R and defining relations

$$(1.1)\qquad \begin{aligned} r^2 &= 1, \qquad r \in \mathrm{R} \\ (rs\ldots)_{n_{rs}} &= (sr\ldots)_{n_{rs}}, \qquad r, s \in \mathrm{R},\ r \neq s, \end{aligned}$$

where $(ab\ldots)_m$ denotes a product of alternating a's and b's with m factors, and n_{rs} is the order of rs in W.

There is a bijection between the double cosets $\mathrm{B}\backslash\mathrm{G}/\mathrm{B}$ and the elements $w \in \mathrm{W}$, given by $w \mapsto \mathrm{B}w\mathrm{B}$, and resulting in the *Bruhat decomposition* of G

$$(1.2)\qquad \mathrm{G} = \bigcup_{w \in \mathrm{W}} \mathrm{B}w\mathrm{B}.$$

More generally, let $\mathrm{J} \subset \mathrm{R}$; then J determines a *parabolic subgroup* $\mathrm{G_J} = \mathrm{BW_JB}$, where $\mathrm{W_J} = \langle \mathrm{J} \rangle$, and for $\mathrm{J}, \mathrm{J}' \subset \mathrm{R}$, there is a bijection between $\mathrm{W_J}\backslash\mathrm{W}/\mathrm{W_{J'}}$ and $\mathrm{G_J}\backslash\mathrm{G}/\mathrm{G_{J'}}$ given by $\Xi \mapsto \mathrm{B}\Xi\mathrm{B}$, $\Xi \in \mathrm{W_J}\backslash\mathrm{W}/\mathrm{W_{J'}}$, of which the Bruhat decomposition is a special case.

Definition (1.3). — Let k be a field of characteristic zero, and let P be a subgroup of a finite group G. Let $e = |\mathrm{P}|^{-1} \sum_{x \in \mathrm{P}} x$ be the idempotent in the group algebra kG affording the 1-representation of P. The *Hecke algebra* $\mathrm{H}_k(\mathrm{G}, \mathrm{P})$ is defined to be the subalgebra of kG given by $e(k\mathrm{G})e$.

The importance of the Hecke algebra, from our point of view, is that it is isomorphic in a natural way with the *centralizer ring* $\mathrm{Hom}_{k\mathrm{G}}(k\mathrm{G}e, k\mathrm{G}e)$ of the left kG-module $k\mathrm{G}e$, which affords the induced representation 1_P^G, where 1_P is the 1-representation of P.

It is easy to check that the " characteristic functions " $\sum_{x \in \Theta} x$ on the double cosets $\Theta \in \mathrm{P}\backslash\mathrm{G}/\mathrm{P}$ form a basis for the Hecke algebra $\mathrm{H}_k(\mathrm{G}, \mathrm{P})$. It will be convenient to refer to the *standard basis* of $\mathrm{H}_k(\mathrm{G}, \mathrm{P})$ as the elements

$$\theta = \frac{1}{|\mathrm{P}|} \sum_{x \in \Theta} x, \qquad \Theta \in \mathrm{P}\backslash\mathrm{G}/\mathrm{P}.$$

The constants of structure $\{c_{\theta\theta'\theta''}\}$ given by

$$\theta\theta' = \sum_{\theta''} c_{\theta\theta'\theta''}\theta''$$

are all integers (for the standard basis).

The structure of the Hecke algebra $\mathrm{H}_k(\mathrm{G}, \mathrm{B})$ of a finite group with a (B, N)-pair, with respect to a Borel subgroup B, was worked out by Iwahori [11] (for the Chevalley groups) and Matsumoto [17] in general. Letting $\{\alpha_w, w \in \mathrm{W}\}$ denote the standard basis, indexed *via* the Bruhat decomposition (1.2) by the elements of the Weyl group, the multiplication in $H_k(G, B)$ is determined by the formulas

$$\text{(1.4)}\qquad \begin{aligned} &\alpha_r \alpha_w = \alpha_{rw}, \quad r\in R, \quad w\in W, \quad \ell(rw)>\ell(w),\\ &\alpha_r \alpha_w = q_r \alpha_{rw} + (q_r - 1)\alpha_w, \quad r\in R, \quad w\in W, \quad \ell(rw)<\ell(w),\end{aligned}$$

where the $\{q_r, r\in R\}$ are the *index parameters*, given by

$$\text{(1.5)}\qquad q_r = [B : (B\cap rBr)] = \operatorname{ind} n_r, \quad n_r \in BrB\cap N.$$

From (1.4) it was proved by Iwahori and Matsumoto that $H_k(G, B)$ has a presentation with generators $\{\alpha_r, r\in R\}$ and relations

$$\text{(1.6)}\qquad \begin{aligned} &\alpha_r^2 = q_r e + (q_r - 1)\alpha_r, \quad r\in R,\\ &(\alpha_r\alpha_s\ldots)_{n_{rs}} = (\alpha_s\alpha_r\ldots)_{n_{rs}}, \quad r, s\in R, \quad r\neq s,\end{aligned}$$

where $e=\alpha_1=|B|^{-1}\sum_{x\in B} x$ is the identity element in $H_k(G, B)$, the $\{q_r\}$ are given by (1.5), and the $\{n_{rs}\}$ by (1.1).

In order to compare the Hecke algebras $H_k(G, B)$ with the group algebra kW, Tits introduced the *generic ring* A corresponding to the Coxeter system (W, R), as follows. Let $\{u_r, r\in R\}$ be indeterminates over k, chosen so that $u_r=u_s$ if and only if r and s are conjugate in W. Let $\mathfrak{O}$ be the polynomial ring $k[u_r; r\in R]$, and let K be the quotient field of $\mathfrak{O}$. Then there exists an associative $\mathfrak{O}$-algebra A, with identity, with a free basis $\{a_w, w\in W\}$ over $\mathfrak{O}$, and multiplication determined by the formulas, for $r\in R$, $w\in W$,

$$\text{(1.7)}\qquad \begin{aligned} &a_r a_w = a_{rw}, \quad \ell(rw)>\ell(w)\\ &a_r a_w = u_r a_{rw} + (u_r - 1)a_w, \quad \ell(rw)<\ell(w).\end{aligned}$$

As in the case of the Hecke algebras, the *generic ring* A has a presentation with generators $\{a_r, r\in R\}$ and relations

$$\text{(1.8)}\qquad \begin{aligned} &a_r^2 = u_r 1 + (u_r - 1)a_r, \quad r\in R,\\ &(a_r a_s\ldots)_{n_{rs}} = (a_s a_r\ldots)_{n_{rs}}, \quad r, s\in R, \quad r\neq s,\end{aligned}$$

with n_{rs} as in (1.1).

Now let L be any field, and let $f:\mathfrak{O}\to L$ be a homomorphism. Then L becomes an $(L, \mathfrak{O})$-bimodule *via* $(\lambda, \lambda', x)\mapsto \lambda\lambda' f(x)$, $\lambda, \lambda'\in L$, $x\in\mathfrak{O}$. Thus we can form the *specialized algebra* $A_{f,L}=L\otimes_{\mathfrak{O}} A$. Then $A_{f,L}$ is an algebra over L with basis $\{a_{wf}=1\otimes a_w\}$, whose constants of structure are obtained by applying f to the constants of structure of A. On the other hand, L can be viewed as an $(\mathfrak{O}, \mathfrak{O})$-bimodule, and $A_{f,L}$ as an algebra over $\mathfrak{O}$, where $x\alpha=f(x)\alpha$, $x\in\mathfrak{O}$, $\alpha\in A_f$. Then f can be extended to a homomorphism of $\mathfrak{O}$-algebras $f: A\to A_{f,L}$, such that $f(a_w)=a_{wf}$ for all $w\in W$.

Some specialized algebras of A are especially noteworthy. Letting $f_0 : \mathfrak{O} \to k$ be defined by $f_0(u_r) = 1$ for all $r \in \mathrm{R}$, one has

(1.9) $$\mathrm{A}_{f_0, k} \cong k\mathrm{W}.$$

Now let $f : \mathfrak{O} \to k$ be defined by $f(u_r) = q_r$, $r \in \mathrm{R}$, where the $\{q_r\}$ are the index parameters (1.5). Then

(1.10) $$\mathrm{A}_{f, k} \cong \mathrm{H}_k(\mathrm{G}, \mathrm{B}).$$

We conclude this preliminary section with a basic theorem, due to Tits. We first define, for a separable algebra S over a field K, the *numerical invariants* of S to be the integers $\{n_i\}$ such that

$$\mathrm{S}^{\overline{\mathrm{K}}} \cong \bigoplus_i \mathrm{M}_{n_i}(\overline{\mathrm{K}}) \qquad \text{(direct sum)},$$

where the $\mathrm{M}_{n_i}(\overline{\mathrm{K}})$ are total matrix algebras over $\overline{\mathrm{K}}$, an algebraic closure of K.

Theorem **(1.11)**. — *Let* A *be an associative algebra over an integral domain* $\mathfrak{O}$ *with quotient field* K, *such that* A *has a finite basis over* $\mathfrak{O}$. *Let* L *be a field, and* $f : \mathfrak{O} \to \mathrm{L}$ *a homomorphism. Let* $\mathrm{A}_{f, \mathrm{L}} = \mathrm{L} \otimes_{\mathfrak{O}} \mathrm{A}$ *be the specialized algebra, and suppose that* $\mathrm{A}_{f, \mathrm{L}}$ *is a separable algebra over* L. *Then* A^{K} *is separable over* K, *and the algebras* A^{K} *and* $\mathrm{A}_{f, \mathrm{L}}$ *have the same numerical invariants.*

For the benefit of the reader who wants to avoid doing exercise 26, p. 56 of Bourbaki [3], and because we shall need to use the details of the proof later, we refer to a proof of this theorem in Steinberg's notes ([20], Lemma 85, p. 249).

From this theorem, together with (1.9) and (1.10), we deduce that the algebras kW and $\mathrm{H}_k(\mathrm{G}, \mathrm{B})$ have the same numerical invariants, and are isomorphic if k is algebraically closed.

2. Hecke algebras corresponding to parabolic subgroups.

The main result of this section is the following theorem which was announced by Iwahori [12].

Theorem **(2.1)**. — *Let k be a field of characteristic zero, and let* G *be a finite group with a* (B, N)*-pair, with Coxeter system* (W, R). *Let* $\mathrm{J} \subset \mathrm{R}$, *and let* $\mathrm{W}_\mathrm{J} = \langle \mathrm{J} \rangle$, $\mathrm{G}_\mathrm{J} = \mathrm{BW}_\mathrm{J}\mathrm{B}$. *Then the Hecke algebras* $\mathrm{H}_k(\mathrm{G}, \mathrm{G}_\mathrm{J})$ *and* $\mathrm{H}_k(\mathrm{W}, \mathrm{W}_\mathrm{J})$ *have the same numerical invariants, and are isomorphic if k is algebraically closed.*

Because of the interpretation of Hecke algebras as centralizer rings of induced permutation representations, this theorem implies the following result.

Corollary **(2.2)**. — *Let* {G, W, J} *be as in Theorem* (2.1), *and let k be a splitting field of characteristic zero for* G *and for* W. *Let*

$$1_{\mathrm{G}_\mathrm{J}}^{\mathrm{G}} = 1_\mathrm{G} + m_1\zeta_1 + \ldots + m_s\zeta_s,$$

and

$$1_{\mathrm{W}_\mathrm{J}}^{\mathrm{W}} = 1_\mathrm{W} + n_1\lambda_1 + \ldots + n_t\lambda_t,$$

where the $\{\zeta_i\}$ *and* $\{\lambda_j\}$ *are distinct irreducible characters. Then* $s=t$, *and for a suitable rearrangement, we have* $m_i=n_i$, $1\leq i\leq s$.

Before giving the proof of Theorem (2.1), we require some lemmas. We assume throughout that k is a field of characteristic zero.

Lemma **(2.3)**. — *Let* G *be a finite group, and let* S *and* T *be subgroups such that* $G>S>T$. *Let* $T\backslash G/T=\{\Theta_\lambda\}_{\lambda\in\Lambda}$, *and let* $\{\theta_\lambda\}_\lambda$ *be the standard basis of* $H_k(G, T)$. *Let* $S\backslash G/S=\{\Xi_\mu\}_{\mu\in M}$. *For each double coset* Ξ_μ, *let* $\xi_\mu=\sum_{\Theta_\lambda\subset\Xi_\mu}\theta_\lambda$. *Then the* $\{\xi_\mu\}_{\mu\in M}$ *form a basis of* $H_k(G, S)$.

The proof is immediate, since each ξ_μ is a multiple of a standard basis element of $H_k(G, S)$.

Now let (W, R) be as in Theorem (2.1), and let $\mathfrak{O}=k[u_r;\, r\in R]$, and A the generic ring with basis $\{a_w\}$, associated with (W, R), and defined in § 1. For $J\subset R$, let

$$W_J\backslash W/W_J=\{\Xi_\lambda\}_{\lambda\in\Lambda},$$

and for each double coset Ξ_λ, set

$$\xi_\lambda=\sum_{w\in\Xi_\lambda}a_w,\qquad \lambda\in\Lambda.$$

Finally, set

$$\varepsilon=\sum_{w\in W_J}a_w.$$

We shall prove that the $\{\xi_\lambda\}$ form a basis of an $\mathfrak{O}$-subalgebra of A which has as specializations the algebras $H_k(G, G_J)$ and $H_k(W, W_J)$. As a first step, we have the following result.

Lemma **(2.4)**. — *Let* Ξ *be a non-empty subset of* W *such that* $W_J\Xi=\Xi$. *Put* $\xi=\sum_{w\in\Xi}a_w$. *Then* $a_r\xi=u_r\xi$, *for all* $r\in J$.

Proof. — For each $r\in J$, there exists a partition

$$W=W_+(r)\cup W_-(r),$$

where $W_+(r)=\{w\in W;\, \ell(rw)>\ell(w)\}$, $W_-(r)=\{w\in W:\ell(rw)<\ell(w)\}$. An element $w\in W$ belongs to $W_+(r)$ if and only if w does not have a reduced expression from R starting with r. Therefore

$$rW_+(r)=W_-(r).$$

The set Ξ also has a partition

$$\Xi=\Xi_+(r)\cup\Xi_-(r),$$

where $\Xi_+(r)=W_+(r)\cap\Xi$, $\Xi_-(r)=W_-(r)\cap\Xi$, and

$$r\Xi_+(r)=\Xi_-(r),\qquad r\in J. \tag{2.5}$$

Putting $\xi'=\sum_{w\in\Xi_+(r)}a_w$, $\xi''=\sum_{w\in\Xi_-(r)}a_w$, and using (1.7) and (2.5), we obtain

$$a_r\xi'=\xi'',$$
$$a_r\xi''=u_r\xi'+(u_r-1)\xi''.$$

Since $\xi = \xi' + \xi''$, it follows that

$$a_r \xi = a_r \xi' + a_r \xi'' = u_r \xi$$

as required.

Corollary **(2.6)**. — *For all* $\lambda \in \Lambda$ *and* $r \in \mathrm{J}$,

$$a_r \xi_\lambda = \xi_\lambda a_r = u_r \xi_\lambda.$$

Lemma **(2.7)**. — *There exists a unique homomorphism of* $\mathfrak{O}$*-algebras* $\nu : \mathrm{A} \to \mathfrak{O}$ *such that* $\nu(a_r) = u_r$ *for* $r \in \mathrm{R}$ *and* $\nu(1) = 1$.

Proof. — Because A has the presentation (1.8), it is sufficient to prove that ν preserves the defining relations. The only one that is not obviously preserved is

$$(a_r a_s \dots)_{n_{rs}} = (a_s a_r \dots)_{n_{rs}},$$

in case n_{rs} is odd. But in that case, r and s are conjugate in W, and $u_r = u_s$, so that the relation is satisfied after ν is applied. This completes the proof.

The next result is an exercise in Bourbaki ([3], Ex. 3, p. 37) and will be used in the proof of Theorem (2.1) and in § 3.

Lemma **(2.8)**. — *Let* $\mathrm{J}, \mathrm{J}' \subset \mathrm{R}$. *There exists a unique element* w^* *of minimal length in* $\mathrm{W_J} w \mathrm{W_{J'}}$. *This element is characterized by the condition that* $\ell(w_1 w^*) = \ell(w_1) + \ell(w^*)$ *and* $\ell(w^* w_2) = \ell(w^*) + \ell(w_2)$ *for all* $w_1 \in \mathrm{W_J}$, $w_2 \in \mathrm{W_{J'}}$. *In particular, by letting* J *or* J′ *be empty, it follows that a coset* $\mathrm{W_J} w$ *(for example) contains a unique element* w^* *of minimal length, which is such that* $\ell(ww^*) = \ell(w) + \ell(w^*)$ *for all* $w \in \mathrm{W_J}$.

Lemma **(2.9)**. — *Let* $\mathrm{N} = \operatorname{Max}_{w \in \mathrm{W_J}} \ell(w)$. *Then* $\mathrm{E} = \nu(\varepsilon)$ *is a monic polynomial in* $\mathfrak{O}$ *of total degree* N. *In particular* $\mathrm{E} \neq 0$.

We can now prove the important result that the $\{\xi_\lambda ; \lambda \in \Lambda\}$ form a basis for a subalgebra of A.

Lemma **(2.10)**. — *The elements* $\{\xi_\lambda ; \lambda \in \Lambda\}$ *form a basis for an* $\mathfrak{O}$*-subalgebra* $\mathrm{H}(\mathrm{R}, \mathrm{J})$ *of* A. *An element* $a \in \mathrm{A}$ *belongs to* $\mathrm{H}(\mathrm{R}, \mathrm{J})$ *if and only if* $\varepsilon a = a\varepsilon = \mathrm{E}a$.

Proof. — We first prove that for all $w \in \mathrm{W_J}$ and $\lambda \in \Lambda$,

$$a_w \xi_\lambda = \nu(a_w) \xi_\lambda. \tag{2.11}$$

Let $w = r_1 \dots r_t$, $r_j \in \mathrm{J}$, be a reduced expression. Then $a_w = a_{r_1} \dots a_{r_t}$, and by Lemma (2.4), we have

$$a_w \xi_\lambda = \nu(a_{r_1}) \dots \nu(a_{r_t}) \xi_\lambda = \nu(a_w) \xi_\lambda.$$

From (2.11) it follows that if $a \in \sum_\lambda \mathfrak{O} \xi_\lambda$, then $\varepsilon a = a\varepsilon = \mathrm{E}a$.

In order to prove the Lemma, it is sufficient to prove that, conversely, if $\varepsilon a = a\varepsilon = \mathrm{E}a$, then $a \in \sum_\lambda \mathfrak{O} \xi_\lambda$. Put $a = \sum_w x_w a_w$, $x_w \in \mathfrak{O}$, and for $r \in \mathrm{J}$, put

$$a' = \sum_{w \in \mathrm{W}_+(r)} x_w a_w, \qquad a'' = \sum_{w \in \mathrm{W}_-(r)} x_w a_w.$$

Since $\varepsilon a = \mathrm{E}a$, and $a_r \varepsilon = u_r \varepsilon$ by (2.11), we have

$$a_r \varepsilon a = a_r \mathrm{E}a = u_r \mathrm{E}a, \qquad r \in \mathrm{J}$$

and, upon cancelling E (because A is a free module over an integral domain) we have

$$(2.12) \qquad a_r a = u_r a, \qquad r \in \mathrm{J}.$$

Now, from $r\mathrm{W}_+(r) = \mathrm{W}_-(r)$, it follows that

$$a_r a' = \sum_{w \in \mathrm{W}_+(r)} x_w a_{rw} = \sum_{w \in \mathrm{W}_-(r)} x_{rw} a_w$$

and

$$\begin{aligned} a_r a'' &= \sum_{w \in \mathrm{W}_-(r)} x_w (u_r a_{rw} + (u_r - 1) a_w) \\ &= u_r \Big(\sum_{w \in \mathrm{W}_+(r)} x_{rw} a_w \Big) + (u_r - 1) \Big(\sum_{w \in \mathrm{W}_-(r)} x_w a_w \Big). \end{aligned}$$

From (2.12), we get $a_r(a' + a'') = u_r(a' + a'')$, and comparison with the equations above implies that

$$x_{rw} = x_w, \quad w \in \mathrm{W}_-(r) \cup \mathrm{W}_+(r).$$

Therefore $x_{rw} = x_w$ for all $w \in \mathrm{W}$ and $r \in \mathrm{J}$. Similarly $x_{wr} = x_w$ for $w \in \mathrm{W}$ and $r \in \mathrm{J}$, and it follows that a is a linear combination of the $\{\xi_\lambda; \lambda \in \Lambda\}$. This completes the proof of the Lemma.

Proof of Theorem (2.1). — We shall apply the deformation theorem (1.11) to the $\mathfrak{O}$-algebra $\mathrm{H}(\mathrm{R}, \mathrm{J})$. Let $f : \mathfrak{O} \to k$ be an epimorphism. Then $\mathrm{H}(\mathrm{R}, \mathrm{J})$ is an $\mathfrak{O}$-subalgebra of A with a basis over $\mathfrak{O}$ consisting of the elements $\{\xi_\lambda; \lambda \in \Lambda\}$. Since f is an epimorphism, $f(\mathrm{H}(\mathrm{R}, \mathrm{J}))$ is a k-subalgebra of $\mathrm{A}_{f,k}$ with k-basis $\{f(\xi_\lambda); \lambda \in \Lambda\}$, and it is clear that $f(\mathrm{H}(\mathrm{R}, \mathrm{J})) \cong \mathrm{H}(\mathrm{R}, \mathrm{J})_{f,k}$ as k-algebras. Now let $f_0 : \mathfrak{O} \to k$ be the map defined by $f_0(u_r) = 1$, $r \in \mathrm{R}$. Since $\mathrm{A}_{f_0,k} \cong k\mathrm{W}$ by (1.10), an application of Lemma (2.3) with $\mathrm{G} \to \mathrm{W}$, $\mathrm{S} \to \mathrm{W}_\mathrm{J}$, $\mathrm{T} \to \{1\}$, shows that $\mathrm{H}(\mathrm{R}, \mathrm{J})_{f_0,k} \cong \mathrm{H}_k(\mathrm{W}, \mathrm{W}_\mathrm{J})$. Now let $f : \mathfrak{O} \to k$ be defined by $f(u_r) = q_r$, $r \in \mathrm{R}$. Then $\mathrm{A}_{f,k} \cong \mathrm{H}_k(\mathrm{G}, \mathrm{B})$ by (1.10). We can now apply Lemma (2.3) with $\mathrm{G} \to \mathrm{G}$, $\mathrm{S} \to \mathrm{G}_\mathrm{J}$, $\mathrm{T} \to \mathrm{B}$. Then $\mathrm{H}(\mathrm{R}, \mathrm{J})_{f,k}$ is isomorphic to the subalgebra of $\mathrm{H}_k(\mathrm{G}, \mathrm{B})$ generated by $f(\xi_\lambda) = \sum_{w \in \Xi_\lambda} \alpha_w$, for $\lambda \in \Lambda$, and $\{\alpha_w\}$ is the standard basis of $\mathrm{H}_k(\mathrm{G}, \mathrm{B})$. From the Bruhat decomposition it follows that $f(\xi_\lambda)$ is the sum of the standard basis elements whose double cosets lie in $\mathrm{B}\Xi_\lambda\mathrm{B} \in \mathrm{G}_\mathrm{J}\backslash\mathrm{G}/\mathrm{G}_\mathrm{J}$. By Lemma (2.3), $\mathrm{H}(\mathrm{R}, \mathrm{J})_{f,k} \cong \mathrm{H}_k(\mathrm{G}, \mathrm{G}_\mathrm{J})$. An application of Theorem (1.11) completes the proof.

3. Commutativity of Hecke algebras corresponding to parabolic subgroups.

As an application of the results in § 2, we give a criterion, in terms of properties of Weyl groups, for the Hecke algebras $\mathrm{H}_k(\mathrm{G}, \mathrm{G}_\mathrm{J})$ to be commutative. This condition is equivalent to the statement that the absolutely irreducible characters ζ appearing in $1_{\mathrm{G}_\mathrm{J}}^\mathrm{G}$ all appear with multiplicity one.

Before proceeding, we first establish some notations. Let $J, J' \subset R$. By Lemma (2.8), there exists a set $W^*_{J,J'}$ of elements w^* such that each w^* is the unique element of minimal length in the double coset $W_J w^* W_{J'}$ containing it. An element $w^* \in W^*_{J,J'}$ if and only if $\ell(rw^*) \geq \ell(w^*)$ for all $r \in J$ and $\ell(w^* r') \geq \ell(w^*)$ for all $r' \in J'$.

Theorem (**3.1**). — *Let k be a field of characteristic zero. The following conditions concerning a finite group with a* (B, N)-*pair* (G, B, N, R), *and a parabolic subgroup* G_J, *for some* $J \subset R$, *are equivalent.*

(i) *The Hecke algebra* $H_k(G, G_J)$ *is commutative.*

(ii) *Each absolutely irreducible character* ζ *appearing in* $1_{G_J}^G$ *has multiplicity one,* $(\zeta, 1_{G_J}^G) = 1$.

(iii) *Each double coset* $\Theta \in G_J \backslash G / G_J$ *has the property that* $\Theta = \Theta^{-1}$.

(iv)-(vi) *Same as* (i)-(iii) *for* $H_k(W, W_J)$, *components* $\zeta \in 1_{W_J}^W$, *and double cosets* $\Xi \in W_J \backslash W / W_J$.

(vii) *Each element of* $W^*_{J,J}$ *is an involution.*

Proof. — Statements (i)-(iii) are equivalent to (iv)-(vi) respectively, by Theorem (2.1) and Corollary (2.2) and the Bruhat decomposition, and we shall now prove the equivalence of (iv)-(vi). (iv) and (v) are well known to be equivalent, and the implication (vi) $\Rightarrow$ (iv) is also known. In more detail the antiautomorphism $\sum_w \lambda_w w \mapsto \sum_w \lambda_w w^{-1}$ of kW restricts to an antiautomorphism of $H_k(W, W_J)$ which maps a standard basis element ξ corresponding to $\Xi \in W_J \backslash W / W_J$ onto the standard basis element associated with Ξ^{-1}. Statement (vi) implies that this antiautomorphism is the identity on $H_k(W, W_J)$, and hence $H_k(W, W_J)$ is commutative.

We next prove that (iv) and (v) imply (vi). For this purpose we use the fact that there exists a mapping c which assigns to each complex character ζ of a finite group G an integer $c(\zeta)$ such that

a) $c(\zeta + \zeta') = c(\zeta) + c(\zeta')$.

b) $c(\zeta)$, for an irreducible character ζ, is either 1, -1 or 0 according as ζ is afforded by a real matrix representation of the group, or $\zeta = \bar{\zeta}$ but ζ is not afforded by a real representation, or $\zeta \neq \bar{\zeta}$ (here $\bar{\zeta}$ is the complex conjugate of ζ).

The function c was shown by Frobenius and Schur to be

$$c(\zeta) = \frac{1}{|G|} \sum_{g \in G} \zeta(g^2).$$

(See Feit [9], § 3.) For another definition of c, see Mackey [16], p. 389.

The next result needed for the proof is the result due to Frame [10] (see also Mackey [16], p. 396) that for a permutation representation 1_P^G,

c) $c(1_P^G) =$ number of double cosets $\Theta \in P \backslash G / P$ such that $\Theta = \Theta^{-1}$.

The final ingredient needed for the proof is the fact that for all irreducible complex characters ζ of a Coxeter group W, *d)* $c(\zeta)=1$.

This is the result of, first, the fact that the characters of a Coxeter group are all real valued, and second, the deeper result (see Benard [1] for the cases E_6, E_7, E_8 and a survey of the investigations of different classes of Coxeter groups) that all the irreducible characters of finite Coxeter groups have Schur indices equal to one.

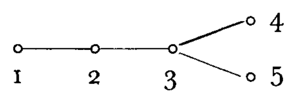
Now assume (v), that $H_k(W, W_J)$ is commutative. By (v), $1_{W_J}^W = 1_W + \zeta_1 + \ldots + \zeta_{t-1}$, where the ζ_i are distinct absolutely irreducible characters, and $t = |W_J \backslash W / W_J|$. Applying *a)*, *b)* and *d)* we have

$$c(1_{W_J}^W) = t$$

which by Frame's result *c)* is the number of self-inverse double cosets. Combining the results, we have (vi).

Finally, the equivalence of (vi) and (vii) is proved as follows. The implication (vii) $\Rightarrow$ (vi) is clear. Assume (vi), and let $w^* \in W_{J,J}^*$. Since $W_J w^* W_J = (W_J w^* W_J)^{-1}$, $(w^*)^{-1} \in W_J w^* W_J$. But $\ell((w^*)^{-1}) = \ell(w^*)$, and by the uniqueness of the element of minimal length in $W_J w^* W_J$ stated in Lemma (2.8), it follows that $w^* = (w^*)^{-1}$. This completes the proof of the Theorem.

Examples. — Using Theorem (3.1), it can be proved that the Hecke algebras $H(G, G_J)$ corresponding to maximal parabolic subgroups are all commutative, in case the Weyl group is of type A_n $(n \geq 1)$, B_n $(n \geq 2)$, C_n $(n \geq 2)$ or D_4. This result is not true for D_5, however. Letting the elements of R be numbered according to the Dynkin diagram



for $J = R - \{r_3\}$, $W_{J,J}^*$ contains an element which is not an involution, namely

$$r_3 r_2 r_1 r_4 r_3 r_5 r_2 r_3.$$

A similar example exists for type F_4.

We shall give a brief outline of the method used to determine the double coset structure $W_J \backslash W / W_{J'}$, in these and other examples.

Starting from a Coxeter system (W, R), and a subset $J \subset R$, we let Ω denote the left coset space W/W_J. If $p \in \Omega$ denotes the left coset W_J, then W acts as a transitive permutation group on Ω, in such a way that the stabilizer of p is W_J.

Any set on which W acts transitively, containing a point p for which the stabilizer is W_J, can be identified with Ω.

We associate a graph Γ with the pair (W, Ω) as follows:

(i) the vertices of Γ are exactly the points of Ω, and

(ii) if $q_1 \in \Omega$, $q_2 \in \Omega$, $r \in R$ satisfy

$$rq_1 = q_2 \quad \text{and} \quad q_1 \neq q_2,$$

then the two vertices q_1, q_2 are connected by an edge marked by r:

$$\underset{q_1}{\circ}\overset{r}{\rule[0.5ex]{4em}{0.4pt}}\underset{q_2}{\circ}$$

When this is the case, we say that q_1 *is adjacent to* q_2 in the graph Γ.

Also we define a map

$$\lambda : \Omega \to \{0, 1, 2, \dots\}$$

as follows: $\lambda(p)=0$; if $q \in \Omega - \{p\}$, then $\lambda(q)$ is the smallest positive integer k such that there exists a sequence $q_0, q_1, \dots, q_k$ of points of Ω satisfying

$$p=q_0, \qquad q=q_k,$$

q_i is adjacent to q_{i+1} $(i=0, 1, \dots, k-1)$. $\lambda(q)$ is called the *distance* of q from p.

Now let K be a subset of R. In order to describe the double coset space $\mathscr{D}_{K, J}=W_K\backslash W/W_J$ in a pictorial way, we can use the partition of vertices of Γ into W_K-orbits. One has then the following facts.

1. Remove all the edges in Γ of the form $\circ\overset{r}{\rule[0.5ex]{2em}{0.4pt}}\circ$ $(r \in R - K)$. Then one obtains a subgraph Γ_K of Γ. Furthermore there is a bijection of the double coset space $\mathscr{D}_{K, J}$ onto the set $\widetilde{\Gamma}_K$ of connected components of the graph Γ_K as follows:

$$\mathscr{D}_{K, J} \to \widetilde{\Gamma}_K,$$

$$W_K w W_J \mapsto \text{the connected component of } \Gamma_K \text{ containing } wp.$$

In fact, two points q_1, q_2 of Ω are in the same W_K-orbit if and only if there exists a sequence $r_1, \dots, r_m$ of elements in K such that $r_m \dots r_1 q_1 = q_2$. However this means that q_1 and q_2 belong to the same connected component of Γ_K.

2. For a point $q=wp$ of Ω, $\lambda(q)$ is given by

$$\lambda(q)=\min_{\sigma \in W_J} \ell(w\sigma),$$

where $\ell(\tau)$ is the length of $\tau \in W$ with respect to R.

This is seen by recalling the definition of $\ell(\tau)$.

3. Let Δ be a connected component of Γ_K and let X be the set of vertices in Δ. Then there is a unique point $x \in X$ such that

$$\lambda(x)=\min_{y \in X} \lambda(y).$$

Furthermore, if $x=r_1 r_2 \dots r_m p$, $m=\lambda(x)$, $r_1 \in R, \dots, r_m \in R$, then $r_1 r_2 \dots r_m = w$ is the unique element in the double coset $W_K w W_J$ for which the length relative to R is the smallest.

Taking 2. into account, this is shown by Lemma (2.8).

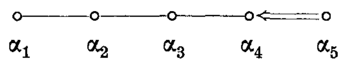
If W_K and W_J are both finite groups, then there is a unique point $z \in X$ such that

$$\lambda(z) = \operatorname*{Max}_{y \in X} \lambda(y)$$

and a similar statement as above is also true.

In the following examples, we consider the case of a Weyl group W. Let $\{\alpha_1, \ldots, \alpha_\ell\}$ be a set of simple roots in some ordering of roots. Denote by w_i the reflection w_{α_i} associated with α_i. Then $R = \{w_1, \ldots, w_\ell\}$ generates W and (W, R) is a Coxeter system.

Example 1. — (W, R) is of type (C_5)



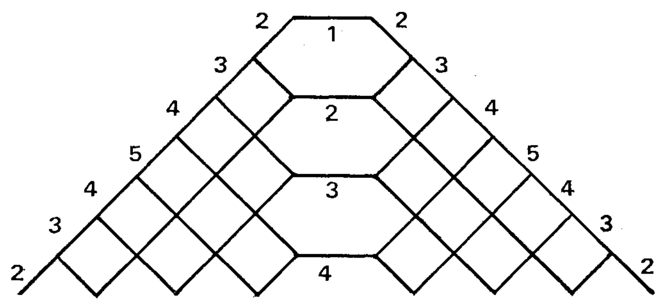
(Dynkin diagram)

Ω = the set of all short roots;

p = the dominant short root;

$J = \{w_2, w_3, w_4, w_5\} = R - \{w_1\}$

Γ:



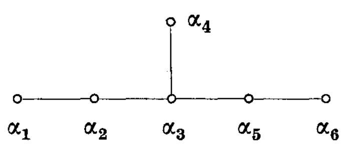
By computing the connected components of Γ_K for $K = \{w_1, w_2, w_3, w_4\}$, one has $|W_K \backslash W / W_J| = 3$. Furthermore, for each connected component of Γ_K, one has the following representative point minimizing the distance from the point p:

$$p, \qquad w_5 w_4 w_3 w_2 p, \qquad w_5 w_4 w_5 w_3 w_4 w_2 w_3 w_1 p.$$

Hence $W_K \backslash W / W_J$ is represented by

$$1, \qquad w_5 w_4 w_3 w_2, \qquad w_5 w_4 w_5 w_3 w_4 w_2 w_3 w_1.$$

Example 2. — (W, S) is of type (E_6)



$$J = \{w_2, w_3, w_4, w_5, w_6\}$$

$$\Omega = W/W_J, \qquad |\Omega| = \frac{|W|}{|W_J|} = \frac{72.6!}{16.5!} = 27$$

(Ω may be taken to be the set of weights in the fundamental irreducible representation of the complex simple Lie algebra (E_6) with the highest weight $\Lambda_1 = \frac{1}{3}(4\alpha_1 + 5\alpha_2 + 6\alpha_3 + 3\alpha_4 + 4\alpha_5 + 2\alpha_6)$.)

A graph Γ can then be constructed as in *Example* 1. Setting $K = J$, the resulting graph Γ_J has 3 connected components.
Hence $|W_J\backslash W/W_J| = 3$ and $W_J\backslash W/W_J$ is represented by

$$1, \quad w_2, \quad w_1 w_2 w_3 w_5 w_4 w_3 w_2 w_1.$$

Since these three elements are all involutive, the Hecke algebra $H(W, W_J)$ is commutative (by Theorem (3.1)).

4. Irreducible characters of parabolic type.

The results on the structure of the Hecke algebras $H_k(G, G_J)$ in §§ 2 and 3, together with the methods of Curtis and Fossum [6], suggest an investigation of the following type of character.

Definition **(4.1)**. — Let G be a finite group with a (B, N)-pair, and let k be a splitting field for G, of characteristic zero. An irreducible character ζ of G afforded by an irreducible kG-module is said to be of *parabolic type* provided that there exists a parabolic subgroup P of G (depending on ζ) such that $(\zeta, 1_P^G) = 1$. Similarly, a character ζ of a Coxeter group (W, R) is of *parabolic type* if for some $J \subset R$, $(\zeta, 1_{W_J}^W) = 1$.

By Frobenius reciprocity, if ζ is of parabolic type, $(\zeta, 1_B^G) > 0$, where B is the Borel subgroup of G. An unsolved problem is whether all irreducible characters ζ such that $(\zeta, 1_B^G) > 0$, are of parabolic type. By Theorem (7.2) below, this problem is equivalent to the corresponding problem for the Weyl group W of G.

The distinctive feature of characters of parabolic type is that their calculation involves only the determination of the one-dimensional representations of the Hecke algebras of the associated parabolic subgroups (see Theorem (4.4) below).

We begin with some preliminary remarks.

Lemma **(4.2)**. — *Let k be a splitting field of characteristic zero for a finite group* G, *and let* P *be a subgroup of* G. *Then k is also a splitting field for the Hecke algebra* $H_k(G, P)$.

Proof. — Let e_1 be a primitive idempotent in $H_k(G, P)$; then since $H_k(G, P) = ekGe$, with $e = |P|^{-1} \sum_{x \in P} x$, and $e_1 e = e e_1 = e_1$,

$$e_1 H_k(G, P) e_1 = e_1 kG e_1.$$

Therefore e_1 is a primitive idempotent in kG, and since k is a splitting field for G, $e_1 kG e_1 = k e_1$. This implies that k is also a splitting field for $H_k(G, P)$, and the Lemma is proved.

Lemma **(4.3)** (Curtis-Fossum [6], Theorem (1.1)). — *Let* G *be a finite group with a* (B, N)*-pair, and let* k *be a splitting field for* G *of characteristic zero. Let* ζ *be a character of parabolic type, such that* $(\zeta, 1_P^G)=1$, *for a parabolic subgroup* P. *Then* $\zeta|H_k(G, P)$ *is a one-dimensional representation of* $H_k(G, P)$, *and conversely, every one-dimensional representation of* $H_k(G, P)$ *is the restriction to* $H_k(G, P)$ *of a unique character of parabolic type* ζ *associated with* P.

It is now necessary to recall some notations used in Curtis-Fossum [6]. Let P be a parabolic subgroup of G, and let $P\backslash G/P=\{B\Xi_\lambda B\}_{\lambda\in\Lambda_P}$, where $W_J\backslash W/W_J=\{\Xi_\lambda\}_{\lambda\in\Lambda_P}$. Let $\{\eta_\lambda\}_{\lambda\in\Lambda_P}$ be the standard basis of $H_k(G, P)$, and for each λ, let $\hat{\eta}_\lambda$ denote the standard basis element corresponding to the double coset $(B\Xi_\lambda B)^{-1}$. For each $\lambda\in\Lambda_P$, let $n_\lambda\in B\Xi_\lambda B$, and set

$$\operatorname{ind} n_\lambda=[P : (P\cap n_\lambda^{-1}Pn_\lambda)],$$

(or $\operatorname{ind}_P n_\lambda$); then $|Pn_\lambda P|=|P| \operatorname{ind} n_\lambda$.

Theorem **(4.4)**. — *Let* P *be a parabolic subgroup of a finite group* G *with a* (B, N)*-pair, and let* k *be a splitting field of characteristic zero for* G. *Let* $\varphi : H_k(G, P)\to k$ *be a one-dimensional representation of the Hecke algebra, with associated character of parabolic type* ζ, *according to Lemma* (4.3). *Then the following statements are valid.*

(i) $\zeta(1)=[G : P]\{\sum_{\lambda\in\Lambda_P}(\operatorname{ind} n_\lambda)^{-1}\varphi(\hat{\eta}_\lambda)\varphi(\eta_\lambda)\}^{-1}$,

(ii) $b=\zeta(1)[G : P]^{-1}\sum_{\lambda\in\Lambda_P}(\operatorname{ind} n_\lambda)^{-1}\varphi(\hat{\eta}_\lambda)\eta_\lambda$

is a primitive idempotent in kG *affording the character* ζ.

(iii) *Let* $\mathfrak{C}$ *be a conjugacy class in* G, *and let* $g\in\mathfrak{C}$. *Then*

$$\zeta(g^{-1})=|C_G(g)|\sum_{\lambda\in\Lambda_P}(\operatorname{ind} n_\lambda)^{-1}\varphi(\hat{\eta}_\lambda)\gamma_\lambda(\mathfrak{C})\{\sum_{\lambda\in\Lambda_P}(\operatorname{ind} n_\lambda)^{-1}\varphi(\hat{\eta}_\lambda)\varphi(\eta_\lambda)\}^{-1},$$

where $\gamma_\lambda(\mathfrak{C})=|P|^{-1}|\mathfrak{C}\cap B\Xi_\lambda B|$, $\lambda\in\Lambda_P$.

Proof. — The first two statements are proved in Curtis and Fossum [6] (Corollary (2.5) and Theorem (3.1)). The third follows from (i), (ii) and the following result of Littlewood [15]:

Lemma **(4.5)**. — *Let* $e=\sum_{g\in G}\lambda_g g$ *be an idempotent in the group algebra* kG *of a finite group* G *over a splitting field* k *of characteristic zero, such that* kGe *is a minimal left ideal affording the character* ζ *of* G. *Let* $\mathfrak{C}$ *be a conjugacy class in* G, *and let* $g\in\mathfrak{C}$. *Then*

$$\zeta(g)=|C_G(g)|\sum_{g\in\mathfrak{C}^{-1}}\lambda_g.$$

This completes the proof of the Theorem.

A similar theorem holds for characters of parabolic type of Coxeter groups.

Examples **(4.6)**. — The simplest examples of characters of parabolic type come from the one-dimensional representations of the Hecke algebra of the parabolic group B itself. A one-dimensional representation φ of $H_k(G, B)$ must preserve the defining relations (1.6), and it follows that

$$\varphi(\alpha_r) = q_r \quad \text{or} \quad -1.$$

Using the fact that $\varphi(\alpha_r) = \varphi(\alpha_s)$, for $r, s \in R$, in case r and s are conjugate in W, it is possible to determine all these homomorphisms (see Iwahori [11], p. 235, for the case of the Chevalley groups). There are two or four such representations according as the irreducible Weyl group W has two or four one-dimensional representations. In this section we discuss two of the representations φ, called Ind and σ, together with the corresponding characters of parabolic type. A full discussion of all one-dimensional representations of $H_k(G, B)$ is given in § 10.

a) The homomorphism Ind is defined by $\mathrm{Ind}(\alpha_w) = \mathrm{ind}\, n_w$, $n_w \in BwB$. It is the restriction to $H_k(G, B)$ of the homomorphism of $kG \to k$ afforded by the trivial representation 1_G of G, and from Theorem (4.4), it follows that the corresponding character of parabolic type is 1_G.

b) The homomorphism σ is defined by the formulas $\sigma(\alpha_w) = (-1)^{\ell(w)}$, $w \in W$. The corresponding character of parabolic type is the Steinberg character χ (Curtis [5]) of G. Using (i) of the Theorem, we have

$$\chi(1) = \mathrm{ind}\, n_{w_0} = [B : (B \cap w_0^{-1} B w_0)]$$

where w_0 is the unique element in W of maximal length. For a saturated (B, N)-pair (Bourbaki [3], Ex. 5, p. 47), we have $\mathrm{ind}\, n_{w_0} = [B : H]$, from Richen [18]. This proof of the formula $\chi(1) = [B : H]$ is independent of Solomon's theorem, used in Curtis [5], that the representation sgn of W given by $\mathrm{sgn}(w) = (-1)^{\ell(w)}$ can be expressed as

$$\mathrm{sgn} = \sum_{J \subset R} (-1)^{|J|} 1_{W_J}^{W}.$$

From (ii) of the Theorem, we obtain for the idempotent b_χ affording χ,

$$b_\chi = \frac{\chi(1)}{[G : B]} \sum_{w \in W} \frac{(-1)^{\ell(w)}}{\mathrm{ind}\, n_w} \alpha_w.$$

5. Generic idempotents and degrees for characters of parabolic type.

The known finite groups of Lie type all occur in families, having the same Weyl group, and parametrized by a set of prime powers. We first axiomatize this situation, and then proceed to derive formulas for generic idempotents and degrees associated with all characters of parabolic type of the groups in a family.

Definition **(5.1)**. — A *system* $\mathcal{S}$ *of* (B, N)*-pairs of type* (W, R) consists of a Coxeter system (W, R), an infinite set $\mathcal{CP}$ of prime powers q, called *characteristic powers*, a set of positive integers $\{c_r; r\in \mathrm{R}\}$, and for each $q\in\mathscr{CP}$, a finite group $\mathrm{G}=\mathrm{G}(q)$ with a (B, N)-pair having (W, R) as its Coxeter system, such that the following conditions are satisfied:

(i) $c_r=c_s$, for $r, s\in\mathrm{R}$, if r and s are conjugate in W;

(ii) for each group $\mathrm{G}=\mathrm{G}(q)\in\mathscr{S}$, the index parameters (see (1.5)) are given by $q_r=q^{c_r}$, for $r\in\mathrm{R}$.

Examples. — Each Chevalley group associated with a finite field of q elements, with Coxeter system (W, R), belongs to a system of type (W, R), in which the set $\mathscr{CP}$ of characteristic powers q is the set of all prime powers q. In this case the index parameters q_r are all q.

Examples of systems for which the parameters c_r are not all equal to one are furnished by the twisted groups of Steinberg, Suzuki, and Ree (see Steinberg [20], § 11). The set of characteristic powers of the system of twisted groups of Suzuki or Ree consist of powers of a fixed prime number.

Lemma **(5.2)**. — *Let* $\mathscr{S}$ *be a system of type* (W, R). *For each group* $\mathrm{G}(q)\in\mathscr{S}$, *and each element* $w\in\mathrm{W}$, *let* $w=r_1\ldots r_e$ *be a reduced expression of* w *from* R. *Then for* $n_w\in\mathrm{B}w\mathrm{B}$,

$$\mathrm{ind}_{\mathrm{B}}n_w=q^{c_{r_1}+\cdots+c_{r_e}}.$$

Proof. — From Example (4.6) *a)*, the mapping $\alpha_w\mapsto\mathrm{ind}_{\mathrm{B}}n_w$ is a homomorphism $\mathrm{H}_k(\mathrm{G},\mathrm{B})\to k$ for any field k of characteristic zero. From (1.4), $w=r_1\ldots r_e$ implies that $\alpha_w=\alpha_{r_1}\ldots\alpha_{r_e}$, and the lemma follows, since $\mathrm{ind}_{\mathrm{B}}n_r=q^{c_r}$, $r\in\mathrm{R}$.

Let $\mathscr{S}$ be a system of type (W, R), and let k be a field of characteristic zero. Let $\mathfrak{O}=k[u]$ be the polynomial ring in one variable over k, with quotient field K. Then there exists an algebra $\mathrm{A}(u)$ over $\mathfrak{O}$, with a basis $\{a_w, w\in\mathrm{W}\}$, and a unique associative multiplication satisfying

$$(5.3)\qquad a_r a_w=\begin{cases} a_{rw}, & \ell(rw)>\ell(w)\\ u^{c_r}a_{rw}+(u^{c_r}-1)a_w, & \ell(rw)<\ell(w).\end{cases}$$

Moreover, this algebra has a presentation with generators $\{a_r; r\in\mathrm{R}\}$ and relations as in (1.8). (See [3], p. 55, Ex. 23; the algebra $\mathrm{A}(u)$ is simply the specialized algebra $\mathrm{A}_{f,\mathfrak{O}}$ of the generic ring A defined in § 1, (1.7), for the specialization f such that $u_r\to u^{c_r}$, $r\in\mathrm{R}$.) The algebra $\mathrm{A}(u)$ will be called the *generic ring of the system* $\mathscr{S}$. For $q\in k$, we shall denote by $\mathrm{A}(q)$ the specialized algebra $\mathrm{A}(u)_{f,k}$ for the specialization $f:u\to q$. Then we have

$$(5.4)\qquad \begin{aligned}&\mathrm{A}(q)\cong\mathrm{H}_k(\mathrm{G}(q),\mathrm{B}(q)),\qquad q\in\mathscr{CP},\\ &\mathrm{A}(1)\cong k\mathrm{W}.\end{aligned}$$

More generally, let $\mathrm{J}\subset\mathrm{R}$, and let $\mathrm{W_J}\backslash\mathrm{W}/\mathrm{W_J}=\{\Xi_\lambda;\lambda\in\Lambda\}$. From § 2, there exists a homomorphism $\nu:\mathrm{A}\to\mathfrak{O}$ such that $\nu(a_r)=u^{c_r}$, $r\in\mathrm{R}$. Letting $\varepsilon=\sum_{w\in\mathrm{W_J}}a_w$, $\mathrm{E}=\nu(\varepsilon)$, there exists an $\mathfrak{O}$-subalgebra $\mathrm{H}(\mathrm{R},\mathrm{J};u)$ of $\mathrm{A}(u)$ with a basis $\{\xi_\lambda,\lambda\in\Lambda\}$, whose elements ξ are characterized by the condition $\xi\varepsilon = \varepsilon\xi = E\xi$. Finally, for $q \in k$, letting $H(R, J; q) = H(R, J; u)_{f,k}$ for the specialization $u \to q$, we have from § 2,

$$(5.5) \qquad H(R, J; q) \cong H_k(G(q), G_J(q)), \quad q \in \mathscr{CP}$$
$$H(R, J; 1) \cong H_k(W, W_J).$$

We are now in a position to state the following result.

Proposition **(5.6)**. — *Let* $\mathscr{S}$ *be a system of type* (W, R), *with generic ring* $A(u)$. *Let* $\overline{K}$ *and* $\overline{k}$ *denote algebraic closures of* K *and* k *respectively, and* $\mathfrak{O}^*$ *the integral closure of* $\mathfrak{O}$ *in* $\overline{K}$. *Suppose the subalgebra* $H(R, J; u)^{\overline{K}}$ *of* $A(u)^{\overline{K}}$ *has a one-dimensional representation* φ. *Then* $\varphi(\xi) \in \mathfrak{O}^*$ *for all* $\xi \in H(R, J; u)$. *Letting* $f: u \to q$ *denote a specialization of* $\mathfrak{O}$, *for* $q \in \mathscr{CP} \cup \{1\}$, *there exists a homomorphism* $f^*: \mathfrak{O}^* \to \overline{k}$ *extending* f. *Then* $\varphi_f: f(\xi) \to f^*(\varphi(\xi))$, *for* $\xi \in H(R, J; u)$, *defines a one-dimensional representation of* $H(R, J; q)^{\overline{k}}$. *Moreover every one-dimensional representation of a specialized algebra* $H(R, J; q)^{\overline{k}}$, *for* $q \in \mathscr{CP} \cup \{1\}$ *is obtained in this way from some one-dimensional representation* φ *of* $H(R, J; u)$ *and some extension* f^* *of* f *to* $\mathfrak{O}^*$.

Proof. — The fact that $\varphi(\xi) \in \mathfrak{O}^*$ follows from a familiar argument concerning group characters (see Curtis-Reiner [7], p. 235) since $H(R, J; u)$ has a basis whose constants of structure belong to $\mathfrak{O}$, and if $\varphi: H(R, J; u) \to \overline{K}$ is a one-dimensional representation, $\varphi(\varepsilon) \neq 0$ since ε is a multiple by $E \neq 0$ of the identity element in $H(R, J; u)^{\overline{K}}$. The existence of an extension f^* of f is proved in Bourbaki [2] (chap. 5, § 2, no. 1, Cor. 4 to Th. 1). The existence of the representations φ_f is clear from the definition. Finally, let $q \in \mathscr{CP} \cup \{1\}$. By (5.5), the specialized algebra $H(R, J; q)$ is separable. By the deformation Theorem (1.11), the algebras $H(R, J; u)^{\overline{K}}$ and $H(R, J; q)^{\overline{k}}$ have the same numerical invariants. It follows that $\varphi \neq \varphi'$ implies $\varphi_f \neq \varphi'_f$ and that every one-dimensional representation of $H(R, J; q)^{\overline{k}}$ is obtained in the manner we have described. This completes the proof.

Theorem **(5.7)**. — *Let* $\mathscr{S}$ *be a system of type* (W, R), *and let* $\overline{k}$, $\overline{K}$ *be as in Proposition* (5.6). *Let* $J \subset R$, *and let* φ *be an arbitrary one-dimensional representation of* $H(R, J; u)^{\overline{K}}$. *Let*

$$e_\varphi = \sum_{\lambda \in \Lambda} \frac{1}{\nu(\xi_\lambda)} \varphi(\hat{\xi}_\lambda) \xi_\lambda \in H(R, J; u)^{\overline{K}},$$

where for $\lambda \in \Lambda$, $\hat{\xi}_\lambda$ *is the basis element of* $H(R, J; u)$ *corresponding to the double coset* Ξ_λ^{-1}. *Then*

$$\varphi(e_\varphi) \neq 0$$

and $\varphi(e_\varphi)^{-1} e_\varphi = e'_\varphi$ *is a central primitive idempotent in* $H(R, J; u)^{\overline{K}}$ *such that* $\xi e'_\varphi = \varphi(\xi) e'_\varphi$, *for* $\xi \in H(R, J; u)$. *Let the* generic degree *associated with* φ *be defined by*

$$(5.8) \qquad d_\varphi = \varphi(e_\varphi)^{-1} \sum_{w \in W} \nu(a_w).$$

Now let $q \in \mathscr{CP} \cup \{1\}$, *let* $f: \mathfrak{O} \to k$ *be the specialization* $u \to q$, *and let* f^* *be an extension of* f *to* $\mathfrak{O}^*$. *Then* $\varphi(e_\varphi)^{-1}$ *and* d_φ *belong to the specialization ring of* f^*, *so that* $f^*(e'_\varphi)$ *and* $f^*(d_\varphi)$ *are* *defined. Moreover,* $f^*(e'_\varphi)$ *is a primitive idempotent in* $k\mathrm{G}(q)$ *or* $k\mathrm{W}$ *affording the character of parabolic type* ζ *associated with* φ_f, *and* $f^*(d_\varphi)=\zeta(1)$.

We first remark that from Proposition (5.6), each one-dimensional representation φ of $\mathrm{H}(\mathrm{R},\mathrm{J};u)$ determines a one-dimensional representation φ_f of $\mathrm{H}(\mathrm{R},\mathrm{J};q)$ for each $q\in\mathscr{C}\mathscr{P}\cup\{1\}$. Since (5.5) asserts that $\mathrm{H}(\mathrm{R},\mathrm{J};q)\cong \mathrm{H}_k(\mathrm{G},\mathrm{G}_\mathrm{J})$ or $\mathrm{H}_k(\mathrm{W},\mathrm{W}_\mathrm{J})$, Theorem (4.4), together with the remark at the end of the proof, imply that there do exist characters of parabolic type associated with all the specialized homomorphisms φ_f. The proof of the theorem depends on the following lemmas.

Lemma **(5.9)**. — *Assume the notation of the theorem. Let* $f:u\to q$ *be a specialization, for some* $q\in\mathscr{C}\mathscr{P}$, *let* G *denote* $\mathrm{G}(q)$, *and let the subgroups* B *and* G_J *be taken relative to* G. *Then*

$$f(\mathrm{E})=[\mathrm{G}_\mathrm{J}:\mathrm{B}] \quad \textit{and} \quad f(\nu(\xi_\lambda))=(\mathrm{ind}_{\mathrm{G}_\mathrm{J}} n_\lambda)f(\mathrm{E}),$$

where $n_\lambda\in \mathrm{B}\Xi_\lambda\mathrm{B}$. *Similarly, if* f_0 *is the specialization* $u\to 1$, *then*

$$f_0(\mathrm{E})=|\mathrm{W}_\mathrm{J}| \quad \textit{and} \quad f_0(\nu(\xi_\lambda))=(\mathrm{ind}_{\mathrm{W}_\mathrm{J}} w_\lambda)f_0(\mathrm{E}),$$

where $w_\lambda\in\Xi_\lambda$.

Proof. — We have, from § 2,

$$\varepsilon=\sum_{w\in \mathrm{W}_\mathrm{J}} a_w,\quad \xi_\lambda=\sum_{w\in\Xi_\lambda} a_w,\quad \Xi_\lambda\in \mathrm{W}_\mathrm{J}\backslash \mathrm{W}/\mathrm{W}_\mathrm{J},$$

and $\mathrm{E}=\nu(\varepsilon)$. Then from the definition of ν and Lemma (5.2), we have, for $n_w\in \mathrm{B}w\mathrm{B}$

$$f(\mathrm{E})=\sum_{w\in\mathrm{W}_\mathrm{J}}\mathrm{ind}_\mathrm{B}\, n_w,\quad f(\nu(\xi_\lambda))=\sum_{w\in\Xi_\lambda}\mathrm{ind}_\mathrm{B}\, n_w.$$

The Bruhat Theorem (1.2) implies that

$$f(\mathrm{E})|\mathrm{B}|=|\mathrm{G}_\mathrm{J}|;\quad f(\nu(\xi_\lambda))|\mathrm{B}|=|\mathrm{B}\Xi_\lambda\mathrm{B}|.$$

Similarly, $$f_0(\mathrm{E})=|\mathrm{W}_\mathrm{J}|;\quad f_0(\nu(\xi_\lambda))=|\Xi_\lambda|.$$

This completes the proof of the Lemma.

Lemma **(5.10)**. — *Define a bilinear form* $\langle\, ,\rangle$ *on* $\mathrm{H}(\mathrm{R},\mathrm{J};u)^{\overline{\mathrm{K}}}$, *whose value at* (ξ,ξ'), *for* $\xi,\xi'\in\mathrm{H}(\mathrm{R},\mathrm{J};u)$, *is the coefficient of* ε *in the expansion of the product* $\xi\xi'$ *in terms of the basis* $\{\xi_\lambda;\lambda\in\Lambda\}$. *Then* $\{\xi_\lambda;\lambda\in\Lambda\}$ *and* $\{\nu(\xi_\lambda)^{-1}\xi_{\hat\lambda}\}$ *are dual bases with respect to the form, and the form is symmetric.*

Proof. — Let

$$\xi_\lambda\xi_{\lambda'}=\sum_{\lambda''} b_{\lambda\lambda'\lambda''}\xi_{\lambda''},\qquad b_{\lambda\lambda'\lambda''}\in\mathfrak{O},\qquad \lambda,\lambda',\lambda''\in\Lambda.$$

Then $\langle\xi_\lambda,\xi_{\lambda'}\rangle=b_{\lambda\lambda'1}$, if $\xi_1=\varepsilon$.

For the purposes of this proof let $\hat\lambda$ be defined by $\Xi_{\hat\lambda}=\Xi_\lambda^{-1}$. The relations

$$b_{\lambda\lambda'1}-\delta_{\hat\lambda\lambda'}\nu(\xi_\lambda)=0$$

are polynomial equations in u, and are satisfied provided we can prove

$$\textbf{(5.11)}\qquad f(b_{\lambda\lambda'1})-\delta_{\hat\lambda\lambda'}f(\nu(\xi_\lambda))=0,\qquad \lambda,\lambda'\in\Lambda,$$

for all specializations $f:u\to q$, $q\in\mathscr{C}\mathscr{P}$, since $\mathscr{C}\mathscr{P}$ is an infinite set. The same argument will prove that the form is symmetric. Letting η_λ denote the standard basis element in $H_k(G, G_J)$ corresponding to Ξ_λ, we have $f(\xi_\lambda)=[G_J : B]\eta_\lambda$, for $\lambda \in \Lambda$. It follows that if

$$\eta_\lambda \eta_{\lambda'} = \sum_{\lambda''} c_{\lambda\lambda'\lambda''} \eta_{\lambda''}$$

are the structure equations in $H_k(G, G_J)$, then by (5.9),

$$f(b_{\lambda\lambda'\lambda''}) = f(E) c_{\lambda\lambda'\lambda''}.$$

It is known that for a Hecke algebra $H_k(G, P)$, with standard basis $\{\eta_\lambda\}$, with $P\backslash G/P = \{\Theta_\lambda\}_{\lambda\in\Lambda}$, that

$$c_{\lambda\lambda' 1} - \delta_{\lambda\hat{\lambda}'} \operatorname{ind}_P g_\lambda = 0, \qquad \lambda, \lambda' \in \Lambda, \qquad g_\lambda \in \Theta_\lambda$$

(see, for example, the computation in Curtis-Fossum [6], p. 404). By this fact and (5.9), it follows that the equations (5.11) are satisfied. This completes the proof.

Corollary **(5.12)**. — $H(R, J; u)^{\overline{K}}$ *is a semi-simple symmetric algebra, with identity* $E^{-1}\varepsilon$, *and dual basis* $\{\xi_\lambda\}$, $\{\nu(\xi_\lambda)^{-1}\hat{\xi}_\lambda\}$.

Proof. — $H(R, J : u)^{\overline{K}}$ is semi-simple because it has separable specializations, and hence is actually separable. The form defined in Lemma (5.10) is symmetric, associative and non-degenerate, so that $H(R, J; u)$ is symmetric (see Curtis-Reiner [7], p. 440).

Lemma **(5.13)**. — *Let* S *be a semi-simple symmetric algebra over a field* K, *with dual bases* $\{a_i\}$ *and* $\{b_i\}$ *with respect to a symmetric associative scalar product. Let* $\varphi : S \to K$ *be a one-dimensional representation. Then* $e_\varphi = \sum_i \varphi(a_i) b_i \neq 0$, $\varphi(e_\varphi) \neq 0$, *and* $\varphi(e_\varphi)^{-1} e_\varphi$ *is a central idempotent in* S *affording the representation* φ.

Proof. — Using the dual bases (see Curtis-Reiner [7], p. 441) it follows that for all $s \in S$,

$$s e_\varphi = e_\varphi s = \varphi(s) e_\varphi.$$

In particular $e_\varphi^2 = \varphi(e_\varphi) e_\varphi$. Therefore e_φ is central, $\neq 0$, and $\varphi(e_\varphi) \neq 0$ since a semi-simple algebra contains no non-zero nilpotent elements in the center. This completes the proof of the Lemma.

Proof of Theorem (5.7). — For the proof we let f denote a specialization $u \to q$, for $q \in \mathscr{CP}$, and f_0 the specialization $u \to 1$. We first recall, from Theorem (4.4) and the corresponding result for W, that if ζ, ζ_0 are the characters of parabolic type of G and W, respectively, associated with φ, then

$$\zeta(1) = [G : G_J]\{\sum_{\lambda\in\Lambda} (\operatorname{ind}_{G_J} n_\lambda)^{-1} \varphi_f(\hat{\eta}_\lambda)\varphi_f(\eta_\lambda)\}^{-1},$$

and

$$\zeta_0(1) = [W : W_J]\{\sum_{\lambda\in\Lambda} (\operatorname{ind}_{W_J} w_\lambda)^{-1} \varphi_{f_0}(\hat{\eta}_\lambda^0)\varphi_{f_0}(\eta_\lambda^0)\}^{-1}$$

where $w_\lambda \in \Xi_\lambda$ and $\{\eta_\lambda^0\}$ is the standard basis of $H_k(W, W_J)$. We have, by (5.9) and the proof of (5.10),

$$f^*(\varphi(e_\varphi)) = \sum_{\lambda\in\Lambda} f(\nu(\xi_\lambda))^{-1} \varphi_f(f(\hat{\xi}_\lambda))\varphi_f(f(\xi_\lambda))$$
$$= f(E) \sum_{\lambda\in\Lambda} (\operatorname{ind}_{G_J} n_\lambda)^{-1} \varphi_f(\hat{\eta}_\lambda)\varphi_f(\eta_\lambda),$$

and

$$f_0^*(\varphi(e_\varphi)) = f_0(E) \sum_{\lambda\in\Lambda} (\operatorname{ind}_{W_J} w_\lambda)^{-1} \varphi_{f_0}(\hat{\eta}_\lambda^0)\varphi_{f_0}(\eta_\lambda^0),$$

where we have extended f^* and f_0^* to their specialization rings (Zariski-Samuel [21], p. 2). These expressions are both $\neq 0$ because of the above formulas for the degrees $\zeta(1)$ and $\zeta_0(1)$. Therefore $\varphi(e_\varphi)\neq 0$, and $\varphi(e_\varphi)^{-1}$ belongs to the specialization rings of f and f_0. The fact that $f^*(d_\varphi)$ gives the degrees now follows from the facts that $\sum_{w\in W} f(\nu(a_w))/f(E)=[G:G_J]$ and $\sum_{w\in W} f_0(\nu(a_w))/f_0(E)=[W:W_J]$.

The statement that $\varphi(e_\varphi)^{-1}e_\varphi$ is a central primitive idempotent affording φ follows from Lemma (5.10) (and its corollary) and (5.13).

Finally, for the specializations of $\varphi(e_\varphi)^{-1}e_\varphi$ we have from the above computations,

$$f^*(\varphi(e_\varphi)^{-1}e_\varphi)=\{\sum_{\lambda\in\Lambda}(\mathrm{ind}_{G_\lambda} n_\lambda)^{-1}\varphi_f(\hat{\eta}_\lambda)\varphi_f(\eta_\lambda)\}^{-1}(\sum_{\lambda\in\Lambda}(\mathrm{ind}_{G_\lambda} n_\lambda)^{-1}\varphi_f(\hat{\eta}_\lambda)\eta_\lambda)$$

and $$f_0^*(\varphi(e_\varphi)^{-1}e_\varphi)=\{\sum_{\lambda\in\Lambda}(\mathrm{ind}_{W_J} w_\lambda)^{-1}\varphi_{f_0}(\hat{\eta}_\lambda^0)\varphi_{f_0}(\eta_\lambda^0)\}^{-1}(\sum_{\lambda\in\Lambda}(\mathrm{ind}_{W_J} w_\lambda)^{-1}\varphi_{f_0}(\hat{\eta}_\lambda^0)\eta_\lambda^0)$$

which are primitive idempotents in kG and kW, respectively, affording ζ and ζ_0, by Theorem (4.4). This completes the proof of the Theorem.

6. Rationality of characters of parabolic type.

Any unexplained notations in this section are all taken from § 5. The main purpose of this section is to show that in case a one-dimensional representation φ of $H(R, J; u)$ is rational in the sense that $\varphi(\xi_\lambda)\in\mathbf{Q}[u]$, then the generic degree d_φ is also a polynomial in u with rational coefficients, and the corresponding characters of parabolic type all take only rational integral values.

Theorem **(6.1)**. — *Let $\mathscr{S}$ be a family of* (B, N)*-pairs of type* (W, R), *and let k be the field of rational numbers* $\mathbf{Q}$. *Let* $J\subset R$, *and let φ be a one-dimensional representation of* $H(R, J; u)^{\bar{K}}$ *such that* $\varphi(\xi_\lambda)\in\mathfrak{O}=\mathbf{Q}[u]$ *for all* $\lambda\in\Lambda$. *Then the generic degree* $d_\varphi\in\mathbf{Q}[u]$. *For every specialization associated with* $q\in\mathscr{CP}\cup\{1\}$, *the character of parabolic type ζ of the group* $G(q)$ (*or* W *if* $q=1$) *is afforded by a rational representation of* $G(q)$ (*or* W), *and hence* $\zeta(g)\in\mathbf{Z}$ *for all elements g of the group.*

Proof. — The formula (5.8) for d_φ shows that $d_\varphi\in\mathbf{Q}(u)$. The last part of Theorem (5.7) implies that $d_\varphi(q)\in\mathbf{Z}$ for all $q\in\mathscr{CP}$. Since the set $\mathscr{CP}$ is infinite, it follows that the rational function $d_\varphi(u)$ must in fact be a polynomial in u (the relevant general theorem about rational functions being left as an exercise for the reader). Since $\varphi(\xi_\lambda)\in\mathbf{Q}[u]$, $\varphi_f(f(\xi_\lambda))\in\mathbf{Q}$ for all λ, and hence the primitive idempotent $f^*(e'_\varphi)$ affording ζ (see Theorem (5.7)) belongs to $\mathbf{Q}G(q)$ (or $\mathbf{Q}W$, respectively). Then ζ is afforded by the rational module $\mathbf{Q}G(q)f^*(e'_\varphi)$ (and similarly for W). This completes the proof.

Theorem **(6.2)**. — *Let $\mathscr{S}$ be a family of type* (W, R) *whose set of characteristic powers $\mathscr{CP}$ contains almost all prime numbers. Let k be the rational field* $\mathbf{Q}$, *let* $J\subset R$, *and let φ be a one-dimensional representation of* $H(R, J; u)^{\bar{K}}$. *A necessary and sufficient condition for* $\varphi(\xi_\lambda)\in\mathbf{Q}[u]$

for all λ *is that* $\varphi_f(f(\xi_\lambda)) \in \mathbf{Q}$ *for all specialized homomorphisms* φ_f, *where* $f: u \to q$ *runs through the specializations for the prime powers* $q \in \mathscr{CP}$.

Proof. — The necessity of the condition is clear, by the definition of the specialized homomorphisms φ_f in Proposition (5.6). For the sufficiency, we introduce indeterminates $\{X_\lambda; \lambda \in \Lambda\}$ over $\overline{K}$, and let

$$\xi = \sum_\lambda X_\lambda \xi_\lambda, \quad X = (X_\lambda)$$

be a generic element of the algebra $H(R, J; u)^{\overline{K}(X)}$. Let $P(t)$ be the characteristic polynomial of ξ; then $P(t) \in \mathbf{Q}(u; X)[t]$ and we have $P(t) = \prod_i P_i(t)^{e_i}$, with the $P_i(t)$ distinct irreducible polynomials in $\mathbf{Q}(u; X)[t]$. Since $P(t)$ is a monic polynomial with coefficients in $\mathbf{Q}[u; X]$, it follows that each $P_i(t) \in \mathbf{Q}[u; X; t]$. Upon extending φ to $H(R, J; u)^{\overline{K}(X)}$, the linear polynomial $t - \varphi(\xi)$ divides $P(t)$, and hence divides some $P_{i_0}(t)$ in $\overline{K}(X)[t]$. If $P_{i_0}(t)$ has degree one, then $\varphi(\xi) \in \mathbf{Q}[u; X]$, and it follows that $\varphi(\xi_\lambda) \in \mathbf{Q}[u]$ for all λ, as required. We now suppose $\deg P_{i_0}(t) > 1$ and will derive a contradiction. We first observe that $\varphi(\xi) \in \mathfrak{O}^*[X]$, where $\mathfrak{O}^*$ is the integral closure of $\mathfrak{O}$ in $\overline{K}$. Now let $q \in \mathscr{CP}$, f the specialization $u \to q$, and extend f to $\mathfrak{O}^*$, and to the integral closure of $\mathfrak{O}[X]$ in $\overline{K}(X)$, which is $\mathfrak{O}^*[X]$ (see Bourbaki [2], chap. V, Prop. 13). Letting f^* be the specialization extended to $(\mathfrak{O}^*[X])[t]$, we have $(t - f^*(\varphi(\xi))) \mid f^*(P_{i_0}(t))$, where

$$f^*(\varphi(\xi)) = \sum_\lambda X_\lambda \varphi_f(f(\xi_\lambda)) \in \mathbf{Q}[X]$$

by the hypothesis of the theorem. Therefore, for every specialization $f: u \to q$, $q \in \mathscr{CP}$, the specialized polynomial $f^*(P_{i_0}(t)) \in \mathbf{Q}(X)[t]$ is reducible in $\mathbf{Q}(X)[t]$. This contradicts the Hilbert Irreducibility Theorem (see Lang [14], chap. VIII, Cor. 3, p. 148). This completes the proof of the Theorem.

Corollary **(6.3)**. — *If* $\varphi_f(f(\xi_\lambda)) \in \mathbf{Q}$ *for all* λ *and all specializations* $f: u \to q$, *with* $q \in \mathscr{CP}$, *then* $\varphi_{f_0}(f_0(\xi_\lambda)) \in \mathbf{Q}$, *for all* X, *for the specialization* $f_0: u \to 1$, *implying rationality of the corresponding character of* W.

7. Relations with the Weyl group.

We return to the general situation described in § 1:

G	a finite group with a (B, N)-pair;
(W, R)	the Coxeter system of G;
k	algebraically closed field of characteristic zero;
$\mathfrak{O} = k[u_r; r \in R]$	as in § 1;
A	generic ring of (W, R), over $\mathfrak{O}$;
K	quotient field of $\mathfrak{O}$;
$\overline{K}$	an algebraic closure of K;
$\mathfrak{O}^*$	integral closure of $\mathfrak{O}$ in K.

Our first result is analogous to Proposition (5.6). Let $f:\mathfrak{O}\to k$ be a surjective homomorphism, such that the specialized algebra $A_{f,k}$ is separable, and let $f^*:\mathfrak{O}^*\to k$ be an extension of f.

Proposition **(7.1)**. — *Let ψ be a character of $A^{\bar{K}}$ afforded by an irreducible $A^{\bar{K}}$-module. Then $\psi(a_w)\in\mathfrak{O}^*$, for all $w\in W$. The linear mapping $\psi_f: A_{f,k}\to k$ defined by $\psi_f(a_{wf})=f^*(\psi(a_w))$ is the character of an irreducible $A_{f,k}$-module. Every irreducible character of $A_{f,k}$ is obtained in this way.*

Proof. — This result is essentially contained in Steinberg's proof ([20], § 14) of the deformation Theorem (1.11). We sketch the steps in the argument, referring the reader to Steinberg [20] for their proofs. Let $\{X_w; w\in W\}$ be indeterminates over $\bar{K}$, and

$$a=\sum_w X_w a_w$$

a generic element of $A^{\bar{K}}$, and $P(t)$ the characteristic polynomial of a. The monic irreducible factors of $P(t)$ correspond, in a bijective fashion, to the irreducible representations of $A^{\bar{K}}$. Moreover if $P_i(t)$ is such a factor, $P_i(t)\in(\mathfrak{O}^*[X])[t]$, where $X=(X_w)$, and if ψ_i is the character of the corresponding module, extended to $A^{\bar{K}(X)}$, then $\psi_i(a)$ is a coefficient of $P_i(t)$. Moreover, $f^*(P_i(t))$ is an irreducible factor of the characteristic polynomial of a generic element $f(a)$ of $A_{f,k}$, and $\pm\psi_{if}(f(a))$ is the coefficient of the highest power but one of t in $f^*(P_i(t))$. Since the irreducible factors of the characteristic polynomial of $f(a)$ are all of this form, the result follows.

Using Proposition (7.1), we may speak of characters of A and corresponding characters of A_f, and of corresponding characters of two different separable specializations of A.

The main result of this section is the following one.

Theorem **(7.2)**. — *Let G be a finite group with a* (B, N)*-pair and* (W, R) *the Coxeter system of* G. *Let k be an algebraically closed field of characteristic zero. There is a natural bijective correspondence $\zeta\leftrightarrow\zeta_0$ between the irreducible k-characters ζ of G such that $(\zeta, 1_B^G)>0$ and the irreducible k-characters ζ_0 of W, satisfying $(\zeta, 1_{G_J}^G)=(\zeta_0, 1_{W_J}^W)$, for all $J\subset R$. In particular, a character ζ of G (with $(\zeta, 1_B^G)>0$) is of parabolic type if and only if the corresponding character of W is of parabolic type.*

Proof. — There is a natural correspondence described above between the characters of $A_{f,k}\cong H_k(G, B)$ and those of $A_{f_0,k}\cong kW$. The characters of $H_k(G, B)$ are restrictions of characters ζ of G such that $(\zeta, 1_B^G)>0$, in a bijective fashion, by Theorem (1.1) of Curtis-Fossum [6]. Before giving the rest of the proof, we require some preliminary results.

Lemma **(7.3)**. — *Keeping the notation as above, let $J\subset R$, and let ψ be an irreducible character of $A^{\bar{K}}$. Let ε, $E=\nu(\varepsilon)$ etc. be as in § 2. Then $\psi(\varepsilon)=mE$ for some non-negative integer m. Letting $f:\mathfrak{O}\to k$ be a specialization such that A_f is separable, and ψ_f defined as in Proposition (7.1), we have $\psi_f(f(\varepsilon))=mf(E)$.*

Proof. — From Lemma (2.10), we have $\varepsilon^2 = E\varepsilon$. It follows that $E^{-1}\varepsilon$ is an idempotent, and hence $\psi(E^{-1}\varepsilon)$ is a non-negative integer for any character ψ of a representation. The rest of the lemma is now clear.

The next result is an immediate consequence of the Frobenius Reciprocity Theorem.

Lemma **(7.4)**. — *Let* P *be a subgroup of a finite group* G, *and* k *an algebraically closed field of characteristic zero. Let* ζ *be a* k*-character of* G, *and* $e = |P|^{-1} \sum_{x \in P} x$. *Then* $\zeta(e) = (\zeta, 1_P^G)$.

Proof of Theorem (7.2). — Let ψ be a character of $A^{\bar{K}}$ as in Lemma (7.3), and let $J \subset R$, and m as in the Lemma. It is sufficient to prove that

a) if f is the specialization $u_r \to q_r$, and ζ the character in 1_B^G corresponding to ψ, then $m = (\zeta, 1_{G_J}^G)$; and

b) if f_0 is the specialization $u_r \to 1$, $r \in R$, and ζ_0 the character of W corresponding to ψ, then $m = (\zeta_0, 1_{W_J}^W)$.

To prove *a)*, it suffices, by (7.4), to calculate $\zeta(e_J)$, where $e_J = |G_J|^{-1} \sum_{x \in G_J} x$. From Theorem (1.1) of Curtis-Fossum [6], we have

$$\zeta(e_J) = \psi_f(e_J). \tag{7.5}$$

From Lemma (7.3) we have

$$\psi_f(f(\varepsilon)) = mf(E),$$

and by (5.9), $f(E) = [G_J : B]$, while $f(\varepsilon) = [G_J : B]e_J$. Cancelling $[G_J : B]$, we have, from (7.5), $\zeta(e_J) = m$.

To prove *b)*, let $e_J^0 = |W_J|^{-1} \sum_{x \in W_J} x$. Then $\zeta^0(e_J^0) = \psi_{f_0}(e_J^0)$, and $\psi_{f_0}(f_0(\varepsilon)) = mf_0(E)$. Then by (5.9), $f_0(E) = |W_J|$ and $f_0(\varepsilon) = |W_J| e_J^0$. Thus $\zeta^0(e_J^0) = m$ and *b)* is proved. This completes the proof of the theorem.

8. Representations of the generic ring corresponding to a Coxeter system of dihedral type.

The purpose of this section is to construct all the irreducible representations of the generic ring corresponding to a dihedral group. The method was suggested by the known representation theory of the dihedral group itself. For an application of the results of this section, see a forthcoming paper by Kilmoyer and Solomon on the Feit-Higman theorem.

Theorem **(8.1)**. — *Let* W *be a dihedral group of order* $2n$ *having the presentation* $W = \langle r, s;\, r^2 = s^2 = (rs)^n = 1 \rangle$ *and let* A *be the generic ring of the Coxeter system* $(W, \{r, s\})$ *over* $\mathfrak{O} = \mathbf{Q}[u_r, u_s]$ *as in* § 1. *Let* K *be the quotient field of* $\mathfrak{O}$, $\bar{K}$ *an algebraic closure of* K, *and* $\mathfrak{O}^*$ *the integral closure of* $\mathfrak{O}$ *in* $\bar{K}$. *For any* $c \in \bar{K}$ *let*

$$R(c) = \begin{pmatrix} -1 & c \\ 0 & u_r \end{pmatrix}, \qquad S(c) = \begin{pmatrix} u_s & 0 \\ c & -1 \end{pmatrix}.$$

Let $\rho\in\mathfrak{O}^*$ *be such that* $\rho^2=u_r u_s$, *and* $\rho=u_r$ *if n is odd,* $\theta_j=\frac{2\pi j}{n}$, *and let* c_j, d_j *be any elements of* $\overline{\mathrm{K}}$ *such that*

$$c_j d_j = u_r + u_s + 2\rho\cos\theta_j. \tag{8.2}$$

If $n=2m$ *is even, then* $\mathrm{A}^{\overline{\mathrm{K}}}$ *has four representations of degree* 1, $\lambda_1=\nu$, $\lambda_2=\sigma$, λ_3, λ_4 *given by*

$$\lambda_1(a_r)=u_r,\quad \lambda_1(a_s)=u_s,\qquad \lambda_2(a_r)=-1,\quad \lambda_2(a_s)=-1$$
$$\lambda_3(a_r)=u_r,\quad \lambda_3(a_s)=-1,\qquad \lambda_4(a_r)=-1,\quad \lambda_4(a_s)=u_s$$

and $m-1$ *inequivalent irreducible representations* $\pi_1,\ldots,\pi_{m-1}$ *of degree* 2 *given by*

$$\pi_j(a_r)=\mathrm{R}(c_j),\qquad \pi_j(a_s)=\mathrm{S}(d_j).$$

If $n=2m+1$ *is odd, then* $\mathrm{A}^{\overline{\mathrm{K}}}$ *has two representations* $\nu=\lambda_1$, $\sigma=\lambda_2$, *of degree* 1 *given by*

$$\lambda_1(a_r)=u_r,\quad \lambda_1(a_s)=u_s,\qquad \lambda_2(a_r)=-1,\quad \lambda_2(a_s)=-1,$$

and m inequivalent irreducible representations $\pi_1,\ldots,\pi_m$ *of degree* 2 *given by*

$$\pi_j(a_r)=\mathrm{R}(c_j),\qquad \pi_j(a_s)=\mathrm{S}(d_j).$$

These are, up to equivalence, all the irreducible representations of $\mathrm{A}^{\overline{\mathrm{K}}}$.

Proof. — There is nothing to be said about the representations h of degree 1 since the relations (1.8) are satisfied with $\lambda(a_r)$ and $\lambda(a_s)$ in place of a_r and a_s. Let $\theta=\theta_j$, $c=c_j$, $d=d_j$ for $0<j<n/2$, and $\mathrm{R}=\mathrm{R}(c)$, $\mathrm{S}=\mathrm{S}(d)$. It follows from (8.2) that RS and SR have the same eigenvalues, namely $\rho\exp(\pm i\theta)$. Let P be an invertible 2×2 matrix such that

$$\mathrm{P}^{-1}\mathrm{RSP}=\mathrm{D}=\begin{pmatrix}\rho e^{i\theta} & 0\\ 0 & \rho e^{-i\theta}\end{pmatrix} \tag{8.3}$$

and let $\mathrm{R}'=\mathrm{P}^{-1}\mathrm{RP}$, $\mathrm{S}'=\mathrm{P}^{-1}\mathrm{SP}$. Suppose that

$$\mathrm{R}'=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix},\qquad \alpha,\beta,\gamma,\delta\in\overline{\mathrm{K}}. \tag{8.4}$$

Taking account of the fact that $\det\mathrm{R}'=\det\mathrm{R}=-u_r$ one has

$$\mathrm{S}'=\mathrm{R}'^{-1}\mathrm{D}=\begin{pmatrix}-u_r^{-1}\rho e^{i\theta}\delta & u_r^{-1}\rho e^{-i\theta}\beta\\ u_r^{-1}\rho e^{i\theta}\gamma & -u_r^{-1}\rho e^{-i\theta}\alpha\end{pmatrix}. \tag{8.5}$$

From the fact that $\operatorname{Trace}\mathrm{R}'=u_r-1$, $\operatorname{Trace}\mathrm{S}'=u_s-1$ it follows that

$$\begin{aligned}\alpha &= (2i\rho\sin\theta)^{-1}[u_r(u_s-1)+\rho(u_r-1)e^{i\theta}]\\ \delta &= -(2i\rho\sin\theta)^{-1}[u_r(u_s-1)+\rho(u_r-1)e^{-i\theta}].\end{aligned} \tag{8.6}$$

Now if $n=2m$ is even, then $e^{in\theta}=1$, $e^{im\theta}=\pm 1$, and hence $(RS)^m=(SR)^m=\pm\rho^m.I$, where I is the 2×2 identity matrix. On the other hand if $n=2m+1$ is odd, then $u_r=u_s$ and it follows from (8.6) that $\delta=-e^{-i\theta}\alpha$. Thus from (8.5)

$$S'=\begin{pmatrix}\alpha & \beta e^{-i\theta}\\ \gamma e^{i\theta} & \delta\end{pmatrix}.$$

But then since $e^{(2m+1)i\theta}=1$, one has $D^mR'D^{-m}=S'$, $D^mR'=S'D^m$, $(R'S')^mR'=(S'R')^mS'$, and hence $(RS)^mR=(SR)^mS$. Thus in either case the relations (1.8) are satisfied with R and S in place of a_r and a_s. This shows that the representations π_j may be defined as in the statement of the theorem. Let φ_j be the character of π_j. Since $\varphi_j(a_r a_s)=2\rho\cos\theta_j$, distinct j give rise to distinct φ_j and hence to inequivalent representations π_j. An easy computation shows that π_j is irreducible. The sum of the squares of the degrees of the representations we have constructed is $4.1^2+(m-1).2^2=4m=2n$ if n is even, and is $2.1^2+m.2^2=2(2m+1)=2n$ if n is odd. Since $\dim A=2n$ it follows that $A^{\overline{K}}$ is semi-simple, and we have constructed all the irreducible representations of $A^{\overline{K}}$. This completes the proof of the theorem.

Remark **(8.7)**. — The irreducible characters of A determined in theorem (8.1) are all of parabolic type in the sense that the corresponding irreducible characters of W are of parabolic type. In fact, it is obvious that the characters λ of degree 1 are of parabolic type. If $\varphi=\varphi_j$ is an irreducible character of degree 2, let $J=\{r\}$, $e_J=(1+u_r)^{-1}(a_1+a_r)$. One has $\varphi(a_1)=2$, $\varphi(a_r)=u_r-1$, thus $\varphi(e_J)=1$ and φ is of parabolic type (see Lemmas (7.3), (7.4) and the proof of Theorem (7.2)).

9. The reflection representation and its compounds.

Every finite irreducible Coxeter system (W, R) has a natural faithful representation as a group generated by reflections on a finite dimensional Euclidean vector space. Moreover, it is known that the exterior powers of this representation are distinct and irreducible. The main object of this section is to construct the analogues of these representations for the generic ring of (W, R) and to show that the corresponding irreducible characters are all of the parabolic type.

We use the notation of § 1 except that our base field k is the field $\mathbf{Q}$ of rational numbers:

(W, R) = a finite irreducible Coxeter system
$\ell=|R|$
$\mathbf{Q}$ = the field of rational numbers
$\mathfrak{O}=\mathbf{Q}[u_r;\, r\in R]$ as in § 1
K = the quotient field of $\mathfrak{O}$
$\overline{K}$ = an algebraic closure of K
$\mathfrak{O}^*$ = the integral closure of $\mathfrak{O}$ in $\overline{K}$
A = the generic ring of (W, R) over $\mathfrak{O}$ as in § 1.

Let V be an ℓ-dimensional vector space over K having the basis $\{\alpha_r; r\in \mathrm{R}\}$, and put $\mathrm{M}=\mathrm{V}^{\overline{\mathrm{K}}}$. Let $\{c_{r,s}; r, s\in\mathrm{R}\}$ be elements of $\overline{\mathrm{K}}$ such that for all $r, s\in \mathrm{R}$ one has

$$
\begin{aligned}
&c_{r,r}=u_r+1\\
(9.1)\qquad &c_{r,s}=c_{s,r}=0 \quad \text{if } n_{r,s}=2\\
&c_{r,s}c_{s,r}=u_r+u_s+2\sqrt{u_ru_s}\cos\frac{2\pi}{n_{r,s}} \quad \text{if } n_{r,s}>2.
\end{aligned}
$$

By $\sqrt{u_ru_s}$ we mean a fixed square root of u_ru_s in $\mathfrak{O}^*$ such that $\sqrt{u_ru_s}=u_r=u_s$ if $u_r=u_s$.

Lemma **(9.2)**. — *There exists a nonzero symmetric bilinear form* B *on* M, *unique up to a scalar multiple, such that*

$$
(9.3)\qquad c_{r,s}=\frac{(u_r+1)\mathrm{B}(\alpha_r, \alpha_s)}{\mathrm{B}(\alpha_r, \alpha_r)}.
$$

Proof. — We argue by induction on the rank of the Coxeter system (W, R). If $|\mathrm{R}|=1$, the lemma is clear. Suppose $|\mathrm{R}|>1$ and let J be a maximal proper subset of R, with the property that the Coxeter graph of $(\mathrm{W_J}, \mathrm{J})$ is connected. Let $r_0\in\mathrm{R}-\mathrm{J}$ and assume that (9.3) is satisfied for all $r, s\in \mathrm{J}$. Since the Coxeter graph of (W, R) is a tree there exists a unique element s_0 of J such that $n_{r_0,s_0}>2$. Thus we define

$$
\begin{aligned}
&\mathrm{B}(\alpha_{r_0}, \alpha_{s_0})=\mathrm{B}(\alpha_{s_0}, \alpha_{r_0})=(u_{s_0}+1)^{-1}\mathrm{B}(\alpha_{s_0}, \alpha_{s_0})c_{s_0,r_0}\\
(9.4)\qquad &\mathrm{B}(\alpha_{r_0}, \alpha_{r_0})=(u_{r_0}+1)\mathrm{B}(\alpha_{r_0}, \alpha_{s_0})c^{-1}_{r_0,s_0}\\
&\mathrm{B}(\alpha_{r_0}, \alpha_r)=\mathrm{B}(\alpha_r, \alpha_{r_0})=0, \qquad r\in\mathrm{R}, \qquad r\neq r_0, s_0.
\end{aligned}
$$

For each $r\in\mathrm{R}$ let the linear operator X_r on M be defined by

$$
(9.5)\qquad \mathrm{X}_r.\alpha=u_r\alpha-(u_r+1)\frac{\mathrm{B}(\alpha_r, \alpha)}{\mathrm{B}(\alpha_r, \alpha_r)}\alpha_r.
$$

Since α_r is non-isotropic relative to the form B, the space M is the direct sum of the line $\overline{\mathrm{K}}\alpha_r$ and the hyperplane M_r orthogonal to $\overline{\mathrm{K}}\alpha_r$. It is clear from (9.5) that X_r is equal to -1 on $\overline{\mathrm{K}}\alpha_r$ and is equal to u_r on H_r. Since -1 and u_r are the only eigenvalues of X_r, one sees that

$$
(9.6)\qquad \mathrm{X}_r^2=u_r.\mathrm{I}+(u_r-1)\mathrm{X}_r.
$$

Lemma **(9.7)**. — *Let* $\mathrm{M}_{r,s}$ *be the subspace of* M *spanned by* α_r *and* α_s; *then the restriction of* B *to* $\mathrm{M}_{r,s}$ *is nondegenerate.*

Proof. — From (9.3) it suffices to observe that the matrix

$$
\begin{pmatrix} c_{r,r} & c_{r,s}\\ c_{s,r} & c_{s,s}\end{pmatrix}
$$

is non-singular. By (9.1) the determinant of this matrix is equal to $(u_r+1)(u_s+1)$ if $n_{r,s}=2$ and is equal to

$$u_r u_s - 2\sqrt{u_r u_s}\cos\frac{2\pi}{n_{r,s}}+1$$

if $n_{r,s}>2$.

Proposition **(9.8)**. — *There exists a unique representation* $\pi : A^{\overline{K}} \to \operatorname{End} M$ *such that* $\pi(a_r)=X_r$ *for all* $r\in R$.

Proof. — Let r and s be distinct elements of R. By Lemma (9.7) the space M is the direct sum of the subspace $M_{r,s}$ and its orthogonal complement $M_{r,s}^{\perp}$. If $\alpha\in M_{r,s}^{\perp}$, one has $X_r.\alpha=u_r\alpha$, $X_s.\alpha=u_s\alpha$. Thus the relations (1.8) are satisfied by the restrictions of X_r and X_s to $M_{r,s}^{\perp}$ in place of a_r and a_s. Now the matrices of the restrictions of X_r and X_s to $M_{r,s}$ in the basis $\{\alpha_r, \alpha_s\}$ are respectively

$$\begin{pmatrix} -1 & c_{r,s} \\ 0 & u_r \end{pmatrix}, \quad \begin{pmatrix} u_s & 0 \\ c_{s,r} & -1 \end{pmatrix}.$$

Thus by (9.1) and Theorem (8.1) the relations (1.8) are also satisfied by the restrictions of X_r and X_s to $M_{r,s}$ in place of a_r and a_s. Hence X_r and X_s satisfy the relations (1.8) in place of a_r and a_s. This shows that the representation π may be defined as in the statement of the proposition.

We call π the *reflection representation* of $A^{\overline{K}}$ because the specialization $u_r\to 1$ results in the natural representation of W as a group generated by reflections. The next proposition shows that with two exceptions this representation has an $\mathfrak{O}$-form.

Proposition **(9.9)**. — *Let* $\{b_{r,s}; r, s\in R\}$ *be elements of* $\overline{K}$ *which satisfy the equations* (9.1) *with* $b_{r,s}$ *in place of* $c_{r,s}$. *Then there exist* $d_r\in\overline{K}$, $r\in R$, *such that if* $\alpha'_r=d_r\alpha_r$, *then the action of the basis elements* $\{a_r\}$ *of* A *on the new basis of* M *is given by* $a_r.\alpha'_s=u_r\alpha'_s-b_{r,s}\alpha'_r$.

Proof. — Let d_r, $r\in R$ be the elements of $\overline{K}$ to be chosen to satisfy the proposition. Put $\alpha'_r=d_r\alpha_r$. Then $a_r\alpha'_s=u_r\alpha'_s-d_s d_r^{-1}c_{r,s}\alpha'_r$. Thus it suffices to show that the d_r can be chosen so that $b_{r,s}=d_s d_r^{-1}c_{r,s}$. The fact that this is possible follows by induction on the rank of (W, R) as in the proof of Lemma (9.2).

Definition **(9.10)**. — Let N be a $\overline{K}$-vector space and $\rho : A^{\overline{K}}\to\operatorname{End} N$ a representation of $A^{\overline{K}}$. Let $\mathfrak{O}'$ be a subring of $\overline{K}$ which contains $\mathfrak{O}$. We say that ρ has an $\mathfrak{O}'$-form or that ρ is defined over $\mathfrak{O}'$ provided that there exists a basis of N such that the coefficients of the matrices of $\rho(a_w)$, $w\in W$ relative to this basis all lie in $\mathfrak{O}'$.

Corollary **(9.11)**. — *Let* (W, R) *be the Coxeter system of a finite irreducible* (B, N)*-pair. Put* $\mathfrak{O}'=\mathfrak{O}[\sqrt{u_r u_s}]$ *if* (W, R) *is of type* (G_2), $\mathfrak{O}'=\mathfrak{O}[\sqrt{2u_r u_s}]$ *if* (W, R) *is dihedral of order* 16, *and* $\mathfrak{O}'=\mathfrak{O}$ *in all other cases. Then the reflection representation* π *is defined over* $\mathfrak{O}'$.

Proof. — By the theorem of Feit-Higman [8] W is either the Weyl group of a simple complex Lie algebra or is equal to the dihedral group of order 16. If (W, R) is not of type (G_2) or dihedral of order 16, then $n_{r,s}=1, 2, 3,$ or 4, for all $r, s\in R$. If (W, R) is of type (G_2), $n_{r,s}=6$ for $r \neq s$, and if (W, R) is dihedral of order 16, $n_{r,s}=8$ for $r \neq s$. For these values of $n_{r,s}$ equations (9.1) become

$$c_{r,r}=u_r+1$$
$$c_{r,s}=c_{s,r}=0, \quad n_{r,s}=2$$
$$c_{r,s}c_{s,r}=\begin{cases} u_r=u_s, & n_{r,s}=3 \\ u_r+u_s, & n_{r,s}=4 \\ u_r+u_s+\sqrt{u_r u_s}, & n_{r,s}=6 \\ u_r+u_s+\sqrt{2u_r u_s}, & n_{r,s}=8. \end{cases}$$

Thus the corollary follows from Proposition (9.9).

Let $\overset{k}{\wedge}\mathrm{M}$ be the k-fold exterior product of M. We consider $\overset{k}{\wedge}\mathrm{M}$ as a subspace of the exterior algebra of M. For each $r \in \mathrm{R}$ define the linear operator $\mathrm{X}_r^{(k)}$ on $\overset{k}{\wedge}\mathrm{M}$ by

$$\mathrm{X}_r^{(k)}(\xi_1 \wedge \xi_2 \wedge \ldots \wedge \xi_k)=u_r^{-(k-1)}\mathrm{X}_r(\xi_1) \wedge \ldots \wedge \mathrm{X}_r(\xi_k).$$

Proposition **(9.12)**. — *There exists a unique representation* $\pi^{(k)}: \mathrm{A}^{\overline{\mathrm{K}}} \rightarrow \mathrm{End}(\overset{k}{\wedge}\mathrm{M})$ *such that* $\pi^{(k)}(a_r)=\mathrm{X}_r^{(k)}$. *Moreover in case* (W, R) *is the Coxeter system of an irreducible* (B, N)*-pair, the representations* $\pi^{(k)}$ *are defined over the same ring extension* $\mathfrak{O}'$ *of* $\mathfrak{O}$ *as the reflection representation* π (see Corollary (9.11)).

Proof. — Let $\{c_1, \ldots, c_\ell\}$ be the eigenvalues of X_r counted with multiplicity. Then the eigenvalues of $\mathrm{X}_r^{(k)}$ are $\{u_r^{-(k-1)}c_{i_1}c_{i_2}\ldots c_{i_k}\}$ where $(i_1, i_2, \ldots, i_k)$ runs over the set of all sequences of positive integers such that $1 \leq i_1 < i_2 < \ldots < i_k \leq \ell$. Since the -1 and u_r-eigenspaces of X_r are 1 and $\ell-1$ dimensional respectively it follows that -1 and u_r are also the only eigenvalues of $\mathrm{X}_r^{(k)}$, the -1 eigenspace of $\mathrm{X}_r^{(k)}$ being $\binom{\ell-1}{k-1}$ dimensional and the u_r-eigenspace of $\mathrm{X}_r^{(k)}$ being $\binom{\ell-1}{k}$ dimensional. In particular, we have

$$(\mathrm{X}_r^{(k)})^2=u_r.\mathrm{I}+(u_r-1)\mathrm{X}_r^{(k)}.$$

It is immediate from the definition that $(\mathrm{X}_r^{(k)}\mathrm{X}_s^{(k)}\cdots)_{n_{r,s}}=(\mathrm{X}_s^{(k)}\mathrm{X}_r^{(k)}\cdots)_{n_{r,s}}$ because this relation is satisfied by $\{\mathrm{X}_r\}$. Thus the relations (1.8) are satisfied with $\mathrm{X}_r^{(k)}$ and $\mathrm{X}_s^{(k)}$ in place of a_r and a_s. This shows that the representations $\pi^{(k)}$ may be defined as in the statement of the proposition. By Corollary (9.11) we may choose a basis $\{\beta_r; r \in \mathrm{R}\}$ of M such that $\mathrm{X}_r(\beta_s)=u_r\beta_s-b_{r,s}\beta_r$ for all $r, s \in \mathrm{R}$ and $b_{r,s} \in \mathfrak{O}'$. For convenience of notation let $\mathrm{R}=\{1, 2, \ldots, \ell\}$. Then the set $\mathscr{B}=\{\beta_{i_1} \wedge \beta_{i_2} \wedge \ldots \wedge \beta_{i_k}\}$ forms a basis of $\overset{k}{\wedge}\mathrm{M}$, where $1 \leq i_1 < i_2 < \ldots < i_k \leq \ell$. Now

$$\mathrm{X}_r^{(k)}(\beta_{i_1} \wedge \ldots \wedge \beta_{i_k})=u_r^{-(k-1)}\{(u_r\beta_{i_1}+b_{r,i_1}\beta_r) \wedge \ldots \wedge (u_r\beta_{i_k}+b_{r,i_k}\beta_r)\}$$
$$=u_r\beta_{i_1} \wedge \ldots \wedge \beta_{i_k}+\sum_{j=1}^{k}(-1)^{j+1}b_{r,i_j}\beta_r \wedge \beta_{i_1} \wedge \ldots \wedge \check{\beta}_{i_j} \wedge \ldots \wedge \beta_{i_k},$$

where the notation $\check{\beta}_{i_j}$ means that the factor β_{i_j} has been omitted. Since $\beta_r \wedge \beta_{i_1} \wedge \ldots \wedge \check{\beta}_{i_j} \wedge \ldots \wedge \beta_{i_k}$ is either equal to zero or is $\pm$ an element of $\mathscr{B}$, the representation $\pi^{(k)}$ is defined over $\mathfrak{O}'$. This completes the proof of the proposition.

We call the $\pi^{(k)}$ $(0 \leq k \leq \ell)$ the *compounds* of the reflection representation $\pi = \pi^{(1)}$. Note that $\pi^{(0)} = \nu$ and $\pi^{(\ell)} = \sigma$, where $\sigma(a_w) = (-1)^{\ell(w)}$ for all $w \in \mathrm{W}$.

Theorem **(9.13)**. — *The bilinear form* B *is nondegenerate. The representations* $\pi^{(k)}$ *are irreducible and pairwise inequivalent.*

Proof. — We argue by induction on the rank of (W, R). If $|\mathrm{R}| = 1$ the conclusion of the theorem is obviously valid. Assume $|\mathrm{R}| > 1$, let J be a maximal connected proper subset of R and put $\{r_0\} = \mathrm{R} - \mathrm{J}$. Let $\mathrm{M_J}$ be the subspace of M having the basis $\{\alpha_r; r \in \mathrm{J}\}$ and let $\mathrm{A}^{\overline{\mathrm{K}}}_{\mathrm{J}}$ be the subalgebra of $\mathrm{A}^{\overline{\mathrm{K}}}$ generated by $\{a_r; r \in \mathrm{J}\}$. By the induction assumption the restriction of B to $\mathrm{M_J}$ is nondegenerate so that M is the direct sum of the subspace $\mathrm{M_J}$ and its orthogonal complement $\mathrm{M_J^\perp}$ relative to B. $\mathrm{M_J^\perp}$ is one dimensional and if $\alpha \in \mathrm{M_J^\perp}$ we have $a_r . \alpha = u_r \alpha$ for all $r \in \mathrm{J}$. Now

$$(9.14) \qquad \overset{k}{\wedge}\mathrm{M} = \overset{k}{\wedge}\mathrm{M_J} \oplus [(\overset{k-1}{\wedge}\mathrm{M_J}) \wedge \mathrm{M_J^\perp}].$$

It is clear that $\overset{k-1}{\wedge}\mathrm{M_J} \wedge \mathrm{M_J^\perp}$ and $\overset{k-1}{\wedge}\mathrm{M_J}$ are isomorphic as $\mathrm{A}^{\overline{\mathrm{K}}}_{\mathrm{J}}$-modules. By the induction assumption $\overset{k}{\wedge}\mathrm{M_J}$ and $\overset{k-1}{\wedge}\mathrm{M_J}$ are distinct and irreducible as $\mathrm{A}^{\overline{\mathrm{K}}}_{\mathrm{J}}$-modules. Thus as an $\mathrm{A}^{\overline{\mathrm{K}}}$-module either $\overset{k}{\wedge}\mathrm{M}$ is irreducible or (9.14) is the decomposition of $\overset{k}{\wedge}\mathrm{M}$ into distinct irreducible $\mathrm{A}^{\overline{\mathrm{K}}}$-components. But one can easily see that $\overset{k}{\wedge}\mathrm{M_J}$ is not stable under the action of $\pi^{(k)}(a_{r_0})$. Hence $\overset{k}{\wedge}\mathrm{M}$ is irreducible. The proof of proposition (9.12) shows that the dimension of the u_r-eigenspace of $\pi^{(k)}(a_r)$ is $\binom{\ell-1}{k}$. Thus if $\overset{k}{\wedge}\mathrm{M}$ and $\overset{k'}{\wedge}\mathrm{M}$ are $\mathrm{A}^{\overline{\mathrm{K}}}$-isomorphic we must have $\binom{\ell}{k} = \binom{\ell}{k'}$ and $\binom{\ell-1}{k} = \binom{\ell-1}{k'}$ whence $k = k'$. The form B is nondegenerate for if there exists $\alpha \in \mathrm{M}$ such that $\mathrm{B}(\alpha, \alpha_r) = 0$ for all $r \in \mathrm{R}$, then $a_r . \alpha = u_r \alpha$ for all $r \in \mathrm{R}$ which contradicts the irreducibility of M. This completes the proof of the theorem.

Theorem **(9.15)**. — *Let* $\varphi^{(k)}$ *be the character of* $\pi^{(k)}$ $(0 \leq k \leq \ell)$. *Let* $\mathrm{J} \subset \mathrm{R}$, ε, $\mathrm{E} = \nu(\varepsilon)$ *be as in* § 2. *Then* $\varphi^{(k)}(\varepsilon) = m\mathrm{E}$ *where* $m = \binom{|\mathrm{R}-\mathrm{J}|}{k}$.

Proof. — Let $\mathrm{M_J}$, $\mathrm{A}^{\overline{\mathrm{K}}}_{\mathrm{J}}$ be as in the proof of Theorem (9.13). By the proof of Lemma (7.3) $\eta = \mathrm{E}^{-1}\varepsilon$ is an idempotent. Moreover, it is clear from Lemma (2.10) that $\pi^{(k)}(\eta)$ is the projection of $\overset{k}{\wedge}\mathrm{M}$ onto the subspace consisting of all vectors ξ such that $a . \xi = \nu(a)\xi$ for all $a \in \mathrm{A}^{\overline{\mathrm{K}}}_{\mathrm{J}}$. Thus it suffices to prove that the dimension of this subspace is m. Let $\mathrm{N} = \mathrm{M_J}$ and $\mathrm{P} = \mathrm{M_J^\perp}$, the orthogonal complement of $\mathrm{M_J}$ relative to the form B. Since B is nondegenerate on $\mathrm{N} \times \mathrm{N}$, $\mathrm{M} = \mathrm{N} \oplus \mathrm{P}$, and thus

$$(9.16) \qquad \overset{k}{\wedge}\mathrm{M} = \bigoplus_{i=0}^{k} ((\overset{i}{\wedge}\mathrm{N}) \wedge (\overset{k-i}{\wedge}\mathrm{P})).$$

Since P affords $|\mathrm{R}-\mathrm{J}|$ copies of the one-dimensional representation $\nu = \nu_{\mathrm{J}}$ of $\mathrm{A}^{\overline{\mathrm{K}}}_{\mathrm{J}}$ it is easily seen that $(\overset{i}{\wedge}\mathrm{N}) \wedge (\overset{k-i}{\wedge}\mathrm{P})$ and the direct sum of m_i copies of $\overset{i}{\wedge}\mathrm{N}$, $(\overset{i}{\wedge}\mathrm{N})^{m_i}$, are equivalent as $A_J^{\bar K}$-modules, where $m_i = \binom{|R-J|}{k-i}$. We assert that $\overset{i}{\wedge} N$ does not contain an element ξ such that $a.\xi = \nu(a)\xi$ for all $a \in A_J^{\bar K}$ if $i > 0$. Indeed, identify J with the corresponding set of points of the Coxeter graph of (W_J, J) and let $J = J_1 \cup J_2 \cup \ldots \cup J_t$ be the decomposition of J into pairwise disjoint connected subsets. Then

$$\overset{i}{\wedge} M_J = \bigoplus (\overset{i_1}{\wedge} M_{J_1}) \wedge (\overset{i_2}{\wedge} M_{J_2}) \wedge \ldots \wedge (\overset{i_t}{\wedge} M_{J_t}),$$

the summation being extended over all sequences $(i_1, i_2, \ldots, i_t)$ of non negative integers such that $i_1 + i_2 + \ldots i_t = i$. Suppose that $\xi = \xi_{i_1} \wedge \xi_{i_2} \wedge \ldots \wedge \xi_{i_t}$, $(\xi_{i_j} \in \overset{i_j}{\wedge} M_{J_j})$ is a nonzero vector. Then

$$a_r . \xi = u_r^{-(t-1)} a_r . \xi_{i_1} \wedge a_r . \xi_{i_2} \wedge \ldots \wedge a_r . \xi_{i_t},$$

for all $r \in R$. Thus if $a_r . \xi = u_r \xi$ for all $r \in J$, it follows from the pairwise orthogonality of the M_{J_j} that $a_r . \xi_{i_j} = u_r \xi_{i_j}$ for all $r \in J$. But then by Theorem (9.13) we must have $i_j = 0$ $(1 \leq j \leq t)$, and hence $i = 0$. Thus the subspace of $\overset{k}{\wedge} M$ consisting of all vectors ξ such that $a.\xi = \nu(a)\xi$ for all $a \in A_J^{\bar K}$ is just the zeroth summand of (9.16) which has dimension $m = \binom{|R-J|}{k}$. This completes the proof of the theorem.

Corollary **(9.17)**. — *Let* (G, B, N, R) *be a finite* (B, N)*-pair whose Coxeter system is* (W, R). *Let* $\zeta^{(k)}$ (*respectively* $\zeta_0^{(k)}$) *be the irreducible character of* G (*respectively* W) *corresponding to* $\varphi^{(k)}$ *in the sense of Theorem* (7.2). *Then one has*

$$\textbf{(9.18)} \qquad (\zeta^{(k)}, (1_{G_J})^G) = (\zeta_0^{(k)}, (1_{W_J})^W) = \binom{|R-J|}{k}.$$

In particular, these characters are all of parabolic type.

Proof. — (9.18) is immediate from Theorems (7.2) and (9.15). To see that the characters are parabolic type it suffices to take $|J| = |R| - k$.

If (G, B, N, R) is a finite (B, N) pair whose Coxeter system is (W, R), we call $\zeta = \zeta^{(1)}$ the reflection character of G. ζ is the irreducible character of G which corresponds to the *reflection character* φ of the generic ring $A^{\bar K}$ of (W, R) in the sense of Theorem (7.2). We have computed the generic degree d_φ of the reflection character by using the fact that φ is of parabolic type. The details of the method used to obtain the formula for d_φ will appear elsewhere. It turns out that $d_\varphi \in \mathbf{Q}[u]$ for each system $\mathscr{S}$ except when $\mathscr{S}$ is of type (F_4^1) in which case $d_\varphi \in \mathbf{Q}(\sqrt{2u})$. If $G(q) \in \mathscr{S}$, then $d_\varphi(q)$ is the degree of the reflection character of $G(q)$, and $d_\varphi(1)$ is the degree of the reflection character of the Weyl group of $G(q)$.

It is a curious fact that if $G(q)$ is an (untwisted) Chevalley group, and if the Coxeter graph of the Weyl group W of $G(q)$ is simply laced, then $d_\varphi(u) = u^{m_1} + u^{m_2} + \ldots + u^{m_\ell}$ where $\{m_1, m_2, \ldots, m_\ell\}$ are the exponents of W.

The following is the list of the generic degrees d_φ for the various systems $\mathscr{S}$ of (B, N)-pairs. The notations for the groups belonging to the different systems is taken from Carter [4], p. 239.

$\underline{G(q)}$	$\underline{d_{\varphi}(u)}$
$A_{\ell}(q)$	$\dfrac{u(u^{\ell}-1)}{u-1}$
$B_{\ell}(q)$	$\dfrac{u(u^{\ell}-1)(u^{\ell-1}+1)}{2(u-1)}$
$D_{\ell}(q)$	$\dfrac{u(u^{\ell}-1)(u^{\ell-2}+1)}{(u-1)(u+1)}$
$E_6(q)$	$\dfrac{u(u^4+1)(u^9-1)}{(u-1)}$
$E_7(q)$	$\dfrac{u(u^6+1)(u^{14}-1)}{(u^2+1)(u^2-1)}$
$E_8(q)$	$\dfrac{u(u^{10}+1)(u^{24}-1)}{u^6-1}$
$F_4(q)$	$\dfrac{u(u^3+1)^2(u^4+1)}{2}$
$G_2(q)$	$\dfrac{u(u+1)^2(u^2+u+1)}{6}$
$A^1_{2\ell}(q^2)$	$\dfrac{u^3(u^{2\ell}-1)(u^{2\ell-1}+1)}{(u+1)(u^2-1)}$
$A^1_{2\ell-1}(q^2)$	$\dfrac{u^2(u^{2\ell}-1)(u^{2\ell-3}+1)}{(u+1)(u^2-1)}$
$D^1_{\ell}(q^2)$	$\dfrac{u^2(u^{\ell-1}-1)(u^{\ell-1}+1)}{(u-1)(u+1)}$
$E^1_6(q^2)$	$\dfrac{u^2(u^4+1)(u^5+1)(u^6+1)}{(u+1)}$
$D^2_4(q^3)$	$\dfrac{u^3(u+1)(u^6-1)}{2(u-1)}$
$B^1_2(q)$ $(q=2^{2n+1})$	u^2
$G^1_2(q)$ $(q=3^{2n+1})$	u^3
$F^1_4(q)$ $(q=2^{2n+1})$	$\dfrac{u^2(u+1)(u^2+1)(u^9+u^6+u^3+1)}{4(u+\sqrt{2u}+1)(u^3-u\sqrt{2u}+1)}$

10. The one-dimensional representation of the generic ring.

Let (W, R) be a finite irreducible Coxeter system and A the generic ring of (W, R) over $\mathfrak{D}$ as in § 1. We discuss in this section all the one-dimensional representations of $A^{\bar{K}}$ and give formulas for the generic degrees of these representations.

Lemma (**10.1**). — *Let* (W, R) *be a Coxeter system and let* $\sim$ *be the equivalence relation on* R *defined by* $r \sim s$ *if and only if there exists a sequence* $r = r_1, r_2, \ldots, r_p = s$ *of elements of* R *such that* $n_{r_i, r_{i+1}}$ *is odd* $(1 \leq i \leq p-1)$; *then* $r \sim s$ *if and only if r is conjugate to s in* W. *Moreover, if t is the number of conjugacy classes of the elements of* R *in* W, *then* $|W/W'| = 2^t$ *where* W′ *is the commutator subgroup of* W (see [3], p. 12).

Proof. — Let $R_1, \ldots, R_m$ be the equivalence classes of R modulo $\sim$. If $r, s \in R$ are such that $n_{r,s} = 2k+1$ is odd, the equation $(rs\ldots)_{n_{r,s}} = (sr\ldots)_{n_{r,s}}$ may be written in the form $(rs)^k r = s(rs)^k$. Thus $r \sim s$ implies that r is conjugate to s in W and consequently $t \leq m$. For each i $(1 \leq i \leq m)$ let

$$\varphi_i(r) = \begin{cases} -1, & r \in R_i \\ +1 & r \notin R_i. \end{cases}$$

Then one easily sees that $(\varphi_i(r)\varphi_i(s)\ldots)_{n_{r,s}} = (\varphi_i(s)\varphi_i(r)\ldots)_{n_{r,s}}$ so that φ_i defines a one-dimensional representation of W. Since $\varphi_1^{\varepsilon_1}\varphi_2^{\varepsilon_2}\ldots\varphi_m^{\varepsilon_m}$ are distinct ($\varepsilon_i = 0$ or 1, $1 \leq i \leq m$), it follows that $2^m \leq |W/W'| \leq 2^t$. Hence $t = m$ and $|W/W'| = 2^t$ as asserted.

By the preceding lemma the conjugacy classes of the elements of R in W can be found by examining the Coxeter graph of (W, R). If (W, R) is the Coxeter system of a finite irreducible (B, N)-pair, there is only one such class if the diagram is simply laced, while if the diagram is not simply laced there are two such classes corresponding to the points of Coxeter graph which lie on opposite sides of a multiple bond. Let R_1, R_2 be the two classes. Put $R_2 = \emptyset$ if there is only one such class. Let $u_r = u_1$ for all $r \in R_1$ and $u_r = u_2$ for all $r \in R_2$.

Proposition (**10.2**). — *Let the notation be as above. If* R_2 *is empty,* $A^{\bar{K}}$ *has exactly two representations* ν *and* σ *of degree* 1. *If* R_2 *is nonempty,* $A^{\bar{K}}$ *has exactly four representations* ν, σ, σ_1 *and* σ_2 *of degree* 1. *These representations are given by*

$$\begin{aligned} \nu(a_r) &= u_r, & r \in R \\ \sigma(a_r) &= -1, & r \in R \\ \sigma_1(a_r) &= \begin{cases} u_1, & r \in R_1 \\ -1, & r \in R_2 \end{cases} \\ \sigma_2(a_r) &= \begin{cases} -1, & r \in R_1 \\ u_2, & r \in R_2. \end{cases} \end{aligned} \tag{10.3}$$

Proof. — The fact that the representations ν, σ, σ_1, σ_2 exist follows immediately from Lemma (10.1) together with the generators and relations (1.8). Lemma (10.1) also implies that the number of representations of W of degree 1 is equal to 2 or 4 according to whether R_2 is empty or not. But $A^{\bar{K}}$ has the same numerical invariants as $\mathbf{Q}W$, the group algebra of W over the rational number field. Hence we have described all the representations of $A^{\bar{K}}$ of degree 1.

Keeping the above notation, let $w \in W$ and let $w = r_1 r_2 \dots r_p$ be a reduced expression for w $(\ell(w)=p)$. Define the functions ℓ_1 and ℓ_2 on W by

$$\text{(10.4)} \qquad \begin{aligned} \ell_1(w) &= |\{i \mid 1 \le i \le p,\ r_i \in R_1\}| \\ \ell_2(w) &= |\{i \mid 1 \le i \le p,\ r_i \in R_2\}|. \end{aligned}$$

Corollary **(10.5)**. — *The integers $\ell_1(w)$ and $\ell_2(w)$ depend only on w, not on the choice of reduced expression for w. Moreover, one has:*

$$\text{(10.6)} \qquad \begin{aligned} \nu(a_w) &= u_1^{\ell_1(w)} u_2^{\ell_2(w)} \\ \sigma(a_w) &= (-1)^{\ell(w)} \\ \sigma_1(a_w) &= u_1^{\ell_1(w)}(-1)^{\ell_2(w)} \\ \sigma_2(a_w) &= (-1)^{\ell_1(w)} u_2^{\ell_2(w)}. \end{aligned}$$

Proof. — (10.6) is obvious from (10.3) and (10.4). The expressions for σ_1 and σ_2 show that $\ell_1(w)$ and $\ell_2(w)$ do not depend on the choice of reduced expression for w.

For any representation φ of $A^{\bar{K}}$ of degree 1, put

$$\text{(10.7)} \qquad g_\varphi(u_1, u_2) = \{\sum_{w \in W} \nu(a_w)^{-1} \varphi(a_w) \varphi(a_{w^{-1}})\}^{-1} \sum_{w \in W} \nu(a_w).$$

Thus if $\mathscr{S}$ is a system of (B, N)-pairs as in Definition (5.1), one has by (5.8) that the generic degree d_φ is given by $d_\varphi(u) = g_\varphi(u^{c_1}, u^{c_2})$, where we have put $c_1 = c_r$, $r \in R_1$ and $c_2 = c_r$, $r \in R_2$.

Definition **(10.8)**. — *Put* $P(u_1, u_2) = \sum_{w \in W} \nu(a_w)$. *We call* P *the* Poincaré polynomial *of the Coxeter system* (W, R).

Proposition **(10.9)**. — *Let* ν, σ, σ_1, σ_2 *be as in* (10.3); *then one has*

$$\begin{aligned} g_\nu(u_1, u_2) &= P(u_1, u_2)/P(u_1, u_2) = 1 \\ g_\sigma(u_1, u_2) &= P(u_1, u_2)/P(u_1^{-1}, u_2^{-1}) \\ g_{\sigma_1}(u_1, u_2) &= P(u_1, u_2)/P(u_1, u_2^{-1}) \\ g_{\sigma_2}(u_1, u_2) &= P(u_1, u_2)/P(u_1^{-1}, u_2). \end{aligned}$$

Proof. — These formulas follow directly upon substituting (10.6) in (10.7).

We can calculate g_σ explicitly from these formulas: Let $w \in W$ be arbitrary and let w_0 be the unique element of maximal length in W, then $\ell(w_0) = \ell(w_0 w^{-1}) + \ell(w)$ so that $\nu(a_{w_0})\nu(a_w)^{-1} = \nu(a_{w_0 w^{-1}})$. Thus

$$\nu(a_{w_0}) P(u_1^{-1}, u_2^{-1}) = \nu(a_{w_0}) \sum_{w \in W} \nu(a_w)^{-1} = \sum_{w \in W} \nu(a_{w_0 w^{-1}}) = P(u_1, u_2),$$

and by Proposition (10.9), $g_\sigma(u_1, u_2) = \nu(a_{w_0})$. Now one knows [19] that $\ell_i(w_0) = \ell_i h/2$ where $\ell_i = |R_i|$, $i = 1, 2$ and h is the Coxeter number of (W, R). Thus

$$g_\sigma(u_1, u_2) = (u_1^{\ell_1} u_2^{\ell_2})^{h/2}.$$

We give a list below of d_{σ_1}, d_{σ_2}, d_σ for each system $\mathscr{S}$ of (B, N)-pairs. The determination of d_{σ_1} and d_{σ_2} depends upon knowledge of the Poincaré polynomial P. The calculation of the Poincaré polynomial is done using the combinatorial method at the end of § 3. We shall omit the details of this calculation.

$A_\ell(q)$ $\quad d_\sigma = u^{\ell(\ell+1)/2}$

$B_\ell(q)$ $\quad d_\sigma = u^{\ell^2}$

$$d_{\sigma_1} = \frac{u(u^{\ell-1}+1)(u^\ell+1)}{2(u+1)}$$

$$d_{\sigma_2} = \frac{u^{(\ell-1)^2}(u^{\ell-1}+1)(u^\ell+1)}{2(u+1)}$$

$D_\ell(q)$ $\quad d_\sigma = u^{\ell(\ell-1)}$

$E_6(q)$ $\quad d_\sigma = u^{36}$

$E_7(q)$ $\quad d_\sigma = u^{63}$

$E_8(q)$ $\quad d_\sigma = u^{120}$

F_4 $\quad d_\sigma = u^{24}$

$$d_{\sigma_1} = \frac{u^4(u^2+1)(u^3+1)^2(u^4+1)(u^6+1)}{8(u+1)^2}$$

$$d_{\sigma_2} = d_{\sigma_1}$$

$G_2(q)$ $\quad d_\sigma = u^6$

$$d_{\sigma_1} = \frac{u(u^4+u^2+1)}{3}$$

$$d_{\sigma_2} = d_{\sigma_1}$$

$A^1_{2\ell}(q^2)$ $\quad d_\sigma = u^{\ell(2\ell+1)}$

$$d_{\sigma_1} = \frac{u^4(u^{2\ell-3}+1)(u^{2\ell-1}+1)(u^{2\ell+1}+1)}{(u+1)^2(u^3+1)}$$

$$d_{\sigma_2} = \frac{u^{2\ell^2-5\ell+4}(u^{2\ell-3}+1)(u^{2\ell-1}+1)(u^{2\ell+1}+1)}{(u+1)^2(u^3+1)}$$

$A^1_{2\ell-1}(q^2)$ $\quad d_\sigma = u^{\ell(2\ell-1)}$

$$d_{\sigma_1} = \frac{u(u^{2\ell-1}+1)}{(u+1)}$$

$$d_{\sigma_2} = \frac{u^{(\ell-1)(2\ell-1)}(u^{2\ell-1}+1)}{(u+1)}$$

$\mathrm{D}^1_\ell(q^2)$

$$d_\sigma = u^{(\ell-1)(\ell-2)}$$

$$d_{\sigma_1} = \frac{u^3(u^{\ell-3}+1)(u^{\ell-2}+1)(u^{\ell-1}+1)(u^\ell+1)}{2(u+1)^2(u^2+1)}$$

$$d_{\sigma_2} = \frac{u^{\ell^2-5\ell+7}(u^{\ell-3}+1)(u^{\ell-2}+1)(u^{\ell-1}+1)(u^\ell+1)}{2(u+1)^2(u^2+1)}$$

$\mathrm{E}^1_6(q^2)$

$$d_\sigma = u^{36}$$

$$d_{\sigma_1} = \frac{u^{15}(u^4+1)(u^5+1)(u^6+1)(u^9+1)}{2(u+1)(u^2+1)(u^3+1)}$$

$$d_{\sigma_2} = \frac{u^3(u^4+1)(u^5+1)(u^6+1)(u^9+1)}{2(u+1)(u^2+1)(u^3+1)}$$

$\mathrm{D}^2_4(q^3)$

$$d_\sigma = u^{12}$$

$$d_{\sigma_1} = \frac{u^7(u^6+1)}{(u^2+1)}$$

$$d_{\sigma_2} = \frac{u(u^6+1)}{(u^2+1)}$$

$\mathrm{B}^1_2(q)$

$$d_\sigma = u^2$$

$\mathrm{G}^1_2(q)$

$$d_\sigma = u^3$$

$\mathrm{F}^1_4(q)$

$$d_\sigma = u^{12}$$

$$d_{\sigma_1} = \frac{u^5(u^6+1)(u^3+1)}{(u+1)(u^2+1)}$$

$$d_{\sigma_2} = \frac{u(u^6+1)(u^3+1)}{(u+1)(u^2+1)}$$

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