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HECKE ALGEBRAS, $U_q sl_n$, AND THE DONALD–FLANIGAN CONJECTURE FOR S_n

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Abstract. The Donald–Flanigan conjecture asserts that the integral group ring $\mathbb{Z}G$ of a finite group G can be deformed to an algebra A over the power series ring $\mathbb{Z}[[t]]$ with underlying module $\mathbb{Z}G[[t]]$ such that if p is any prime dividing $\#G$ then $A \otimes_{\mathbb{Z}[[t]]} \overline{\mathbb{F}_p((t))}$ is a direct sum of total matric algebras whose blocks are in natural bijection with and of the same dimensions as those of $\mathbb{C}G$. We prove this for $G = S_n$ using the natural representation of its Hecke algebra $\mathcal{H}$ by quantum Yang-Baxter matrices to show that over $\mathbb{Z}[q]$ localized at the multiplicatively closed set generated by q and all $i_{q^2} = 1+q^2+q^4+\cdots+q^{2(i-1)}, i = 1, 2, \ldots, n$, the Hecke algebra becomes a direct sum of total matric algebras. The corresponding "canonical" primitive idempotents are distinct from Wenzl's and in the classical case ($q = 1$), from those of Young.

1. Introduction

The original Donald–Flanigan conjecture [DF, 1974] asserts that if G is a group of finite order divisible by a prime p then $\mathbb{F}_pG$ can be deformed to an $\mathbb{F}_p[[t]]$–algebra which becomes separable when coefficients are extended to the Laurent series field $\mathbb{F}_p((t))$. Such an algebra will be called a *solution* to the Donald–Flanigan problem for *G at the prime p*. Extending coefficients to the algebraic closure of $\mathbb{F}_p((t))$ gives an algebra which is a direct sum of total matric algebras called its "blocks". Different solutions to the Donald–Flanigan problem at the same prime may have different block sizes. For example, if $G = \mathbb{Z}/2 \times \mathbb{Z}/2$ then $\mathbb{F}_2G \cong \mathbb{F}_2[x]/(x^2) \otimes_{\mathbb{F}_2} \mathbb{F}_2[y]/(y^2)$, which deforms to $\mathbb{F}_2[[t]][x]/(x^2+tx)\otimes_{\mathbb{F}_2[[t]]}\mathbb{F}_2[y]/(y^2+ty)$. This commutative algebra becomes separable when coefficients are extended to $\mathbb{F}_2((t))$, and its blocks are all one-dimensional. These are the same as for the complex group algebra $\mathbb{C}G$. On the other hand, there is also a non-commutative solution. Let $\mathbb{F}_2\langle x, y\rangle$ denote the non–commutative polynomial ring in two variables over the field of two elements. Then $\mathbb{F}_2G \cong \mathbb{F}_2\langle x, y\rangle/(x^2, y^2, xy - yx)$, which deforms to $\mathbb{F}_2\langle x, y\rangle/(x^2, y^2, xy - yx - t)$. Over $\mathbb{F}_2((t))$ the latter is just a 2×2 total matric algebra whose blocks do not correspond to those of $\mathbb{C}G$. To insure the desired correspondence we make the following

Definition. A global solution to the Donald–Flanigan problem for a finite group G is a deformation A_t of the integral group ring $\mathbb{Z}G$, together with a multiplicatively

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closed subset S of the coefficient ring $\mathbb{Z}[[t]]$ such that i) $S^{-1}A_t$ is separable over $S^{-1}\mathbb{Z}[[t]]$, and ii) S contains no rational prime dividing the order $\#G$ of G.

If ii) is weakened to allow that S contain certain rational primes $p_1, p_2, \ldots$ dividing $\#G$, then we say that we have a global solution *away from* $p_1, p_2, \ldots$. When S is understood, we generally refer to the $\mathbb{Z}[[t]]$ algebra A_t itself as the global solution.

Now the composite map $\mathbb{Z}[[t]] \to \mathbb{F}_p((t))$ consisting of reduction modulo p and inversion of t (which may be done in either order) can be factored through $S^{-1}\mathbb{Z}[[t]]$. Therefore, the separable algebras $\mathbb{F}_p((t)) \otimes_{S^{-1}\mathbb{Z}[[t]]} S^{-1}A_t$ for the various primes p dividing $\#G$ are all quotients of the same separable algebra $S^{-1}\mathbb{Z}((t)) \otimes_{S^{-1}\mathbb{Z}[[t]]} S^{-1}A_t$ and their blocks are therefore in natural correspondence with the blocks of the latter. But the latter has characteristic 0, so extending its coefficients to include $\mathbb{C}$ we see that a global solution not only gives a "local" solution at each prime (or in the weaker case, away from primes $p_1, p_2, \ldots,$) but naturally identifies the blocks of each of the local solutions which it generates with those of the complex group algebra $\mathbb{C}G$. It is a scheme-theoretic solution generalizing in the strongest possible way Maschke's theorem, that a group algebra kG over a ring k is separable whenever the order of G is invertible in k – provided it exists! Note that the concept of a global solution is important even when G is a p–group.

Our main result here is that the Donald–Flanigan problem has a global solution for every symmetric group S_n. In the proof we are led to reexamine the relation between (non-increasing) partititons $\mathfrak{p}$ of n and representations of S_n. In the "quantized" case, where the group algebra of S_n is replaced by its Hecke algebra $\mathcal{H}_n$, this yields the following: To each $\mathfrak{p}$ there is canonically associated (i.e., without choice of any Young tableau) a simple module with a non-degenerate, symmetric bilinear form (for brevity called an inner product). This has a canonical orthogonal basis indexed by the various Young tableaux associated to $\mathfrak{p}$. The rank of this "canonical" module is the number of such diagrams. That these canonical modules are up to isomorphism all the simple modules and that the Hecke algebra operates as the full ring of linear endomorphisms of each will be evident. In the unquantized case our procedure produces primitive idempotents which, although indexed by Young tableaux and necessarily conjugate to the classical ones computed using Young symmetrizers, are distinct from them.

By Maschke's theorem, kS_n will be separable whenever k contains $\mathbb{Z}_{1,n} := \mathbb{Z}[1/n!]$. The symmetric group is unusual, however, in that $\mathbb{Z}_{1,n}S_n$ is not only separable but already a direct sum of total matric algebras over $\mathbb{Z}_{1,n}$. We need the quantum analog. Set $\mathbb{Z}_q := \mathbb{Z}[q, q^{-1}]$ where q is a variable. The **Hecke algebra** $\mathcal{H}_n$ of S_n is a free module of rank $n!$ over $\mathbb{Z}_q$ with basis elements T_w indexed by the elements $w \in S_n$ and multiplication given as follows: The *length* $\ell(w)$ is the number of factors in a shortest expression of w as a product of generators $s_i := (i, i+1), i = 1, \ldots, n-1$, of S_n. Multiplication is determined by setting (i) $T_sT_w = T_{sw}$ if s is one of these generators and $w \in S_n$ is an element with $\ell(sw) > \ell(w)$, and (ii) $T_s^2 = (q - q^{-1})T_s + 1$. This implies that $T_sT_w = (q - q^{-1})T_w + T_{sw}$ when $\ell(sw) < \ell(w)$. (Often instead of (ii) one takes $T_s^2 = (q-1)T_s + q$, cf. [Hu]; that definition can be transformed into ours by substituting q^2 for q and dividing the generators by the new q. The present form is more useful when dealing with quantum groups.) When necessary to indicate the dependence on the parameter q we may write $\mathcal{H}_n(q)$. Writing $1 + t$ for q one sees that $\mathcal{H}_n(1 + t)$ is a deformation of $\mathbb{Z}S_n$.

Set $i_q := (1-q^i)/(1-q)$ and similarly $i_{q^2} := (1-q^{2i})/(1-q^2)$. These are the "$q$–numbers". For i an integer they are polynomials in q (or q^{-1} for $i<0$). We call i the "argument " and q the "parameter" of i_q. Set $n_{q^2}! := n_{q^2}(n-1)_{q^2}\dots 2_{q^2}$ and $\mathbb{Z}_{q,n} = \mathbb{Z}[q, q^{-1}, 1/n_{q^2}!]$. We will prove the following, from which one recovers the corresponding assertion for $\mathbb{Z}[1/n!]S_n$ by letting $q \to 1$.

Theorem 1.1. *Over $\mathbb{Z}_{q,n}$ the Hecke algebra $\mathcal{H}_n$ becomes a direct sum of total matric algebras.*

With our definition of a global solution we then have

Corollary 1.2. *Setting $q = 1+t$, the Hecke algebra $\mathcal{H}_n(1+t)$ together with the multiplicatively closed subset of $\mathbb{Z}[[t]]$ generated by $1/n_{q^2}! = 1/n_{(1+t)^2}!$ is a global solution to the Donald–Flanigan problem for the symmetric group S_n.*

Another immediate corollary to Theorem 1.1 is that $\mathcal{H}_n(q)$ "splits", i.e., becomes a direct sum of total matric algebras, over any field k in which q is not a $2i$–th root of unity for any $i = 2, \dots n$. One will find the proposition in this form in Dipper and James [DJ]. Previously Curtis, Iwahori, and Kilmoyer [CIK], working over $\mathbb{C}$, showed that if q is not a root of unity in the usual sense then the Hecke algebra is isomorphic to $\mathbb{C}S_n$. The hypothesis that k be a field is, however, too strong to permit one to conclude that $\mathcal{H}_n$ is a global solution to the Donald–Flanigan conjecture. It does show that it is one at each individual prime but gives no way of linking the primes to show that the matrix blocks correspond. Wenzl [W] proved a result closer to our first theorem by showing explicitly, using "quantized" Young symmetrizers, how to construct the idempotents of the localized Hecke algebra away from the prime 2. That is, his method requires extension of the coefficients to $\mathbb{Z}[q, q^{-1}, 1/2, 1/n_{q^2}!]$. (It fails at 2 because Wenzl uses square roots, which he says could be avoided.) This gives a global solution away from 2, but in group theory the prime 2 is indispensable since by the Feit–Thompson theorem a group of odd order is solvable, [FT].

Rather than amend either [W] or [DJ], we examine the tensor powers $V^{\otimes n}$ of a free module V, on which the symmetric group is represented by permutations of the tensor factors. In the "quantized" case, the generators of the Hecke algebra are represented by quantum Yang–Baxter matrices. These generate the commutant of the standard quantized universal enveloping algebra $U_q sl_n$ operating on the same space, extending to the quantized case a central observation of Schur's thesis [S]. One immediately recovers, amongst others, the basic result of Lusztig [L] and Rosso [R] that $U_q sl_n$ has over $\mathbb{C}$ essentially the same representation theory as $U sl_n$ provided that q is not a root of unity. For generic q the cohomology theory of Hopf algebras [GSk1], [GSk2], [Shn] gives the stronger result that any quantization, not merely the standard one, in fact has this property. For other relevant work, cf. also [GGSk1], [CFW], [F], as well as the thesis of P.N. Hoefsmit [Hoe] which, although often cited, unfortunately remains unpublished.

An important question which we have not been able to answer is, What is the separability idempotent of $\mathcal{H}_n(q)$ over $\mathbb{Z}_{q,n}$? Clearly what we have done is to invert the "quantum dimension" to obtain a separable algebra. Knowing that idempotent might be helpful in extending our results to all Weyl groups.

2. Some elementary representation theory of S_n

As an introduction to our approach, we review some of the elementary representation theory of S_n. Let V be a vector space of dimension d over $\mathbb{Q}$ with a basis $x_1, \dots, x_d$ except that when $d = 2$ we will write x for x_1 and y for x_2. The n–th tensor power of V will be denoted simply V^n. On this S_n operates by permutation of the tensor factors, inducing an operation of its group algebra $\mathbb{Q}S_n$, and $\operatorname{End} V$ operates diagonally. These operations clearly commute, and Schur [S] showed that they are mutual commutants inside $\operatorname{End} V^n$, i.e., each consists of all operators commuting with the other. From a modern viewpoint this is clear, since it is evident that any operator commuting with S_n must act diagonally, and since $\mathbb{Q}S_n$ is separable its second commutant inside $\operatorname{End} V^n$ must be itself.

Tensor products of elements, like $x \otimes y$, will be denoted simply by concatenation, xy. Note that if $d \geq n$ then the representation of $\mathbb{Q}S_n$ on V^n is faithful, since the $n!$ images of $x_1 x_2 \dots x_n$ are linearly independent. Having chosen a basis for V we can make it into an inner product space by taking these basis elements to be orthonormal. This induces an inner product (a, b) on V^n in which the monomials of total degree n in the basis elements of V form an orthonormal basis for V^n. The adjoint of an $L \in \operatorname{End} V^n$ will be denoted L^t, so by definition $(La, b) = (a, L^t b)$. Our inner product is symmetric, so the matrix of L^t relative to an orthonormal basis is just the transpose of that of L. The following is then obvious.

Lemma 2.1. *If $\sigma \in S_n$ then $\sigma^t = \sigma^{-1}$; i.e., the elements of S_n are orthogonal operators.*

Any real representation of a finite group is equivalent to one by orthogonal matrices since we can introduce an invariant metric on the representation space by first choosing an inner product arbitrarily and then averaging with respect to the group operations. The foregoing is, however, both canonical and extendable to the quantized case. It yields

Theorem 2.2. *Central elements of $\mathbb{Q}S_n$ are self–adjoint.*

Proof. The sum of all elements in any conjugacy class is self–adjoint because in S_n every element is conjugate to its inverse, and the central elements are linear combinations of these sums. □

Notice that the orthogonal projection of one S_n submodule of V^n on another is a submodule of the latter which is a quotient module of the former. We therefore have the following, which extends to the quantized case.

Theorem 2.3. *The isotypical components of V^n are mutually orthogonal.*

On the tensor algebra TV of V, of which V^n is a homogeneous component, denote the derivation $\partial/\partial x_i$ by ∂_i. We then have a representation of the Lie algebra sl_n: letting $H_i, X_i, Y_i, i = 1, \dots, d-1$, be a Cartan basis, send X_i to $x_i \partial_{i+1}, Y_i$ to $x_{i+1}\partial_i$ and H_i to their commutator $x_i\partial_i - x_{i+1}\partial_{i+1}$. These are the "infinitesimal generators" of the special linear group $Sl(V)$ acting diagonally, they commute with the action of S_n, and any linear operator commuting with all of them lies in $\mathbb{Q}S_n$. (To see this, extend coefficients to $\mathbb{R}$ and note that the special linear group is generated by the exponentials of its infinitesimal generators.) The representation of sl_d gives rise to one of its universal enveloping algebra $U := Usl_d$. For the proofs of the theorems asserted in the Introduction it will never be necessary to use the Casimir operator, but it is interesting to consider it. Since any element of the center of U is *a fortiori* in its centralizer, such an element must operate like a central element of $\mathbb{Q}S_n$, so we have

Theorem 2.4. *Central elements of U, and in particular the quadratic and higher Casimir operators, act self-adjointly on V^n. Submodules of distinct eigenspaces of a central element of U are never isomorphic.*

For sl_2 the quadratic Casimir operator is $\mathcal{C} = \frac{1}{2} + \frac{1}{2}H^2 + XY + YX$. (The constant term $\frac{1}{2}$ is not important for us but arises naturally and is essential in certain parts of the theory of quantum groups when one must exponentiate the Casimir element.) It is easy to check in this case that $\mathcal{C}$ is the essentially unique quadratic central element of U. The eigenspaces of $\mathcal{C}$ are S_n submodules of V^n and distinct eigenspaces of a self–adjoint operator are mutually orthogonal, so we have immediately a decomposition of V^n into an orthogonal direct sum of submodules. This is generally not a full isotypical decomposition since the quadratic Casimir of a simple Lie algebra $\mathfrak{g}$ may have the same eigenvalue on non–isomorphic simple $\mathfrak{g}$ modules. The center of the universal enveloping algebra of a simple Lie algebra of rank r is a polynomial ring in r elements, the quadratic and higher Casimir operators; together they give the full isotypical decomposition. For $d = 2$ the rank is 1 and all central elements are polynomials in the quadratic Casimir, so the problem does not arise; and it is trivial in this case that $\mathcal{C}$ has distinct eigenvalues on non-isomorphic simple modules. (This is *a fortiori* true in the quantized case; see the next section.)

Letting $d = 2$, denote the basis elements of V by x, y and those of the Lie algebra sl_2 by H, X, Y, where $Xx = 0, Xy = x, Yx = y, Yy = 0, Hx = x$, and $Hy = -y$. In this case we consider only partitions of the form $(n-i, i), i = 0, \dots, n$, and denote the corresponding submodule of V^n by $V(n-i, i)$. This is the span of all monomials a of degree $n-i$ in x and i in y. An $a \in V(n-i, i)$ will be called *homogeneous of weight* $|a| := n-2i$; one then has $Ha = (n-2i)a = |a|a$. It is easy to see (and will in any case be shown in the quantized case) that $\ker X|V(n-i, i) \neq 0$ if and only if $n - 2i \geq 0$, in which case we denote this kernel by $V(n-i, i; 0)$. The distinct simple sl_2 modules are the symmetric powers $V^{\odot r}, r = 0, 1, \dots,$ of V, defined as follows: For $r = 0$ this is the "trivial" one-dimensional module annihilated by all elements of sl_2, for $r = 1$ it is V itself, the "vector representation", and for larger r it is the $r+1$–dimensional space spanned by the ordinary monomials (i.e., in commuting variables) $x^{r-i}y^i, i = 0, \dots, r$ on which H, X, Y act as $x\partial_x - y\partial_y, x\partial_y, y\partial_x$, respectively. The eigenvalue of the Casimir on $V^{\odot r}$ is $\frac{1}{2}(r+1)^2$. These are different for non-negative values of r, so the decomposition of any finite-dimensional sl_2 module into eigenspaces of the Casimir is already a decomposition into isotypical sl_2 submodules.

The various $V^{\odot r}$ can be distinguished by the index of nilpotence of Y (or X) when it acts on them, the index being $r+1$. The index of nilpotence of Y on V^n is $n+1$, so the highest $V^{\odot r}$ is that with $r = n$; it obviously occurs exactly once and is generated by its highest weight element, x^n. Note that $V(n-i, i; 0)$, the kernel of X in $V(n-i, i)$, consists of all the highest weight vectors for those submodules of V^n which are isomorphic to $V^{\odot(n-2i)}$. Setting $V(n-i-r, i+r; r) := Y^r V(n-i, i; 0)$ it follows that $Y \mid V(n-i-r+1, i+r-1; r-1) \to V(n-i-r, i+r; r)$ is an isomorphism for $r = 0, \dots, n-2i-1$ and is the zero mapping for $r = n-i$. Therefore, every simple S_n–submodule of V^n is isomorphic to a submodule of some $V(n-i, i; 0)$, but the latter, we will see, are all simple. The essential step in proving both this and that

$\mathbb{Q}S_n$ operates on each simple module as its full ring of linear endomorphisms is to see how $V(n-i,i;0)$ is constructed from simple S_{n-1} submodules of V^{n-1}. Observe that if $n-2i>0$, so $V(n-i-1,i;0)$ is defined, then $a \in V(n-i-1,i;0)$ implies $xa \in V(n-i,i;0)$. Now suppose that $b \in V(n-i,i-1;0)$. Since by hypothesis $Xb=0$, we have $XYb = Hb = |b|b$, where $|b| = n-2i+1 > 0$. It follows that $X(yb - |b|^{-1}xYb) = 0$, so we make the following definition, to be generalized later:

$$\mathcal{P}b = yb - |b|^{-1}xYb.$$

Then we have also $\mathcal{P}b \in V(n-i,i;0)$. View S_{n-1} as the subgroup of S_n permuting $2,\dots,n$ and leaving 1 fixed. Then it is clear that left multiplication by x viewed as a mapping $V(n-i-1,i;0) \to V(n-i,i;0)$ (where $n-2i>0$) and the mapping $\mathcal{P}: V(n-i,i-1;0) \to V(n-i,i;0)$ (where $n-2i \geq 0$) are both S_{n-1}–module monomorphisms.

Theorem 2.5. *Suppose that $n-2i \geq 0$. Then $V(n-i,i;0)$ is the orthogonal direct sum of $\mathcal{P}V(n-i,i-1;0)$ and $xV(n-i-1,i;0)$, the latter being omitted if $n-2i=0$. Moreover $V(n-i,i;0)$ is simple and $\mathbb{Q}S_n$ operates on it as its full ring of linear endomorphisms.*

Proof. First we must show that every element $c \in V(n-i,i;0)$ actually has the form $c = xa + \mathcal{P}b$ for some $a \in V(n-i-1,i;0)$ and $b \in V(n-i,i-1;0)$. Write $c = xc_0 + yc_1$. Since $Xc = xXc_0 + xc_1 + yXc_1 = 0$ we must have $Xc_1 = 0$, so $c_1 \in V(n-i,i-1;0)$. Then $c - \mathcal{P}c_1 = x(c_0 + |c_1|^{-1}Yc_1), |c_1| = n-2i+1$, so the desired $a = c_0 + |c_1|^{-1}Yc_1$. Now make the inductive assumption that the assertion is true for all smaller values of n, there being nothing to prove when $n=1$. Suppose that $a \in V(n-i-1,i;0)$ and $b \in V(n-i,i-1;0)$. Clearly xa is orthogonal to yb so it will be orthogonal to $\mathcal{P}b$ if the inner product (xa, xYb) vanishes. But $(xa,xYb) = (a,Yb)$ and $a \in V(n-i-1,i;0)$ while $Yb \in YV(n-i,i-1;0) = V(n-i-1,i;1)$. There are two ways to see that these are orthogonal. First, they lie in different eigenspaces of the Casimir. Without invoking the Casimir, however, observe that by the inductive hypothesis $xV(n-i-1,i;0)$ and $\mathcal{P}V(n-i,i-1;0)$ are non-isomorphic simple $\mathbb{Q}S_{n-1}$–submodules of $V(n-i,i;0)$ considered as an S_{n-1}–module. They are therefore orthogonal since, as remarked, the orthogonal projection of one module on another is a submodule of the second. To see that $\mathbb{Q}S_{n-1}$ acts on each as its full ring of linear endomorphisms, suppose that $V(n-i-1,i;0)$ has rank r and $V(n-i,i-1;0)$ has rank s. These are the same as the ranks of their images in $V(n-i,i;0)$, so taking a basis of $V(n-i,i;0)$ formed by combining bases of these images, one can view the representation of $\mathbb{Q}S_n$ on $V(n-i,i;0)$ as one by $(r+s)\times(r+s)$ matrices which already contains the direct sum of the $r\times r$ and $s \times s$ matrices. But neither $xV(n-i-1,1;0)$ nor $\mathcal{P}V(n-i,i-1;0)$ is an S_n submodule. It follows that we must have the full ring of $(r+s)\times(r+s)$ matrices. □

The orthogonality of the summands out of which $V(n-i,i;0)$ is built gives the inductive construction of its canonical orthogonal basis: combine the images of those of $V(n-i,i-1;0)$ and (if $n-2i>0$) $V(n-i-1,i;0)$ under $\mathcal{P}$ and left multiplication by x, respectively. The elements of the resulting basis may therefore be indexed by sequences of length n in x and $\mathcal{P}$, where x appears $n-i$ times and $\mathcal{P}$ appears i times and where, in each terminal segment (subsequence consisting of the last j elements for each j) the number of $\mathcal{P}$'s does not exceed the number of x's. Instead of sequences we may take the index set to be two-rowed Young tableaux. For if the sequence (which must end with x) is given then the associated Young tableau is constructed as follows: Put "1" in the first row and first column. The position of "2" is determined by the next–to–last symbol in the sequence; if it is an x then put "2" in the next position in the first row, if it is a $\mathcal{P}$ put it in the first position in the second row. If the integers through i have been positioned using the last i entries in the sequence, then $i+1$ is put in the first open position of the first or second row according as the $n-i$th entry in the sequence is an x or a $\mathcal{P}$. The condition on the sequence is precisely that this should give a Young tableau, so we have

Proposition 2.6. *The canonical orthogonal basis of $V(n-i, i; 0)$ is indexed by the (standard) two-rowed Young tableaux in which the first row has length $n-i$ and the second row has length i. In particular, the rank of the module is the number of such tableaux.*

In view of this, we will call a sequence of x's and $\mathcal{P}$'s in which every terminal segment contains no more $\mathcal{P}$'s than x's a *generating sequence.* For two variables, the number of generating sequences is the difference of the binomial coefficients $\binom{n}{i} - \binom{n}{i-1}$, which we denote by $\langle n, i\rangle$. The recursion formula for these is essentially the same as for the binomial coefficients themselves: $\langle n, i\rangle = \langle n-1, i-1\rangle + \langle n-1, i\rangle$, where $n-2i \geq 0$ and the second summand is omitted if $n-2i = 0$.

As an elementary example, consider the simple two-dimensional representation $V(2, 1; 0)$ of S_3. Its orthogonal basis is indexed by the two sequences $x\mathcal{P}x, \mathcal{P}xx$. The first basis element is thus $x(\mathcal{P}x) = x(yx - xy)$ and the second is $\mathcal{P}(xx) = yxx - \frac{1}{2}x(yx + xy)$. The two basis elements correspond, respectively, to the Young tableaux having $1, 2$ in the first row (second row empty) and having $1, 2$ in the first column (second column empty). It is important, as we shall prove later in the quantized case, that the norm of any element of the canonical orthogonal basis of any $V(n-i, i; 0)$ (i.e., the sum of the squares of its coefficients when expressed in terms of the original monomial basis) must be a unit in the ring $\mathbb{Z}[1/n!]$. In the small example with $n = 3$ just given, the norms of the two basis elements are 2 and $3/2$, respectively. For the case of arbitrary d we shall have to generalize $\mathcal{P}$ to a sequence of d operators. (Here we actually have two; the first is left multiplication by x.) We turn now to the quantized case.

3. Representation of the Hecke algebra on V^n

Recall that the Hecke algebra $\mathcal{H}_n$ of S_n, whose representations we must study, is the algebra generated over $\mathbb{Z}[q, q^{-1}]$ by elements $T_{s_i}, i = 1, \ldots, n-1$, corresponding to the generators $s_i = (i, i+1)$ of S_n. Its multiplication is given by $T_s T_w = T_{sw}$ if s is one of these generators and $w \in S_n$ is an element with length $\ell(sw) > \ell(w)$, and $T_s^2 = (q - q^{-1})T_s + 1$. It is a free module of rank n over $\mathbb{Z}[q, q^{-1}]$. The familiar "Artin presentation" of S_n is by generators $s_i, i = 1, \ldots, n-1$, with

$$\begin{aligned} s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, \\ s_i s_j &= s_j s_i \qquad \text{if } |i-j| > 1, \\ s_i^2 &= 1. \end{aligned}$$

It follows that for the special case of S_n we obtain an equivalent definition of the Hecke algebra $\mathcal{H}_n$ by requiring that the generators $T_i := T_{s_i}$ satisfy the "braid" and "Hecke" relations

$$\begin{aligned} T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1}, \\ T_iT_j &= T_jT_i \qquad \text{if } |i-j|>1, \\ T_i^2 &= (q-q^{-1})T_i + 1. \end{aligned}$$

For the Hecke algebra of S_n certainly satisfies these relations, so if the coefficient ring were a field it would be sufficient to show that the algebra these define has dimension no greater than that of the Hecke algebra. For this it is sufficient to show that if a formal product w of generators of S_n gives a non-reduced expression for some element of S_n, then the corresponding product τ of generators $T_i := T_{s_i}$ satisfying the second set of relations can also be shortened. But w can be shortened only if by applying the braid relations it can be rewritten to contain the square of a generator, in which case τ can be shortened also. Over a domain we must note further only that the Hecke algebra, which is in any case a quotient of the algebra defined by the above relations, is a free module, so the quotient map splits. Many authors, e.g., [W], simply adopt the second definition.

Now let V be a free module of rank d over $\mathbb{Z}[q,q^{-1}]$. There is then a natural representation of $\mathcal{H}_n$ on V^n given as follows. The standard $d^2\times d^2$ quantum Yang–Baxter matrix

$$R = \sum_{\substack{i\neq j\\ i,j=1}}^{d} e_{ii}\otimes e_{jj} + q\sum_{i=1}^{d} e_{ii}\otimes e_{ii} + (q-q^{-1})\sum_{1\le j<i\le n} e_{ij}\otimes e_{ji}$$

for the simple Lie algebra sl_d (cf. [FRT]) may be viewed as having coefficients in $\mathbb{Z}[q,q^{-1}]$. We view this as operating on V^2 so: Let the basis of V be $x_1,\dots,x_d$, so that of V^2 consists of the x_ix_j $(= x_i\otimes x_j)$ in lexicographic order. Then set $(e_{ij}\otimes e_{kl})x_rx_s = x_ix_k$ if $j=r, l=s$, and 0 otherwise. Let (12) operate as the interchange of tensor factors in V^2. As a matrix, we have

$$(12) = \sum_{i,j} e_{ij}\otimes e_{ji}.$$

Set

$$\overline{R} = (12)R = \sum_{\substack{i\neq j\\ i,j=1}}^{d} e_{ji}\otimes e_{ij} + q\sum_{i=1}^{d} e_{ii}\otimes e_{ii} + (q-q^{-1})\sum_{1\le j<i\le n} e_{jj}\otimes e_{ii}.$$

This is symmetric. (The transpose of $e_{ij}\otimes e_{kl}$ is $e_{ji}\otimes e_{lk}$, the pair (i,k) being the row index and (j,l) the column index.) It is also "*balanced*", i.e., in every term the sum of the column indices equals the sum of the row indices and it is therefore a direct sum of matrices in each of which these "*weights*" are constant. (This concept is meaningful for an arbitrary tensor power of a matric algebra; for the first power, a balanced matrix is simply diagonal.)

Let $\overline{R}_i, i=1,\dots,n-1$, denote the operation of $\overline{R}$ in tensor factors $i,i+1$ of V^n. It is a basic fact that these satisfy the braid relations, cf. [FRT]:

$$\begin{aligned} \overline{R}_i\overline{R}_{i+1}\overline{R}_i &= \overline{R}_{i+1}\overline{R}_i\overline{R}_{i+1}, \\ \overline{R}_i\overline{R}_j &= \overline{R}_j\overline{R}_i \qquad \text{if } |i-j|>1. \end{aligned}$$

Sending T_i to $\overline{R}_i$ therefore induces a representation of $\mathcal{H}_n$ on V^n. Note that $\overline{R}$ is self-adjoint – its matrix is symmetric relative to the standard basis of V^2 consisting of the x_ix_j in lexicographic order – and therefore so are all the $\overline{R}_i$. Here, for example, is the operation of $\overline{R}$ on V^2 in the case where $d = 2$:

$$\begin{aligned}\overline{R}x^2 &= qx^2,\\ \overline{R}xy &= (q - q^{-1})xy + yx,\\ \overline{R}yx &= xy,\\ \overline{R}y^2 &= qy^2.\end{aligned}$$

As in the classical case, we have the following basic

Lemma 3.1. *The orthogonal projection of one $\mathcal{H}_n$–submodule of V^n on another is a submodule of the second.*

Proof. This would be trivial if, as in the classical case, the representation were generated by orthogonal transformations. However, viewing q as a real parameter there are constants C and S such that $C\overline{R} + S$ is orthogonal. In fact, set

$$\gamma = \sum_{i<j} e_{ij} \wedge e_{ji} \qquad \text{where} \qquad e_{ij} \wedge e_{kl} = \frac{1}{2}(e_{ij} \otimes e_{kl} - e_{kl} \otimes e_{ij}).$$

This is the infinitesimal of the deformation from Usl_n to U_qsl_n. Set $q = \sec t - \tan t$. Then $(\cos t)\overline{R} + \sin t = e^{-t\gamma}(12)e^{t\gamma}$ in which all the factors are orthogonal; here $C = \cos t, S = \sin t$. (Cf. [GGSk2] where the interchange of tensor factors "(12)"is denoted by P.) It follows in this case that the representation of the Hecke algebra is generated by orthogonal transformations of V^n, so here the orthogonal projection of one $\mathcal{H}_n$–submodule on another is indeed a submodule of the second. This, however, is a purely formal property and therefore holds generally. □

The Hecke algebra may be viewed as a "quantization" of the group ring $\mathbb{Z}S_n$ and the commutant of its representation on V^n is a representation of a particular quantization $U_q := U_qsl_d$, so we describe that which we use starting with the case $d = 2$. Observe first that the unquantized U is a Hopf algebra with primitive generators H, X, and Y. The Drinfel'd-Jimbo quantization (cf. [D], [J1], [J3]) replaces H by a pair of invertible group–like generators K, K^{-1} (i.e., $\Delta K = K \otimes K, \Delta K^{-1} = K^{-1} \otimes K^{-1}$). We want, in effect, that $K = q^H$, so K will act on the tensor algebra TV as the automorphism sending x to qx and y to $q^{-1}y$, which forces the commutation relations $KX = q^2XK, KY = q^{-2}YK$. The remaining multiplication and comultiplication rules are given by

$$qXY - q^{-1}YX = (q^{-1} - q)^{-1}(1 - K^2),$$
$$\Delta X = X \otimes 1 + K \otimes X, \qquad \Delta Y = Y \otimes 1 + K \otimes Y.$$

(For a full definition one needs, in addition, the $q-$analogues of the Serre relations giving, in particular, the nilpotence of $\operatorname{ad} X$ and $\operatorname{ad} Y$, but we do not need these here, nor shall we need the quantized antipode.) The right side of the first relation, which has the correct quasi–classical limit as $q \to 1$, namely H, may seem to pose a problem since we do not assume that $q - q^{-1}$ is invertible. We therefore take the left side, which is well-defined, as an additional generator, denoted simply by H. This "quantized" H should be distinguished from the original unquantized one which, if we should need it, would be denoted H_0. Both K and the quantized H will be seen to be well-defined operators on V^n.

The operations of X, Y on V itself are the same as before quantization, that is, $Xx = 0, Xy = x, Yx = y, Yy = 0$. However, the new comultiplication (which defines the tensor product of modules, in particular extending the operation of U_q to V^n for every n) now specifies that for homogeneous elements $\alpha, \beta \in V^n$ we have

$$X(\alpha\beta) = (X\alpha)\beta + q^{|\alpha|}\alpha X\beta, \qquad Y(\alpha\beta) = (Y\alpha)\beta + q^{|\alpha|}\alpha Y\beta.$$

From the definitions one can also readily deduce that

$$H\alpha = q|\alpha|_{q^2}\alpha.$$

It follows, in particular, that if $X\alpha = 0$ then $XY\alpha = |\alpha|_{q^2}\alpha$. Most important, as the reader should check, the operations of U_q so defined on V^n commute with those of the $\overline{R}_i$ and therefore are $\mathcal{H}$–module morphisms. The second cohomology of a simple Lie algebra taken with coefficients in itself vanishes, so a simple Lie algebra admits only trivial deformations of either itself as a Lie algebra or of its universal enveloping algebra, giving the following quantized form of Schur's theorem in the generic case, cf. [J2].

Theorem 3.2. *Setting $q = 1 + t$ and extending coefficients to $\mathbb{Q}[[t]]$, the operations of $\mathcal{H}_n$ and U_q on V^n are mutual commutants.*

Proof. This is an exercise in deformation theory: Note that after the extension $\mathcal{H}_n$ is isomorphic to $\mathbb{Q}S_n[[t]]$ and, because sl_2 is simple, U_q likewise becomes isomorphic to U with coefficients extended. □

Since $U_q sl_d$ is a trivial deformation of Usl_d it has (after extension of coefficients) a Casimir operator. Although we shall not need it, for $d = 2$ and our specific quantization, this is given by

$$\mathcal{C} = (q - q^{-1})^{-2}(qK + q^{-1}K^{-1} - 2) + q^{-1}K^{-1}YX.$$

(This is adapted from Rosso, [R].) The strange form is forced by our quantization, which in turn is forced by the requirement that U_q be in the commutant of $\mathcal{H}$. It is easy to verify directly that $\mathcal{C}$ is in the center of U_q. Its limit as $q \to 1$ is half our previous classical Casimir operator, which we now would denote by $\mathcal{C}_0$. Recall that we still have a symmetric inner product on V^n.

Theorem 3.3. *The quadratic and higher Casimir operators are self-adjoint.*

Proof. The assertion will hold if it does so generically, and is not affected by extension of coefficients, so we may replace q by $1 + t$ and take coefficients to be in $\mathbb{Q}[[t]]$. The Casimir is then a central element of $\mathcal{H}_n$, which, since coefficients have been extended, is isomorphic to $\mathbb{Q}S_n[[t]]$. The generators of $\mathcal{H}_n$ were self-adjoint. Therefore transposition carries it as a whole into itself. Since the center is preserved, transposition can only act as a permutation on the central idempotents. But when $t = 0$ the central elements are self-adjoint, so this permutation, which is a continuous function of t, is the identity when $t = 0$. Therefore it is the identity on the center for all t, so the Casimir is preserved. □

The case $d = 2$ proceeds exactly as in the classical case. Recall that $\mathbb{Z}_{q,n} := \mathbb{Z}[q, q^{-1}, 1/n_{q^2}!]$ which we henceforth always take as coefficients. The quantized H operates semisimply on V^n which is the direct sum of its eigenspaces $V(n - i, i)$; the eigenvalue on this is $q(n - 2i)_{q^2}$, which is a unit. In fact, any eigenvalue of H on a module of finite rank must be of the form $q\lambda_{q^2}$ with λ a non-negative integer. For if α is an eigenvector and we write the eigenvalue formally as $q\lambda_{q^2}$ with $\lambda_{q^2} = (1-q^{2\lambda})/(1-q^2)$, then it is easy to verify that $HX^m\alpha = q(\lambda+2m)_{q^2}\alpha$ and $HY^m\alpha = q(\lambda-2m)_{q^2}\alpha$. A simple telescoping induction on the relation $qXY - q^{-1}YX = H$ gives

$$q^mXY^m - q^{-m}Y^mX = q^{m-1}HY^{m-1} + q^{m-3}YHY^{m-2} + \cdots + q^{-m+1}Y^{m-1}H,$$

so we have

Lemma 3.4. *If $H\alpha = q\lambda_{q^2}\alpha$ and $X\alpha = 0$, then*

$$XY^m\alpha = q^{-2m+2}(\lambda - m + 1)_{q^2}m_{q^2}Y^{m-1}\alpha.$$

The coefficient on the right will not vanish for any m unless λ is a non-negative integer, which therefore must be the case if the module is of finite rank. Moreover, one then has that if $X\alpha = 0$ and $H\alpha = 0$ then $Y\alpha = 0$ as well. The simple modules of finite rank over U_q thus look precisely like those over U, provided the non-zero coefficients which appear in the analogues of the usual formulas are units, which is exactly what we have supposed.

Returning to V^n, the submodule $\ker X$ is obviously homogeneous, i.e., the direct sum of its components in each of the $V(n-i,i)$. As before, write $\ker X|V(n-i,i) = V(n-i,i;0)$. If $\alpha \in V(n-i-1,i;0)$ then $x\alpha \in V(n-i,i;0)$; if $\beta \in V(n-i,i-1;0)$ set $\mathcal{P}\beta = -(q|\beta|_{q^2})^{-1}xY\beta + y\beta$ for $|\beta| \neq 0$ and $\mathcal{P}\beta = 0$ if $|\beta| = 0$. (This is the quantized version of our earlier $\mathcal{P}$.) It is easy to check that this is also in $V(n-i,i;0)$. Exactly as in the unquantized case, set $V(n-i,i;r) = Y^rV(n-i+r,i-r;0)$ for $r = 0,\dots,i$ (they will be seen to vanish for larger r). These are all $\mathcal{H}_n$-submodules.

Lemma 3.5.

1. *If $0 \le r < n-2i$, then*

   $$Y : V(n-i-r,i+r;r) \to V(n-i-r-1,i+r+1;r+1)$$

   and

   $$X : V(n-i-r-1,i+r+1;r+1) \to V(n-i-r,i+r;r)$$

   are isomorphisms and the compositions XY and YX are multiplication by an invertible constant.
2. *$Y^rV(n-i,i;0) = 0$ for $r > n-2i$, so $V(n-i-r,i+r;r) = 0$ for $r > n-2i$, and $V(n-i,i;0) = 0$ for $n-i < i$.*

Proof. (1) Suppose that $1 \le r \le n-2i$ and that $\alpha \in V(n-i,i;0)$, so $\beta = Y^{r-1}\alpha \in V(n-i-r+1,i+r-1;r-1)$. Since $X\alpha = 0$, by Lemma 3.4 we have

$$XY\beta = XY^r\alpha = q^{-2r+2}(|\alpha| - r + 1)_{q^2}r_{q^2}\beta.$$

Then the coefficient $c := q^{-2r+2}(|\alpha| - r + 1)_{q^2}r_{q^2}$ is invertible, so XY is an automorphism, whence X is onto and Y is one–to–one. However, by definition Y is onto, thus invertible, so X and Y are both isomorphisms. Now $XY\beta = c\beta$ implies $YXY\beta = cY\beta$. Since Y is onto, YX is just multiplication by the same c. (2) Clearly some power of Y annihilates $V(n-i,i;0)$. If r is the first such, then Lemma 3.4 with $\lambda = n-2i$ implies that $n-2i-r+1 = 0$, else Y^{r-1} would already annihilate $V(n-i,i;0)$. □

Using the quantized $\mathcal{P}$ and exactly the same proof as in the classical case, we now have

Lemma 3.6. *If $n-2i \geq 0$, then $V(n-i,i;0)$ is the orthogonal direct sum of $xV(n-i-1,i;0)$ and $\mathcal{P}V(n-i,i-1;0)$, where the first summand appears only if $n-2i>0$.*

Again, with arguments identical to those in the classical case we have, when $d=2$,

Theorem 3.7. *Every simple $\mathcal{H}_n$–submodule of V^n is isomorphic to one of the $V(n-i,i;0)$. Each of the latter has a canonical orthogonal basis indexed by the two-rowed Young tableaux with rows of length $n-i$ and i, and $\mathcal{H}_n$ acts on each as its full ring of linear endomorphisms.*

It follows, as before, that the norms of the basis elements are units. Also, $V(n-i,i)$ is the orthogonal direct sum of its submodules $V(n-i,i;r)$, each of which is simple.

4. The case of arbitrary rank

Suppose now that V is a free module of rank d over $Z_{q,n} = \mathbb{Z}[q,q^{-1},1/n_{q^2}!]$ with basis $x_1,\dots,x_d$. The "standard" quantization $U_q = U_q sl_n$ (cf. [D]) may be viewed as generated by elements $X_i, Y_i, K_i, H_i, i=1,\dots,d-1$, which for fixed i generate a sub-bialgebra isomorphic to $U_q sl_2$. Here X_i and Y_i act on V like $x_i\partial/\partial x_{i+1}$ and $x_{i+1}\partial/\partial x_i$, respectively. These act on V^n so: If α is homogeneous in variables x_i and x_{i+1} of degrees n_i and n_{i+1} in each, respectively, then set $|\alpha|_i = n_i - n_{i+1}$ and

$$X_i(\alpha\beta) = (X_i\alpha)\beta + q^{|\alpha|_i}\alpha(X_i\beta); \quad Y_i(\alpha\beta) = (Y_i\alpha)\beta + q^{|\alpha|_i}\alpha(Y_i\beta).$$

In particular, X_i and Y_i treat all variables x_j with $j \neq i, i+1$ as constants. It is then evident from the preceding formula that the X_i and Y_i act as $\mathcal{H}_n$ morphisms under the induced representation of $\mathcal{H}_n$ on V^n. Moreover, if $|i-j|>1$ then X_i and Y_j commute, as do X_i and X_j, and Y_i and Y_j, but unlike the classical case, X_i and $Y_{i\pm 1}$ only "q–commute", i.e., we have

$$X_i Y_{i\pm 1} = q Y_{i\pm 1} X_i.$$

Each X_i, Y_i, K_i and (quantized) H_i together generate a subalgebra of $U_q sl_n$ isomorphic to the subalgebra of $U_q sl_2$ generated by X, Y, K and H.

If $\mathfrak{p} = (n_1,\dots,n_d)$ is a non-increasing partition of n into d parts (some of which may be zero), then we will write $\mathfrak{p}! = n_1! n_2! \cdots n_d!$, and the usual multinomial coefficient $n!/\mathfrak{p}!$ will be denoted simply by $\binom{n}{\mathfrak{p}}$. When $\mathfrak{p}'$ is a non-increasing partition of $n-1$ into d parts differing in exactly one place from $\mathfrak{p}$, then we write $\mathfrak{p}' \to \mathfrak{p}$. Denote by $V(\mathfrak{p})$ the submodule of V^n spanned by all monomials of degree n_i in $x_i, i=1,\dots,d$, where the n_i are the parts of $\mathfrak{p}$. (This is meaningful even if $\mathfrak{p}$ fails to be non–increasing.) In analogy with the preceding section, we set

$$V(\mathfrak{p};0) = \bigcap_{i=1}^{d-1} \ker X_i.$$

(This, too, is meaningful even if $\mathfrak{p}$ fails to be non–increasing, but from the preceding section it then vanishes.) For the rest of this section, unless specified, we assume the coefficient ring to be $\mathbb{Z}_{q,n} = \mathbb{Z}[q,q^{-1},1/n_{q^2}!]$. We can now construct the $V(\mathfrak{p};0)$ from the $V(\mathfrak{p}';0)$ with $\mathfrak{p}' \to \mathfrak{p}$ in a way generalizing that of the preceding section. Suppose that $\mathfrak{p} = (n_1,\dots,n_d)$ and that $\mathfrak{p}' = (n_1,\dots,n_r-1,\dots,n_d)$, where the latter is still non-increasing. Define inductively polynomials in $Y_1, \dots, Y_{r-1}$ (whose dependence on $\mathfrak{p}$ and r we momentarily suppress) by $\widehat{P}_0 = 1$, $\widehat{P}_1 = Y_{r-1}$, and

$$\widehat{P}_i = (n_{r-i+1} - n_r + i)_{q^2} Y_{r-i} \widehat{P}_{i-1} - q(n_{r-i+1} - n_r + i - 1)_{q^2} \widehat{P}_{i-1} Y_{r-i}, \quad 1 < i < r.$$

It follows that $\widehat{P}_i$ is a polynomial in the operators $Y_{r-i}, Y_{r-i+1}, \dots, Y_{r-1}$ only, and is homogeneous of degree 1 in each. Applied to a homogeneous α, it decreases its degree in x_{r-i} by one and increases its degree in x_r by one. Set $P_0 = 1$ and for $i > 0$ set

$$P_i = (-1)^i q^{-i} [(n_{r-1} - n_r + 1)_{q^2} \dots (n_{r-i} - n_r + i)_{q^2}]^{-1} \widehat{P}_i.$$

Note that the "arguments" of the coefficients never exceed n. Finally, set

$$\mathcal{P}(\mathfrak{p}' \to \mathfrak{p}) = x_r P_0 + x_{r-1} P_1 + \dots + x_1 P_{r-1}$$

which we view as a mapping $V(\mathfrak{p}') \to V(\mathfrak{p})$.

Lemma 4.1. *Suppose* $\mathfrak{p}' = (n_1, \dots, n_r - 1, \dots, n_d) \to \mathfrak{p} = (n_1, \dots, n_r, \dots, n_d)$.

1. *For all* $i = 1, \dots, r-1$, *if* $\alpha \in V(\mathfrak{p}')$ *is in* $\ker X_j$, *then*

(a) $$X_j \widehat{P}_i \alpha = 0 \qquad \textit{for} \qquad j \neq r - i,$$

(b) $$X_{r-i} \widehat{P}_i \alpha = (n_{r-i} - n_r + i)_{q^2} \widehat{P}_{i-1} \alpha.$$

2. *If* $\alpha \in V(\mathfrak{p}'; 0) = \bigcap_{j=1}^{d-1} \ker X_j \mid V(\mathfrak{p}')$, *then* $\mathcal{P}(\mathfrak{p} \to \mathfrak{p}')\alpha \in V(\mathfrak{p}; 0)$.
3. $V(\mathfrak{p}; 0) = \sum_{\mathfrak{p}' \to \mathfrak{p}} \mathcal{P}(\mathfrak{p}' \to \mathfrak{p}) V(\mathfrak{p}'; 0)$.

Proof. (1) When $i = 1$ statement (a) is immediate because X_j commutes with Y_{r-1} and $X_j \alpha = 0$; (b) asserts simply that if $X_{r-1}\alpha = 0$ then $X_{r-1} Y_{r-1} \alpha = (n_{r-1} - n_r + 1)_{q^2} \alpha$. This holds from the corresponding assertion for $d = 2$. Suppose now that $i > 1$ and that the assertions hold for all smaller i. For simplicity, write $(n_{r-i+1} - n_r + i - 1) = \lambda$. Then

(*) $$X_j \widehat{P}_i \alpha = (\lambda + 1)_{q^2} X_j Y_{r-i} \widehat{P}_{i-1} \alpha - q \lambda_{q^2} X_j \widehat{P}_{i-1} Y_{r-i} \alpha.$$

Suppose first that $j > r - i + 1$ or $j < r - i - 1$. Then X_j commutes with Y_{r-i} so the first term vanishes by the inductive hypothesis. So does the second, since $Y_{r-i}\alpha$ is still in $\ker X_j$. For $j = r - i + 1$, using the commutation relations and the inductive hypothesis, including that on (b), one finds that the first term is

$$q(\lambda + 1)_{q^2} Y_{r-i} X_{r-i+1} \widehat{P}_{i-1} \alpha = q(\lambda + 1)_{q^2} \lambda_{q^2} Y_{r-i} \widehat{P}_{i-2} \alpha.$$

Since Y_{r-i} commutes with all the Y in $\widehat{P}_{i-2}$, to show that the whole expression vanishes we must show only that $X_{r-i+1} \widehat{P}_{i-1} Y_{r-i} \alpha = (\lambda + 1)_{q^2} \widehat{P}_{i-2} Y_{r-i} \alpha$. As before, $Y_{r-i}\alpha$ is still in $\ker X_{r-i+1}$, so we may again apply the inductive hypothesis on (b) noting, however, that the degree of $Y_{r-i}\alpha$ in x_{r-i+1} is greater by 1 than the corresponding degree of α. When $j = r - i - 1$, X_j commutes or q–commutes with $\widehat{P}_{i-1}$ and Y_{r-i}, so $X_j \alpha = 0$ implies that in (*) both factors are zero. This proves (1a). For (b), setting $j = r - i$ in (*), note that X_{r-i} annihilates $\widehat{P}_{i-1}\alpha$, the degree of which in x_{r-i+1} is one less than that of α. If we let $\mu = n_{r-i} - n_{r-i+1}$ be the $r - i$ norm of α, then

$$X_{r-i} Y_{r-i} \alpha = q^{-1} H_{r-i} \alpha = \mu_{q^2} \alpha$$

and

$$X_{r-i} Y_{r-i} \widehat{P}_{i-1} \alpha = q^{-1} H_{r-i} \widehat{P}_{i-1} \alpha = (\mu + 1)_{q^2} \widehat{P}_{i-1} \alpha.$$

The first term on the right of (*) is $(\lambda+1)_{q^2}(\mu+1)_{q^2}\widehat{P}_{i-1}\alpha$. For the second term, since $\widehat{P}_{i-1}$ is homogeneous of degree 1 in Y_{r-i+1} and all its other factors commute with X_{r-i} one gets $-q^2\lambda_{q^2}\mu_{q^2}\widehat{P}_{i-1}\alpha$. That the sum of the coefficients of $\widehat{P}_{i-1}\alpha$ which appear is indeed $(\lambda+\mu+1)_{q^2}$ is an exercise in the definition of the "q–numbers". (Note that it is true when $q=1$.)

(2) Suppose that $\alpha \in V(\mathfrak{p}';0)$ and consider $X_j x_i P_{r-i}\alpha$. If $j \neq i, i+1$ then this is a multiple of $x_i X_j P_{r-i}\alpha$ and therefore vanishes, by (1a). But

$$X_i(x_{i+1}P_{r-i-1}+x_iP_{r-i})\alpha = (x_iP_{r-i-1}+q^{-1}x_{i+1}X_iP_{r-i-1}+qx_iX_iP_{r-i})\alpha.$$

The middle term vanishes by (1a), so this is $x_i(P_{r-i-1}+qX_iP_{r-i})\alpha$ which vanishes by (1b) and the definition of the P_i.

(3) Every $b \in V(\mathfrak{p};0)$ can be written in the form $x_1a_1+\cdots+x_da_d$. If $a_r \neq 0$ while $a_{r+1}=\cdots=a_d=0$, then we say that b has *length* r. It is sufficient to show that b can be written as the sum of an element of $\sum \mathcal{P}(\mathfrak{p}' \to \mathfrak{p})V(\mathfrak{p}';0)$ and one of shorter length. Now $X_ia_r=0$ for all i since X_ib contains a non-zero multiple of $x_rX_ia_r$ and no other term beginning with x_r. Since $a_r \neq 0$ yet lies in $\bigcap \ker X_i$, it follows that $n_r > n_{r+1}$, so $\mathfrak{p}'=(n_1,\dots,n_r-1,\dots,n_d) \to \mathfrak{p}$. But then $b-\mathcal{P}(\mathfrak{p}' \to \mathfrak{p})a_r$ is a shorter element of $V(\mathfrak{p};0)$. □

There will be no ambiguity if we now denote $\mathcal{P}(\mathfrak{p}' \to \mathfrak{p}) : V(\mathfrak{p}';0) \to V(\mathfrak{p};0)$ by $\mathcal{P}_r$ where r is the unique place in which $\mathfrak{p}'$ and $\mathfrak{p}$ differ. More generally, we may view it as an operator defined on any $V(\mathfrak{p}';0)$ where $\mathfrak{p}'$ is a non-increasing partition of some $n' \leq n$ into d parts with the property that its rth part can be increased by 1 without losing the property of being non-increasing. Now suppose that we have a sequence

$$\mathfrak{p}^{(n)}=(0^d) \to (1,0^{d-1})=\mathfrak{p}^{(n-1)} \to \cdots \to \mathfrak{p}'' \to \mathfrak{p}' \to \mathfrak{p}$$

where each $\mathfrak{p}^{(i)}$ is a non–increasing partition of $n-i$. For each "morphism" $\mathfrak{p}^{(i)} \to \mathfrak{p}^{(i-1)}$ there is a unique place r_i in which $\mathfrak{p}^{(i)}$ and $\mathfrak{p}^{(i-1)}$ differ, so the sequence $(r_1,r_2,\dots,r_n)$ determines the sequence of morphisms and therefore an element $\mathcal{P}_{r_1}\mathcal{P}_{r_2}\dots\mathcal{P}_{r_n}1 \in V(\mathfrak{p};0)$. Here necessarily $r_n=1$ so $\mathcal{P}_{r_n}1=x_1$. The sequence is not arbitrary: in any "terminal segment" $(r_j,r_{j+1},\dots,r_n)$ the number of times any integer k appears amongst the $r_i, i=j,\dots,n$, cannot exceed the number of times that $k-1$ appears, since the partition $\mathfrak{p}^{(j-1)}$ would otherwise fail to be non-increasing. However, that is the only restriction. These generating sequences $(r)=(r_1,\dots,r_n)$ correspond to d–rowed Young tableaux: the value of r_{n-i} is the row in which $i+1$ is placed, and it is put to the immediate right of any other integers previously placed. Since $r_n=1$, we must put "1" in the first position of the first row. Then r_{n-1}, whose value can be either 1 or 2, determines the placement of "2". If it is 1 then "2" goes in the first row to the right of "1"; if it is 2 then "2" begins the second row, etc., generalizing the discussion of the previous section. The correspondence between generating sequences and d–rowed Young tableaux is clearly a bijection. Recall that our coefficient ring is $\mathbb{Z}_{q,n}=\mathbb{Z}[q,q^{-1},1/n_{q^2}!]$.

Theorem 4.2. (i) *Each $V(\mathfrak{p};0)$ is a simple $\mathcal{H}_n$ module, no two are isomorphic, and $\mathcal{H}_n$ acts on each as its full ring of $Z_{q,n}$–linear endomorphisms. It has a canonical orthogonal basis indexed by the d–rowed Young tableaux associated with the partition $\mathfrak{p}$.* (ii) *$\mathcal{H}_n$ is a direct sum of total matric algebras.* (iii) *Every simple submodule of V^n is isomorphic to exactly one of the $V(\mathfrak{p};0)$. If $d \geq n$ then every simple $\mathcal{H}_n$ module is isomorphic to one of these.*

Proof. Make the inductive assumption that the assertions hold for all smaller values of n. (i) As an S_{n-1}–module, $V(\mathfrak{p};0)$ is, by the preceding, a sum of mutually non-isomorphic simple modules which must therefore be mutually orthogonal inside $V(\mathfrak{p};0)$. None of these summands is an S_n–module and $\mathcal{H}_{n-1}$ already acts on each as its full ring of linear endomorphisms. Suppose that the ranks of these submodules are $m_1, \dots, m_s$, so that of $V(\mathfrak{p};0)$ is $m := m_1 + \dots + m_s$. Then $\mathcal{H}_n$ operates on $V(\mathfrak{p};0)$ as a subalgebra of an $m \times m$ matric algebra which already contains the direct sum, denote it D, of $m_1 \times m_1, \dots, m_s \times m_s$ total matric algebras and in which none of these is a submodule. Over a field it is easy to see that $\mathcal{H}_n$ must then operate as the full $m \times m$ matric algebra. (Pass to the skeleton of the induced operation of $\mathcal{H}_n$, which permits one to assume that all the $m_i = 1$.) Hence it holds if we map $\mathbb{Z}_{q,n}$ into any field and take as coefficients the subfield generated by the image, but this implies that it holds for $\mathbb{Z}_{q,n}$ itself. It follows that $V(\mathfrak{p};0)$ is a simple $\mathcal{H}_n$ module. If any two were isomorphic, say $V(\mathfrak{p}_1;0)$ and $V(\mathfrak{p}_2;0)$, then they would already be isomorphic as $\mathcal{H}_{n-1}$ modules. But for distinct non-increasing partitions $\mathfrak{p}$ of n, the sets of $\mathfrak{p}' \to \mathfrak{p}$ are distinct although generally not disjoint. It follows that $V(\mathfrak{p}_1;0)$ and $V(\mathfrak{p}_2;0)$ cannot be isomorphic as $\mathcal{H}_{n-1}$ modules. The assertion about the canonical orthogonal basis is now evident from the remarks preceding the theorem.

(ii), (iii) As in the case $d = 2$, for each $i = 1, \dots, d-1$ it is the case that $V(\mathfrak{p})$ is the orthogonal direct sum of $\ker X_i$ and the image of Y_i. Therefore, $V(\mathfrak{p})$ is the orthogonal direct sum of $V(\mathfrak{p};0)$ and images of the various Y_i. If $\mathfrak{p} = (n_1, \dots, n_d)$ and $\mathfrak{q} = (n_1, \dots, n_{i-1} - 1, n_i + 1, \dots, n_d)$ are both non–increasing partitions of n into d parts differing only in places $i-1$ and i, then we write $Y_i\mathfrak{q} = \mathfrak{p}$. The case $d = 2$ shows that $Y_i|V(\mathfrak{q};0) \to V(\mathfrak{p};0)$ is then an isomorphism. Therefore no simple modules can appear in the orthogonal complement of $V(\mathfrak{p};0)$ inside $V(\mathfrak{p})$ except those which are images of simple modules inside the various $V(\mathfrak{q})$ for which there is a Y_i with $Y_i\mathfrak{q} = \mathfrak{p}$. Every simple module must therefore come from some $V(\mathfrak{p};0)$. This shows moreover that V^n is a direct sum of simple submodules; it is "semisimple" or "completely reducible". To show that $\mathcal{H}_n$ is a direct sum of total matric algebras it is therefore sufficient to show that for $d \geq n$ its representation on V^n is faithful, which is true in the classical case and therefore also true for generic q. Now the ranks of the simple $V(\mathfrak{p};0)$ modules do not depend on q, so for $d = n$ we already know that $\mathcal{H}_n$ has a homomorphic image with total rank equal to that of $\mathcal{H}_n$ itself, namely, $n!$. Over a field we would be done. As our coefficient ring is a domain, it follows that the kernel contains only torsion elements of which there are none since $\mathcal{H}_n$ is free. □

We have thus come to one of our main conclusions: The Hecke algebra $\mathcal{H}_n$ "splits" over $\mathbb{Z}[q, q^{-1}, 1/n_{q^2}!]$, i.e., over that ring of coefficients, it becomes a direct sum of total matric algebras. This much could already have been obtained by repairing the omissions in [DJ] or [W], but the association to each non-increasing partition of n of a canonical simple module with an inner product and canonical orthogonal basis indexed by Young tableaux is not immediately deducible from either. Since our "canonical" orthogonal basis is indeed a basis, we have the following number–theoretic assertion.

Proposition 4.3. *(i) The norms of all canonical basis elements of the canonical $\mathcal{H}_n$ modules are units in $\mathbb{Z}[q, q^{-1}, 1/n_{q^2}!]$; (ii) (The case $q = 1$) The norms of all canonical basis elements of the canonical S_n modules are units in $Z[1/n!]$.*

5. The Donald-Flanigan conjecture

Consider $\mathcal{H}_n = \mathcal{H}_n(q)$ for the moment as defined over $\mathbb{Z}_q = \mathbb{Z}[q, q^{-1}]$ and replace q by $1+t$. Then $\mathbb{Z}_q \subset \mathbb{Z}[[t]]$, so we can consider $\mathcal{H}_n(1+t)$ as an algebra over the latter, and as such it is obviously a deformation of $\mathbb{Z}S_n$.

Corollary 1.2. *Setting $q = 1+t$, the Hecke algebra $\mathcal{H}_n(1+t)$ together with the multiplicatively closed subset S of $\mathbb{Z}[[t]]$ generated by $1/n_{q^2}! = 1/n_{(1+t)^2}!$ is a global solution to the Donald–Flanigan problem for the symmetric group S_n.*

Proof. Not only is $S^{-1}\mathcal{H}_n$ separable, but we have seen that it is already a direct sum of total matric algebras. Now observe that $n_{q^2}!$ with q replaced by $1+t$ becomes an element of $\mathbb{Z}[[t]]$ which does not vanish after reduction modulo any prime, for no factor $i_{q^2} = (1-q^{2i})/(1-q^2)$ can vanish. (If $p|i$ then the constant term of $i_{q^2}!$ vanishes, but never the entire expression, so it remains invertible as an element of $\mathbb{F}_p((t))$ for every rational prime p.) It follows that S, considered as a subset of $\mathbb{Z}[[t]]$, contains no rational prime. □

6. Idempotents

Having completed our principal task, we briefly discuss primitive idempotents, mainly to show that even in the classical case the "canonical" idempotents generated by our procedure differ from those constructed using Young symmetrizers. Since the quantized case will take us too far afield, we restrict attention to $\mathbb{Z}[1/n!]S_n$. Suppose that the rank of V is now also precisely n. Then we now know that the foregoing group ring is just $\bigoplus \operatorname{End} V(\mathfrak{p}; 0)$, where $\mathfrak{p}$ runs over the non-increasing partitions of n and the endomorphisms are with respect to the coefficient ring $\mathbb{Z}[1/n!]$. Each $V(\mathfrak{p}; 0)$ is free with a canonical orthogonal basis indexed by Young tableaux, and the norm of each basis element is a unit. If $v \in V(\mathfrak{p}; 0)$ is such a basis element, then the projection of an arbitrary $u \in V(\mathfrak{p}; 0)$ is $((u,v)/(v,v))v$, which is again in $V(\mathfrak{p}; 0)$ since the denominator is a unit. Therefore, associated to each Young tableau there is a primitive idempotent of $\mathbb{Z}[1/n!]S_n$, namely, the projection on the corresponding basis vector. We call this the *canonical* idempotent associated to the Young tableau.

Consider the simplest non-trivial case, $n = 3$, where we denote the basis vectors of V by x_1, x_2, x_3. We have three simple modules, $V((3); 0), V((1^3); 0)$, and $V((2,1); 0)$. The first two each have rank one and are spanned, respectively, by x_1^3 (not the "symmetrized" element $\sum_{\sigma \in S_n} x_{\sigma 1} x_{\sigma 2} x_{\sigma 3}$, which lies in $V((1^3)))$ and by the skew element $\sum_{\sigma \in S_n} (-1)^\sigma x_{\sigma 1} x_{\sigma 2} x_{\sigma 3}$. The module $V((2,1); 0)$ has rank two. Writing x for x_1 and y for x_2, it has, as we have seen, the canonical basis

$$v_1 = yx^2 - \frac{1}{2}x(xy + yx), \qquad v_2 = x(xy - yx)$$

corresponding to the Young tableaux which we may write, in obvious notation, as $[[1,2],[3]]$ and $[[1,3],[2]]$. The classical Young idempotents corresponding to these are, respectively,

$$\frac{1}{3}(1 - (13))(1 + (12)) = \frac{1}{3}(1 + (12) - (13) - (123)),$$

and

$$\frac{1}{3}(1 - (12))(1 + (13)) = \frac{1}{3}(1 + (13) - (12) - (132)).$$

Their sum is the central idempotent $\frac{1}{3}(2-(123)-(132))$, which is the projection operator of $\bigoplus V(\mathfrak{p};0)$ on $V((2,1);0)$. This depends only on the partition $(2,1)$ of 3, so we may denote it $e_{(2,1)}$. Then the projection on v_1 is $\frac{1}{2}(1+(23))e_{(2,1)}$ and that on v_2 is $\frac{1}{2}(1-(23))e_{(2,1)}$, as one can immediately verify. These are the "canonical" primitive idempotents associated to $[[12],3]$ and $[[13],2]$ and are clearly not the same as the Young idempotents (although they necessarily have the same sum as the corresponding Young idempotents).

Notice that once we know the central idempotents of $\mathbb{Z}[1/n!]S_n$, then the canonical idempotents can be computed inductively from the case of $n-1$ (with S_{n-1} acting on the last $n-1$ variables) because of the way that $V(\mathfrak{p};0)$ is constructed from the $V(\mathfrak{p}';0)$ with $\mathfrak{p}'\to\mathfrak{p}$. This is illustrated here. The basis element v_1 has come from the unique canonical basis element xx of $V((2);0)$ and v_2 from the unique basis element $xy-yx$ of $V((1,1);0)$. The projections on these are $\frac{1}{2}(1+(12))$ and $\frac{1}{2}(1-(12))$, respectively. These give rise to the projections on v_1 and v_2 where (12) must now be replaced by (23) since S_2 is now operating on the last two variables. The central idempotents can be computed using the independent Casimir operators of degrees $2,\dots,n$ of Usl_n. This is not difficult since we only need to know how they operate on the $V(\mathfrak{p};0)$, which is given readily by a beautiful theorem of Harish–Chandra (cf. [K, p.308]). Details, particularly in the quantized case, are the subject for another paper.

7. Afterword: Status of the Donald–Flanigan conjecture and applications to group theory

The general Donald–Flanigan conjecture, even in its original form, has proven remarkably resilient. It has been characterized as a "modular form of Maschke's theorem." The statement is straightforward but seems to go deep into the structure of finite groups. It encodes certain assertions about the cohomology of groups which imply that for any finite group G and prime p dividing $\#G$ there exists an element g for whose centralizer $C=C_G(g)$ one has $H^1(C,\mathbb{F}_p)\neq 0$. This can be restated as a previously unknown "dual" to Cauchy's theorem: If $p|\#G$ then there is an element $g\in G$ whose centralizer $C_G(g)$ contains a normal subgroup of index p, cf. [GG]. This has been verified by Fleischmann, Janiszczak and Lempken [FJL], who proved the stronger "weak non–Schur property": There exists in G an element g whose order is divisible by p and whose centralizer C has the property that its commutator subgroup C' does not contain the p–part of g. This proposition can be reduced to the case of simple G, and they prove it using the most difficult result so far known in group theory – the classification of the finite simple groups – by showing that it is indeed true for all of them. There is some delicacy to the choice of hypothesis; the "strong non-Schur property" which asserts that the preceding g can be taken to be a p–element fails for three of the exceptional groups. It is an interesting and not too difficult exercise to prove the weak non–Schur property directly for the alternating and symmetric groups. The existence of a global solution may have even more cohomological and group–theoretic consequences.

Various cases of the Donald–Flanigan conjecture are known, but so far, no other global solutions are known except the one in [ESps]. Donald and Flanigan in their original 1974 paper [DF] settled only the case of abelian groups. There was no further progress until 1988, when the second author [Sps] proved the conjecture for the group algebra over $\mathbb{F}_p$ of groups with cyclic p–Sylow subgroups (a condition equivalent, by Higman's theorem, to the group algebra having finite representation type). The present authors [GSps] also settled the case of groups with abelian normal p–Sylow. The Donald–Flanigan conjecture has been verified for all groups of order less than 32 except for the quaternion group and the "extra special" group of order 27. In [Sps2], the second author showed that for p–solvable groups with cyclic p–Sylow, the integral group ring and the p–modular semisimple deformation can be achieved by a single deformation with a "discrete" and a "continuous axis". Such a deformation has been called *liftable*. Michler [M] proved a local version of this result for blocks of cyclic defect group. In [ESps], the conjecture is confirmed for all blocks with dihedral defect group, in [Sps3], the deformations of the dihedral 2-groups are shown to be liftable, and in [ESps] this result is extended to other dihedral groups. One might expect that the Hecke algebra of a finite Coxeter group is always a global solution to the Donald–Flanigan problem, but this fails, e.g., for the dihedral group D_n of order $2n$. In that case, however, there is a suitable deformation of the Hecke algebra which serves, suggesting that for finite Coxeter groups there is always a global solution which is at least a deformation of its Hecke algebra.

The Donald–Flanigan conjecture is a tantalizing open problem in finite group theory. If it is true for all finite groups then the reasons would seem to lie deep, and if not, deciding for which groups it does hold may be equally hard.

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