HECKE ALGEBRAS WITH UNEQUAL PARAMETERS

G. Lusztig

Introduction

This is revised version of my book "Hecke algebra with unequal parameters", CRM monograph series vol.18, Amer.Math.Soc. 2003. (That book was based on the Aisenstadt lectures given at the CRM, Université de Montréal, in May/June 2002; and included material from lectures given at MIT during the Fall of 1999 [L15].)

This version updates the book by taking into account the recent results of Elias and Williamson [EW] which allow us to present the results in a more general setup. In particular the chapter on the quasisplit case (§16) is more general than that in the book; it makes use of results in the Appendix. In §9 a discussion of double cosets in a Coxeter group with respect to two parabolic subgroups has been added. The definition of the ring J (§18) is now done in more generality than in the book. Since in that generality J may not have a unit element, an imbedding of J into a ring with unit is described. A discussion of a (tensor) category version of J is given in 18.15-18.20 where it is shown how this leads to a new construction of a simple algebraic group from an affine Weyl group which, unlike earlier constructions, does not use perverse sheaves on an affine Grassmannian.

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Hecke algebras arise as endomorphism algebras of representations of groups induced by representations of subgroups. In these notes we are mainly interested in a particular kind of Hecke algebras, which arise in the representation theory of reductive algebraic groups over finite or p-adic fields (see 0.3, 0.6). These Hecke algebras are specializations of certain algebras (known as Iwahori-Hecke algebras) which can be defined without reference to algebraic groups, namely by explicit generators and relations (see 3.2) in terms of a Coxeter group W (see 3.1) and a weight function $L : W \to \mathbf{Z}$ (see 3.1), that is, a *weighted Coxeter group*. An Iwahori-Hecke algebra is completely specified by a *weighted Coxeter graph*, that is, the Coxeter graph of W (see 1.1) where for each vertex we specify the value of L at the corresponding simple reflection.

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A particularly simple kind of Iwahori-Hecke algebras corresponds to the case where the weight function is constant on the set of simple reflections (equal parameter case). In this case one has the theory of the "new basis" [KL1] and cells [KL1], [L7], [L9]. The main goal of these notes is to try to extend as much as possible the theory of the new basis to the general case (of not necessarily equal parameters). We give a number of conjectures for what should happen in the general case and we present some evidence for these conjectures.

We now review the contents of these notes.

§1 introduces Coxeter groups following [Bo]. We also give a realization of the classical affine Weyl groups as periodic permutations of $\mathbf{Z}$ following an idea of [L4]. §2 contains some standard results on the partial order on a Coxeter group. In §3 we introduce the Iwahori-Hecke algebra attached to a weighted Coxeter group. Useful references for this are [Bo],[GP]. In §4 we define the bar operator following [KL1]. This is used in §5 to define the "new basis" (c_w) of an Iwahori-Hecke algebra following [KL1] for equal parameters and [L3] in general. In §6 we study some multiplicative properties of the new basis, following [KL1] and [L3]. In §7 we compute explicitly the "new basis" in the case of dihedral groups. In §8 we define left, right and two-sided cells. In §9 we study the behaviour of the new basis in relation to a given parabolic subgroup. In §10,§12 we study a "basis" dual to the new basis. In §11 we consider the case of finite Coxeter groups. In §13 we study the function $\mathbf{a}$ on certain weighted Coxeter groups following an idea from [L7]. In §14 we present a list of conjectures concerning cells and the function $\mathbf{a}$ and we show that they can be deduced from a much shorter list of conjectures. These conjectures are established in a "split case" in §15 (following [L9]), in a "quasisplit case" in §16 and for an infinite dihedral group in §17. Note that in the first two cases the proof requires arguments from the theory of Soergel modules while in the third case the argument is computational. In §18, assuming the truth of these conjectures we develop the theory of J-rings in the weighted case, following an idea from [L9]; we also discuss a tensor category version of a J-ring. §19,§20,§21 (where W is assumed to be a Weyl group) are in preparation for §22 where the class of constructible representations of W is introduced and studied in the weighted case (conjecturally these are the representations of W carried by left cells), for §23 where two-sided cells are discussed and for §24 where certain virtual representations of W ("virtual cells) are discussed. In §25 we discuss the weighted Coxeter groups which arise in the examples 0.3 and 0.6. We formulate a conjecture (25.3) which relates the two-sided cells of such a weighted Coxeter group to the two-sided cells of a larger Coxeter group with the weight function given by the length. In §26 we state (following [L16]) the classification of irreducible representations of Hecke algebras of the type discussed in 0.6 in terms of the geometry of the dual group. In §27 we give a new realization of a Hecke algebra as in 0.3 or 0.6 as a space of functions on the rational points of an algebraic variety defined over $\mathbf{F}_q$. This leads us to a (partly conjectural) geometrical interpretation of the coefficients $p_{y,w}$ of the new basis of the Hecke algebra in terms of intersection cohomology, generalizing the results of [KL2]. We expect that this geometrical interpretation should play a role in the proof of the conjectures in §14 in the cases arising from algebraic groups as in 0.3, 0.6. In the Appendix we discuss Coxeter groups with a given automorphism which preserve the set of simple reflections.

0.1. In 0.1-0.8 we give a survey of the theory of Hecke algebras arising from reductive groups.

Let Γ be a group acting transitively on a set X. If $\mathbf{E}$ is a Γ-equivariant $\mathbf{C}$-vector bundle over X (with discrete topology) then the fibre $\mathbf{E}_x$ of $\mathbf{E}$ at $x \in X$ is naturally a representation of $\Gamma_x = \{g \in \Gamma; gx = x\}$. Moreover, for $x \in X$, $\mathbf{E} \mapsto \mathbf{E}_x$ is an equivalence from the category of Γ-equivariant vector bundles on X of finite dimension and that of finite dimensional Γ_x-modules over $\mathbf{C}$.

Let $\mathbf{E}$ be a Γ-equivariant $\mathbf{C}$-vector bundle of finite dimension over X. Then Γ acts naturally on the vector space $\oplus_{x\in X}\mathbf{E}_x$. (This is the representation of Γ induced by the representation of Γ_x on $\mathbf{E}_x$, for any $x \in X$.) The $\mathbf{C}$-algebra

$$H = H(\Gamma, X, \mathbf{E}) = \mathrm{End}_\Gamma(\oplus_{x\in X}\mathbf{E}_x)$$

is called the *Hecke algebra*. The image of the obvious imbedding

$$H \subset \prod_{(x,x')\in X\times X} \mathrm{Hom}(\mathbf{E}_x, \mathbf{E}_{x'}), \quad \phi \mapsto (\phi^x_{x'})_{(x,x')\in X\times X}$$

consists of all $(f^x_{x'}) \in \prod_{(x,x')\in X\times X} \mathrm{Hom}(\mathbf{E}_x, \mathbf{E}_{x'})$ such that

for any $x \in X$ we have $f^x_{x'} = 0$ for all but finitely many $x' \in X$;

for any $g \in \Gamma$ and any $(x, x') \in X \times X$, the compositions $\mathbf{E}_x \xrightarrow{f^x_{x'}} \mathbf{E}_{x'} \xrightarrow{g} \mathbf{E}_{gx'}$, $\mathbf{E}_x \xrightarrow{g} \mathbf{E}_{gx} \xrightarrow{f^{gx}_{gx'}} \mathbf{E}_{gx'}$ coincide.

For any Γ-orbit $\mathcal{C}$ in $X \times X$ we set

$$H_\mathcal{C} = H(\Gamma, X, \mathbf{E})_\mathcal{C} = \{\phi \in H; \phi^x_{x'} \neq 0 \implies (x', x) \in \mathcal{C}\}.$$

Then $H_\mathcal{C} = 0$ unless $\mathcal{C}$ is *finitary* in the following sense:

for some (or any) $x \in X$, the set $\{x' \in X; (x', x) \in \mathcal{C}\}$ is finite,

in which case

$$H_\mathcal{C} \xrightarrow{\sim} \mathrm{Hom}_{\Gamma_x\cap\Gamma_{x'}}(\mathbf{E}_x, \mathbf{E}_{x'}), \quad \phi \mapsto \phi^x_{x'}$$

for $(x', x) \in \mathcal{C}$. Moreover,

(a) $H = \oplus_{\mathcal{C} \text{ finitary}} H_\mathcal{C}$.

0.2. To explain how Hecke algebras arise from reductive algebraic groups we need the notion of "unipotent cuspidal representation".

Let p be a prime number and let $\mathbf{F}$ be an algebraic closure of the finite field with p elements. Let q be a power of p and let $\mathbf{F}_q$ be the subfield of $\mathbf{F}$ with q elements. Let G be a connected reductive algebraic group over $\mathbf{F}$ with a fixed $\mathbf{F}_q$ structure and let $F : G \to G$ be the corresponding Frobenius map.

We refer to [DL] for the notion of unipotent cuspidal representation of the finite group $G^F = G(\mathbf{F}_q)$. We will only give here the definition assuming that q is sufficiently large. Let E be an irreducible representation over $\mathbf{C}$ of G^F and let $\chi_E : G^F \to \mathbf{C}$ be its character. We say that E is unipotent if, for any F-stable maximal torus T of G, the restriction of χ_E to the set of regular elements in T^F is a constant, say $c_T \in \mathbf{Z}$. We say that E is unipotent cuspidal if, in addition, for any T as above that is contained in some proper F-stable parabolic subgroup of G, we have $c_T = 0$.

The unipotent cuspidal representations of G^F are classified in [L6]. For example, if G is a torus times a symplectic group of rank $n \geq 0$ then G^F has (up to isomorphism) a unique unipotent cuspidal representation if $n = k^2 + k$ for some integer $k \geq 0$, and none, otherwise.

0.3. Let G, F be as in 0.2. For any parabolic subgroup P of G let U_P be the unipotent radical of P and let $\bar{P} = P/U_P$. Let $\mathcal{P}$ be an F-stable G-conjugacy class of parabolic subgroups of G and let $\mathbf{E}$ be a G^F-equivariant vector bundle over $\mathcal{P}^F$ (a G^F-homogeneous space) such that for some (or any) $P \in \mathcal{P}^F$, the P^F-action on the fibre $\mathbf{E}_P$ of $\mathbf{E}$ at P factors through a unipotent, cuspidal $\bar{P}^F$-module. (To give such $\mathbf{E}$ is the same as to give, for some $P \in \mathcal{P}^F$, a unipotent cuspidal representation of $\bar{P}^F$.) The Hecke algebra $H(G^F, \mathcal{P}^F, \mathbf{E})$ is defined.

Let W be the set of G-orbits on the set of ordered pairs of Borel subgroups in G; it is known that W may be naturally regarded as a finite Coxeter group (see 1.1) with a set S of simple reflections. Now any Borel subgroup of G is contained in a unique subgroup in $\mathcal{P}$; this defines a (surjective) map from W to $G\backslash(\mathcal{P} \times \mathcal{P})$, the set of G-orbits in $\mathcal{P} \times \mathcal{P}$. The inverse image of the diagonal orbit under this map is the subgroup W_J of W generated by a subset J of S and $W \to G\backslash(\mathcal{P} \times \mathcal{P})$ factors through a bijection

(a) $$W_J\backslash W/W_J \xrightarrow{\sim} G\backslash(\mathcal{P} \times \mathcal{P}).$$

Let $\mathcal{W}$ be the set of all $w \in W$ such that $wW_J = W_J w$ and w has minimal length in $wW_J = W_J w$. Then $\mathcal{W}$ is a subgroup of W. The Frobenius map $u : W \to W$ restricts to an isomorphism $u : \mathcal{W} \to \mathcal{W}$ whose fixed point set $\mathcal{W}^u$ is naturally a Coxeter group with simple reflections indexed by $u\backslash(S - J)$ (set of orbits of $u : S - J \to S - J$). See 25.1(a). A G-orbit $\mathcal{O}$ on $\mathcal{P} \times \mathcal{P}$ is said to be *good* if for $(P, P') \in \mathcal{O}$ we have $(P \cap P')U_P = P$ or equivalently $(P \cap P')U_{P'} = P'$. Otherwise, $\mathcal{O}$ is said to be *bad*. If $\mathcal{O}$ is a good, F-stable G-orbit on $\mathcal{P} \times \mathcal{P}$ then $\mathcal{O}^F$ is a G^F-orbit on $\mathcal{P}^F \times \mathcal{P}^F$ and $\dim H(G^F, \mathcal{P}^F, \mathbf{E})_{\mathcal{O}^F} = 1$. If $\mathcal{O}$ is an F-stable bad G-orbit on $\mathcal{P} \times \mathcal{P}$ then $\mathcal{O}^F$ is a G^F-orbit on $\mathcal{P}^F \times \mathcal{P}^F$ and $\dim H(G^F, \mathcal{P}^F, \mathbf{E})_{\mathcal{O}^F} = 0$. Now the bijection (a) restricts (via the imbedding $\mathcal{W}^u \subset W_J\backslash W/W_J$, $w \mapsto W_J w W_J$) to a bijection $w \mapsto \mathcal{O}_w$ of $\mathcal{W}^u$ onto the set of good, F-stable G-orbits on $\mathcal{P} \times \mathcal{P}$. It follows that 0.1(a) becomes in our case

(b) $$H(G^F, \mathcal{P}^F, \mathbf{E}) = \oplus_{w \in \mathcal{W}^u} H(G^F, \mathcal{P}^F, \mathbf{E})_{\mathcal{O}_w}$$

with

(c) $$\dim H(G^F,\mathcal{P}^F,\mathbf{E})_{\mathcal{O}_w}=1 \text{ for all } w\in\mathcal{W}^u.$$

Let τ_k be the generator of $\mathcal{W}^u$ corresponding to $k\in u\backslash(S-J)$. There is a unique basis element T_{τ_k} of $H(G^F,\mathcal{P}^F,\mathbf{E})_{\mathcal{O}_{\tau_k}}$ such that

(d) $$(T_{\tau_k}+q^{-N_k/2})(T_{\tau_k}-q^{N_k/2})=0$$

for some $N_k\in\mathbf{Z}_{>0}$. (N_k is uniquely determined.) The elements $T_{\tau_k}(k\in u\backslash(S-J))$ generate the $\mathbf{C}$-algebra $H(G^F,\mathcal{P}^F,\mathbf{E})$. They satisfy identities of the form

(e) $$T_{\tau_k}T_{\tau_{k'}}T_{\tau_k}\dots=T_{\tau_{k'}}T_{\tau_k}T_{\tau_{k'}}\dots$$

for $k\neq k'$ in $u\backslash(S-J)$; both products have a number of factors equal to the order of $\tau_k\tau_{k'}$ in $\mathcal{W}^u$. Now $T_{\tau_k}\mapsto T_{\tau_k}$ gives an isomorphism from an Iwahori-Hecke algebra (see 3.2) specialized at $v=\sqrt{q}$ to the algebra $H(G^F,\mathcal{P}^F,\mathbf{E})$.

The function $k\mapsto N_k$ coincides with the function $k\mapsto L(\tau_k)$ in 25.2.

(The results in this subsection appeared in [L2,L1]. In the special case where $\mathcal{P}$ is the set of Borel subgroups of G and $\mathbf{E}$ is the trivial vector bundle $\mathbf{C}$, they were first proved by Iwahori [I]; if, in addition, $u=1$ on W then $N_k=1$ for all k.)

0.4. Let V be an $\mathbf{F}_q$-vector space of dimension $n\geq 2$. Then $G=SL(\mathbf{F}\otimes V)$ has a natural $\mathbf{F}_q$-structure. Let $\mathcal{P}$ be the set of all Borel subgroups of G. Then $\mathcal{P}^F$ may be identified as a set with G^F-action with the set $\mathcal{F}$ of all flags $V_*=(V_0\subset V_1\subset V_2\subset\dots\subset V_n)$ of subspaces of V ($\dim V_i=i$ for all i).

Let $V_*=(V_0\subset V_1\subset V_2\subset\dots\subset V_n)$, $V'_*=(V'_0\subset V'_1\subset V'_2\subset\dots\subset V'_n)$ be flags in $\mathcal{F}$. For $i\in[0,n],j\in[1,n]$ we set $d_{ij}=\dim\frac{V'_i\cap V_j}{V'_i\cap V_{j-1}}\in\{0,1\}$. For $i\in[0,n]$ we set $X_i=\{j\in[1,n];d_{ij}=1\}$. Then $\emptyset=X_0\subset X_1\subset X_2\subset\dots\subset X_n=[1,n]$ and for $i\in[1,n]$ there is a unique $a_i\in[1,n]$ such that $X_i=X_{i-1}\sqcup\{a_i\}$. Also, $i\mapsto a_i$ is a permutation of $[1,n]$. Now $(V_*,V'_*)\mapsto(a_i)$ defines a bijection of $G^F\backslash(\mathcal{P}^F\times\mathcal{P}^F)=G^F\backslash(\mathcal{F}\times\mathcal{F})$ with the symmetric group $\mathfrak{S}_n$. Let $\mathbf{E}$ be the trivial G^F-equivariant vector bundle $\mathbf{C}$ on $\mathcal{P}^F=\mathcal{F}$. Then $H(G^F,\mathcal{P}^F,\mathbf{E})$ is defined. In our case we have $W=\mathcal{W}=\mathcal{W}^u=\mathfrak{S}_n$.

0.5. Let V,n be as in 0.4. Assume that $n=2m$ and that V has a fixed non-degenerate symplectic form $\langle,\rangle:V\times V\to\mathbf{F}_q$. Then $G=Sp(\mathbf{F}\otimes V)$ has a natural $\mathbf{F}_q$-structure. Assume that $m=r+k^2+k$ where $k\in\mathbf{N},r\in\mathbf{Z}_{>0}$. Let $\mathcal{F}$ be the set of all flags $V_*=(V_0\subset V_1\subset V_2\subset\dots\subset V_r)$ of isotropic subspaces of V ($\dim V_i=i$ for all i). There is a unique G-conjugacy class $\mathcal{P}$ of parabolic subgroups of G such that, if $V_*\in\mathcal{F}$, then

$$\{g\in G;g(\mathbf{F}\otimes V_j)=\mathbf{F}\otimes V_j\quad\forall j\}\in\mathcal{P}.$$

We may identify $\mathcal{P}^F=\mathcal{F}$ as spaces with G^F-action.

Let $U \mapsto \mathcal{D}_k(U)$ be a functor from the category of symplectic vector spaces of dimension $2k^2+2k$ over $\mathbf{F}_q$ (and isomorphisms between them) to the category of $\mathbf{C}$-vector spaces (and isomorphisms between them) such that for any U, the $Sp(U)$-module $\mathcal{D}_k(U)$ is unipotent, cuspidal. (Such a functor exists and is unique up to isomorphism.) Let $\mathbf{E}$ be the vector bundle over $\mathcal{P}^F$ (or equivalently $\mathcal{F}$) whose fibre at $V_* = (V_0 \subset V_1 \subset V_2 \subset \dots \subset V_r) \in \mathcal{F}^F$ is $\mathcal{D}_k(V_r^\perp/V_r)$. (Here $V_s^\perp = \{x \in V; \langle x, V_s\rangle = 0\}$.)

This vector bundle is naturally G^F-equivariant (since $\mathcal{D}_k$ is a functor). Hence $H(G^F, \mathcal{P}^F, \mathbf{E})$ is defined.

Let $V_* = (V_0 \subset V_1 \subset V_2 \subset \dots \subset V_r)$, $V'_* = (V'_0 \subset V'_1 \subset V'_2 \subset \dots \subset V'_r)$ be flags in $\mathcal{F}$. The G-orbit of the point of $\mathcal{P} \times \mathcal{P}$ corresponding to (V_*, V'_*) is good if the following three equivalent conditions hold:

$V_r \cap V'_r = V_r \cap V'^\perp_r$,

$V_r \cap V'_r = V_r^\perp \cap V'_r$,

$V_r \cap V'_r = (V_r^\perp \cap V'^\perp_r) \cap (V_r + V'_r)$.

If these conditions hold, we can define an isomorphism $\psi^{V_r}_{V'_r} : V_r^\perp/V_r \to V'^\perp_r/V'_r$ by requiring that the diagram

$$\begin{array}{ccc} (V_r^\perp \cap V'^\perp_r)/(V_r \cap V'_r) & \xrightarrow{=} & (V_r^\perp \cap V'^\perp_r)/(V_r \cap V'_r) \\ \downarrow & & \downarrow \\ V_r^\perp/V_r & \xrightarrow{\psi^{V_r}_{V'_r}} & V'^\perp_r/V'_r \end{array}$$

(where the vertical maps are the isomorphisms induced by the inclusion) is commutative. In this case, to (V_*, V'_*) we associate an element σ of the group W_r of permutations of $\mathcal{S} = \{1, 2, \dots, r, r', \dots, 2', 1'\}$ that commute with the involution $f : \mathcal{S} \to \mathcal{S}, j \mapsto j', j' \mapsto j$. For $j \in [1, r]$ we set

$$A_j = \{h \in [1, r]; V_{h-1} \cap V'_j \ne V_h \cap V'_j\}, \quad B_j = f\{h \in [1, r]; V_{h-1}^\perp \cap V'_j \ne V_h^\perp \cap V'_j\}.$$

Then $\sharp(A_j \cap B_j) = j, A_1 \cap B_1 \subset A_2 \cap B_2 \subset \dots \subset A_s \cap B_s$ and $h \in A_j \implies h' \notin B_j$. For $j \in [1, r]$ define $a_j \in \mathcal{S}$ by $A_j \cup B_j = \{a_1, a_2, \dots, a_j\}$. Then σ is defined by the condition that $\sigma(j) = a_j$ for $j \in [1, r]$. We see that in our case, $\mathcal{W} = \mathcal{W}^u$ may be identified with W_r. In our case, the Iwahori-Hecke algebra corresponds to the weighted Coxeter graph

$$\bullet_{2k+1} = \bullet_1 - \bullet_1 - \dots - \bullet_1$$

(r vertices); in the case where $r = 1$ this should be interpreted as a graph with one vertex marked by $2k+1$.

0.6. Let ϵ be an indeterminate. Let $\mathbf{K}$ be the subfield of $\mathbf{F}((\epsilon))$ generated by $\mathbf{F}_q((\epsilon))$ and $\mathbf{F}$. Let $\mathbf{G}$ be a split connected simply connected almost simple algebraic group over $\mathbf{K}$ with a fixed $\mathbf{F}_q((\epsilon))$-rational structure. We identify $\mathbf{G}$ with its group of $\mathbf{K}$-points. There is a "Frobenius map" $F : \mathbf{G} \to \mathbf{G}$ whose fixed point set is $\mathbf{G}(\mathbf{F}_q((\epsilon)))$. Let $\mathcal{B}$ be the set of all Iwahori subgroups of $\mathbf{G}$. (This concept will be illustrated in 0.7.) A subgroup of $\mathbf{G}$ is said to be a parahoric subgroup if it is $\neq \mathbf{G}$ and it contains some Iwahori subgroup. If P is a parahoric subgroup then P has a "pro-unipotent radical" U_P and $\bar{P} = P/U_P$ is a connected, reductive group over $\mathbf{F}$. Let $\mathcal{P}$ be an F-stable $\mathbf{G}$-conjugacy class of parahoric subgroups of $\mathbf{G}$ and let $\mathbf{E}$ be a $\mathbf{G}^F$-equivariant vector bundle over $\mathcal{P}^F$ (a $\mathbf{G}^F$-homogeneous space) such that for some (or any) $P \in \mathcal{P}^F$, the P^F-action on the fibre $\mathbf{E}_P$ of $\mathbf{E}$ at P factors through a unipotent, cuspidal $\bar{P}^F$-module. (To give such $\mathbf{E}$ is the same as to give, for some $P \in \mathcal{P}^F$, a unipotent cuspidal representation of $\bar{P}^F$.) The Hecke algebra $H(\mathbf{G}^F, \mathcal{P}^F, \mathbf{E})$ is defined.

Let W be the set of $\mathbf{G}$-orbits on $\mathcal{B} \times \mathcal{B}$; it is known that W may be naturally regarded as a Coxeter group (more precisely, an affine Weyl group, see 1.15) with a set S of simple reflections. Now any Iwahori subgroup of $\mathbf{G}$ is contained in a unique subgroup in $\mathcal{P}$; this defines a (surjective) map from W to $\mathbf{G}\backslash(\mathcal{P} \times \mathcal{P})$, the set of $\mathbf{G}$-orbits in $\mathcal{P} \times \mathcal{P}$. The inverse image of the diagonal orbit under this map is the subgroup W_J of W generated by a subset J of S and $W \to \mathbf{G}\backslash(\mathcal{P} \times \mathcal{P})$ factors through a bijection

(a) $$W_J\backslash W/W_J \xrightarrow{\sim} \mathbf{G}\backslash(\mathcal{P} \times \mathcal{P}).$$

Let $\mathcal{W}$ be the set of all $w \in W$ such that $wW_J = W_J w$ and w has minimal length in $wW_J = W_J w$. Then $\mathcal{W}$ is a subgroup of W. The Frobenius map $u : W \to W$ restricts to an isomorphism $u : \mathcal{W} \to \mathcal{W}$ whose fixed point set $\mathcal{W}^u$ is naturally an infinite Coxeter group with simple reflections indexed by $u\backslash(S-J)$ (set of orbits of $u : S - J \to S - J$). (We make the additional assumption that $\sharp(u\backslash(S-J)) \geq 2$.) See 25.1(a). A $\mathbf{G}$-orbit $\mathcal{O}$ on $\mathcal{P} \times \mathcal{P}$ is said to be *good* if for $(P, P') \in \mathcal{O}$ we have $(P \cap P')U_P = P$ or equivalently $(P \cap P')U_{P'} = P'$. Otherwise, $\mathcal{O}$ is said to be *bad.* If $\mathcal{O}$ is a good, F-stable $\mathbf{G}$-orbit on $\mathcal{P} \times \mathcal{P}$ then $\mathcal{O}^F$ is a $\mathbf{G}^F$-orbit on $\mathcal{P}^F \times \mathcal{P}^F$ and $\dim H(\mathbf{G}^F, \mathcal{P}^F, \mathbf{E})_{\mathcal{O}^F} = 1$. If $\mathcal{O}$ is an F-stable bad $\mathbf{G}$-orbit on $\mathcal{P} \times \mathcal{P}$ then $\mathcal{O}^F$ is a $\mathbf{G}^F$-orbit on $\mathcal{P}^F \times \mathcal{P}^F$ and $\dim H(\mathbf{G}^F, \mathcal{P}^F, \mathbf{E})_{\mathcal{O}^F} = 0$. Now the bijection (a) restricts (via the imbedding $\mathcal{W}^u \subset W_J\backslash W/W_J$, $w \mapsto W_J w W_J$) to a bijection $w \mapsto \mathcal{O}_w$ of $\mathcal{W}^u$ onto the set of good, F-stable $\mathbf{G}$-orbits on $\mathcal{P} \times \mathcal{P}$. It follows that 0.1(a) becomes in our case

(b) $$H(\mathbf{G}^F, \mathcal{P}^F, \mathbf{E}) = \oplus_{w \in \mathcal{W}^u} H(\mathbf{G}^F, \mathcal{P}^F, \mathbf{E})_{\mathcal{O}_w}$$

with

(c) $$\dim H(\mathbf{G}^F, \mathcal{P}^F, \mathbf{E})_{\mathcal{O}_w} = 1 \text{ for all } w \in \mathcal{W}^u.$$

Let τ_k be the generator of $\mathcal{W}^u$ corresponding to $k \in u\backslash(S-J)$. There is a unique basis element T_{τ_k} of $H(\mathbf{G}^F, \mathcal{P}^F, \mathbf{E})_{\mathcal{O}_{\tau_k}}$ such that

(d) $$(T_{\tau_k} + q^{-N_k/2})(T_{\tau_k} - q^{N_k/2}) = 0$$

for some $N_k \in \mathbf{Z}_{>0}$. (N_k is uniquely determined.) The elements $T_{\tau_k}(k \in u\backslash(S-J))$ generate the $\mathbf{C}$-algebra $H(\mathbf{G}^F, \mathcal{P}^F, \mathbf{E})$. They satisfy identities of the form

$$T_{\tau_k} T_{\tau_{k'}} T_{\tau_k} \cdots = T_{\tau_{k'}} T_{\tau_k} T_{\tau_{k'}} \ldots$$

for $k \neq k'$ in $u\backslash(S-J)$ with $\tau_k \tau_{k'}$ of order $m_{k,k'} < \infty$ in $\mathcal{W}^u$ (both products have $m_{k,k'}$ factors). Now $T_{\tau_k} \mapsto T_{\tau_k}$ gives an isomorphism from an Iwahori-Hecke algebra (see 3.2) specialized at $v = \sqrt{q}$ to the algebra $H(\mathbf{G}^F, \mathcal{P}^F, \mathbf{E})$. The function $k \mapsto N_k$ coincides with the function $k \mapsto L(\tau_k)$ in 25.2.

(The results in this subsection appeared in [L11,L13]. In the special case where $\mathcal{P} = \mathcal{B}$, $u = 1$ and $\mathbf{E}$ is the trivial vector bundle $\mathbf{C}$, they were first proved by Iwahori-Matsumoto [IM]; in this case, $N_k = 1$ for all k.)

0.7. Let V be an $\mathbf{F}_q((\epsilon))$-vector space of dimension $n \in [2, \infty)$ with a fixed volume form ω. Then $G = SL(\mathbf{F} \otimes_{\mathbf{F}_q} V)$ is as in 0.6. Let $\mathfrak{A} = \mathbf{F}_q[[\epsilon]]$. A lattice in V is an $\mathfrak{A}$-submodule of V of rank n which generates V. For a lattice $\mathcal{L}$ in V we set $vol(\mathcal{L}) = r$ where $m \in \mathbf{Z}$ is defined by the condition that the n-th exterior power of $\mathcal{L}$ (an $\mathfrak{A}$-module) has $\epsilon^{-r}\omega$ as basis element. Let $\mathcal{F}$ be the set of all sequences of lattices $(\mathcal{L}_j)_{j \in \mathbf{Z}}$ such that

$$\mathcal{L}_{j-1} \subset \mathcal{L}_j, vol(\mathcal{L}_j) = j, \epsilon\mathcal{L}_j = \mathcal{L}_{j-n}$$

for all j. We may identify $\mathcal{B}^F$ with $\mathcal{F}$ as sets with (transitive) G^F-action.

Let $\mathcal{L}_* = (\mathcal{L}_j)_{j \in \mathbf{Z}}, \mathcal{L}'_* = (\mathcal{L}'_j)_{j \in \mathbf{Z}}$ be elements of $\mathcal{F}$. For $i, j \in \mathbf{Z}$ we set

$$d_{ij} = \dim \frac{\mathcal{L}'_i \cap \mathcal{L}_j}{\mathcal{L}'_i \cap \mathcal{L}_{j-1}} \in \{0, 1\}.$$

For $i \in \mathbf{Z}$ let $X_i = \{j \in \mathbf{Z}; d_{ij} = 1\}$. Then $X_{i-1} \subset X_i$ for all i and $\sharp(X_i - X_{i-1}) = 1$ for all i. Define $a_i \in \mathbf{Z}$ by $X_i = X_{i-1} \sqcup \{a_i\}$. Now $X_{i-n} = X_i - n$. Hence

(a) $$a_{i+n} = a_i + n \text{ for all } i \in \mathbf{Z}.$$

One can check that $i \mapsto a_i$ is a bijection $\mathbf{Z} \to \mathbf{Z}$. Using the fact that $vol(\mathcal{L}_j) = vol(\mathcal{L}'_j) = j$ we see that

(b) $$\sum_{i=1}^{n} (a_i - i) = 0.$$

This gives a bijection of W onto the group of all bijections $\mathbf{Z} \xrightarrow{\sim} \mathbf{Z}$ that satisfy (a),(b) (see 1.12).

0.8. Let V, n be as in 0.7. Assume that $n = 2m$ and that V has a fixed non-degenerate symplectic form $\langle,\rangle : V \times V \to \mathbf{F}_q((\epsilon))$. Then $G = Sp(\mathbf{F} \otimes_{\mathbf{F}_q} V)$ is as in 0.6. If $\mathcal{L}$ is a lattice in V then $\mathcal{L}^\sharp = \{x \in V; \langle x, \mathcal{L}\rangle \in \mathfrak{A}\}$ is again a lattice; moreover, $(\mathcal{L}^\sharp)^\sharp = \mathcal{L}$. Assume that $m = r + k^2 + k + l^2 + l$ where $k, l, r \in \mathbf{N}, r \geq 1$. Let $\mathcal{N}$ be the set of all integers of the form $a + 2mt$ where $t \in \mathbf{Z}$ and

$$k^2 + k \leq a \leq k^2 + k + r \text{ or } -(k^2 + k + r) \leq a \leq -(k^2 + k).$$

Let $\mathcal{F}$ be the set of all sequences of lattices $(\mathcal{L}_j)_{j\in\mathcal{N}}$ such that
$\mathcal{L}_j \subset \mathcal{L}_{j'}$ if $j \leq j'$ in $\mathcal{N}$,
$\mathcal{L}_j^\sharp = \mathcal{L}_{-j}$ for all $j \in \mathcal{N}$,
$\epsilon\mathcal{L}_j = \mathcal{L}_{j-2m}$ for all $j \in \mathcal{N}$,
$vol(\mathcal{L}_j) = j$ for all $j \in \mathcal{N}$.
Here the volume of a lattice is defined in terms of the volume form on V attached to the symplectic form. There is a unique G-conjugacy class $\mathcal{P}$ of parahoric subgroups of G such that, if $(\mathcal{L}_j)_{j\in\mathcal{N}} \in \mathcal{F}$, then

$$\{g \in G; g(\mathbf{F} \otimes \mathcal{L}_j) = \mathbf{F} \otimes \mathcal{L}_j \quad \forall j \in \mathcal{N}\} \in \mathcal{P}.$$

We may identify $\mathcal{P}^F$ and $\mathcal{F}$ as sets with G^F-action. If $(\mathcal{L}_j)_{j\in\mathcal{N}} \in \mathcal{F}$, then the $\mathbf{F}_q$-vector space $\mathcal{L}_{k^2+k}/\mathcal{L}_{-k^2-k}$ (of dimension $2k^2 + 2k$) has a natural non-degenerate symplectic form induced by $x, y \mapsto Res\langle x, y\rangle$ and the $\mathbf{F}_q$-vector space $\mathcal{L}_{2m-k^2-k-r}/\mathcal{L}_{k^2+k+r}$ (of dimension $2l^2 + 2l$) has a natural non-degenerate symplectic form induced by $x, y \mapsto Res\langle x, \epsilon y\rangle$. Here $Res : \mathbf{F}_q((\epsilon)) \to \mathbf{F}_q$ denotes residue at 0.

Let $\mathcal{D}_k$ be a functor as in 0.5 and let $\mathcal{D}_l$ be an analogous functor obtained by replacing k by l. Let $\mathbf{E}$ be the vector bundle over $\mathcal{P}^F$ (or equivalently $\mathcal{F}$) whose fibre at $(\mathcal{L}_j)_{j\in\mathcal{N}} \in \mathcal{F}$ is

$$\mathcal{D}_k(\mathcal{L}_{k^2+k}/\mathcal{L}_{-k^2-k}) \otimes \mathcal{D}_l(\mathcal{L}_{2m-k^2-k-r}/\mathcal{L}_{k^2+k+r}).$$

This vector bundle is naturally G^F-equivariant (since $\mathcal{D}_k, \mathcal{D}_l$ are functors). Hence $H(G^F, \mathcal{P}^F, \mathbf{E})$ is defined. In our case, the Iwahori-Hecke algebra corresponds to the weighted Coxeter graph

$$\bullet_{2k+1} = \bullet_1 - \bullet_1 - \cdots - \bullet_1 = \bullet_{2l+1}$$

($r + 1$ vertices); in the case where $r = 1$ this should be interpreted as a Coxeter graph with 2 vertices marked by $2k + 1, 2l + 1$, joined by a quadruple edge.

0.9. Notation. We set $[a, b] = \{z \in \mathbf{Z}; a \leq z \leq b\}$, $[a, b) = \{z \in \mathbf{Z}; a \leq z < b\}$. If X is a subset of a group G, we denote by $\langle X\rangle$ the subgroup of G generated by X.

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Appendix

1. Coxeter groups

1.1. Let S be a finite set and let $(m_{s,s'})_{(s,s')\in S\times S}$ be a matrix with entries in $\mathbf{N}\cup\{\infty\}$ such that $m_{s,s}=1$ for all s and $m_{s,s'}=m_{s',s}\geq 2$ for all $s\neq s'$. (A *Coxeter matrix.*) In the case where $m_{s,s'}\in\{2,3,4,6,\infty\}$ for all $s\neq s'$, the matrix $(m_{s,s'})_{(s,s')\in S\times S}$ is completely described by a graph (the *Coxeter graph*) with set of vertices in bijection with S where the vertices corresponding to $s\neq s'$ are joined by an edge if $m_{s,s'}=3$, by a double edge if $m_{s,s'}=4$, by a triple edge if $m_{s,s'}=6$, by a quadruple edge if $m_{s,s'}=\infty$.

Let W be the group defined by the generators $s (s\in S)$ and relations

$$(ss')^{m_{s,s'}}=1$$

for any s, s' in S such that $m_{s,s'} < \infty$. We say that W, S is a *Coxeter group*. Note that the Coxeter matrix is uniquely determined by W, S (see 1.3(b) below). We sometimes refer to W itself as a Coxeter group. In W we have $s^2 = 1$ for all s. Clearly, there is a unique homomorphism

$$\mathrm{sgn} : W \to \{1, -1\}$$

such that $\mathrm{sgn}(s) = -1$ for all s. (”*Sign* representation”.)

For $w \in W$ let $l(w)$ be the smallest integer $q \geq 0$ such that $w = s_1 s_2 \dots s_q$ with $s_1, s_2, \dots, s_q$ in S. (We then say that $w = s_1 s_2 \dots s_q$ is a *reduced expression* and $l(w)$ is the *length* of w.) Now $l(1) = 0, l(s) = 1$ for $s \in S$. (Indeed, $s \neq 1$ in W since $\mathrm{sgn}(s) = -1, \mathrm{sgn}(1) = 1$.)

Lemma 1.2. *Let $w \in W, s \in S$.*

(a) We have either $l(sw) = l(w) + 1$ or $l(sw) = l(w) - 1$.

(b) We have either $l(ws) = l(w) + 1$ or $l(ws) = l(w) - 1$.

Clearly, $\mathrm{sgn}(w) = (-1)^{l(w)}$. Since $\mathrm{sgn}(sw) = -\mathrm{sgn}(w)$, we have $(-1)^{l(sw)} = -(-1)^{l(w)}$. Hence $l(sw) \neq l(w)$. This, together with the obvious inequalities $l(w) - 1 \leq l(sw) \leq l(w) + 1$ gives (a). The proof of (b) is similar.

Proposition 1.3. *Let E be an $\mathbf{R}$-vector space with basis $(e_s)_{s \in S}$. For $s \in S$ define a linear map $\sigma_s : E \to E$ by $\sigma_s(e_{s'}) = e_{s'} + 2\cos\frac{\pi}{m_{s,s'}} e_s$ for all $s' \in S$.*

(a) There is a unique homomorphism $\sigma : W \to GL(E)$ such that $\sigma(s) = \sigma_s$ for all $s \in S$.

(b) If $s \neq s'$ in S, then ss' has order $m_{s,s'}$ in W. In particular, $s \neq s'$ in W.

We have $\sigma_s(e_s) = -e_s$ and σ_s induces the identity map on $E/\mathbf{R}e_s$. Hence $\sigma_s^2 = 1$. Now let $s \neq s'$ in S, $m = m_{s,s'}$, $\Phi = \sigma_s \sigma_{s'}$. We have
$\Phi(e_s) = (4\cos^2\frac{\pi}{m} - 1)e_s + 2\cos\frac{\pi}{m} e_{s'}$,
$\Phi(e_{s'}) = -2\cos\frac{\pi}{m} e_s - e_{s'}$.
Hence Φ restricts to an endomorphism ϕ of $\mathbf{R}e_s \oplus \mathbf{R}e_{s'}$ whose characteristic polynomial is

$$X^2 - 2\cos\frac{2\pi}{m} X + 1 = (X - e^{2\pi\sqrt{-1}/m})(X - e^{-2\pi\sqrt{-1}/m}).$$

It follows that, if $2 < m < \infty$, then $1 + \phi + \phi^2 + \dots + \phi^{m-1} = 0$. The same holds if $m = 2$ (in this case we see directly that $\phi = -1$). Since Φ induces the identity map on $E/(\mathbf{R}e_s \oplus \mathbf{R}e_{s'})$, it follows that $\Phi : E \to E$ has order m (if $m < \infty$). If $m = \infty$, we have $\phi \neq 1$ and $(\phi - 1)^2 = 0$, hence ϕ has infinite order and Φ has also infinite order. Now (a), (b) follow.

Corollary 1.4. *Let $s_1 \neq s_2$ in S. For $k \geq 0$ let $1_k = s_1 s_2 s_1 \dots$ (k factors), $2_k = s_2 s_1 s_2 \dots$ (k factors).*

(a) Assume that $m = m_{s_1,s_2} < \infty$. Then $\langle s_1, s_2 \rangle$ consists of the elements $1_k, 2_k$ $(k = 0, 1, \dots, m)$; these elements are distinct except for the equalities $1_0 = 2_0, 1_m = 2_m$. For $k \in [0, m]$ we have $l(1_k) = l(2_k) = k$.

(b) Assume that $m_{s_1,s_2} = \infty$. *Then* $\langle s_1, s_2\rangle$ *consists of the elements* $1_k, 2_k$ *(*$k = 0, 1, \dots$*); these elements are distinct except for the equality* $1_0 = 2_0$. *For all* $k \ge 0$ *we have* $l(1_k) = l(2_k) = k$.

This follows immediately from 1.3(b).

We identify S with a subset of W (see 1.3(b)) said to be the set of *simple reflections*. Let

$$T = \cup_{w\in W} wSw^{-1} \subset W.$$

Proposition 1.5. *Let* $R = \{1, -1\} \times T$. *For* $s \in S$ *define* $U_s : R \to R$ *by* $U_s(\epsilon, t) = (\epsilon(-1)^{\delta_{s,t}}, sts)$ *where* δ *is the Kronecker symbol. There is a unique homomorphism* U *of* W *into the group of permutations of* R *such that* $U(s) = U_s$ *for all* $s \in S$.

We have $U_s^2(\epsilon, t) = (\epsilon(-1)^{\delta_{s,t}+\delta_{s,sts}}, t) = (\epsilon, t)$ since the conditions $s = t$, $s = sts$ are equivalent. Thus, $U_s^2 = 1$. For $s \ne s'$ in S with $m = m_{s,s'} < \infty$ we have

$$U_sU_{s'}(\epsilon, t) = (\epsilon(-1)^{\delta_{s',t}+\delta_{s,s'ts'}}, ss'ts's)$$

hence

$$(U_sU_{s'})^m(\epsilon, t) = (\epsilon(-1)^\kappa, (ss')^m t(s's)^m) = (\epsilon(-1)^\kappa, t)$$

where

$$\kappa = \delta_{s',t} + \delta_{s,s'ts'} + \delta_{s',ss'ts's} + \dots = \delta_{s',t} + \delta_{s'ss',t} + \delta_{s'ss'ss',t} + \dots$$

(both sums have exactly $2m$ terms). It is enough to show that κ is even, or that t appears an even number of times in the $2m$-term sequence $s', s'ss', s'ss'ss', \dots$. This follows from the fact that in this sequence the k-th term is equal to the $(k + m)$-th term for $k = 1, 2, \dots, m$.

Proposition 1.6. *Let* $w \in W$. *Let* $w = s_1s_2 \dots s_q$ *be a reduced expression.*

(a) The elements $s_1, s_1s_2s_1, s_1s_2s_3s_2s_1, \dots, s_1s_2 \dots s_q \dots s_2s_1$ *are distinct.*

(b) These elements form a subset of T *that depends only on* w, *not on the choice of reduced expression for it.*

Assume that $s_1s_2 \dots s_i \dots s_2s_1 = s_1s_2 \dots s_j \dots s_2s_1$ for some $1 \le i < j \le q$. Then $s_i = s_{i+1}s_{i+2} \dots s_j \dots s_{i+2}s_{i+1}$ hence

$$\begin{aligned} s_1s_2 \dots s_q &= s_1s_2 \dots s_{i-1}(s_{i+1}s_{i+2} \dots s_j \dots s_{i+2}s_{i+1})s_{i+1} \dots s_js_{j+1} \dots s_q \\ &= s_1s_2 \dots s_{i-1}s_{i+1}s_{i+2} \dots s_{j-1}s_{j+1} \dots s_q, \end{aligned}$$

which shows that $l(w) \le q - 2$, contradiction. This proves (a).

For $(\epsilon, t) \in R$ we have (see 1.5) $U(w^{-1})(\epsilon, t) = (\epsilon\eta(w, t), w^{-1}tw)$ where $\eta(w, t) = \pm 1$ depends only on w, t. On the other hand,

$$\begin{aligned} U(w^{-1})(\epsilon, t) &= U_{s_q} \dots U_{s_1}(\epsilon, t) \\ &= (\epsilon(-1)^{\delta_{s_1,t}+\delta_{s_2,s_1ts_1}+\dots+\delta_{s_q,s_{q-1}\dots s_1ts_1\dots s_{q-1}}}, w^{-1}tw) \\ &= (\epsilon(-1)^{\delta_{s_1,t}+\delta_{s_1s_2s_1,t}+\dots+\delta_{s_1\dots s_q\dots s_1,t}}, w^{-1}tw). \end{aligned}$$

Thus, $\eta(w,t)=(-1)^{\delta_{s_1,t}+\delta_{s_1s_2s_1,t}+\dots+\delta_{s_1\dots s_q\dots s_1,t}}$. Using (a), we see that for $t\in T$, the sum $\delta_{s_1,t}+\delta_{s_1s_2s_1,t}+\dots+\delta_{s_1\dots s_q\dots s_1,t}$ is 1 if t belongs to the subset in (b) and is 0, otherwise. Hence the subset in (b) is just $\{t\in T;\eta(w,t)=-1\}$. This completes the proof.

Proposition 1.7. *Let $w\in W, s\in S$ be such that $l(sw)=l(w)-1$. Let $w=s_1s_2\dots s_q$ be a reduced expression. Then there exists $j\in[1,q]$ such that*
$ss_1s_2\dots s_{j-1}=s_1s_2\dots s_j$.

Let $w'=sw$. Let $w'=s'_1s'_2\dots s'_{q-1}$ be a reduced expression. Then $w=ss'_1s'_2\dots s'_{q-1}$ is another reduced expression. By 1.6(b), the q-term sequences

$$s_1, s_1s_2s_1, s_1s_2s_3s_2s_1,\dots \text{ and } s, ss'_1s, ss'_1s'_2s'_1s,\dots$$

coincide up to rearranging terms. In particular, $s=s_1s_2\dots s_j\dots s_2s_1$ for some $j\in[1,q]$. The proposition follows.

1.8. Let X be the set of all sequences $(s_1,s_2,\dots,s_q)$ in S such that $s_1s_2\dots s_q$ is a reduced expression in W. We regard X as the vertices of a graph in which $(s_1,s_2,\dots,s_q),(s'_1,s'_2,\dots,s'_{q'})$ are joined if one is obtained from the other by replacing m consecutive entries of form $s,s',s,s',\dots$ by the m entries $s',s,s',s,\dots$; here $s\ne s'$ in S are such that $m=m_{s,s'}<\infty$. We use the notation

$$(s_1,s_2,\dots,s_q)\sim(s'_1,s'_2,\dots,s'_{q'})$$

for "$(s_1,s_2,\dots,s_q),(s'_1,s'_2,\dots,s'_{q'})$ are in the same connected component of X". (When this holds we have necessarily $q=q'$ and $s_1s_2\dots s_q=s'_1s'_2\dots s'_q$ in W.)

The following result is due to Matsumoto and Tits.

Theorem 1.9. *Let $\mathbf{s}=(s_1,s_2,\dots,s_q),\mathbf{s}'=(s'_1,s'_2,\dots,s'_q)$ in X be such that $s_1s_2\dots s_q=s'_1s'_2\dots s'_q=w\in W$. Then $\mathbf{s}\sim\mathbf{s}'$.*

Let C (resp. C') be the connected component of X that contains $\mathbf{s}$ (resp. $\mathbf{s}'$). For $i\in[1,q]$ we set
$\mathbf{s}(i)=(\dots,s'_1,s_1,s'_1,s_1,s_2,s_3,\dots,s_i)$ (a q-element sequence in S),
$\underline{\mathbf{s}}(i)=\dots s'_1s_1s'_1s_1s_2s_3\dots s_i\in W$ (the product of this sequence).
Let C_i be the connected component of X that contains $\mathbf{s}(i)$. Then $\mathbf{s}=\mathbf{s}(q)$. Hence $C=C_q$.

We argue by induction on q. The theorem is obvious for $q\in\{0,1\}$. We now assume that $q\ge 2$ and that the theorem is known for $q-1$. We first prove the following weaker statement.

(*A*) *In the setup of the theorem we have either* $\mathbf{s}\sim\mathbf{s}'$ *or*

(a) $s_1s_2\dots s_q=s'_1s_1s_2\dots s_{q-1}$ and $(s'_1,s_1,s_2,\dots,s_{q-1})\sim(s'_1,s'_2,\dots,s'_q)$.

We have $l(s'_1w)=l(w)-1$. By 1.7 we have $s'_1s_1s_2\dots s_{i-1}=s_1s_2\dots s_i$ for some $i\in[1,q]$, so that $w=s'_1s_1s_2\dots s_{i-1}s_{i+1}\dots s_q$. In particular,

$$(s'_1,s_1,s_2,\dots,s_{i-1},s_{i+1},\dots,s_q)\in X.$$

By the induction hypothesis, $(s_1,s_2,\dots,s_{i-1},s_{i+1},\dots,s_q)\sim(s'_2,\dots,s'_q)$. Hence

(b) $$(s'_1,s_1,s_2,\dots,s_{i-1},s_{i+1},\dots,s_q)\sim(s'_1,s'_2,\dots,s'_q).$$

Assume first that $i<q$. Then from $s'_1s_1s_2\dots s_{i-1}s_{i+1}\dots s_{q-1}=s_1s_2\dots s_{q-1}$ and the induction hypothesis we deduce that
$(s'_1,s_1,s_2,\dots,s_{i-1},s_{i+1},\dots,s_{q-1})\sim(s_1,s_2,\dots,s_{q-1})$, hence
$(s'_1,s_1,s_2,\dots,s_{i-1},s_{i+1},\dots,s_{q-1},s_q)\in C$.
Combining this with (b) we deduce that $C=C'$.

Assume next that $i=q$ so that $s_1s_2\dots s_q=s'_1s_1s_2\dots s_{q-1}$. Then (b) shows that (a) holds. Thus, (A) is proved.

Next we prove for $p\in[0,q-2]$ the following generalization of (A).

(A'_p) *In the setup of the theorem we have either $C=C'$ or:*
for $i\in[q-p-1,q]$ we have $\mathbf{s}(i)\in X,\underline{\mathbf{s}}(i)=w$, $C_i=C$ if $i-q\in2\mathbf{Z}$, $C_i=C'$ if $i-q\notin2\mathbf{Z}$.
For $p=0$ this reduces to (A). Assume now that $p>0$ and that (A'_{p-1}) is already known. We prove that (A'_p) holds.

If $C=C'$, then we are done. Hence by (A'_{p-1}) we may assume that: for $i\in[q-p,q]$ we have $\mathbf{s}(i)\in X,\underline{\mathbf{s}}(i)=w$, $C_i=C$ if $i-q\in2\mathbf{Z}$, $C_i=C'$ if $i-q\notin2\mathbf{Z}$.

Applying (A) to $\mathbf{s}(q-p),\mathbf{s}(q-p+1)$ (instead of $\mathbf{s},\mathbf{s}'$), we see that either $C_{q-p}=C_{q-p+1}$ or:
$\mathbf{s}(q-p),\mathbf{s}(q-p-1)$ are in X, $\underline{\mathbf{s}}(q-p)=\underline{\mathbf{s}}(q-p-1)$ and $C_{q-p-1}=C_{q-p+1}$.
In both cases, we see that (A'_p) holds.

This completes the inductive proof of (A'_p). In particular, (A'_{q-2}) holds. In other words, in the setup of the theorem, either $C=C'$ holds or:

(c) *for $i\in[1,q]$ we have $\mathbf{s}(i)\in X,\underline{\mathbf{s}}(i)=w$, $C_i=C$ if $i-q\in2\mathbf{Z}$, $C_i=C'$ if $i-q\notin2\mathbf{Z}$.*
If $C=C'$, then we are done. Hence we may assume that (c) holds. In particular,

(d) $$\mathbf{s}(2)\in X,\mathbf{s}(1)\in X,\underline{\mathbf{s}}(2)=\underline{\mathbf{s}}(1).$$

From $\mathbf{s}(1)\in X$ and $q\ge2$ we see that $s'_1\ne s_1$ and that $q\le m=m_{s_1,s'_1}$. From $\underline{\mathbf{s}}(2)=\underline{\mathbf{s}}(1)$ we see that $s_2\in\langle s_1,s'_1\rangle$, hence s_2 is either s_1 or s'_1. In fact we cannot have $s_2=s_1$ since this would contradict $\mathbf{s}(2)\in X$. Hence $s_2=s'_1$. We see that $\mathbf{s}(2)=(\dots,s'_1,s_1,s'_1,s_1,s'_1)$ (the number of terms is $q,q\le m$). Since $\underline{\mathbf{s}}(2)=\underline{\mathbf{s}}(1)$, it follows that $q=m$, so that $\mathbf{s}(2),\mathbf{s}(1)$ are joined in X. It follows that $C_2=C_1$. By (c), for some permutation a,b of $1,2$ we have $C_a=C,C_b=C'$. Since $C_a=C_b$ it follows that $C=C'$. The theorem is proved.

Proposition 1.10. *Let $w \in W$ and let $s, t \in S$ be such that $l(swt) = l(w), l(sw) = l(wt)$. Then $sw = wt$.*

Let $w = s_1 s_2 \dots s_q$ be a reduced expression.

Assume first that $l(wt) = q+1$. Then $s_1 s_2 \dots s_q t$ is a reduced expression for wt. Now $l(swt) = l(wt)-1$ hence by 1.7 there exists $i \in [1, q]$ such that $ss_1 s_2 \dots s_{i-1} = s_1 s_2 \dots s_i$ or else $ss_1 s_2 \dots s_q = s_1 s_2 \dots s_q t$. If the second alternative occurs, we are done. If the first alternative occurs, we have $sw = s_1 s_2 \dots s_{i-1} s_{i+1} \dots s_q$ hence $l(sw) \le q-1$. This contradicts $l(sw) = l(wt)$.

Assume next that $l(wt) = q-1$. Let $w' = wt$. Then $l(sw't) = l(w'), l(sw') = l(w't)$. We have $l(w't) = l(w') + 1$ hence the first part of the proof applies and gives $sw' = w't$. Hence $sw = wt$. The proposition is proved.

1.11. We can regard S as the set of vertices of a graph in which s, s' are joined if $m_{s,s'} > 2$. We say that W is *irreducible* if this graph is connected. It is easy to see that in general, W is naturally a product of irreducible Coxeter groups, corresponding to the connected components of S.

In the setup of 1.3, let $(,) : E \times E \to \mathbf{R}$ be the symmetric $\mathbf{R}$-bilinear form given by $(e_s, e_{s'}) = -\cos \frac{\pi}{m_{s,s'}}$. Then $\sigma(w) : E \to E$ preserves $(,)$ for any $w \in W$.

We say that W is *tame* if $(e, e) \ge 0$ for any $e \in E$. It is easy to see that, if W is finite then W is tame.

We say that W is *integral* if, for any $s \ne s'$ in S, we have $4\cos^2 \frac{\pi}{m_{s,s'}} \in \mathbf{N}$ (or equivalently $m_{s,s'} \in \{2, 3, 4, 6, \infty\}$).

We will be mainly interested in the case where W is tame. The tame, irreducible W are of three kinds:

(a) finite, integral;
(b) finite, non-integral;
(c) tame, infinite (and automatically integral).

1.12. For $k \in \mathbf{Z}$ define $\rho_k : \mathbf{Z} \to \mathbf{Z}$ by $\rho_k(z) = z + k$. Let $n \ge 2$. Let $\tilde{W}$ be the group of all permutations $\sigma : \mathbf{Z} \to \mathbf{Z}$ such that $\sigma\rho_n = \rho_n\sigma$. Define $\chi : \tilde{W} \to \mathbf{Z}$ by $\chi(\sigma) = \sum_{k \in X} (\sigma(k) - k)$ where X is a set of representatives for the residue classes $\mod n$ in $\mathbf{Z}$. One checks that χ does not depend on the choice of X and χ is a group homomorphism with image $n\mathbf{Z}$. Now $\tilde{W}' = \ker(\chi)$ is generated by $\{s_m; m \in \mathbf{Z}/n\mathbf{Z}\}$ where $s_m : \mathbf{Z} \to \mathbf{Z}$ is defined by

$s_m(z) = z + 1$ if $z = m \mod n$,
$s_m(z) = z - 1$ if $z = m + 1 \mod n$,
$s_m(z) = z$ for all other $z \in \mathbf{Z}$.

It is a Coxeter group on these generators, said to be of type $\tilde{A}_{n-1}$. (This description of $\tilde{W}'$ appears in [L4].) For $n \ge 3$, $m, m' \in \mathbf{Z}/n\mathbf{Z}$ are joined by a single edge in the Coxeter graph if $m - m' = 1 \mod n$ and are not joined otherwise. For $n = 2$, $0, 1 \in \mathbf{Z}/2\mathbf{Z}$ are joined by a quadruple edge in the Coxeter graph. The length function on $\tilde{W}'$ is given by

$$l(\sigma) = \sharp(Y_\sigma/\tau_n)$$

where, for $\sigma \in \tilde{W}'$,

$$Y_\sigma = \{(i,j) \in \mathbf{Z} \times \mathbf{Z}; i < j, \sigma(i) > \sigma(j)\}$$

and Y_σ/τ_n is the (finite) set of orbits of $\tau_n : Y_\sigma \to Y_\sigma$, $(i,j) \mapsto (i+n, j+n)$.

1.13. Assume now that $n = 2p \ge 4$, where $p \in \mathbf{N}$. Let W be the subgroup of $\tilde{W}$ consisting of all $\sigma \in \tilde{W}$ that commute with the involution $\mathbf{Z} \to \mathbf{Z}, z \mapsto 1 - z$. We compute $\chi(\sigma)$ for $\sigma \in W$, taking $X = \{-(p-1), \dots, -1, 0, 1, 2, \dots, p\}$:

$$\chi(\sigma) = \sum_{k \in [1,p]} (\sigma(k) - k) + \sum_{k \in [1,p]} (\sigma(1-k) - (1-k))$$
$$= \sum_{k \in [1,p]} (\sigma(k) - k) + \sum_{k \in [1,p]} (1 - \sigma(k) - (1-k)) = 0.$$

Thus, W is a subgroup of $\tilde{W}'$. Now W is generated by $s'_0, s'_1, \dots, s'_p$ where

$$s'_0 = s_0, s'_1 = s_1 s_{-1}, s'_2 = s_2 s_{-2}, \dots, s'_{p-1} = s_{p-1} s_{1-p}, s'_p = s_p.$$

It is a Coxeter group on these generators, said to be of type $\tilde{C}_p$. The Coxeter graph is

$$\bullet = \bullet - \bullet - \dots - \bullet = \bullet$$

with vertices corresponding to $0, 1, 2, \dots, p-1, p$.

Let $\sigma \in W$. We have a partition $Y_\sigma = Y_\sigma^0 \sqcup Y_\sigma^1$ where

$$Y_\sigma^0 = \{(i,j) \in \mathbf{Z} \times \mathbf{Z}; i < j, \sigma(i) > \sigma(j), i + j \ne 1 \mod 2p\},$$
$$Y_\sigma^1 = \{(i,j) \in \mathbf{Z} \times \mathbf{Z}; i < j, \sigma(i) > \sigma(j), i + j = 1 \mod 2p\}.$$

Now Y_σ^1/τ_n is the fixed point set of the involution of Y_σ/τ_n induced by the involution $(i,j) \mapsto (1-j, 1-i)$ of Y_σ. Hence we have $\sharp(Y_\sigma/\tau_n) = 2l^0(\sigma) + l^1(\sigma)$ where $l^0(\sigma) = \sharp(Y_\sigma^0/\tau_n)/2$, $l^1(\sigma) = \sharp(Y_\sigma^1/\tau_n)$ are integers. Now $(i,j) \mapsto (i, \frac{i+j-1}{2p})$ is a bijection of Y_σ^1 with

$$\{(i,h) \in \mathbf{Z} \times \mathbf{Z}; 2i < 1 + 2ph, 2\sigma(i) > 1 + 2ph\} = \{(i,h) \in \mathbf{Z} \times \mathbf{Z}; i \le ph < \sigma(i)\}.$$

It follows that

$l^1(\sigma) = \sum_{i \in [1-p,p]; i < \sigma(i)} f(i)$

where

$f(i) = \sharp(x \in p\mathbf{Z}; i \le x < \sigma(i))$.

Let $\mathbf{Z}' = [1,p] + 2p\mathbf{Z}, \mathbf{Z}'' = [1-p, 0] + 2p\mathbf{Z}$; then $\mathbf{Z} = \mathbf{Z}' \sqcup \mathbf{Z}''$. We have $l^1(\sigma) = l'(\sigma) + l''(\sigma)$ where

$$l'(\sigma) = \sum_{i \in [1-p,0]; \sigma(i) \in \mathbf{Z}'', i < \sigma(i)} \frac{f(i)}{2} + \sum_{i \in [1,p]; \sigma(i) \in \mathbf{Z}', i < \sigma(i)} \frac{f(i)}{2}$$
$$+ \sum_{i \in [1-p,0]; \sigma(i) \in \mathbf{Z}', i < \sigma(i)} \frac{f(i)+1}{2} + \sum_{i \in [1,p]; \sigma(i) \in \mathbf{Z}'', i < \sigma(i)} \frac{f(i)-1}{2},$$

$$l''(\sigma)=\sum_{i\in[1-p,0];\sigma(i)\in\mathbf{Z}'',i<\sigma(i)}\frac{f(i)}{2}+\sum_{i\in[1,p];\sigma(i)\in\mathbf{Z}',i<\sigma(i)}\frac{f(i)}{2}$$
$$+\sum_{i\in[1-p,0];\sigma(i)\in\mathbf{Z}',i<\sigma(i)}\frac{f(i)-1}{2}+\sum_{i\in[1,p];\sigma(i)\in\mathbf{Z}'',i<\sigma(i)}\frac{f(i)+1}{2}.$$

These are integers since $f(i)$ is even if $i,\sigma(i)$ are in the same set $\mathbf{Z}'$ or $\mathbf{Z}''$ and is odd otherwise. We see that the length of σ in $\tilde{W}'$ is $2l^0(\sigma)+l'(\sigma)+l''(\sigma)$. On the other hand, the length of σ in W is

$$l^0(\sigma)+l'(\sigma)+l''(\sigma).$$

Now $l'(\sigma)$ (resp. $l''(\sigma)$) is the number of times that s'_0 (resp. s'_p) appears in a reduced expression of σ in W.

One can show that l^0,l',l'' are weight functions on W in the sense of 3.1.

Define $\chi':W\to\{\pm1\}$ and $\chi'':W\to\{\pm1\}$ by $\chi'(\sigma)=(-1)^{l'(\sigma)}$, $\chi''(\sigma)=(-1)^{l''(\sigma)}$. Then χ',χ'' are group homomorphisms.

Assuming that $p\ge 3$, let $W'=\ker(\chi')$. This is the subgroup of W generated by

$$s'_0s'_1s'_0,s'_1,s'_2,\dots,s'_{p-1},s'_p.$$

It is a Coxeter group on these generators, said to be of type $\tilde{B}_p$. The Coxeter graph has vertices $\tilde{1},1,2,\dots,(p-1),p$ corresponding to $s'_0s'_1s'_0,s'_1,s'_2,\dots,s'_{p-1},s'_p$ and edges

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Assuming that $p\ge 4$, let $W''=\ker(\chi')\cap\ker(\chi'')$. This is the subgroup of W (or W') generated by

$$s'_0s'_1s'_0,s'_1,s'_2,\dots,s'_{p-1},s'_ps'_{p-1}s'_p.$$

It is a Coxeter group on these generators, said to be of type $\tilde{D}_p$. The Coxeter graph has vertices $\tilde{1},1,2,\dots,(p-1),\widetilde{(p-1)}$ and edges

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1.14. Let $p\ge q\ge r\ge 1$ be integers such that $p^{-1}+q^{-1}+r^{-1}=1$. Then p,q,r is $3,3,3$ or $4,4,2$ or $6,3,2$. Thus, q and r divide p. Consider the graph with vertices

$$\{(\frac{ip}{p},0,0);i\in[1,p]\}\cup\{(0,\frac{ip}{q},0);i\in[1,q]\}\cup\{(0,0,\frac{ip}{r});i\in[1,r]\}$$

where $(p,0,0),(0,p,0),(0,0,p)$ are identified; the edges are
$(\frac{ip}{p},0,0)$—$(\frac{(i+1)p}{p},0,0), 1 \le i < p$,
$(0,\frac{ip}{q},0)$—$(0,\frac{(i+1)p}{q},0), 1 \le i < q$,
$(0,0,\frac{ip}{r})$—$(0,0,\frac{(i+1)p}{r}), 1 \le i < r$.
The Coxeter group W corresponding to this graph is said to be of type $\tilde{E}_n$ where $n = p+q+r-3$. Thus, $n \in \{6,7,8\}$.

Let W be of type $\tilde{E}_6$. Let W' be the subgroup of W generated by

$$s_{1,0,0}, s_{2,0,0}, s_{3,0,0}, s_{0,2,0}s_{0,0,2}, s_{0,1,0}s_{0,0,1}.$$

(The index of a generator of W is the corresponding vertex of the graph.) Then W' is a Coxeter group on these generators, said to be of type $\tilde{F}_4$. The Coxeter graph is

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Let W be of type $\tilde{D}_4$. The standard generators may be denoted by s_0, s_1, s_2, s_3, s_4 where the Coxeter graph has vertices $0,1,2,3,4$ with four edges joining 0 with $1,2,3,4$. Let W' be the subgroup of W generated by $s_1, s_0, s_2s_3s_4$. Then W' is a Coxeter group on these generators, said to be of type $\tilde{G}_2$. The Coxeter graph is

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1.15. The collection of Coxeter groups of type $\tilde{A}_{n-1}(n \ge 2)$, $\tilde{D}_n(n \ge 4)$, $\tilde{C}_n(n \ge 2)$, $\tilde{B}_n(n \ge 3)$, $\tilde{E}_n(n = 6,7,8)$, $\tilde{F}_4, \tilde{G}_2$ (see 1.12-1.14) coincides with the collection of infinite, tame, irreducible Coxeter groups (or *affine Weyl groups*).

1.16. Let W, S be an affine Weyl group. Let $\mathcal{T}$ be the union of all finite conjugacy classes in W. Then $\mathcal{T}$ is a normal, finitely generated free abelian subgroup of W of finite index. Let S_{min} be the set of all $s \in S$ such that the obvious composition $\langle S - \{s\}\rangle \to W \to W/\mathcal{T}$ is an isomorphism. (This composition is injective for any $s \in S$.) Now $S_{min} \ne \emptyset$. We describe S_{min} in each case.

If W is of type $\tilde{A}_{n-1}$ we have $S_{min} = S$. In the setup of 1.13, S_{min} corresponds to the following vertices of the Coxeter graph: $1, p$, if W is of type $\tilde{C}_p$; $1, \tilde{1}$, if W is of type $\tilde{B}_p$; $1, \tilde{1}, (p-1), \widetilde{(p-1)}$, if W is of type $\tilde{D}_p$. In the setup of 1.14, if W is of type $\tilde{E}_n$ then S_{min} corresponds to the following vertices of the Coxeter graph: $(1,0,0),(0,1,0),(0,0,1)$ if W is of type $\tilde{E}_6$; $(1,0,0),(0,1,0)$ if W is of type $\tilde{E}_7$; $(1,0,0)$ if W is of type $\tilde{E}_8$. If W is of type $\tilde{F}_4$ then $S_{min} = \{s_{1,0,0}\}$; if W is of type $\tilde{G}_2$ then $S_{min} = \{s_1\}$.

1.17. For a Coxeter group W, S we denote by A_W the group of all automorphisms of W that map S into itself. (This is also the group of automorphisms of the corresponding Coxeter graph.)

1.18. Let W,S be an affine Weyl group. Let $\mathcal{T}\subset W$ be as in 1.16. Let Ω be the set of all $a\in A_W$ such that there exists $w\in W$ with $a(t)=wtw^{-1}$ for all $t\in\mathcal{T}$. Now Ω is a commutative normal subgroup of A_W. The action of A_W on W restricts to an action of Ω on S_{min} which is simply transitive.

1.19. For any $I\subset S$, let $W_I=\langle I\rangle$. Then (W_I,I) is a Coxeter group whose Coxeter matrix is a submatrix of that of W,S. See §9.

1.20. Let W,S be an affine Weyl group. Let $s\in S_{min}$. Then $W_{S-\{s\}},S-\{s\}$ is a finite Coxeter group.

A finite Coxeter group is said to be of type $A_{n-1}(n\ge 2)$ (resp. $C_n(n\ge 2)$, $B_n(n\ge 3)$, $D_n(n\ge 4)$, $E_n(n=6,7,8)$, F_4,G_2) if it is isomorphic to $W_{S-\{s\}}$ for some W,S,s as above, where W has type $\tilde A_{n-1}(n\ge 2)$ (resp. $\tilde C_n(n\ge 2)$, $\tilde B_n(n\ge 3)$, $\tilde D_n(n\ge 4)$, $\tilde E_n(n=6,7,8)$, $\tilde F_4,\tilde G_2$).

The collection of Coxeter groups of type $A_{n-1}(n\ge 2)$, $C_n(n\ge 2)$, $B_n(n\ge 3)$, $D_n(n\ge 4)$, $E_n(n=6,7,8)$, F_4,G_2 coincides with the collection of finite, integral, irreducible Coxeter groups $\ne\{1\}$ (or *Weyl groups*). The group $W=\{1\}$ with $S=\emptyset$ is also considered to be a Weyl group. Note that the types C_n and B_n coincide for $n\ge 3$.

For a Weyl group W,S we set $n(W)=2\sharp(T)/\sharp(S)^2$ where T is as in 1.4. We list below the numbers $n(W)$ for W of various types:

$A_{n-1}: n(W)=1+\frac{1}{n-1}$
$D_n: n(W)=2-\frac{2}{n}$
$B_n: n(W)=2$
$E_6: n(W)=2$
$E_7: n(W)=2.57\dots$
$F_4: n(W)=3$
$G_2: n(W)=3$
$E_8: n(W)=3.75.$

We see that the maximum value of $n(W)$ is achieved in type E_8.

1.21. Let W,S be a Weyl group of type E_8. Let W' be the subgroup of W generated by

$$s_{2,0,0}s_{0,2,0},\ s_{3,0,0}s_{0,4,0},\ s_{4,0,0}s_{6,0,0},\ s_{5,0,0}s_{0,0,3}.$$

(The index of a generator of W is the corresponding vertex of the graph, see 1.13.) Then W' is a (finite, non-integral) Coxeter group on these generators, said to be of type H_4. This description of H_4 appeared in [L4].

2. Partial order on W

2.1. Let W,S be a Coxeter group. Let y,w be two elements of W. Following Chevalley, we say that $y\le w$ if there exists a sequence $y=y_0,y_1,y_2,\dots,y_n=w$ in W such that $l(y_k)-l(y_{k-1})=1$ for $k\in[1,n]$ and $y_ky_{k-1}^{-1}\in T$ (or equivalently $y_{k-1}y_k^{-1}\in T$, or $y_k^{-1}y_{k-1}\in T$, or $y_{k-1}^{-1}y_k\in T$) for $k\in[1,n]$. This is a partial order

on W. Note that $y \le w$ implies $y^{-1} \le w^{-1}$ and $l(y) \le l(w)$. We write $y < w$ or $w > y$ instead of $y \le w, y \ne w$. If $w \in W, s \in S$ then,
$sw < w$ if and only if $l(sw) = l(w) - 1$;
$sw > w$ if and only if $l(sw) = l(w) + 1$.

Lemma 2.2. *Let $w = s_1 s_2 \dots s_q$ be a reduced expression in W and let $t \in T$. The following are equivalent:*
(i) $U(w^{-1})(\epsilon, t) = (-\epsilon, w^{-1}tw)$ for $\epsilon = \pm 1$;
(ii) $t = s_1 s_2 \dots s_i \dots s_2 s_1$ for some $i \in [1, q]$;
(iii) $l(tw) < l(w)$.

The equivalence of (i),(ii) has been proved in 1.6.

Proof of (ii) $\implies$ (iii). Assume that (ii) holds. Then $tw = s_1 \dots s_{i-1} s_{i+1} \dots s_q$ hence $l(tw) < q$ and (iii) holds.

Proof of (iii) $\implies$ (i). First we check that

(a) $$U(t)(\epsilon, t) = (-\epsilon, t).$$

If $t \in S$, (a) is clear. If (a) is true for t then it is also true for sts where $s \in S$. Indeed,

$$U(sts)(\epsilon, sts) = U_s U(t) U_s(\epsilon, sts) = U_s U(t)(\epsilon(-1)^{\delta_{s,sts}}, t) = U_s(-\epsilon(-1)^{\delta_{s,sts}}, t)$$
$$= (-\epsilon(-1)^{\delta_{s,sts} + \delta_{s,t}}, sts) = (-\epsilon, sts);$$

(a) follows. Assume now that (i) does not hold; thus, $U(w^{-1})(\epsilon, t) = (\epsilon, w^{-1}tw)$. Then

$$U((tw)^{-1})(\epsilon, t) = U(w^{-1})U(t)(\epsilon, t) = U(w^{-1})(-\epsilon, t) = (-\epsilon, w^{-1}tw)$$
$$= (-\epsilon, (tw)^{-1}t(tw)).$$

Since (i) $\implies$ (iii) we deduce that $l(w) < l(tw)$; thus, (iii) does not hold. The lemma is proved.

Lemma 2.3. *Let $y, z \in W$ and let $s \in S$. If $sy \le z < sz$, then $y \le sz$.*

We argue by induction on $l(z) - l(sy)$. If $l(z) - l(sy) = 0$ then $z = sy$ and the result is clear. Now assume that $l(z) > l(sy)$. Then $sy < z$. We can assume that $sy < y$ (otherwise the result is trivial). We can find $t \in T$ such that $sy < tsy \le z$ and $l(tsy) = l(sy) + 1$. If $t = s$, then $y \le z$ and we are done. Hence we may assume that $t \ne s$. We show that

(a) $$y < stsy.$$

Assume that (a) does not hold. Then $y, tsy, sy, stsy$ have lengths $q+1, q+1, q, q$. We can find a reduced expression $y = ss_1 s_2 \dots s_q$. Since $l(stsy) < l(y)$, we see from 2.2 that either $sts = ss_1 \dots s_i \dots s_1 s$ for some $i \in [1, q]$ or $sts = s$. (This last case has been excluded.) It follows that

$$tsy = s_1 \dots s_i \dots s_1 s s s_1 s_2 \dots s_q = s_1 \dots s_{i-1} s_{i+1} \dots s_q.$$

Thus, $l(tsy) \le q - 1$, a contradiction. Thus, (a) holds. Let $y' = stsy$. We have $sy' \le z < sz$ and $l(z) - l(sy') < l(z) - l(sy)$. By the induction hypothesis, $y' \le sz$. We have $y < y'$ by (a), hence $y \le sz$. The lemma is proved.

Proposition 2.4. *The following three conditions on $y, w \in W$ are equivalent:*
(i) $y \le w$;
(ii) for any reduced expression $w = s_1 s_2 \dots s_q$ there exists a subsequence $i_1 < i_2 < \dots < i_r$ of $1, 2, \dots, q$ such that $y = s_{i_1} s_{i_2} \dots s_{i_r}$, $r = l(y)$;
(iii) there exists a reduced expression $w = s_1 s_2 \dots s_q$ and a subsequence $i_1 < i_2 < \dots < i_r$ of $1, 2, \dots, q$ such that $y = s_{i_1} s_{i_2} \dots s_{i_r}$.

Proof of (i) $\implies$ (ii). We may assume that $y < w$. Let $y = y_0, y_1, y_2, \dots, y_n = w$ be as in 2.1. Let $w = s_1 s_2 \dots s_q$ be a reduced expression. Since $y_{n-1} y_n^{-1} \in T$, $l(y_{n-1}) = l(y_n) - 1$, we see from 2.2 that there exists $i \in [1, q]$ such that $y_{n-1} y_n^{-1} = s_1 s_2 \dots s_i \dots s_2 s_1$ hence $y_{n-1} = s_1 s_2 \dots s_{i-1} s_{i+1} \dots s_q$ (a reduced expression). Similarly, since $y_{n-2} y_{n-1}^{-1} \in T$, $l(y_{n-2}) = l(y_{n-1}) - 1$, we see from 2.2 (applied to y_{n-1}) that there exists $j \in [1, q] - \{i\}$ such that y_{n-2} equals

$$s_1 s_2 \dots s_{i-1} s_{i+1} \dots s_{j-1} s_{j+1} \dots s_q \text{ or } s_1 s_2 \dots s_{j-1} s_{j+1} \dots s_{i-1} s_{i+1} \dots s_q$$

(depending on whether $i < j$ or $i > j$). Continuing in this way we see that y is of the required form.

The proof of (ii) $\implies$ (iii) is trivial.

Proof of (iii) $\implies$ (i). Assume that $w = s_1 s_2 \dots s_q$ (reduced expression) and $y = s_{i_1} s_{i_2} \dots s_{i_r}$ where $i_1 < i_2 < \dots < i_r$ is a subsequence of $1, 2, \dots, q$. We argue by induction on q. If $q = 0$ there is nothing to prove. Now assume $q > 0$.

If $i_1 > 1$, then the induction hypothesis is applicable to $y, w' = s_2 \dots s_q$ and yields $y \le w'$. But $w' \le w$ hence $y \le w$. If $i_1 = 1$ then the induction hypothesis is applicable to $y' = s_{i_2} \dots s_{i_r}, w' = s_2 \dots s_q$ and yields $y' \le w'$. Thus, $s_1 y \le s_1 w < w$. By 2.3 we then have $y \le w$. The proposition is proved.

Corollary 2.5. *Let $y, z \in W$ and let $s \in S$.*
(a) Assume that $sz < z$. Then $y \le z \Leftrightarrow sy \le z$.
(b) Assume that $y < sy$. Then $y \le z \Leftrightarrow y \le sz$.

We prove (a). We can find a reduced expression of z of form $z = ss_1 s_2 \dots s_q$. Assume that $y \le z$. By 2.4 we can find a subsequence $i_1 < i_2 < \dots < i_r$ of $1, 2, \dots, q$ such that either $y = s_{i_1} s_{i_2} \dots s_{i_r}$ or $y = ss_{i_1} s_{i_2} \dots s_{i_r}$. In the first case we have $sy = ss_{i_1} s_{i_2} \dots s_{i_r}$ and in the second case we have $sy = s_{i_1} s_{i_2} \dots s_{i_r}$. In both cases, $sy \le z$ by 2.4. The same argument shows that, if $sy \le z$ then $y \le z$. This proves (a).

We prove (b). Assume that $y \le z$. We must prove that $y \le sz$. If $z < sz$, this is clear. Thus we may assume that $sz < z$. We can find a reduced expression of z of form $z = ss_1s_2 \dots s_q$. By 2.4 we can find a subsequence $i_1 < i_2 < \dots < i_r$ of $1, 2, \dots, q$ such that either

$$y = s_{i_1}s_{i_2} \dots s_{i_r}, l(y) = r \text{ or } y = ss_{i_1}s_{i_2} \dots s_{i_r}, l(y) = r + 1.$$

In the second case we have $l(sy) = r < l(y)$, contradicting $y < sy$. Thus we are in the first case. Hence y is the product of a subsequence of $s_1, s_2, \dots, s_q$ and using again 2.4, we deduce that $y \le sz$ (note that $sz = s_1s_2 \dots s_q$ is a reduced expression). The lemma is proved.

3. The algebra $\mathcal{H}$

3.1. Let W, S be a Coxeter group. A map $L : W \to \mathbf{Z}$ is said to be a *weight function* for W if $L(ww') = L(w) + L(w')$ for any $w, w' \in W$ such that $l(ww') = l(w) + l(w')$. We will assume that a weight function $L : W \to \mathbf{Z}$ is fixed; we then say that W, L is a *weighted Coxeter group.* (For example we could take $L = l$; in that case we say that we are in the *split case.*) Note that L is determined by its values on S which are subject only to the condition

$L(s) = L(s')$ for any $s \ne s'$ in S such that $m_{s,s'}$ is finite and odd.

We have $L(w) = L(w^{-1})$ for all $w \in W$.

Let $\mathcal{A} = \mathbf{Z}[v, v^{-1}]$ where v is an indeterminate. For $s \in S$ we set $v_s = v^{L(s)} \in \mathcal{A}$.

3.2. Let $\mathcal{H}$ be the $\mathcal{A}$-algebra with 1 defined by the generators $T_s (s \in S)$ and the relations

(a) $$(T_s - v_s)(T_s + v_s^{-1}) = 0 \text{ for } s \in S$$

(b) $$T_sT_{s'}T_s \dots = T_{s'}T_sT_{s'} \dots$$

(both products have $m_{s,s'}$ factors) for any $s \ne s'$ in S such that $m_{s,s'} < \infty$; $\mathcal{H}$ is called the *Iwahori-Hecke algebra.*

For $w \in W$ we define $T_w \in \mathcal{H}$ by $T_w = T_{s_1}T_{s_2} \dots T_{s_q}$, where $w = s_1s_2 \dots s_q$ is a reduced expression in W. By (b) and 1.9, T_w is independent of the choice of reduced expression. We have $T_1 = 1$. From the definitions it is clear that for $s \in S, w \in W$ we have

$$T_sT_w = T_{sw} \text{ if } l(sw) = l(w) + 1,$$

$$T_sT_w = T_{sw} + (v_s - v_s^{-1})T_w \text{ if } l(sw) = l(w) - 1.$$

In particular, the $\mathcal{A}$-submodule of $\mathcal{H}$ generated by $\{T_w; w \in W\}$ is a left ideal of $\mathcal{H}$. It contains $1 = T_1$ hence it is the whole of $\mathcal{H}$. Thus $\{T_w; w \in W\}$ generates the $\mathcal{A}$-module $\mathcal{H}$.

Proposition 3.3. *$\{T_w; w \in W\}$ is an $\mathcal{A}$-basis of $\mathcal{H}$.*

We follow the lines of the proof in [Bo, Ex.23, p.55].

We consider the free $\mathcal{A}$-module $\mathcal{E}$ with basis $(e_w)_{w\in W}$. For any $s \in S$ we define $\mathcal{A}$-linear maps $P_s, Q_s : \mathcal{E} \to \mathcal{E}$ by

$P_s(e_w) = e_{sw}$ if $l(sw) = l(w) + 1$,

$P_s(e_w) = e_{sw} + (v_s - v_s^{-1})e_w$ if $l(sw) = l(w) - 1$;

$Q_s(e_w) = e_{ws}$ if $l(ws) = l(w) + 1$,

$Q_s(e_w) = e_{ws} + (v_s - v_s^{-1})e_w$ if $l(ws) = l(w) - 1$.

We shall continue the proof assuming that

(a) *$P_sQ_t = Q_tP_s$ for any s, t in S.*

Let $\mathfrak{A}$ be the $\mathcal{A}$-subalgebra with 1 of $\mathrm{End}(\mathcal{E})$ generated by $\{P_s; s \in S\}$. The map $\mathfrak{A} \to \mathcal{E}$ given by $\pi \mapsto \pi(e_1)$ is surjective. Indeed, if $w = s_1s_2 \dots s_q$ is a reduced expression in W, then $e_w = P_{s_1} \dots P_{s_q} e_1$. Assume now that $\pi \in \mathfrak{A}$ satisfies $\pi(e_1) = 0$. Let $\pi' = Q_{s_q} \dots Q_{s_1}$. By (a) we have $\pi\pi' = \pi'\pi$ hence

$$0 = \pi'\pi(e_1) = \pi\pi'(e_1) = \pi(Q_{s_q} \dots Q_{s_1}(e_1)) = \pi(e_w).$$

Since w is arbitrary, it follows that $\pi = 0$. We see that the map $\mathfrak{A} \to \mathcal{E}$ is injective, hence an isomorphism of $\mathcal{A}$-modules. Using this isomorphism we transport the algebra structure of $\mathfrak{A}$ to an algebra structure on $\mathcal{E}$ with unit element e_1. For this algebra structure we have $P_s(e_1)\pi(e_1) = P_s(\pi(e_1))$ for $s \in S, \pi \in \mathfrak{A}$. Hence $e_se_w = P_s(e_w)$ for any $w \in W, s \in S$. It follows that

(b) $e_se_w = e_{sw}$ if $l(sw) = l(w) + 1$,

(c) $e_se_w = e_{sw} + (v_s - v_s^{-1})e_w$ if $l(sw) = l(w) - 1$.

From (b) it follows that, if $w = s_1s_2 \dots s_q$ is a reduced expression, then $e_w = e_{s_1}e_{s_2} \dots e_{s_q}$. In particular, if $s \neq s'$ in S are such that $m = m_{s,s'} < \infty$ then $e_se_{s'}e_s \dots = e_{s'}e_se_{s'} \dots$ (both products have m factors); indeed, this follows from the equality $e_{ss's\dots} = e_{s'ss'\dots}$ (see 1.4). From (c) we deduce that $e_s^2 = 1 + (v_s - v_s^{-1})e_s$ for $s \in S$, or that $(e_s - v_s)(e_s + v_s^{-1}) = 0$. We see that there is a unique algebra homomorphism $\mathcal{H} \to \mathcal{E}$ preserving 1 such that $T_s \mapsto e_s$ for all $s \in S$. It takes T_w to e_w for any $w \in W$. Assume now that $a_w \in \mathcal{A}$ ($w \in W$) are zero for all but finitely many w and that $\sum_w a_wT_w = 0$ in $\mathcal{H}$. Applying $\mathcal{H} \to \mathcal{E}$ we obtain $\sum_w a_we_w = 0$. Since (e_w) is a basis of $\mathcal{E}$, it follows that $a_w = 0$ for all w. Thus, $\{T_w; w \in W\}$ is an $\mathcal{A}$-basis of $\mathcal{H}$. This completes the proof, modulo the verification of (a).

We prove (a). Let $w \in W$. We distinguish six cases.

Case 1. swt, sw, wt, w have lengths $q+2, q+1, q+1, q$. Then

$$P_sQ_t(e_w) = Q_tP_s(e_w) = e_{swt}.$$

Case 2. w, sw, wt, swt have lengths $q+2, q+1, q+1, q$. Then

$$P_sQ_t(e_w) = Q_tP_s(e_w)$$
$$= e_{swt} + (v_t - v_t^{-1})e_{sw} + (v_s - v_s^{-1})e_{wt} + (v_t - v_t^{-1})(v_s - v_s^{-1})e_w.$$

Case 3. wt, swt, w, sw have lengths $q+2, q+1, q+1, q$. Then

$$P_sQ_t(e_w) = Q_tP_s(e_w) = e_{swt} + (v_s - v_s^{-1})e_{wt}.$$

Case 4. sw, swt, w, wt have lengths $q+2, q+1, q+1, q$. Then

$$P_sQ_t(e_w) = Q_tP_s(e_w) = e_{swt} + (v_t - v_t^{-1})e_{sw}.$$

Case 5. swt, w, wt, sw have lengths $q+1, q+1, q, q$. Then

$$P_sQ_t(e_w) = e_{swt} + (v_t - v_t^{-1})e_{sw} + (v_t - v_t^{-1})(v_s - v_s^{-1})e_w,$$

$$Q_tP_s(e_w) = e_{swt} + (v_s - v_s^{-1})e_{wt} + (v_t - v_t^{-1})(v_s - v_s^{-1})e_w.$$

Case 6. sw, wt, w, swt have lengths $q+1, q+1, q, q$. Then

$$P_sQ_t(e_w) = e_{swt} + (v_s - v_s^{-1})e_{wt},$$

$$Q_tP_s(e_w) = e_{swt} + (v_t - v_t^{-1})e_{sw}.$$

In cases 5, 6 we have $sw = wt$ by 1.10. In case 5 we have $L(t) + L(wt) = L(w) = L(swt) = L(s) + L(wt)$ hence $L(t) = L(s)$ and $v_s = v_t$. In case 6 we have $L(t) + L(swt) = L(sw) = L(wt) = L(s) + L(swt)$, hence $L(t) = L(s)$ and $v_s = v_t$. Hence $P_sQ_t(e_w) = Q_tP_s(e_w)$ in each case. The proposition is proved.

3.4. There is a unique involutive antiautomorphism $h \mapsto h^\flat$ of the algebra $\mathcal{H}$ which carries T_s to T_s for any $s \in S$. It carries T_w to $T_{w^{-1}}$ for any $w \in W$.

3.5. For $s \in S$, the element $T_s \in \mathcal{H}$ is invertible: $T_s^{-1} = T_s - (v_s - v_s^{-1})$. It follows that T_w is invertible for each $w \in W$; if $w = s_1s_2 \dots s_q$ is a reduced expression in W, then $T_w^{-1} = T_{s_q}^{-1} \dots T_{s_2}^{-1}T_{s_1}^{-1}$.

There is a unique algebra involution of $\mathcal{H}$ denoted $h \mapsto h^\dagger$ such that $T_s^\dagger = -T_s^{-1}$ for any $s \in S$. We have $T_w^\dagger = \text{sgn}(w)T_{w^{-1}}^{-1}$ for any $w \in W$.

4. The bar operator

4.1. We preserve the setup of 3.1. Let $^-\colon \mathcal{A} \to \mathcal{A}$ be the ring involution which takes v^n to v^{-n} for any $n \in \mathbf{Z}$.

Lemma 4.2. *(a) There is a unique ring homomorphism* $^-\colon \mathcal{H} \to \mathcal{H}$ *which is* $\mathcal{A}$*-semilinear with respect to* $^-\colon \mathcal{A} \to \mathcal{A}$ *and satisfies* $\overline{T}_s = T_s^{-1}$ *for any* $s \in S$.

(b) This homomorphism is involutive. It takes T_w *to* $T_{w^{-1}}^{-1}$ *for any* $w \in W$.

The following identities can be deduced easily from 3.2(a),(b),(d):
$(T_s^{-1} - v_s^{-1})(T_s^{-1} + v_s) = 0$ for $s \in S$,
$T_s^{-1}T_{s'}^{-1}T_s^{-1} \dots = T_{s'}^{-1}T_s^{-1}T_{s'}^{-1} \dots$
(both products have $m_{s,s'}$ factors) for any $s \neq s'$ in S such that $m_{s,s'} < \infty$; (a) follows.

We prove (b). Let $s \in S$. Applying $^-$ to $T_s\bar{T}_s = 1$ gives $\bar{T}_s\bar{\bar{T}}_s = 1$. We have also $\bar{T}_sT_s = 1$ hence $\bar{\bar{T}}_s = T_s$. It follows that the square of $^-$ is 1. The second assertion of (b) is immediate. The lemma is proved.

4.3. For any $w \in W$ we can write uniquely $\overline{T}_w = \sum_{y \in W} \overline{r}_{y,w} T_y$ where $r_{y,w} \in \mathcal{A}$ are zero for all but finitely many y. Note that $r_{w,w} = 1$.

Lemma 4.4. *Let $w \in W$ and $s \in S$ be such that $w > sw$. For $y \in W$ we have*
$r_{y,w} = r_{sy,sw}$ *if* $sy < y$,
$r_{y,w} = r_{sy,sw} + (v_s - v_s^{-1}) r_{y,sw}$ *if* $sy > y$.

We have

$$\begin{aligned}\overline{T}_w = T_s^{-1}\overline{T}_{sw} &= (T_s - (v_s - v_s^{-1}))\sum_y \overline{r}_{y,sw} T_y \\ &= \sum_y \overline{r}_{y,sw} T_{sy} - \sum_y (v_s - v_s^{-1})\overline{r}_{y,sw} T_y + \sum_{y;sy<y} (v_s - v_s^{-1})\overline{r}_{y,sw} T_y \\ &= \sum_y \overline{r}_{sy,sw} T_y - \sum_{y;sy>y} (v_s - v_s^{-1})\overline{r}_{y,sw} T_y.\end{aligned}$$

The lemma follows.

Lemma 4.5. *For any y, w we have* $\overline{r}_{y,w} = \operatorname{sgn}(yw) r_{y,w}$.

We argue by induction on $l(w)$. If $l(w) = 0$, then $w = 1$ and the result is obvious. Assume now that $l(w) \geq 1$. We can find $s \in S$ such that $w > sw$. Assume first that $sy < y$. From 4.4 we see, using the induction hypothesis, that

$$\overline{r}_{y,w} = \overline{r}_{sy,sw} = \operatorname{sgn}(sysw) r_{sy,sw} = \operatorname{sgn}(yw) r_{y,w}.$$

Assume next that $sy > y$. From 4.4 we see, using the induction hypothesis, that

$$\begin{aligned}\overline{r}_{y,w} &= \overline{r}_{sy,sw} + (v_s^{-1} - v_s)\overline{r}_{y,sw} = \operatorname{sgn}(sysw) r_{sy,sw} + (v_s^{-1} - v_s)\operatorname{sgn}(ysw) r_{y,sw} \\ &= \operatorname{sgn}(yw)(r_{sy,sw} + (v_s - v_s^{-1}) r_{y,sw}) = \operatorname{sgn}(yw) r_{y,w}.\end{aligned}$$

The lemma is proved.

Lemma 4.6. *For any $x, z \in W$ we have* $\sum_y \overline{r}_{x,y} r_{y,z} = \delta_{x,z}$.

Using the fact that $^-$ is an involution, we have

$$T_z = \overline{\overline{T}}_z = \overline{\sum_y \overline{r}_{y,z} T_y} = \sum_y r_{y,z} \overline{T}_y = \sum_y \sum_x r_{y,z} \overline{r}_{x,y} T_x.$$

We now compare the coefficients of T_x on both sides. The lemma follows.

Proposition 4.7. *Let $y, w \in W$.*
(a) If $r_{y,w} \neq 0$, then $y \leq w$.
(b) Assume that $L(s) > 0$ for all $s \in S$. If $y \leq w$, then
$r_{y,w} = v^{L(w)-L(y)} \mod v^{L(w)-L(y)-1}\mathbf{Z}[v^{-1}]$,
$r_{y,w} = \operatorname{sgn}(yw) v^{-L(w)+L(y)} \mod v^{-L(w)+L(y)+1}\mathbf{Z}[v]$.

(c) Without assumption on L, $r_{y,w} \in v^{L(w)-L(y)}\mathbf{Z}[v^2, v^{-2}]$.

We prove (a) by induction on $l(w)$. If $l(w) = 0$ then $w = 1$ and the result is obvious. Assume now that $l(w) \geq 1$. We can find $s \in S$ such that $w > sw$. Assume first that $sy < y$. From 4.4 we see that $r_{sy,sw} \neq 0$ hence, by the induction hypothesis, $sy \leq sw$. Thus $sy \leq sw < w$ and, by 2.3, we deduce $y \leq w$. Assume next that $sy > y$. From 4.4 we see that either $r_{sy,sw} \neq 0$ or $r_{y,sw} \neq 0$ hence, by the induction hypothesis, $sy \leq sw$ or $y \leq sw$. Combining this with $y < sy$ and $sw < w$ we see that $y \leq w$. This proves (a).

We prove the first assertion of (b) by induction on $l(w)$. If $l(w) = 0$ then $w = 1$ and the result is obvious. Assume now that $l(w) \geq 1$. We can find $s \in S$ such that $w > sw$. Assume first that $sy < y$. Then we have also $sy < w$ and, using 2.5(b), we deduce $sy \leq sw$. By the induction hypothesis,

$$\begin{aligned} r_{sy,sw} &= v^{L(sw)-L(sy)} + \text{strictly lower powers} \\ &= v^{L(w)-L(y)} + \text{strictly lower powers.} \end{aligned}$$

But $r_{y,w} = r_{sy,sw}$ and the result follows. Assume next that $sy > y$. From $y < sy, y \leq w$ we deduce using 2.5(b) that $y \leq sw$. By the induction hypothesis, we have $r_{y,sw} = v^{L(sw)-L(y)} +$ strictly lower powers. Hence

$$\begin{aligned} &(v_s - v_s^{-1})r_{y,sw} = v^{L(s)}v^{L(sw)-L(y)} + \text{strictly lower powers} \\ &= v^{L(w)-L(y)} + \text{strictly lower powers.} \end{aligned}$$

On the other hand, if $sy \leq sw$, then by the induction hypothesis,

$$\begin{aligned} r_{sy,sw} &= v^{L(sw)-L(sy)} + \text{strictly lower powers} \\ &= v^{L(w)-L(y)-2L(s)} + \text{strictly lower powers} \end{aligned}$$

while if $sy \not\leq sw$ then $r_{sy,sw} = 0$ by (a). Thus, in $r_{y,w} = r_{sy,sw} + (v_s - v_s^{-1})r_{y,sw}$, the term $r_{sy,sw}$ contributes only powers of v which are strictly smaller than $L(w) - L(y)$ hence $r_{y,w} = v^{L(w)-L(y)} +$ strictly lower powers. This proves the first assertion of (b). The second assertion of (b) follows from the first using 4.5.

We prove (c) by induction on $l(w)$. If $l(w) = 0$ then $w = 1$ and the result is obvious. Assume now that $l(w) \geq 1$. We can find $s \in S$ such that $w > sw$. Assume first that $sy < y$. By the induction hypothesis,

$$r_{y,w} = r_{sy,sw} \in v^{L(sw)-L(sy)}\mathbf{Z}[v^2, v^{-2}] = v^{L(w)-L(y)}\mathbf{Z}[v^2, v^{-2}]$$

as required. Assume next that $sy > y$. By the induction hypothesis,

$$\begin{aligned} &r_{y,w} = r_{sy,sw} + (v_s - v_s^{-1})r_{y,sw} \\ &\in v^{L(sw)-L(sy)}\mathbf{Z}[v^2, v^{-2}] + v^{L(s)}v^{L(sw)-L(y)}\mathbf{Z}[v^2, v^{-2}] = v^{L(w)-L(y)}\mathbf{Z}[v^2, v^{-2}], \end{aligned}$$

as required. The proposition is proved.

Proposition 4.8. *For $x < z$ in W we have $\sum_{y;x\le y\le z}\mathrm{sgn}(y)=0$ (D.N. Verma).*

Using 4.5 we can rewrite 4.6 (in our case) in the form

(a) $$\sum_y \mathrm{sgn}(xy) r_{x,y} r_{y,z} = 0.$$

Here we may restrict the summation to y such that $x \le y \le z$. In the rest of the proof we shall take $L = l$. Then 4.7(b) holds and we see that if $x \le y \le z$, then

$r_{x,y}r_{y,z} = v^{l(y)-l(x)}v^{l(z)-l(y)}+$ strictly lower powers of v.

Hence (a) states that

$\sum_{y;x\le y\le z}\mathrm{sgn}(xy)v^{l(z)-l(x)}+$ strictly lower powers of v is 0.

In particular $\sum_{y;x\le y\le z}\mathrm{sgn}(xy)=0$. The proposition is proved.

4.9. Now $^-\colon \mathcal{H}\to\mathcal{H}$ commutes with $h \mapsto h^\flat$. Hence

(a) $r_{y^{-1},w^{-1}} = r_{y,w}$

for any $y, w \in W$. On the other hand, it is clear that $^-\colon \mathcal{H}\to\mathcal{H}$ and $^\dagger : \mathcal{H}\to\mathcal{H}$ commute.

5. The elements c_w

5.1. We preserve the setup of 3.1. For any $n \in \mathbf{Z}$ let

$$\mathcal{A}_{\le n} = \oplus_{m;m\le n}\mathbf{Z}v^m, \mathcal{A}_{\ge n} = \oplus_{m;m\ge n}\mathbf{Z}v^m,$$

$$\mathcal{A}_{< n} = \oplus_{m;m< n}\mathbf{Z}v^m, \mathcal{A}_{> n} = \oplus_{m;m> n}\mathbf{Z}v^m,$$

$$\mathcal{H}_{\le 0} = \oplus_w \mathcal{A}_{\le 0}T_w, \mathcal{H}_{<0} = \oplus_w \mathcal{A}_{<0}T_w.$$

We have $\mathcal{A}_{\le 0} = \mathbf{Z}[v^{-1}]$, $\mathcal{H}_{<0}\subset\mathcal{H}_{\le 0}\subset\mathcal{H}$.

Theorem 5.2. *(a) Let $w \in W$. There exists a unique element $c_w \in \mathcal{H}_{\le 0}$ such that $\bar{c}_w = c_w$ and $c_w = T_w \mod \mathcal{H}_{<0}$.*

(b) $\{c_w; w \in W\}$ is an $\mathcal{A}_{\le 0}$-basis of $\mathcal{H}_{\le 0}$ and an $\mathcal{A}$-basis of $\mathcal{H}$.

We prove the existence part of (a). We will construct, for any x such that $x \le w$, an element $u_x \in \mathcal{A}_{\le 0}$ such that

(c) $u_w = 1$,

(d) $u_x \in \mathcal{A}_{<0}, \bar{u}_x - u_x = \sum_{y;x<y\le w} r_{x,y}u_y$ for any $x < w$.

We argue by induction on $l(w)-l(x)$. If $l(w)-l(x)=0$ then $x = w$ and we set $u_x = 1$. Assume now that $l(w)-l(x) > 0$ and that u_z is already defined whenever $z \le w, l(w)-l(z) < l(w)-l(x)$ so that (c) holds and (d) holds if x is replaced by any such z. Then the right hand side of the equality in (d) is defined. We denote it by $a_x \in \mathcal{A}$. We have

$$\begin{aligned} a_x + \overline{a}_x &= \sum_{y;x<y\le w} r_{x,y}u_y + \sum_{y;x<y\le w} \overline{r}_{x,y}\overline{u}_y \\ &= \sum_{y;x<y\le w} r_{x,y}u_y + \sum_{y;x<y\le w} \overline{r}_{x,y}(u_y + \sum_{z;y<z\le w} r_{y,z}u_z) \\ &= \sum_{z;x<z\le w} r_{x,z}u_z + \sum_{z;x<z\le w} \overline{r}_{x,z}u_z + \sum_{z;x<z\le w}\sum_{y;x<y<z} \overline{r}_{x,y}r_{y,z}u_z \\ &= \sum_{z;x<z\le w}\sum_{y;x\le y\le z} \overline{r}_{x,y}r_{y,z}u_z = \sum_{z;x<z\le w} \delta_{x,z}u_z = 0. \end{aligned}$$

(We have used 4.6 and the equality $r_{y,y} = 1$.) Since $a_x + \overline{a}_x = 0$, we have $a_x = \sum_{n\in\mathbf{Z}} \gamma_n v^n$ (finite sum) where $\gamma_n \in \mathbf{Z}$ satisfy $\gamma_n + \gamma_{-n} = 0$ for all n and in particular, $\gamma_0 = 0$. Then $u_x = -\sum_{n<0} \gamma_n v^n \in \mathcal{A}_{<0}$ satisfies $\overline{u}_x - u_x = a_x$. This completes the inductive construction of the elements u_x. We set $c_w = \sum_{y;y\le w} u_y T_y \in \mathcal{H}_{\le 0}$. It is clear that $c_w = T_w \mod \mathcal{H}_{<0}$. We have

$$\begin{aligned} \overline{c}_w &= \sum_{y;y\le w} \overline{u}_y\overline{T}_y = \sum_{y;y\le w} \overline{u}_y \sum_{x;x\le y} \overline{r}_{x,y}T_x = \sum_{x;x\le w} (\sum_{y;x\le y\le w} \overline{r}_{x,y}\overline{u}_y)T_x \\ &= \sum_{x;x\le w} u_xT_x = c_w. \end{aligned}$$

(We have used the fact that $r_{x,y} \ne 0$ implies $x \le y$, see 4.7, and (d).) The existence of the element c_w is established.

To prove uniqueness, it suffices to verify the following statement:

(e) *If $h \in \mathcal{H}_{<0}$ satisfies $\overline{h} = h$ then $h = 0$.*

We can write uniquely $h = \sum_{y\in W} f_y T_y$ where $f_y \in \mathcal{A}_{<0}$ are zero for all but finitely many y. Assume that not all f_y are 0. Then we can find $l_0 \in \mathbf{N}$ such that

$$Y_0 := \{y \in W; f_y \ne 0, l(y) = l_0\} \ne \emptyset \text{ and } \{y \in W; f_y \ne 0, l(y) > l_0\} = \emptyset.$$

Now $\sum_y f_yT_y = \overline{\sum_y f_yT_y}$ implies

$$\sum_{y\in Y_0} f_yT_y = \sum_{y\in Y_0} \overline{f_y}T_y \mod \sum_{y;l(y)<l_0} \mathcal{A}T_y$$

hence $\overline{f_y} = f_y$ for any $y \in Y_0$. Since $f_y \in \mathcal{A}_{<0}$, it follows that $f_y = 0$ for any $y \in Y_0$, a contradiction. We have proved that $f_y = 0$ for all y; (e) is verified and (a) is proved.

The elements c_w constructed in (a) (for various w) are related to the basis T_w by a triangular matrix (with respect to $\le$) with 1 on the diagonal. Hence these elements satisfy (b). The theorem is proved.

5.3. For any $w \in W$ we set $c_w = \sum_{y \in W} p_{y,w} T_y$ where $p_{y,w} \in \mathcal{A}_{\le 0}$. By the proof of 5.2 we have

$p_{y,w} = 0$ unless $y \le w$,

$p_{w,w} = 1$,

$p_{y,w} \in \mathcal{A}_{<0}$ if $y < w$.

Moreover, for any $x \le w$ in W we have

$\bar{p}_{x,w} = \sum_{y; x \le y \le w} r_{x,y} p_{y,w}$.

Proposition 5.4. *(a) Assume that $L(s) > 0$ for all $s \in S$. If $x \le w$, then*

$p_{x,w} = v^{-L(w)+L(x)} \mod v^{-L(w)+L(x)+1}\mathbf{Z}[v]$.

(b) Without assumption on L, for $x \le w$ we have $p_{x,w} \in v^{L(w)-L(x)}\mathbf{Z}[v^2, v^{-2}]$.

We prove (a) by induction on $l(w) - l(x)$. If $l(w) - l(x) = 0$ then $x = w$, $p_{x,w} = 1$ and the result is obvious. Assume now that $l(w) - l(x) > 0$. Using 4.7(b) and the induction hypothesis, we see that $\sum_{y; x < y \le w} r_{x,y} p_{y,w}$ is equal to

$$\sum_{y; x < y \le w} \operatorname{sgn}(x)\operatorname{sgn}(y) v^{-L(y)+L(x)} v^{-L(w)+L(y)} = \sum_{y; x < y \le w} \operatorname{sgn}(x)\operatorname{sgn}(y) v^{-L(w)+L(x)}$$

plus strictly higher powers of v. Using 4.8, we see that this is $-v^{-L(w)+L(x)}$ plus strictly higher powers of v. Thus,

$\bar{p}_{x,w} - p_{x,w} = -v^{-L(w)+L(x)} +$ strictly higher powers of v.

Since $\bar{p}_{x,w} \in v\mathbf{Z}[v]$, it is in particular a $\mathbf{Z}$-linear combination of powers of v strictly higher than $-L(w) + L(x)$. (We use that, if $a \le b$ in W then $L(a) \le L(b)$ which follows from 2.4.) Hence

$-p_{x,w} = -v^{-L(w)+L(x)} +$ strictly higher powers of v.

This proves (a).

We prove (b) by induction on $l(w) - l(x)$. If $l(w) - l(x) = 0$, then $x = w$, $p_{x,w} = 1$ and the result is obvious. Assume now that $l(w) - l(x) > 0$. Using 4.7(c) and the induction hypothesis, we see that

$$\sum_{y; x < y \le w} r_{x,y} p_{y,w} \in \sum_{y; x < y \le w} v^{L(y)-L(x)} v^{L(w)-L(y)} \mathbf{Z}[v^2, v^{-2}] \subset v^{L(w)-L(x)}\mathbf{Z}[v^2, v^{-2}].$$

Thus, $\bar{p}_{x,w} - p_{x,w} \in v^{L(w)-L(x)}\mathbf{Z}[v^2, v^{-2}]$. Hence $p_{x,w} \in v^{L(w)-L(x)}\mathbf{Z}[v^2, v^{-2}]$. The proposition is proved.

5.5. Let $s \in S$. From $T_s^{-1} = T_s - (v_s - v_s^{-1})$ we see that $r_{1,s} = v_s - v_s^{-1}$. We also see that

$\overline{T_s + v_s^{-1}} = T_s - (v_s - v_s^{-1}) + v_s = T_s + v_s^{-1}$,

$\overline{T_s - v_s} = T_s - (v_s - v_s^{-1}) - v_s^{-1} = T_s - v_s$.

If $L(s) = 0$ we have $T_s^{-1} = T_s$. Hence,

$c_s = T_s + v_s^{-1}$ if $L(s) > 0$,

$c_s = T_s - v_s$ if $L(s) < 0$,

$c_s = T_s$ if $L(s) = 0$.

5.6. Now $h \mapsto h^\flat$ carries $\mathcal{H}_{\le 0}$ into itself; moreover, it commutes with $^-\colon \mathcal{H} \to \mathcal{H}$ (as pointed out in 4.9). Hence it carries c_w to $c_{w^{-1}}$ for any $w \in W$. It follows that

(a) $p_{y^{-1},w^{-1}} = p_{y,w}$

for any $y, w \in W$.

6. Left or right multiplication by c_s

6.1. We preserve the setup of 3.1 and we fix $s \in S$. Assume first that $L(s) = 0$. In this case we have $c_s = T_s$; moreover, for any $y \in W$ we have $T_s T_y = T_{sy}$. Hence for $w \in W$ we have

$$c_s c_w = \sum_y p_{y,w} T_s T_y = \sum_y p_{y,w} T_{sy} = \sum_y p_{sy,w} T_y.$$

We see that $c_s c_w \in \mathcal{H}_{\le 0}$ and $c_s c_w = T_{sw} \mod \mathcal{H}_{<0}$. Since $\overline{c_s c_w} = c_s c_w$, it follows that, in this case, $c_s c_w = c_{sw}$. Similarly we have $c_w c_s = c_{ws}$.

6.2. *In the remainder of this chapter (except in 6.8) we assume that $L(s) > 0$.*

Proposition 6.3. *To any $y, w \in W$ such that $sy < y < w < sw$ one can assign uniquely an element $\mu^s_{y,w} \in \mathcal{A}$ so that*

(i) $\overline{\mu}^s_{y,w} = \mu^s_{y,w}$ and

(ii) $\sum_{z; y \le z < w; sz < z} p_{y,z} \mu^s_{z,w} - v_s p_{y,w} \in \mathcal{A}_{<0}$

for any $y, w \in W$ such that $sy < y < w < sw$.

Let y, w be as above. We may assume that $\mu^s_{z,w}$ are already defined for all z such that $y < z < w; sz < z$. Then condition (ii) is of the form:

$\mu^s_{y,w}$ *equals a known element of $\mathcal{A}$ modulo $\mathcal{A}_{<0}$.*

This condition determines uniquely the coefficients of v^n with $n \ge 0$ in $\mu^s_{y,w}$. Then condition (i) determines uniquely the coefficients of v^n with $n < 0$ in $\mu^s_{y,w}$. The proposition is proved.

Proposition 6.4. *Let $y, w \in W$ be such that $sy < y < w < sw$. Then $\mu^s_{y,w}$ is a $\mathbf{Z}$-linear combination of powers v^n with $-L(s) + 1 \le n \le L(s) - 1$ and $n = L(w) - L(y) - L(s) \mod 2$.*

We may assume that this is already known for all $\mu^s_{z,w}$ with z such that $y < z < w; sz < z$. Using 6.3(ii) and 5.4(b), we see that $\mu^s_{y,w}$ is a $\mathbf{Z}$-linear combination of powers v^n such that, whenever $n \ge 0$, we have $n \le L(s) - 1$ and $n = L(w) - L(y) - L(s) \mod 2$. Using now 6.3(i), we deduce the remaining assertions of the proposition.

Corollary 6.5. *Assume that $L(s) = 1$. Let $y, w \in W$ be such that $sy < y < w < sw$. Then $\mu^s_{y,w}$ is an integer, equal to the coefficient of v^{-1} in $p_{y,w}$. In particular, it is 0 unless $L(w) - L(y)$ is odd.*

In this case, the inequalities of 6.4 become $0 \le n \le 0$. They imply $n = 0$. Thus, $\mu^s_{y,w} \in \mathbf{Z}$. Picking up the coefficient of v^0 in the two sides of 6.3(ii), we see that $\mu^s_{y,w}$ is equal to the coefficient of v^{-1} in $p_{y,w}$. The last assertion follows from 5.4.

Theorem 6.6. *Let $w \in W$.*

(a) If $w < sw$, then $c_s c_w = c_{sw} + \sum_{z;sz<z<w} \mu^s_{z,w} c_z$.

(b) If $sw < w$, then $c_s c_w = (v_s + v_s^{-1})c_w$.

Since $c_s = T_s + v_s^{-1}$ (see 5.5), we see that (b) is equivalent to $(T_s - v_s)c_w = 0$, or to

(c) $p_{x,w} = v_s^{-1} p_{sx,w}$

(where $sw < w$ and $x < sx$). We prove the theorem by induction on $l(w)$. If $l(w) = 0$, then $w = 1$ and the result is obvious. Assume now that $l(w) \geq 1$ and that the result holds when w is replaced by w' with $l(w') < l(w)$.

Case 1. Assume that $w < sw$. Using $c_s = T_s + v_s^{-1}$, we see that the coefficient of T_y in the left hand side minus the right hand side of (a) is

$$f_y = v_s^\sigma p_{y,w} + p_{sy,w} - p_{y,sw} - \sum_{z;y\leq z<w;sz<z} p_{y,z}\mu^s_{z,w}$$

where $\sigma = 1$ if $sy < y$ and $\sigma = -1$ if $sy > y$. We must show that $f_y = 0$. We first show that

(d) $f_y \in \mathcal{A}_{<0}$.

If $sy < y$ this follows from 6.3(ii). (The contribution of $p_{sy,w} - p_{y,sw}$ is in $\mathcal{A}_{<0}$ if $sy \neq w$ and is $1 - 1 = 0$ if $sy = w$.)

If $sy > y$ then, by (c) (applied to z in the sum, instead of w), we have

$$\begin{aligned} f_y &= v_s^{-1} p_{y,w} + p_{sy,w} - p_{y,sw} - \sum_{z;y\leq z<w;sz<z} v_s^{-1} p_{sy,z}\mu^s_{z,w} \\ &= v_s^{-1} f_{sy} + v_s^{-1} p_{sy,sw} - p_{y,sw} \end{aligned}$$

(the second equality holds by 2.5(a)) and this is in $\mathcal{A}_{<0}$ since $f_{sy} \in \mathcal{A}_{<0}$ (by the previous paragraph), $v_s^{-1} \in \mathcal{A}_{<0}$ and since $y \neq sw$. Thus, (d) is proved.

Since both sides of (a) are fixed by $\bar{}$, the sum $\sum_y f_y T_y$ is fixed by $\bar{}$. From (d) and 5.2(e) we see that $f_y = 0$ for all y, as required.

Case 2. Assume that $w > sw$. Then case 1 is applicable to sw (by the induction hypothesis). We see that

$$c_w = (T_s + v_s^{-1})c_{sw} - \sum_{z;sz<z<sw} \mu^s_{z,sw} c_z.$$

Now $(T_s - v_s)(T_s + v_s^{-1}) = 0$ and $(T_s - v_s)c_z = 0$ for each z in the sum (by the induction hypothesis). Hence $(T_s - v_s)c_w = 0$. The theorem is proved.

Corollary 6.7. *Let $w \in W$.*

(a) If $w < ws$, then $c_w c_s = c_{ws} + \sum_{z;zs<z<w} \mu^s_{z^{-1},w^{-1}} c_z$.

(b) If $ws < w$, then $c_w c_s = (v_s + v_s^{-1})c_w$.

We write the equalities in 6.6(a),(b) for w^{-1} instead of w and we apply to these equalities $h \mapsto h^\flat$ which carries c_w to $c_{w^{-1}}$; the corollary follows.

6.8. Now 6.3, 6.6, 6.7 remain valid when $L(s) < 0$ provided that we replace in their statements and proofs v_s by $-v_s^{-1}$.

7. Dihedral groups

7.1. We preserve the setup of 3.1; we assume that S consists of two elements s_1, s_2. For $i = 1, 2$, let $L_i = L(s_i), T_i = T_{s_i}, c_i = c_{s_i}$. We assume that $L_1 > 0, L_2 > 0$. Let $\zeta = v^{L_1-L_2} + v^{L_2-L_1} \in \mathcal{A}$. Let $m = m_{s_1,s_2}$. Let $1_k, 2_k$ be as in 1.4. For $w \in W$ we set
$\Gamma_w = \sum_{y;y\le w} v^{-L(w)+L(y)} T_y$.

Lemma 7.2. *We have*
$c_1\Gamma_{2_k} = \Gamma_{1_{k+1}} + v^{L_1-L_2}\Gamma_{1_{k-1}}$ *if* $k \in [2, m)$,
$c_2\Gamma_{1_k} = \Gamma_{2_{k+1}} + v^{-L_1+L_2}\Gamma_{2_{k-1}}$ *if* $k \in [2, m)$,
$c_1\Gamma_{2_k} = \Gamma_{1_{k+1}}$ *if* $k = 0, 1$,
$c_2\Gamma_{1_k} = \Gamma_{2_{k+1}}$ *if* $k = 0, 1$.

Since $c_i = T_i + v^{-L_i}$, the proof is an easy exercise.

Proposition 7.3. *Assume that* $L_1 = L_2$. *For any* $w \in W$ *we have* $c_w = \Gamma_w$.

This is clear when $l(w) \le 1$. In the present case Lemma 7.2 gives

(c) $$\Gamma_{1_{k+1}} = c_1\Gamma_{2_k} - \Gamma_{1_{k-1}}, \quad \Gamma_{2_{k+1}} = c_2\Gamma_{1_k} - \Gamma_{2_{k-1}}$$

for $k \in [2, m)$. This and 7.2 shows by induction on k that $\bar\Gamma_w = \Gamma_w$ for all $w \in W$. Clearly, $\Gamma_w = T_w \mod \mathcal{H}_{<0}$. The lemma follows.

7.4. *In 7.4-7.6 we assume that* $L_2 > L_1$. In this case, if $m < \infty$, then m is even. (See 3.1.) For $2k + 1 \in [1, m)$ we set

$$\Gamma'_{2_{2k+1}} = \sum_{s\in[0,k-1]} (1 - v^{2L_1} + v^{4L_1} - \dots + (-1)^s v^{2sL_1}) v^{-sL_1-sL_2}$$
$$\times (T_{2_{2k-2s+1}} + v^{-L_2}T_{2_{2k-2s}} + v^{-L_2}T_{1_{2k-2s}} + v^{-2L_2}T_{1_{2k-2s-1}})$$
$$+ (1 - v^{2L_1} + v^{4L_1} - \dots + (-1)^k v^{2kL_1}) v^{-kL_1-kL_2} (T_{2_1} + v^{-L_2}T_{2_0}).$$

For $2k + 1 \in [3, m)$ we set

$$\Gamma'_{1_{2k+1}} = T_{1_{2k+1}} + v^{-L_1}T_{1_{2k}} + v^{-L_1}T_{2_{2k}} + v^{-2L_1}T_{2_{2k-1}}$$
$$+ \sum_{\substack{y \\ y\le 1_{2k-1}}} v^{-L(w)+L(y)}(1 + v^{2L_1})T_y$$

where $w = 1_{2k+1}$. For w such that $l(w)$ is even and for $w = 1_1$ we set $\Gamma'_w = \Gamma_w$.

Lemma 7.5. *We have*
(a) $c_1\Gamma'_{2_{k'}} = \Gamma'_{1_{k'+1}}$, *if* $k' \in [0, m)$;
(b) $c_2\Gamma'_{1_{k'}} = \Gamma'_{2_{k'+1}} + \zeta\Gamma'_{2_{k'-1}} + \Gamma'_{2_{k'-3}}$, *if* $k' \in [4, m)$;
(c) $c_2\Gamma'_{1_{k'}} = \Gamma'_{2_{k'+1}} + \zeta\Gamma'_{2_{k'-1}}$, *if* $k' = 2, 3, k' < m$;

(d) $c_2\Gamma'_{1_{k'}} = \Gamma'_{2_{k'+1}}$ *if* $k' = 0, 1$.

From the definitions we have

(e) $\Gamma'_{2_{2k+1}} = \sum_{s\in[0,k]}(-1)^s v^{s(L_1-L_2)}\Gamma_{2_{2k-2s+1}}$ if $2k+1 \in [1, m)$,

(f) $\Gamma'_{1_{2k+1}} = \Gamma_{1_{2k+1}} + v^{L_1-L_2}\Gamma_{1_{2k-1}}$ if $2k+1 \in [3, m)$.

We prove (a) for $k' = 2k+1$. The left hand side can be computed using (e) and 7.2:

$$c_1\Gamma'_{2_{2k+1}} = c_1(\Gamma_{2_{2k+1}} - v^{L_1-L_2}\Gamma_{2_{2k-1}} + v^{2L_1-2L_2}\Gamma_{2_{2k-3}} + \dots)$$
$$= \Gamma_{1_{2k+2}} + v^{L_1-L_2}\Gamma_{1_{2k}} - v^{L_1-L_2}\Gamma_{1_{2k}} - v^{2L_1-2L_2}\Gamma_{1_{2k-2}}$$
$$+ v^{2L_1-2L_2}\Gamma_{1_{2k-2}} - v^{3L_1-3L_2}\Gamma_{1_{2k-4}} + \dots = \Gamma_{1_{2k+2}} = \Gamma'_{1_{2k+2}}.$$

This proves (a) for $k' = 2k+1$. Now (a) for $k' = 0$ is trivial. We prove (a) for $k' = 2k \ge 2$. The left hand side can be computed using 7.2 and (f):

$$c_1\Gamma'_{2_{2k}} = c_1\Gamma_{2_{2k}} = \Gamma_{1_{2k+1}} + v^{L_1-L_2}\Gamma_{1_{2k-1}} = \Gamma'_{1_{2k+1}}.$$

This proves (a) for $k' = 2k$. We prove (b) for $k' = 2k$. The left hand side can be computed using 7.2:

$$c_2\Gamma'_{1_{2k}} = c_2\Gamma_{1_{2k}} = \Gamma_{2_{2k+1}} + v^{-L_1+L_2}\Gamma_{2_{2k-1}}.$$

The right hand side of (b) is (using (e)):

$$\Gamma_{2_{2k+1}} - v^{L_1-L_2}\Gamma_{2_{2k-1}} + v^{2L_1-2L_2}\Gamma_{2_{2k-3}} + \dots$$
$$+ \zeta\Gamma_{2_{2k-1}} - v^{L_1-L_2}\zeta\Gamma_{2_{2k-3}} + v^{2L_1-2L_2}\zeta\Gamma_{2_{2k-5}} + \dots$$
$$+ \Gamma_{2_{2k-3}} - v^{L_1-L_2}\Gamma_{2_{2k-5}} + v^{2L_1-2L_2}\Gamma_{2_{2k-7}} + \dots = \Gamma_{2_{2k+1}} + v^{-L_1+L_2}\Gamma_{2_{2k-1}}.$$

This proves (b) for $k' = 2k$. We prove (b) for $k' = 2k+1$. The left hand side can be computed using (f) and 7.2:

$$c_2\Gamma'_{1_{2k+1}} = c_2(\Gamma_{1_{2k+1}} + v^{L_1-L_2}\Gamma_{1_{2k-1}})$$
$$= \Gamma_{2_{2k+2}} + v^{-L_1+L_2}\Gamma_{2_{2k}} + v^{L_1-L_2}\Gamma_{2_{2k}} + \Gamma_{2_{2k-2}} = \Gamma'_{2_{2k+2}} + \zeta\Gamma'_{2_{2k}} + \Gamma'_{2_{2k-2}}.$$

This proves (b) for $k' = 2k+1$. The proof of (c),(d) is similar to that of (b). This completes the proof.

Proposition 7.6. *For any* $w \in W$ *we have* $c_w = \Gamma'_w$.

Clearly, $\Gamma'_w = T_w \mod \mathcal{H}_{<0}$. From the formulas in 7.5 we see by induction on $l(w)$ that $\bar{\Gamma}'_w = \Gamma'_w$ for all w. The proposition is proved. (This was proved for $m = 4$ in [L7], for $m = 6$ in [X], for general m independently in [L15] and [GP, p.396].)

Proposition 7.7. *Assume that* $m=\infty$. *For* $a\in\{1,2\}$, *let* $f_a=v^{L(a)}+v^{-L(a)}$.

(a) If $L_1=L_2$ *and* $k,k'\ge 0$ *then* $c_{a_{2k+1}}c_{a_{2k'+1}}=f_a\sum_{u\in[0,\min(2k,2k')]}c_{a_{2k+2k'+1-2u}}$.

(b) If $L_2>L_1$ *and* $k,k'\ge 0$ *then* $c_{2_{2k+1}}c_{2_{2k'+1}}=f_2\sum_{u\in[0,\min(k,k')]}c_{2_{2k+2k'+1-4u}}$.

(c) If $L_2>L_1$ *and* $k,k'\ge 1$ *then*

$$c_{1_{2k+1}}c_{1_{2k'+1}}=f_1\sum_{u\in[0,\min(k-1,k'-1)]}p_u c_{1_{2k+2k'+1-2u}}$$

where $p_u=\zeta$ *for* u *odd,* $p_u\in\mathbf{Z}$ *for* u *even.*

We prove (a). For $k=k'=0$ the equality in (a) is clear. Assume now that $k=0,k'\ge 1$. Using 7.2, 7.3, we have

$$\begin{aligned}c_2c_{2_{2k'+1}}&=c_2(c_2c_{1_{2k'}}-c_{2_{2k'-1}})=f_2c_2c_{1_{2k'}}-f_2c_{2_{2k'-1}}\\&=f_2c_{2_{2k'+1}}+f_2c_{2_{2k'-1}}-f_2c_{2_{2k'-1}}=f_2c_{2_{2k'+1}},\end{aligned}$$

as required. We now prove the equality in (a) for fixed k', by induction on k. The case $k=0$ is already known. Assume now that $k=1$. From 7.2, 7.3 we have $c_{2_3}=c_2c_1c_2-c_2$. Using this and 7.2, 7.3, we have

$$\begin{aligned}c_{2_3}c_{2_{2k'+1}}&=c_2c_1c_2c_{2_{2k'+1}}-c_2c_{2_{2k'+1}}=f_2c_2c_{1_{2k'+2}}+f_2c_2c_{1_{2k'}}-f_2c_{2_{2k'+1}}\\&=f_2c_{2_{2k'+3}}+f_2c_{2_{2k'+1}}+f_2c_{2_{2k'+1}}+(1-\delta_{k',0})f_2c_{2_{2k'-1}}-f_2c_{2_{2k'+1}}\\&=f_2c_{2_{2k'+3}}+f_2c_{2_{2k'+1}}+f_2(1-\delta_{k',0})c_{2_{2k'-1}},\end{aligned}$$

as required. Assume now that $k\ge 2$. From 7.2,7.3 we have

$$c_{2_{2k+1}}=c_2c_1c_{2_{2k-1}}-2c_{2_{2k-1}}-c_{2_{2k-3}}.$$

Using this and the induction hypothesis we have

$$\begin{aligned}c_{2_{2k+1}}c_{2_{2k'+1}}&=c_2c_1c_{2_{2k-1}}c_{2_{2k'+1}}-2c_{2_{2k-1}}c_{2_{2k'+1}}-c_{2_{2k-3}}c_{2_{2k'+1}}\\&=f_2c_1c_2\sum_{u\in[0,\min(2k-2,k')]}c_{2_{2k+2k'-1-2u}}-f_2\sum_{u\in[0,\min(2k-2,k')]}c_{2_{2k+2k'-1-2u}}\\&-f_2\sum_{u\in[0,\min(2k-4,k')]}c_{2_{2k+2k'-3-2u}}.\end{aligned}$$

We now use 7.2,7.3 and (a) follows (for $a=2$). The case $a=1$ is similar.

We prove (b). For $k=k'=0$ the equality in (b) is clear. Assume now that $k=0,k'=1$. Using 7.5, 7.6, we have

$$c_2c_{2_3}=c_2(c_2c_{1_2}-\zeta c_{2_1})=f_2c_2c_{1_2}-f_2\zeta c_{2_1}=f_2c_{2_3}+f_2\zeta c_{2_1}-f_2\zeta c_{2_1}=f_2c_{2_3},$$

as required. Assume next that $k=0, k'\ge 2$. Using 7.5, 7.6, we have

$$c_2c_{2_{2k'+1}}=c_2(c_2c_{1_{2k'}}-\zeta c_{2_{2k'-1}}-c_{2_{2k'-3}})=f_2c_2c_{1_{2k'}}-f_2\zeta c_{2_{2k'-1}}-f_2c_{2_{2k'-3}}$$
$$=f_2c_{2_{2k'+1}}+f_2\zeta c_{2_{2k'-1}}+f_2c_{2_{2k'-3}}-f_2\zeta c_{2_{2k'-1}}-f_2\zeta c_{2_{2k'-3}}=f_2c_{2_{2k'+1}},$$

as required. We now prove the equality in (a) for fixed k', by induction on k. The case $k=0$ is already known. Assume now that $k=1$. From 7.5, 7.6 we have $c_{2_3}=c_2c_1c_2-\zeta c_2$. Using this and 7.5, 7.6, we have

$$c_{2_3}c_{2_{2k'+1}}=c_2c_1c_2c_{2_{2k'+1}}-\zeta c_2c_{2_{2k'+1}}=f_2c_2c_{1_{2k'+2}}-f_2\zeta c_{2_{2k'+1}}$$
$$=f_2c_{2_{2k'+3}}+f_2\zeta c_{2_{2k'+1}}+(1-\delta_{k',0})f_2c_{2_{2k'-1}}-f_2\zeta c_{2_{2k'+1}}$$
$$=f_2c_{2_{2k'+3}}+(1-\delta_{k',0})f_2c_{2_{2k'-1}}$$

as required. Assume now that $k\ge 2$. From 7.5, 7.6 we have

$$c_{2_{2k+1}}=c_2c_1c_{2_{2k-1}}-\zeta c_{2_{2k-1}}-c_{2_{2k-3}}.$$

Using this and the induction hypothesis we have

$$c_{2_{2k+1}}c_{2_{2k'+1}}=c_2c_1c_{2_{2k-1}}c_{2_{2k'+1}}-\zeta c_{2_{2k-1}}c_{2_{2k'+1}}-c_{2_{2k-3}}c_{2_{2k'+1}}$$
$$=f_2c_2c_1\sum_{u\in[0,\min(k-1,k')]}c_{2_{2k+2k'-1-4u}}-f_2\zeta\sum_{u\in[0,\min(k-1,k')]}c_{2_{2k+2k'-1-4u}}$$
$$-f_2\sum_{u\in[0,\min(k-2,k')]}c_{2_{2k+2k'-3-4u}}.$$

We now use 7.5, 7.6 and (b) follows.

The proof of (c) is similar to that of (b). This completes the proof.

Proposition 7.8. *Assume that $4\le m<\infty$ and $L_2>L_1$, so that $m=2k+2$ with $k\ge 1$. Let*
$p_0=(-1)^k(v^{L_2}+v^{-L_2})(v^{k(L_2-L_1)}+v^{(k-2)(L_2-L_1)}+\dots+v^{-k(L_2-L_1)})$.
Then

(a) $$c_{2_{m-1}}c_{2_{m-1}}=pc_{2_{m-1}}+qc_{2_m},$$

for some $p,q\in\mathcal{A}$. Moreover, $p=p_0$.

From 7.5, 7.6, we see that $\mathcal{A}c_{2_{m-1}}+\mathcal{A}c_{2_m}$ is a two-sided ideal of $\mathcal{H}$. Hence (a) holds for some (unknown) $p,q\in\mathcal{A}$. It remains to compute p. Define an algebra homomorphism $\chi:\mathcal{H}\to\mathcal{A}$ by $\chi(T_1)=-v^{-L_1},\chi(T_2)=v^{L_2}$. Since $c_{2_m}=(T_1+v^{-L_1})h$ for some $h\in\mathcal{H}$ (see 7.5,7.6) we see that $\chi(c_{2_m})=0$. Hence applying χ to (a) gives $\chi(c_{2_{m-1}})^2 = p\chi(c_{2_{m-1}})$. It is thus enough to show that $\chi(c_{2_{m-1}}) = p_0$. We verify this for $m = 4$:

$$\begin{aligned}
&\chi(T_2T_1T_2 + v^{-L_2}T_2T_1 + v^{-L_2}T_1T_2 + v^{-2L_2}T_1\\
&+ (v^{-L_1-L_2} - v^{L_1-L_2})T_2 + (v^{-L_1-2L_2} - v^{L_1-2L_2}))\\
&= -v^{-L_1+2L_2} - 2v^{-L_1} - v^{-L_1-2L_2} + (v^{-L_1-L_2} - v^{L_1-L_2})v^{L_2}\\
&+ v^{-L_1-2L_2} - v^{L_1-2L_2} = -v^{-L_1+2L_2} - v^{-L_1} - v^{L_1} - v^{L_1-2L_2}
\end{aligned}$$

and for $m = 6$:

$$\begin{aligned}
&\chi(T_2T_1T_2T_1T_2 + v^{-L_2}T_2T_1T_2T_1 + v^{-L_2}T_1T_2T_1T_2 + v^{-2L_2}T_1T_2T_1\\
&+ (v^{-L_1-L_2} - v^{L_1-L_2})T_2T_1T_2 + (v^{-L_1-2L_2} - v^{L_1-2L_2})T_1T_2\\
&+ (v^{-L_1-2L_2} - v^{L_1-2L_2})T_2T_1 + (v^{-L_1-3L_2} - v^{L_1-3L_2})T_1\\
&+ (v^{-2L_1-2L_2} - v^{-2L_2} + v^{2L_1-2L_2})T_2 + (v^{-2L_1-3L_2} - v^{-3L_2} + v^{2L_1-3L_2}))\\
&= v^{-2L_1+3L_2} + 2v^{-2L_1+L_2} + v^{-2L_1-L_2} - v^{-2L_1+L_2} - 2v^{-2L_1-L_2} - v^{-2L_1-3L_2} + v^{L_2}\\
&+ 2v^{-L_2} + v^{-3L_2} + v^{-2L_1-L_2} - v^{-L_2} + v^{2L_1-L_2} + v^{-2L_1-3L_2} - v^{-3L_2} + v^{2L_1-3L_2}\\
&= v^{-2L_1+3L_2} + v^{-2L_1+L_2} + v^{L_2} + v^{-L_2} + v^{2L_1-L_2} + v^{2L_1-3L_2}. \qquad \blacksquare
\end{aligned}$$

Analogous computations can be carried out for any even m. The proposition is proved.

8. Cells

8.1. We preserve the setup of 3.1. For $z \in W$ define $D_z \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, \mathcal{A})$ by $D_z(c_w) = \delta_{z,w}$ for all $w \in W$. For $w, w' \in W$ we write

$w \leftarrow_{\mathcal{L}} w'$ (or $w' \rightarrow_{\mathcal{L}} w$) if $D_w(c_s c_{w'}) \neq 0$ for some $s \in S$;

$w \leftarrow_{\mathcal{R}} w'$ (or $w' \rightarrow_{\mathcal{R}} w$) if $D_w(c_{w'} c_s) \neq 0$ for some $s \in S$.

If $w, w' \in W$, we say that $w \le_{\mathcal{L}} w'$ (resp. $w \le_{\mathcal{R}} w'$) if there exist $w = w_0, w_1, \dots, w_n = w'$ in W such that for any $i \in [0, n-1]$ we have $w_i \leftarrow_{\mathcal{L}} w_{i+1}$ (resp. $w_i \leftarrow_{\mathcal{R}} w_{i+1}$).

If $w, w' \in W$, we say that $w \le_{\mathcal{LR}} w'$ if there exist $w = w_0, w_1, \dots, w_n = w'$ in W such that for any $i \in [0, n-1]$ we have either $w_i \leftarrow_{\mathcal{L}} w_{i+1}$ or $w_i \leftarrow_{\mathcal{R}} w_{i+1}$.

Clearly $\le_{\mathcal{L}}, \le_{\mathcal{R}}, \le_{\mathcal{LR}}$ are preorders on W. Let $\sim_{\mathcal{L}}, \sim_{\mathcal{R}}, \sim_{\mathcal{LR}}$ be the associated equivalence relations. (For example, we have $w \sim_{\mathcal{L}} w'$ if and only if $w \le_{\mathcal{L}} w'$ and $w' \le_{\mathcal{L}} w$.) The equivalence classes on W for $\sim_{\mathcal{L}}, \sim_{\mathcal{R}}, \sim_{\mathcal{LR}}$ are called respectively *left cells, right cells, two-sided cells* of W. They depend on the weight function L.

If $w, w' \in W$, we say that $w <_{\mathcal{L}} w'$ (resp. $w <_{\mathcal{R}} w'$; $w <_{\mathcal{LR}} w'$) if $w \le_{\mathcal{L}} w'$ and $w \not\sim_{\mathcal{L}} w'$ (resp. $w \le_{\mathcal{R}} w'$ and $w \not\sim_{\mathcal{R}} w'$; $w \le_{\mathcal{LR}} w'$ and $w \not\sim_{\mathcal{LR}} w'$).

For $w, w' \in W$ we have $w \le_{\mathcal{L}} w' \Leftrightarrow w^{-1} \le_{\mathcal{R}} w'^{-1}$ and $w \le_{\mathcal{LR}} w' \Leftrightarrow w^{-1} \le_{\mathcal{LR}} w'^{-1}$.

Hence $w \mapsto w^{-1}$ carries left cells to right cells, right cells to left cells and two-sided cells to two-sided cells.

Lemma 8.2. *Let $w' \in W$.*
(a) $\mathcal{H}_{\leq_{\mathcal{L}} w'} = \oplus_{w; w \leq_{\mathcal{L}} w'} \mathcal{A} c_w$ is a left ideal of $\mathcal{H}$.
(b) $\mathcal{H}_{\leq_{\mathcal{R}} w'} = \oplus_{w; w \leq_{\mathcal{R}} w'} \mathcal{A} c_w$ is a right ideal of $\mathcal{H}$.
(c) $\mathcal{H}_{\leq_{\mathcal{LR}} w'} = \oplus_{w; w \leq_{\mathcal{LR}} w'} \mathcal{A} c_w$ is a two-sided ideal of $\mathcal{H}$.

This follows from the definitions since $c_s (s \in S)$ generate $\mathcal{H}$ as an $\mathcal{A}$-algebra.

8.3. Let Y be a left cell of W. From 8.3(a) we see that for $y \in Y$,

$$\oplus_{w; w \leq_{\mathcal{L}} y} \mathcal{A} c_w / \oplus_{w; w <_{\mathcal{L}} y} \mathcal{A} c_w$$

is a quotient of two left ideals of $\mathcal{H}$ (independent of the choice of y) hence it is naturally a left $\mathcal{H}$-module; it has an $\mathcal{A}$-basis consisting of the images of $c_w (w \in Y)$.

Similarly, if Y' is a right cell of W then, for $y' \in Y'$,

$$\oplus_{w; w \leq_{\mathcal{R}} y'} \mathcal{A} c_w / \oplus_{w; w <_{\mathcal{R}} y'} \mathcal{A} c_w$$

is a quotient of two right ideals of $\mathcal{H}$ (independent of the choice of y') hence it is naturally a right $\mathcal{H}$-module; it has an $\mathcal{A}$-basis consisting of the images of $c_w (w \in Y')$.

If Y'' is a two-sided cell of W then, for $y'' \in Y''$,

$$\oplus_{w; w \leq_{\mathcal{LR}} y''} \mathcal{A} c_w / \oplus_{w; w <_{\mathcal{LR}} y''} \mathcal{A} c_w$$

is a quotient of two two-sided ideals of $\mathcal{H}$ (independent of the choice of y'') hence it is naturally a $\mathcal{H}$-bimodule; it has an $\mathcal{A}$-basis consisting of the images of $c_w (w \in Y'')$.

Lemma 8.4. *Let $s \in S$. Assume that $L(s) > 0$. Let $\mathcal{H}^s = \oplus_{w; sw < w} \mathcal{A} c_w$, ${}^s\mathcal{H} = \oplus_{w; ws < w} \mathcal{A} c_w$.*
(a) $\{h \in \mathcal{H}; (c_s - v_s - v_s^{-1})h = 0\} = \mathcal{H}^s$. Hence $\mathcal{H}^s$ is a right ideal of $\mathcal{H}$.
(b) $\{h \in \mathcal{H}; h(c_s - v_s - v_s^{-1}) = 0\} = {}^s\mathcal{H}$. Hence ${}^s\mathcal{H}$ is a left ideal of $\mathcal{H}$.

We prove the equality in (a). If $h \in \mathcal{H}^s$ then $(c_s - v_s - v_s^{-1})h = 0$ by 6.6(b). Conversely, by 6.6, we have $c_s h \in \mathcal{H}^s$ for any $h \in \mathcal{H}$. Hence, if $h \in \mathcal{H}$ is such that $c_s h = (v_s + v_s^{-1})h$, then $(v_s + v_s^{-1})h \in \mathcal{H}^s$ so that $h \in \mathcal{H}^s$ (since $\mathcal{H}/\mathcal{H}^s$ is a free $\mathcal{A}$-module). This proves (a). The proof of (b) is entirely similar. The lemma is proved.

8.5. For $w \in W$ we set $\mathcal{L}(w) = \{s \in S; sw < w\}, \mathcal{R}(w) = \{s \in S; ws < w\}$.

Lemma 8.6. *Let $w, w' \in W$. Assume that $L(s) > 0$ for all $s \in S$.*
(a) If $w \leq_{\mathcal{L}} w'$, then $\mathcal{R}(w') \subset \mathcal{R}(w)$. If $w \sim_{\mathcal{L}} w'$, then $\mathcal{R}(w') = \mathcal{R}(w)$.
(b) If $w \leq_{\mathcal{R}} w'$, then $\mathcal{L}(w') \subset \mathcal{L}(w)$. If $w \sim_{\mathcal{R}} w'$, then $\mathcal{L}(w') = \mathcal{L}(w)$.

To prove the first assertion of (a), we may assume that $D_w(c_s c_{w'}) \neq 0$ for some $s \in S$. In this case, let $t \in \mathcal{R}(w')$. We must prove that $t \in \mathcal{R}(w)$. We have $c_{w'} \in {}^t\mathcal{H}$. By 8.4, ${}^t\mathcal{H}$ is a left ideal of $\mathcal{H}$. Hence $c_s c_{w'} \in {}^t\mathcal{H}$. By the definition of ${}^t\mathcal{H}$, for $h \in {}^t\mathcal{H}$ we have $D_w(h) = 0$ unless $wt < w$. Hence from $D_w(c_s c_{w'}) \neq 0$ we deduce $wt < w$, as required. This proves the first assertion of (a). The second assertion of (a) follows immediately from the first. The proof of (b) is entirely similar to that of (a). The lemma is proved.

8.7. In the remainder of this chapter we write $\leftarrow, \rightarrow$ instead of $\leftarrow_{\mathcal{L}}, \rightarrow_{\mathcal{L}}$. We describe the left cells of W in the setup of 7.3. From 7.2 and 7.3 we can determine all pairs $y \neq w$ such that $y \leftarrow w$

$1_0 \rightarrow 2_1 \leftrightarrows 1_2 \leftrightarrows 2_3 \leftrightarrows \dots,$
$2_0 \rightarrow 1_1 \leftrightarrows 2_2 \leftrightarrows 1_3 \leftrightarrows \dots,$

if $m = \infty$,

$1_0 \rightarrow 2_1 \leftrightarrows 1_2 \leftrightarrows 2_3 \leftrightarrows \dots \leftrightarrows 2_{m-1} \rightarrow 1_m,$
$2_0 \rightarrow 1_1 \leftrightarrows 2_2 \leftrightarrows 1_3 \leftrightarrows \dots \leftrightarrows 1_{m-1} \rightarrow 2_m,$

if $m < \infty, m$ even,

$1_0 \rightarrow 2_1 \leftrightarrows 1_2 \leftrightarrows 2_3 \leftrightarrows \dots \leftrightarrows 1_{m-1} \rightarrow 2_m,$
$2_0 \rightarrow 1_1 \leftrightarrows 2_2 \leftrightarrows 1_3 \leftrightarrows \dots \leftrightarrows 2_{m-1} \rightarrow 1_m,$

if $m < \infty, m$ odd. Hence the left cells are

$\{1_0\}, \{2_1, 1_2, 2_3, \dots\}, \{1_1, 2_2, 1_3, \dots\},$

if $m = \infty$,

$\{1_0\}, \{2_1, 1_2, 2_3, \dots, 2_{m-1}\}, \{1_1, 2_2, 1_3, \dots, 1_{m-1}\}, \{2_m\},$

if $m < \infty, m$ even,

$\{1_0\}, \{2_1, 1_2, 2_3, \dots, 1_{m-1}\}, \{1_1, 2_2, 1_3, \dots, 2_{m-1}\}, \{2_m\},$

if $m < \infty, m$ odd.

The two-sided cells are $\{1_0\}, W - \{1_0\}$ if $m = \infty$ and $\{1_0\}, \{2_m\}, W - \{1_0, 2_m\}$ if $m < \infty$.

8.8. We describe the left cells of W in the setup of 7.4. From 7.5 and 7.6 we can determine all pairs $y \neq w$ such that $y \leftarrow w$. If $m = \infty$, these pairs are:

$$1_0 \rightarrow 2_1 \leftrightarrows 1_2 \rightarrow 2_3 \leftrightarrows 1_4 \rightarrow \dots, \quad 2_0 \rightarrow 1_1 \rightarrow 2_2 \leftrightarrows 1_3 \rightarrow 2_4 \leftrightarrows \dots,$$

and $2_1 \leftarrow 1_4, 2_2 \leftarrow 1_5, 2_3 \leftarrow 1_6, \dots$.

If $m = 4$, these pairs are:

$$1_0 \rightarrow 2_1 \leftrightarrows 1_2 \rightarrow 2_3 \rightarrow 1_4, \quad 2_0 \rightarrow 1_1 \rightarrow 2_2 \leftrightarrows 1_3 \rightarrow 2_4.$$

If $m = 6$, these pairs are:

$$1_0 \rightarrow 2_1 \leftrightarrows 1_2 \rightarrow 2_3 \leftrightarrows 1_4 \rightarrow 2_5 \rightarrow 1_6, \quad 2_0 \rightarrow 1_1 \rightarrow 2_2 \leftrightarrows 1_3 \rightarrow 2_4 \leftrightarrows 1_5 \rightarrow 2_6,$$

and $2_1 \leftarrow 1_4, 2_2 \leftarrow 1_5$. An analogous pattern holds for any even m.

Hence the left cells are

$$\{1_0\}, \{2_1, 1_2, 2_3, 1_4, \dots\}, \{1_1\}, \{2_2, 1_3, 2_4, 1_5, \dots\},$$

if $m = \infty$,

$$\{1_0\}, \{2_1, 1_2, 2_3, \dots, 1_{m-2}\}, \{2_{m-1}\}, \{1_1\}, \{2_2, 1_3, 2_4, \dots, 1_{m-1}\}, \{2_m\},$$

if $m < \infty$.

The two-sided cells are

$\{1_0\}, \{1_1\}, W - \{1_0, 1_1\}$, if $m = \infty$ and
$\{1_0\}, \{1_1\}, \{2_{m-1}\}, \{2_m\}, W - \{1_0, 1_1, 2_{m-1}, 2_m\}$, if $m < \infty$.

8.9. For further examples of cells (in the case where L is non-costant) see [L3], [B], [G].

9. Cosets of parabolic subgroups

We preserve the setup of 3.1.

Lemma 9.1. *Let $w \in W$. Assume that $w = s_1 s_2 \dots s_q$ with $s_i \in S$. We can find a subsequence $i_1 < i_2 < \dots < i_r$ of $1, 2, \dots, q$ such that $w = s_{i_1} s_{i_2} \dots s_{i_r}$ is a reduced expression in W.*

We argue by induction on q. If $q = 0$ the result is obvious. Assume that $q > 0$. Using the induction hypothesis we can assume that $s_2 \dots s_q$ is a reduced expression. If $s_1 s_2 \dots s_q$ is a reduced expression, we are done. Hence we may assume that $s_1 s_2 \dots s_q$ is not a reduced expression. Then $l(w) \le q - 1$. By 1.7, we can find $j \in [2, q]$ such that $s_1 s_2 \dots s_{j-1} = s_2 s_3 \dots s_j$. Then $w = s_2 s_3 \dots s_{j-1} s_{j+1} \dots s_q$ is a reduced expression. The lemma is proved.

9.2. Let $w \in W$. Let $w = s_1 s_2 \dots s_q$ be a reduced expression of w. Using 1.9, we see that the set $\{s \in S; s = s_i \text{ for some } i \in [1, q]\}$ is independent of the choice of reduced expression. We denote it by S_w.

9.3. In the remainder of this chapter we fix $I \subset S$. Recall that $W_I = \langle I \rangle$.

If $w \in W_I$ then we can find a reduced expression $w = s_1 s_2 \dots s_q$ in W with all $s_i \in I$ (we first write $w = s_1 s_2 \dots s_q$, a not necessarily reduced expression with all $s_i \in I$, and then we apply 9.1). Thus, $S_w \subset I$. Conversely, it is clear that if $w' \in W$ satisfies $S_{w'} \subset I$ then $w' \in W_I$. It follows that

$W_I = \{w \in W; S_w \subset I\}$.

9.4. Replacing $S, (m_{s,s'})_{(s,s') \in S \times S}$ by $I, (m_{s,s'})_{(s,s') \in I \times I}$ in the definition of W we obtain a Coxeter group denoted by W_I^*. We have an obvious homomorphism $f : W_I^* \to W_I$ which takes s to s for $s \in I$.

Proposition 9.5. *$f : W_I^* \to W_I$ is an isomorphism.*

We define $f' : W_I \to W_I^*$ as follows: for $w \in W_I$ we choose a reduced expression $w = s_1 s_2 \dots s_q$ in W; then $s_i \in I$ for all i (see 9.3) and we set $f'(w) = s_1 s_2 \dots s_q$ (product in W_I^*). This map is well defined. Indeed, if $s'_1 s'_2 \dots s'_q$ is another reduced expression for w with all $s'_i \in I$, then we can pass from $(s_1, s_2, \dots, s_q)$ to $(s'_1, s'_2, \dots, s'_q)$ by moving along edges of the graph X (see 1.9); but each edge involved in this move will necessarily involve only pairs (s, s') in I, hence the equation $s_1 s_2 \dots s_q = s'_1 s'_2 \dots s'_q$ must hold in W_I^*. It is clear that $ff'(w) = w$ for all $w \in W_I$. Hence f' is injective.

We show that f' is a group homomorphism. It suffices to show that $f'(sw) = f'(s) f'(w)$ for any $w \in W_I, s \in I$. This is clear if $l(sw) = l(w) + 1$ (in W). Assume now that $l(sw) = l(w) - 1$ (in W). Let $w = s_1 s_2 \dots s_q$ be a reduced expression in W. Then $s_i \in I$ for all i. By 1.7 we have (in W) $sw = s_1 s_2 \dots s_{i-1} s_{i+1} \dots s_q$ for some $i \in [1, q]$. Since $s s_1 s_2 \dots s_{i-1} s_{i+1} \dots s_q$ is a reduced expression for w in W, we have $f'(w) = s s_1 s_2 \dots s_{i-1} s_{i+1} \dots s_q$ (product in W_I^*). We also have $f'(w) = s_1 s_2 \dots s_q$ (product in W_I^*). Hence $s s_1 s_2 \dots s_{i-1} s_{i+1} \dots s_q = s_1 s_2 \dots s_q$ (in W_I^*).

Hence $s_1s_2\dots s_{i-1}s_{i+1}\dots s_q = ss_1s_2\dots s_q$ (in W_I^*). Hence $f'(sw) = f'(s)f'(w)$, as required.

Since the image of f' contains the generators $s \in I$ of W_I^* and f' is a group homomorphism, it follows that f' is surjective. Hence f' is bijective. Since $ff' = 1$ it follows that f is bijective. The proposition is proved.

9.6. We identify W_I^* and W_I via f. Thus, W_I is naturally a Coxeter group. Let $l_I : W_I \to \mathbf{N}$ be the length function of this Coxeter group. Let $w \in W_I$. Let $w = s_1s_2\dots s_q$ be a reduced expression of w (in W). Then $s_i \in I$ for all i (see 9.3). Hence $l_I(w) \le l(w)$. The reverse inequality $l(w) \le l_I(w)$ is obvious. Hence $l_I(w) = l(w)$.

From 2.4 we see that the partial order on W_I defined in the same way as $\le$ on W is just the restriction of $\le$ from W to W_I.

Lemma 9.7. *Let W_Ia be a coset in W.*

(a) This coset has a unique element w of minimal length.

(b) If $y \in W_I$ then $l(yw) = l(y) + l(w)$.

(c) w is characterized by the property that $l(sw) > l(w)$ for all $s \in I$.

Let w be an element of minimal length in the coset. Let $w = s_1s_2\dots s_q$ be a reduced expression. Let $y \in W_I$ and let $y = s'_1s'_2\dots s'_p$ be a reduced expression in W_I. Then $yw = s'_1s'_2\dots s'_ps_1s_2\dots s_q$. By 9.1 we can drop some of the factors in the last product so that we are left with a reduced expression for yw. The factors dropped cannot contain any among the last q since otherwise we would find an element in W_Ia of strictly smaller length than w. Thus, we can find a subsequence $i_1 < i_2 < \dots < i_r$ of $1, 2, \dots, p$ such that $yw = s'_{i_1}s'_{i_2}\dots s'_{i_r}s_1s_2\dots s_q$ is a reduced expression. It follows that $y = s'_1s'_2\dots s'_p = s'_{i_1}s'_{i_2}\dots s'_{i_r}$. Since $p = l(y)$, we must have $r = p$ so that $s'_1s'_2\dots s'_ps_1s_2\dots s_q$ is a reduced expression and $l(yw) = p + q = l(y) + l(w)$.

If now w' is another element of minimal length in W_Ia then $w' = yw$ for some $y \in W_I$. We have $l(w) = l(w') = l(y) + l(w)$ hence $l(y) = 0$ hence $y = 1$ and $w' = w$. This proves (a). Now (b) is already proved. Note that by (b), w has the property in (c). Conversely, let $w' \in W_Ia$ be an element such that $l(sw') > l(w')$ for all $s \in I$. We have $w' = yw$ for some $y \in W_I$. If $y \ne 1$ then for some $s \in I$ we have $l(y) = l(sy) + 1$. By (b) we have $l(w') = l(y) + l(w)$, $l(sw') = l(sy) + l(w)$. Thus $l(w') - l(sw') = l(y) - l(sy) = 1$, a contradiction. Thus $y = 1$ and $w' = w$. The lemma is proved.

We shall denote by ${}^IW$ (resp. W^I) the set of all $w \in W$ such that w has minimal length in W_Iw (resp. in wW_I). If $I \subset J \subset S$ we write ${}^IW_J, W_J^I$ instead of ${}^I(W_J), (W_J)^I$.

Lemma 9.8. *Let W_Ia be a coset in W.*

(a) If W_I is finite, this coset has a unique element w of maximal length. If W_I is infinite, this coset has no element of maximal length.

(b) Assume that W_I is finite. If $y \in W_I$ then $l(yw) = l(w) - l(y)$.

(c) Assume that W_I is finite. Then w is characterized by the property that $l(sw) < l(w)$ for all $s \in I$.

Assume that w has maximal length in $W_I a$. We show that for any $y \in W_I$ we have

(d) $l(yw) = l(w) - l(y)$.

We argue by induction on $l(y)$. If $l(y) = 0$, the result is clear. Assume now that $l(y) = p + 1 \geq 1$. Let $y = s_1 \dots s_p s_{p+1}$ be a reduced expression. By the induction hypothesis, $l(w) = l(s_1 s_2 \dots s_p w) + p$. Hence we can find a reduced expression of w of the form $s_p \dots s_2 s_1 s'_1 s'_2 \dots s'_q$. Since $s_{p+1} \in I$, by our assumption on w we have $l(s_{p+1} w) = l(w) - 1$. Using 1.7, we deduce that either

(1) $s_{p+1} s_p \dots s_{j+1} = s_p \dots s_{j+1} s_j$ for some $j \in [1, p]$ or

(2) $s_{p+1} s_p \dots s_2 s_1 s'_1 s'_2 \dots s'_{i-1} = s_p \dots s_2 s_1 s'_1 s'_2 \dots s'_{i-1} s'_i$ for some $i \in [1, q]$.

In case (1) it follows that $y = s_1 \dots s_p s_{p+1} = s_1 s_2 \dots s_{j-1} s_{j+1} \dots s_p$, contradicting $l(y) = p + 1$. Thus, we must be in case (2). We have

$$yw = s'_1 s'_2 \dots s'_{i-1} s'_{i+1} \dots s'_q \text{ and } l(yw) \leq q - 1 = l(w) - p - 1 = l(w) - l(y).$$

Thus, $l(w) \geq l(yw) + l(y)$. The reverse inequality is obvious. Hence $l(w) = l(yw) + l(y)$. This completes the induction.

From (d) we see that $l(y) \leq l(w)$. Thus $l : W_I \to \mathbf{N}$ is bounded above. Hence there exists $y \in W_I$ of maximal length in W_I. Applying (d) to y, W_I instead of $w, W_I a$ we see that

$$l(y) = l(y'^{-1}) + l(y'^{-1} y) = l(y') + l(y'^{-1} y)$$

for any $y' \in W_I$. Hence a reduced expression of y' followed by a reduced expression of $y'^{-1} y$ gives a reduced expression of y. In particular $y' \leq y$. Since the set $\{y' \in W; y' \leq y\}$ is finite, we see that W_I is finite. Conversely, if W_I is finite then $W_I a$ clearly has some element of maximal length.

If w' is another element of maximal length in $W_I a$ then $w' = yw$ for some $y \in W_I$. We have $l(w) = l(w') = l(w) - l(y)$ hence $l(y) = 0$ hence $y = 1$ and $w' = w$. This proves (a) and (b). The proof of (c) is entirely similar to that of 9.7(c). The lemma is proved.

9.9. Replacing W, L by $W_I, L|_{W_I}$ in the definition of $\mathcal{H}$ we obtain an $\mathcal{A}$-algebra $\mathcal{H}_I$ (naturally a subalgebra of $\mathcal{H}$); instead of $r_{x,y}, p_{x,y}, c_y, \mu^s_{x,y}$ we obtain for $x, y \in W_I$ elements $r^I_{x,y} \in \mathcal{A}, p^I_{x,y} \in \mathcal{A}_{\leq 0}, c^I_y \in \mathcal{H}_I, \mu^{s,I}_{x,y} \in \mathcal{A}$.

Lemma 9.10. *Let $z \in W$ be such that z is the element of minimal length of $W_I z$. Let $x, y \in W_I$. We have*

(a) $\{u' \in W; xz \leq u' \leq yz\} = \{u \in W_I; x \leq u \leq y\} z$;

(b) $r_{xz,yz} = r^I_{x,y}$;

(c) $p_{xz,yz} = p^I_{x,y}$;

(d) $c^I_y = c_y$.

(e) If in addition, $s\in I$, $L(s)>0$ *and* $sx<x<y<sy$, *then* $sxz<xz<yz<syz$ *and* $\mu^{s,I}_{x,y}=\mu^s_{xz,yz}$.

We first prove the following statement.

Assume that z_1,z_2 *have minimal length in* W_Iz_1,W_Iz_2 *respectively, that* $u_1,u_2\in W_I$ *and that* $u_1z_1\le u_2z_2$. *Then*

(f) $z_1\le z_2$; *if in addition,* $z_1=z_2$ *then* $u_1\le u_2$.

Indeed, using 2.4 we see that there exist u'_1,z'_1 such that

$u_1z_1=u'_1z'_1, u'_1\le u_2, z'_1\le z_2$.

Then $u'_1\in W_I$ and $z'_1\in W_Iz_1$ hence $z'_1=wz_1$ where $w\in W_I$, $l(z'_1)=l(w)+l(z_1)$. Hence $z_1\le z'_1$. Since $z'_1\le z_2$, we see that $z_1\le z_2$. If we know that $z_1=z_2$, then $z'_1=z_1$ hence $u_1=u'_1$. Since $u'_1\le u_2$, it follows that $u_1\le u_2$ and (f) is proved.

We prove (a). If $u\in W_I$ and $x\le u\le y$, then $xz\le uz\le yz$ by 2.4 and 9.7(b). Conversely, assume that $u'\in W$ satisfies $xz\le u'\le yz$. Then $u'=uz_1$ where z_1 has minimal length in W_Iu' and $u\in W_I$. Applying (f) to $xz\le uz_1$ and to $uz_1\le yz$ we deduce $z\le z_1\le z$. Hence $z=z_1$. Applying the second part of (f) to $xz\le uz$ and to $uz\le yz$ we deduce $x\le u\le y$. This proves (a).

We prove (b) by induction on $l(y)$. Assume first that $y=1$. Then $r^I_{x,y}=\delta_{x,1}$. Now $r_{xz,z}=0$ unless $xz\le z$ (see 4.7(a)) in which case $x=1$ and $r_{z,z}=1$. Thus, (b) holds for $y=1$. Assume now that $l(y)\ge 1$. We can find $s\in I$ such that $l(sy)=l(y)-1$. We have

$$l(syz)=l(sy)+l(z)=l(y)-1+l(z)=l(yz)-1.$$

If $sx<x$ then we have (as above) $sxz<xz$. Using 4.4 and the induction hypothesis, we have

$$r_{xz,yz}=r_{sxz,syz}=r^I_{sx,sy}=r^I_{x,y}.$$

If $sx>x$ then we have (as above) $sxz>xz$. Using 4.4 and the induction hypothesis, we have

$$r_{xz,yz}=r_{sxz,syz}+(v_s-v_s^{-1})r_{xz,syz}=r^I_{sx,sy}+(v_s-v_s^{-1})r^I_{x,sy}=r^I_{x,y}.$$

This completes the proof of (b).

We prove (c). Using (a), we may assume that $x\le y$ (otherwise, both sides are zero). We argue by induction on $l(y)-l(x)\ge 0$. If $y=x$, the result is clear (both sides are 1). Assume now that $l(y)-l(x)\ge 1$. Using 5.3, then (a),(b) and the induction hypothesis, we have

$$\begin{aligned}\bar p_{xz,yz}&=\sum_{u';xz\le u'\le yz} r_{xz,u'}p_{u',yz}=\sum_{u\in W_I;x\le u\le y} r_{xz,uz}p_{uz,yz}\\&=\sum_{u\in W_I;x\le u\le y} r^I_{x,u}p_{uz,yz}=\sum_{u\in W_I;x<u\le y} r^I_{x,u}p^I_{u,y}+p_{xz,yz}.\end{aligned}$$

Using 5.3 for W_I we have $\bar{p}^I_{x,y} = \sum_{y;x\le u\le y} r^I_{x,u} p^I_{u,y}$. Comparison with the previous equality gives

$$\bar{p}_{xz,yz} - \bar{p}^I_{x,y} = p_{xz,yz} - p^I_{x,y}.$$

The right hand side of this equality is in $\mathcal{A}_{<0}$. Since it is fixed by $\bar{}$, it must be 0. This proves (c). Now (d) is an immediate consequence of (c) (with $z = 1$).

We prove (e). By 6.3(ii),

$$\sum_{u';xz\le u'<yz;su'<u'} p_{xz,u'}\mu^s_{u',yz} - v_s p_{xz,yz} \in \mathcal{A}_{<0}.$$

We rewrite this using (a):

$$\sum_{u\in W_I;x\le u<y;su<u} p_{xz,uz}\mu^s_{uz,yz} - v_s p_{xz,yz} \in \mathcal{A}_{<0}.$$

We may assume that for all u in the sum, other than $u = x$, we have $\mu^s_{uz,yz} = \mu^{s,I}_{u,y}$. Using this and (d), we obtain

$$\mu^s_{xz,yz} + \sum_{u\in W_I;x<u<y;su<u} p^I_{x,u}\mu^{s,I}_{u,y} - v_s p^I_{x,y} \in \mathcal{A}_{<0}.$$

By 6.3(ii) for W_I we have

$$\mu^{s,I}_{x,y} + \sum_{u\in W_I;x<u<y;su<u} p^I_{x,u}\mu^{s,I}_{u,y} - v_s p^I_{x,y} \in \mathcal{A}_{<0}.$$

It follows that $\mu^s_{xz,yz} - \mu^{s,I}_{x,y} \in \mathcal{A}_{<0}$. On the other hand, $\mu^s_{xz,yz} - \mu^{s,I}_{x,y}$ is fixed by $\bar{}$ (see 6.3(i)) hence it is 0. This proves (e). The lemma is proved.

Proposition 9.11. *Assume that $L(s) > 0$ for all $s \in I$.*

(a) Let $z \in W$ be such that z is the element of minimal length of $W_I z$. If x, y in W_I satisfy $x \le_{\mathcal{L}} y$ (relative to W_I), then $xz \le_{\mathcal{L}} yz$ (in W). If x, y in W_I satisfy $x \sim_{\mathcal{L}} y$ (relative to W_I), then $xz \sim_{\mathcal{L}} yz$ (in W).

(b) Let $z \in W$ be such that z is the element of minimal length of zW_I. If x, y in W_I satisfy $x \le_{\mathcal{R}} y$ (relative to W_I), then $zx \le_{\mathcal{R}} zy$ (in W). If x, y in W_I satisfy $x \sim_{\mathcal{R}} y$ (relative to W_I), then $zx \sim_{\mathcal{R}} zy$ (in W).

We prove the first assertion of (a). We may assume that $x \leftarrow_{\mathcal{L}} y$ (relative to W_I) and $x \ne y$. Thus, there exists $s \in I$ such that $sy > y$, $sx < x$ and we have either $x = sy$ or $x < y$ and $\mu^s_{x,y} \ne 0$. If $x = sy$, then $sxz < xz = syz > yz$, hence $xz \leftarrow_{\mathcal{L}} yz$ (in W). Thus, we may assume that $x < y$ and $\mu^s_{x,y} \ne 0$. By 9.10(e) we then have $\mu^s_{xz,yz} \ne 0$, hence $xz \leftarrow_{\mathcal{L}} yz$ (in W). The first assertion of (a) is proved. The second assertion of (a) follows from the first. (b) follows by applying (a) to z^{-1}, x^{-1}, y^{-1} instead of z, x, y.

9.12. Assume that $z \in W$ is such that $W_I z = zW_I$ and z is the element of minimal length of $W_I z = zW_I$. Then $y \mapsto z^{-1}yz$ is an automorphism of W_I. If $s \in I$ then, by 9.7, we have $l(sz) = l(s) + l(z) = 1 + l(z)$; by 9.7 applied to $W_I z^{-1}$ instead of $W_I z$ we have $l((z^{-1}sz)z^{-1}) = l(z^{-1}sz) + l(z^{-1})$ hence $l(z^{-1}s) = l(z^{-1}sz) + l(z^{-1})$; since $l(z^{-1}s) = l(sz)$ and $l(z^{-1}) = l(z)$, it follows that $l(z^{-1}sz) + l(z^{-1}) = 1 + l(z)$, hence $l(z^{-1}sz) = 1$. We see that $y \mapsto z^{-1}yz$ maps I onto itself hence it is an automorphism of W_I as a Coxeter group. This automorphism preserves the function $L|_{W_I}$. Indeed, if $y \in W_I$, then

$$l(zyz^{-1}) + l(z) = l((zyz^{-1})z) = l(zy) = l(y^{-1}z^{-1}) = l(y^{-1}) + l(z^{-1}) = l(y) + l(z)$$

(by 9.7 applied to $W_I z$ and to $W_I z^{-1}$) hence

$$\begin{aligned} &L(zyz^{-1}) + L(z) = L((zyz^{-1})z) = L(zy) = L(y^{-1}z^{-1}) = L(y^{-1}) + L(z^{-1}) \\ &= L(y) + L(z), \end{aligned}$$

so that $L(zyz^{-1}) = L(y)$. In particular, this automorphism respects the preorders $\leq_{\mathcal{L}}, \leq_{\mathcal{R}}, \leq_{\mathcal{LR}}$ of W_I (defined in terms of $L|_{W_I}$) and the associated equivalence relations.

Proposition 9.13. *Assume that $L(s) > 0$ for all $s \in I$. Let z be as in 9.12. If x, y in W_I satisfy $x \leq_{\mathcal{LR}} y$ (relative to W_I), then $xz \leq_{\mathcal{LR}} yz$ (in W). If x, y in W_I satisfy $x \sim_{\mathcal{LR}} y$ (relative to W_I), then $xz \sim_{\mathcal{LR}} yz$ (in W).*

We prove the first assertion. We may assume that either $x \leq_{\mathcal{L}} y$ (in W_I) or $x \leq_{\mathcal{R}} y$ (in W_I). In the first case, by 9.11(a), we have $xz \leq_{\mathcal{L}} yz$ (in W) hence $xz \leq_{\mathcal{LR}} yz$ (in W). In the second case, by 9.12, we have $z^{-1}xz \leq_{\mathcal{R}} z^{-1}yz$. Applying 9.11(b) to $z^{-1}xz, z^{-1}yz$ instead of x, y we see that $xz \leq_{\mathcal{R}} yz$ (in W) hence $xz \leq_{\mathcal{LR}} yz$ (in W). This proves the first assertion. The second assertion follows from the first.

9.14. In the remainder of this chapter we fix two subsets K, K' of S and a $(W_K, W_{K'})$-double coset Ω in W. We have the following result.

Proposition 9.15. *(a) Ω contains a unique element b of minimal length.*

(b) Setting $J = K \cap bK'b^{-1}$, we have $W_K \cap bW_{K'}b^{-1} = W_J$.

(c) The map $W_K^J \times W_{K'} \to \Omega$, $(a, c) \mapsto abc$ is a bijection.

(d) For any $a \in W_K^J$, $c \in W_{K'}$, we have $l(abc) = l(a) + l(b) + l(c)$.

(e) If W_K and $W_{K'}$ are finite then Ω contains a unique element $\tilde{b}$ of maximal length. We have $\tilde{b} = w_0^K w_0^J b w_0^{K'}$ where $w_0^K, w_0^J, w_0^{K'}$ is the unique element of maximal length of $W_K, W_J, W_{K'}$ respectively.

Note that (a) is stated in [Bo, Ch.IV,§1, Ex.3]; (b)-(d) are due to Kilmoyer [Ki] under the assumption that W is a Weyl group. The proof of the proposition is given in 9.16.

9.16. Let b be an element of minimal length in Ω (it clearly exists). We fix a reduced expression $s_1 s_2 \dots s_q$ for b. We show:

(a) *If $w \in \Omega$, then w has a reduced expression of the form*

$$s'_1 s'_2 \dots s'_p s_1 s_2 \dots s_q \tilde{s}_1 \tilde{s}_1 \dots \tilde{s}_r$$

where $s'_i \in K$ for $i \in [1,p]$, $\tilde{s}_i \in K'$ for $i \in [1,r]$.

We can find a not necessarily reduced expression $w = s'_1 s'_2 \dots s'_p s_1 s2 \dots s_q \tilde{s}_1 \tilde{s}_2 \dots \tilde{s}_r$ with $s'_i, \tilde{s}_i$ as in the Lemma. Using 9.1, we can drop some of the simple reflections in this expression so that the resulting expression is a reduced expression for w; none of the dropped reflections can be among $s_1, s_2, \dots, s_p$ (otherwise the minimality of $l(b)$ would be contradicted). The result follows.

From (a) we see that b above is unique (hence 9.15(a) holds) and any $w \in \Omega$ is of the form (b) $w = abc$ with $a \in W_K$, $c \in W_{K'}$, $l(w) = l(a) + l(b) + l(c)$.

We show:

(c) *Let $y \in W_K \cap bW_{K'}b^{-1}$ and let $y = s'_1 s'_2 \dots s'_p$ be a reduced expression in W_K. Then for any $i \in [1,p]$ we have $b^{-1}s'_i b \in W_{K'}$.*

We argue by induction on p. When $p = 0$ there is nothing to prove. We now assume that $p \geq 1$. Let $\tilde{y} = b^{-1}yb \in W_{K'}$. Since $b \in W^{K'}$ and $\tilde{y} \in W_{K'}$, we see from 9.7 that $l(b\tilde{y}) = l(b) + l(\tilde{y})$. Similarly, since $b \in {}^K W$ and $y \in W_K$, we have $l(yb) = l(y) + l(b)$. Since $yb = b\tilde{y}$ it follows that $l(y) + l(b) = l(b) + l(\tilde{y})$ hence $l(y) = l(\tilde{y})$. Let $y = s'_1 s'_2 \dots s'_p$ be a reduced expression in W_K (it is also a reduced expression in W). Let $\tilde{y} = \tilde{s}_1 \tilde{s}_2 \dots \tilde{s}_p$ be a reduced expression in $W_{K'}$ (it is also a reduced expression in W). Then $s'_1 s'_2 \dots s'_p s_1 s_2 \dots s_q$ and $s_1 s_2 \dots s_q \tilde{s}_1 \tilde{s}_2 \dots \tilde{s}_p$ are reduced expressions in W for the same element $yb = b\tilde{y}$. Now

$$s'_1(s_1 s_2 \dots s_q \tilde{s}_1 \tilde{s}_2 \dots \tilde{s}_p) < s_1 s_2 \dots s_q \tilde{s}_1 \tilde{s}_2 \dots \tilde{s}_p$$

hence by 1.7 we have either

$$s'_1 s_1 s_2 \dots s_{i-1} = s_1 s_2 \dots s_{i-1} s_i$$

for some $i \in [1,q]$ or

$$s'_1 s_1 s_2 \dots s_q \tilde{s}_1 \tilde{s}_2 \dots \tilde{s}_{j-1} = s_1 s_2 \dots s_q \tilde{s}_1 \tilde{s}_2 \dots \tilde{s}_{j-1} s_j$$

for some $j \in [1,p]$. In the first case we have

$$s'_2 \dots s'_p s_1 s_2 \dots s_q = s_1 \dots s_{i-1} s_{i+1} \dots s_q \tilde{s}_1 \tilde{s}_2 \dots \tilde{s}_p$$

, so that Ω contains the element $s_1 \dots s_{i-1} s_{i+1} \dots s_q$ of length $< q$, contradicting the definition of b. Thus we must be in the second case so that

$$s'_2 \dots s'_p s_1 s_2 \dots s_q = s_1 s_2 \dots s_q \tilde{s}_1 \dots \tilde{s}_{j-1} \tilde{s}_{j+1} \dots \tilde{s}_p.$$

We then have $s'_2 \dots s'_p b \in bW_{K'}$. We set $y' = s'_2 \dots s'_p$. We have $y' \in W_K \cap bW_{K'}b^{-1}$ and $l(y') = p-1$. By the induction hypothesis we have $b^{-1}s'_i b \in W_{K'}$ for $i \in [2,p]$. We have

$$b^{-1}s'_1 b = (b^{-1}yb)(b^{-1}s'_p b)\dots(b^{-1}s'_2 2b) \in W_{K'}.$$

This proves (c).

We show:

(d) *Let $s' \in K$ be such that $b^{-1}s'b \in W_{K'}$. Then $b^{-1}s'b \in K'$.*

We have $s'b = bz$ where $z \in W_{K'}$. The proof of the equality $l(y) = l(\tilde{y})$ in (c) can be repeated with $y, \tilde{y}$ replaced by s', z; it yields $l(z) = l(s') = 1$ hence $z \in K'$ as required.

Now 9.15(b) follows immediately from (c),(d).

We show:

(e) *Let $w \in \Omega$. We can find $a \in W_K^J, c \in W_{K'}$ so that (b) holds.*

By (b) we can write $w = \tilde{a}b\tilde{c}$ with $\tilde{a} \in W_K, \tilde{c} \in W_{K'}$, $l(w) = l(\tilde{a}) + l(b) + l(\tilde{c})$. Using 9.7 with W, W_I replaced by W_K, W_J, we see that we can write $\tilde{a} = az$ where $a \in W_K^J$, $z \in W_J$ and we have $l(\tilde{a}) = l(a) + l(z)$. We have $w = azb\tilde{c}$ where $b^{-1}zb \in W_{K'}$. Moreover, since $b^{-1}Jb \subset K'$ we have $l(z) = l(b^{-1}zb)$ and $l(w) = l(a) + l(z) + l(b) + l(\tilde{c}) = l(a) + l(b) + l(b^{-1}zb) + l(\tilde{c})$. From $l(w) = l(a) + l(b) + l(b^{-1}zb) + l(\tilde{c})$, $w = ab(b^{-1}zb)\tilde{c}$, we deduce that $l(b^{-1}zb) + l(\tilde{c}) = l(c)$ where $c = (b^{-1}zb)\tilde{c}$. We have $c \in W_{K'}$ and $w = abc$, $l(w) = l(a) + l(b) + l(c)$. This proves (e).

We show:

(f) *Let $a, a' \in W_K$, $c, c' \in W_{K'}$ be such that $a \in W_K^J, a' \in W_K^J$ and $abc = a'bc'$. Then $a = a'$ and $c = c'$.*

We have $a^{-1}a' = bcc'^{-1}b^{-1} \in W_K \cap bW_{K'}b^{-1} = W_J$ (see 9.15(b)). Thus $a' \in aW_J$. Since a', a belong to W_K^J, we have $a = a'$ (we use 9.7(a)). It follows that $bc = bc'$ hence $c = c'$. This proves (f).

From (e),(f) we see that 9.15(c) holds.

We show:

(g) *Let $a \in W_K$, $c \in W_{K'}$ be such that $a \in W_K^J$. Let $w = abc$. Then $l(w) = l(a) + l(b) + l(c)$.*

By (e) we can write $w = a'bc'$ where $a' \in W_K^J, c' \in W_{K'}$ and $l(w) = l(a') + l(b) + l(c')$. We have $abc = a'bc'$. By (f) we have $a = a'$ and $c = c'$. This proves (g).

We prove 9.15(e). Let $w \in \Omega$. Write $w = abc$ with $a \in W_K^J$, $c \in W_{K'}$. We have $l(w_0^K) \ge l(aw_0^J) = l(a) + l(w_0^J)$ hence $l(a) \le l(w_0^K) - l(w_0^J) = l(w_0^K w_0^J)$ and $l(c) \le l(w_0^{K'})$ hence $l(w) = l(a) + l(b) + l(c) \le l(w_0^K w_0^J) + l(b) + l(w_0^{K'}) = l(w_0^K w_0^J b w_0^{K'})$. Moreover if $l(w) = l(w_0^K w_0^J b w_0^{K'})$ then $w_0^K = aw_0^J$, $c = w_0^{K'}$. This proves 9.15(e). Proposition 9.15 is proved.

10. Inversion

10.1. We preserve the setup of 3.1. For $y, w \in W$ we set

$$q'_{y,w} = \sum (-1)^n p_{z_0,z_1} p_{z_1,z_2} \dots p_{z_{n-1},z_n} \in \mathcal{A}$$

(sum over all sequences $y = z_0 < z_1 < z_2 < \cdots < z_n = w$ in W) and

$$q_{y,w} = \mathrm{sgn}(y)\mathrm{sgn}(w) q'_{y,w}.$$

We have
$q_{w,w} = 1$,
$q_{y,w} \in \mathcal{A}_{<0}$ if $y \neq w$,
$q_{y,w} = 0$ unless $y \le w$.

Proposition 10.2. *For any $y, w \in W$ we have $\overline{q}_{y,w} = \sum_{z; y \le z \le w} q_{y,z} r_{z,w}$.*

The (triangular) matrices $Q' = (q'_{y,w}), P = (p_{y,w}), R = (r_{y,w})$ are related by
(a) $Q'P = PQ' = 1$, $\overline{P} = RP$, $\overline{R}R = R\overline{R} = 1$
where $^-$ over a matrix is the matrix obtained by applying $^-$ to each entry. (Although the matrices may be infinite, the products are well defined as each entry of a product is obtained by finitely many operations.) The last three equations in (a) are obtained from 5.3, 4.6; the equations involving Q' follow from the definition. From (a) we deduce $Q'P = 1 = \overline{Q}'\overline{P} = \overline{Q}' RP$. Hence $Q'P = \overline{Q}' RP$. Multiplying on the right by Q' and using $PQ' = 1$ we deduce $Q' = \overline{Q}' R$. Multiplying on the right by $\overline{R}$ gives
(b) $\overline{Q}' = Q'\overline{R}$.
Let $\mathbf{s}$ be the matrix whose y, w entry is $\mathrm{sgn}(y)\delta_{y,w}$. We have $\mathbf{s}^2 = 1$. Let Q be the triangular matrix $(q_{y,w})$. Note that $Q = \mathbf{s}Q'\mathbf{s}$. By 4.5 we have $\overline{R} = \mathbf{s}R\mathbf{s}$. Hence by multiplying the two sides of (b) on the left and right by $\mathbf{s}$ we obtain $\overline{Q} = QR$. The proposition is proved.

10.3. Define an $\mathcal{A}$-linear map $\tau : \mathcal{H} \to \mathcal{A}$ by $\tau(T_w) = \delta_{w,1}$ for $w \in W$.

Lemma 10.4. *(a) For $x, y \in W$ we have $\tau(T_x T_y) = \delta_{xy,1}$.*
(b) For $h, h' \in \mathcal{H}$ we have $\tau(hh') = \tau(h'h)$.
(c) Assume that $L(s) \ge 0$ for all $s \in S$. Let $x, y, z \in W$ and let $M = \min(L(x), L(y), L(z))$. We have $\tau(T_x T_y T_z) \in v^M \mathbf{Z}[v^{-1}]$.

We prove (a) by induction on $l(y)$. If $l(y) = 0$, the result is clear. Assume now that $l(y) \ge 1$. If $l(xy) = l(x) + l(y)$ then $T_x T_y = T_{xy}$ and the result is clear. Hence we may assume that $l(xy) \neq l(x) + l(y)$. Then $l(xy) < l(x) + l(y)$. Let $y = s_1 s_2 \dots s_q$ be a reduced expression. We can find $i \in [1, q]$ such that

(d) $$l(x) + i - 1 = l(x s_1 s_2 \dots s_{i-1}) > l(x s_1 s_2 \dots s_{i-1} s_i).$$

We show that

(e) $$x s_1 s_2 \dots s_{i-1} s_{i+1} \dots s_q \neq 1.$$

If (e) does not hold, then $x = s_q \dots s_{i+1} s_{i-1} \dots s_1$, so that

$$l(x s_1 s_2 \dots s_{i-1}) = l(s_q \dots s_{i+1} s_{i-1} \dots s_1 s_1 \dots s_{i-1}) = l(s_q \dots s_{i+1}) = q - i,$$

$$l(xs_1s_2\dots s_{i-1}s_i)=l(s_q\dots s_{i+1}s_{i-1}\dots s_1s_1\dots s_i)=l(s_q\dots s_{i+1}s_i)=q-i+1,$$

contradicting (d). Thus (e) holds. We have

$$\begin{aligned}&\tau(T_xT_y)=\tau(T_xT_{s_1}T_{s_2}\dots T_{s_q})=\tau(T_{xs_1s_2\dots s_{i-1}}T_{s_i}T_{s_{i+1}\dots s_q})\\&=\tau(T_{xs_1s_2\dots s_{i-1}s_i}T_{s_{i+1}\dots s_q})+(v_s-v_s^{-1})\tau(T_{xs_1s_2\dots s_{i-1}}T_{s_{i+1}\dots s_q}).\end{aligned}$$

By the induction hypothesis and (e), this equals

$$\delta_{xs_1s_2\dots s_{i-1}s_is_{i+1}\dots s_q,1}+(v_s-v_s^{-1})\delta_{xs_1s_2\dots s_{i-1}s_{i+1}\dots s_q,1}=\delta_{xy,1}.$$

This completes the proof of (a). To prove (b), we may assume that $h=T_x,h'=T_y$ for $x,y\in W$; we then use (a) and the obvious equality $\delta_{xy,1}=\delta_{yx,1}$.

We prove (c). Using (b) we see that $\tau(T_xT_yT_z)=\tau(T_yT_zT_x)=\tau(T_zT_xT_y)$. Hence it is enough to show that, for any x,y,z we have

$$\tau(T_xT_yT_z)\in v^{L(x)}\mathbf{Z}[v^{-1}].$$

We argue by induction on $l(x)$. If $l(x)=0$, then $x=1$ and the result follows from (a). Assume now that $l(x)\ge 1$. We can find $s\in S$ such that $xs<x$. If $sy>y$, then by the induction hypothesis,

$$\tau(T_xT_yT_z)=\tau(T_{xs}T_{sy}T_z)\in v^{L(x)-L(s)}\mathbf{Z}[v^{-1}]\subset v^{L(x)}\mathbf{Z}[v^{-1}].$$

If $sy<y$, then by the induction hypothesis,

$$\begin{aligned}&\tau(T_xT_yT_z)=\tau(T_{xs}T_{sy}T_z)+(v_s-v_s^{-1})\tau(T_{xs}T_yT_z)\\&\in v^{L(x)-L(s)}\mathbf{Z}[v^{-1}]+v_sv^{L(x)-L(s)}\mathbf{Z}[v^{-1}]\subset v^{L(x)}\mathbf{Z}[v^{-1}].\end{aligned}$$

The lemma is proved.

10.5. Let $\mathcal{H}'=\mathrm{Hom}_{\mathcal{A}}(\mathcal{H},\mathcal{A})$. We regard $\mathcal{H}'$ as a left $\mathcal{H}$-module where, for $h\in\mathcal{H},\phi\in\mathcal{H}'$ we have $(h\phi)(h_1)=\phi(h_1h)$ for all $h_1\in\mathcal{H}$ and as a right $\mathcal{H}$-module where, for $h\in\mathcal{H},\phi\in\mathcal{H}'$, we have $(\phi h)(h_1)=\phi(hh_1)$ for all $h_1\in\mathcal{H}$.

10.6. We sometimes identify $\mathcal{H}'$ with the set of all formal sums $\sum_{x\in W}a_xT_x$ with $a_x\in\mathcal{A}$; to $\phi\in\mathcal{H}'$ corresponds the formal sum $\sum_{x\in W}\phi(T_{x^{-1}})T_x$. Since $\mathcal{H}$ is contained in the set of such formal sums (it is the set of sums such that $a_x=0$ for all but finitely many x), we see that $\mathcal{H}$ is naturally a subset of $\mathcal{H}'$. Using 10.4(a) we see that the imbedding $\mathcal{H}\subset\mathcal{H}'$ is an imbedding of $\mathcal{H}$-bimodules; it is an equality if W is finite.

10.7. Let $z\in W$. Recall that in 8.1 we have defined $D_z\in\mathcal{H}'$ by $D_z(c_w)=\delta_{z,w}$ for all w. An equivalent definition is

(a) $D_z(T_y)=q'_{z,y}$

for all $y\in W$. Indeed, assuming that (a) holds, we have

$$D_z(c_w)=\sum_y q'_{z,y}p_{y,w}=\delta_{z,w}.$$

Proposition 10.8. *Let $z \in W, s \in S$. Assume that $L(s) > 0$.*
(a) If $zs < z$, then $c_s D_z = (v_s + v_s^{-1})D_z + D_{zs} + \sum_{u;z<u<us} \mu^s_{z^{-1},u^{-1}} D_u$.
(b) If $zs > z$, then $c_s D_z = 0$.

For $a, b \in W$ we define $\delta_{a<b}$ to be 1 if $a < b$ and 0 otherwise. Let $w \in W$. If $ws > w$, then by 6.7(a), we have

$$(c_s D_z)(c_w) = D_z(c_w c_s) = D_z(c_{ws} + \sum_{\substack{x \\ xs<x<w}} \mu^s_{x^{-1},w^{-1}} c_x)$$

(c) $$= \delta_{z,ws} + \sum_{\substack{x \\ xs<x<w}} \mu^s_{x^{-1},w^{-1}} \delta_{z,x}.$$

If $ws < w$, then by 6.7(b), we have

(d) $$(c_s D_z)(c_w) = D_z(c_w c_s) = (v_s + v_s^{-1})D_z(c_w) = (v_s + v_s^{-1})\delta_{z,w}.$$

If $zs < z, ws > w$, then by (c):

$$(c_s D_z)(c_w) = \delta_{zs,w} + \delta_{z<w}\mu^s_{z^{-1},w^{-1}} = (v_s + v_s^{-1})D_z + D_{zs} + \sum_{\substack{u \\ z<u<us}} \mu^s_{z^{-1},u^{-1}} D_u)(c_w).$$

If $zs < z, ws < w$, then by (d):

$$(c_s D_z)(c_w) = (v_s + v_s^{-1})\delta_{z,w} = (v_s + v_s^{-1})D_z + D_{zs} + \sum_{\substack{u \\ z<u<us}} \mu^s_{z^{-1},u^{-1}} D_u)(c_w).$$

If $zs > z, ws > w$, then by (c), we have $(c_s D_z)(c_w) = 0$. If $zs > z, ws < w$, then by (d), we have $(c_s D_z)(c_w) = 0$. Since (c_w) is an $\mathcal{A}$-basis of $\mathcal{H}$, the proposition follows.

10.9. We show that

(a) $$\{\pm c_w; w \in W\} = \{h \in \mathcal{H}; \tau(hh^\flat) \in 1 + \mathcal{A}_{<0}; \bar{h} = h\}.$$

For any $w, w' \in W$ we have

(b) $$\tau(c_w c^\flat_{w'}) = \tau(\sum_{y,y'} p_{y,w} p_{y',w'} T_y T_{y'^{-1}}) = \sum_{y,y'} p_{y,w} p_{y',w'} \delta_{y,y'} = \delta_{w,w'} + z_{w,w'}$$

where $z_{w,w'} \in \mathcal{A}_{<0}$. In particular, $\tau(c_w c^\flat_w) \in 1 + \mathcal{A}_{<0}$.

Conversely, assume that $h \in \mathcal{H}$ satisfies $\tau(hh^\flat) \in 1 + \mathcal{A}_{<0}, \bar{h} = h$. We have $h = \sum_{w \in W} x_w c_w$ where $x_w \in \mathcal{A}$ are 0 for all but finitely many w. We can find

$t \in \mathbf{Z}$ such that $x_w = b_w v^t \mod \mathcal{A}_{<t}$ where $b_w \in \mathbf{Z}$ for all w and $b_w \neq 0$ for some w. Using (b), we have

$$\tau(hh^\flat) = \tau(\sum_{w,w'} x_w x_{w'} c_w c^\flat_{w'}) = \sum_{w,w'} x_w x_{w'} (\delta_{w,w'} + z_{w,w'})$$
$$= \sum_w x_w^2 + \sum_{w,w'} x_w x_{w'} z_{w,w'}.$$

This equals $\sum_w b_w^2 v^{2t}$ modulo $\mathcal{A}_{<2t}$ and also equals 1 modulo $\mathcal{A}_{<0}$. It follows that $t = 0$ and $\sum_w b_w^2 = 1$. Since b_w are integers, there exists $u \in W$ such that $b_u = \pm 1$ and $b_w = 0$ for $w \neq u$. Thus $x_w \in \mathcal{A}_{\leq 0}$ for all w. Since $\bar{h} = h$ we have $\bar{x}_w = x_w$ for all w. It follows that $x_w = b_w$ for all w. Thus $h = \pm c_u$. This proves (a).

10.10. The interest of 10.9(a) is that it provides a definition of c_w (up to sign) without using the basis T_w of $\mathcal{H}$ (instead, it uses $\tau : \mathcal{H} \to \mathcal{A}$). The equality 10.9(a) could be used to give a definition of c_w (up to sign) in more general situations than that considered above, when the basis (T_w) is not defined but $\tau : \mathcal{H} \to \mathcal{A}$ is defined (it is known that τ is defined when W is replaced by certain complex reflection groups, see [BM]). This should lead to a definition of cells for complex reflection groups.

11. The longest element for a finite W

11.1. We preserve the setup of 3.1. Let $I \subset S$ be such that W_I is finite. By 9.8, there is a unique element of maximal length of W_I. We denote it by w_0^I. If w_1 has minimal length in $W_I a$ then $w_0^I w_1$ has maximal length in $W_I a$.

11.2. In the remainder of this chapter we assume that W is finite. Then $w_0 := w_0^S$, the unique element of maximal length of W, is well defined. Since $l(w_0^{-1}) = l(w_0)$, we must have $w_0^{-1} = w_0$. By the argument in the proof of 9.8 we have $w \leq w_0$ for any $w \in W$. By 9.8 we have

(a) $l(ww_0) = l(w_0) - l(w)$

for any $w \in W$. Applying this to w^{-1} and using the equalities $l(w^{-1}w_0) = l(w_0^{-1}w) = l(w_0 w), l(w^{-1}) = l(w)$, we deduce that

(b) $l(w_0 w) = l(w_0) - l(w)$.

We can rewrite (a),(b) as $l(w_0) = l(w^{-1}) + l(ww_0)$, $l(w_0) = l(w_0 w) + l(w^{-1})$. Using this and the definition of L we deduce that

$L(w^{-1}) + L(ww_0) = L(w_0) = L(w_0 w) + L(w^{-1})$,

hence $L(ww_0) = L(w_0 w)$. This implies $L(w_0 w w_0) = L(w)$ for all w. Replacing L by l gives $l(w_0 w w_0) = l(w)$. Thus, the involution $w \mapsto w_0 w w_0$ of W maps S into itself hence is a Coxeter group automorphism preserving the function L.

Lemma 11.3. *Let $y, w \in W$. We have*

(a) $y \leq w \Leftrightarrow w_0 w \leq w_0 y \Leftrightarrow ww_0 \leq yw_0$;

(b) $r_{y,w} = r_{ww_0, yw_0} = r_{w_0 w, w_0 y}$;

(c) $\bar{p}_{ww_0,yw_0} = \sum_{z;y\le z\le w} p_{zw_0,yw_0} r_{z,w}$.

We prove (a). To prove that $y \le w \implies ww_0 \le yw_0$, we may assume that $l(w) - l(y) = 1, yw^{-1} \in T$. Then

$$l(yw_0) - l(ww_0) = l(w_0) - l(y) - (l(w_0) - l(w)) = l(w) - l(y) = 1$$

and $(ww_0)(yw_0)^{-1} = wy^{-1} \in T$. Hence $ww_0 \le yw_0$. The opposite implication is proved in the same way. The second equivalence in (a) follows from the last sentence in 11.2.

We prove the first equality in (b) by induction on $l(w)$. If $l(w) = 0$ then $w = 1$. We have $r_{y,1} = \delta_{y,1}$. Now r_{w_0,yw_0} is zero unless $w_0 \le yw_0$ (see 4.7). On the other hand we have $yw_0 \le w_0$ (see 11.2). Hence r_{w_0,yw_0} is zero unless $yw_0 = w_0$, that is unless $y = 1$ in which case it is 1. Thus the desired equality holds when $l(w) = 0$. Assume now that $l(w) \ge 1$. We can find $s \in S$ such that $sw < w$. Then $sww_0 > ww_0$ by (a).

Assume first that $sy < y$ (hence $syw_0 > yw_0$). By 4.4 and the induction hypothesis we have

$$r_{y,w} = r_{sy,sw} = r_{sww_0,syw_0} = r_{ww_0,yw_0}.$$

Assume next that $sy > y$ (hence $syw_0 < yw_0$.) By 4.4 and the induction hypothesis we have

$$\begin{aligned} r_{y,w} &= r_{sy,sw} + (v_s - v_s^{-1}) r_{y,sw} = r_{sww_0,syw_0} + (v_s - v_s^{-1}) r_{sww_0,yw_0} \\ &= r_{sww_0,syw_0} + (v_s - v_s^{-1}) r_{ww_0,syw_0} = r_{ww_0,yw_0}. \end{aligned}$$

This proves the first equality in (b). The second equality in (b) follows from the last sentence in 11.2.

We prove (c). We may assume that $y \le w$. By 5.3 (for ww_0, yw_0 instead of y, w) we have $\bar{p}_{ww_0,yw_0} = \sum_{z;y\le z\le w} r_{ww_0,zw_0} p_{zw_0,yw_0}$ (we have used (a)). Here we substitute $r_{ww_0,zw_0} = r_{z,w}$ (see (b)) and the result follows. The lemma is proved.

Proposition 11.4. *For any $y, w \in W$ we have $q_{y,w} = p_{ww_0,yw_0} = p_{w_0w,w_0y}$.*

The second equality follows from the last sentence in 11.2. We prove the first equality. We may assume that $y \le w$. We argue by induction on $l(w) - l(y) \ge 0$. If $l(w) - l(y) = 0$ we have $y = w$ and both sides are 1. Assume now that $l(w) - l(y) \ge 1$. Subtracting the identity in 11.3(c) from that in 10.2 and using the induction hypothesis, we obtain

$$\overline{q}_{y,w} - \bar{p}_{ww_0,yw_0} = q_{y,w} - p_{ww_0,yw_0}.$$

The right hand side is in $\mathcal{A}_{<0}$; since it is fixed by $\bar{\ }$, it is 0. The proposition is proved.

Proposition 11.5. *We identify $\mathcal{H} = \mathcal{H}'$ as in 10.6. If $z \in W$, then $D_{z^{-1}} \in \mathcal{H}'$ (see 10.7) becomes an element of $\mathcal{H}$. We have $D_{z^{-1}} T_{w_0}^{-1} = \mathrm{sgn}(zw_0) c_{zw_0}^\dagger$, † as in 3.5.*

By definition, $D_{z^{-1}} \in \mathcal{H}$ is characterized by

$$\tau(D_{z^{-1}} T_{y^{-1}}) = q'_{z^{-1},y^{-1}}$$

for all $y \in W$. Here τ is as in 10.3. Hence, by 10.4(a), we have $D_{z^{-1}} = \sum_y q'_{z^{-1},y^{-1}} T_y$. Using 11.4, we deduce

$$D_{z^{-1}} = \sum_y \mathrm{sgn}(yz) p_{w_0 y^{-1}, w_0 z^{-1}} T_y.$$

Multiplying on the right by $T_{w_0}^{-1}$ gives

$$D_{z^{-1}} T_{w_0}^{-1} = \sum_y \mathrm{sgn}(yz) p_{w_0 y^{-1}, w_0 z^{-1}} T_{w_0 y^{-1}}^{-1}$$

since $T_{w_0 y^{-1}} T_y = T_{w_0}$. On the other hand,

$$\mathrm{sgn}(zw_0) c_{zw_0}^\dagger = \sum_x \mathrm{sgn}(zw_0 x) p_{x, zw_0} T_{x^{-1}}^{-1} = \sum_y \mathrm{sgn}(zw_0 y w_0) p_{yw_0, zw_0} T_{w_0 y^{-1}}^{-1}.$$

We now use the identity $p_{yw_0,zw_0} = p_{w_0 y^{-1}, w_0 z^{-1}}$. The proposition follows.

Proposition 11.6. *Let $u, z \in W, s \in S$ be such that $sz < z < u < su$. Assume that $L(s) > 0$. Then $suw_0 < uw_0 < zw_0 < szw_0$ and $\mu^s_{uw_0,zw_0} = -\mathrm{sgn}(uz) \mu^s_{z,u}$.*

Using 10.8(a), we see that

$$(c_s - (v_s + v_s^{-1})) D_{z^{-1}} T_{w_0}^{-1} = D_{z^{-1}s} T_{w_0}^{-1} + \sum_{u; z^{-1} < u^{-1} < u^{-1}s} \mu^s_{z,u} D_{u^{-1}} T_{w_0}^{-1},$$

hence, using 11.5, we have

$$(c_s - (v_s + v_s^{-1})) \mathrm{sgn}(zw_0) c_{zw_0}^\dagger = \mathrm{sgn}(zsw_0) c_{szw_0}^\dagger + \sum_{u; z<u<su} \mu^s_{z,u} \mathrm{sgn}(uw_0) c_{uw_0}^\dagger.$$

Applying † to both sides and using $(c_s - (v_s + v_s^{-1}))^\dagger = -c_s$ gives

(a) $$-c_s c_{zw_0} = -c_{szw_0} + \sum_{u; z<u<su} \mu^s_{z,u} \mathrm{sgn}(uz) c_{uw_0}.$$

Since $szw_0 > zw_0$, we can apply 6.6(a) and we get

$$c_s c_{zw_0} = c_{szw_0} + \sum_{u'; su' < u' < zw_0} \mu^s_{u', zw_0} c_{u'}$$

or equivalently

$$c_s c_{zw_0} = c_{szw_0} + \sum_{u; z<u<su} \mu^s_{uw_0, zw_0} c_{uw_0}.$$

Comparison with (a) gives

$$-\sum_{u; z<u<su} \mu^s_{z,u} \mathrm{sgn}(uz) c_{uw_0} = \sum_{u; z<u<su} \mu^s_{uw_0, zw_0} c_{uw_0};$$

the proposition follows.

Corollary 11.7. *Assume that $L(s) > 0$ for all $s \in S$. Let $y, w \in W$.*

(a) $y \le_{\mathcal{L}} w \Leftrightarrow ww_0 \le_{\mathcal{L}} yw_0 \Leftrightarrow w_0w \le_{\mathcal{L}} w_0y$;

(b) $y \le_{\mathcal{R}} w \Leftrightarrow ww_0 \le_{\mathcal{R}} yw_0 \Leftrightarrow w_0w \le_{\mathcal{R}} w_0y$;

(c) $y \le_{\mathcal{LR}} w \Leftrightarrow ww_0 \le_{\mathcal{LR}} yw_0 \Leftrightarrow w_0w \le_{\mathcal{LR}} w_0y$.

(d) Left multiplication by w_0 carries left cells to left cells, right cells to right cells, two-sided cells to two-sided cells. The same holds for right multiplication by w_0.

We prove the first equivalence in (a). It is enough to show that $y \le_{\mathcal{L}} w \implies ww_0 \le_{\mathcal{L}} yw_0$. We may assume that $y \leftarrow_{\mathcal{L}} w$ and $y \ne w$. Then there exists $s \in S$ such that $sw > w, sy < y$ and $D_y(c_sc_w) \ne 0$. We have $syw_0 > yw_0, sww_0 < ww_0$. From 6.6 we see that either $y = sw$ or $y < w$ and $\mu^s_{y,w} \ne 0$. In the first case we have $ww_0 = syw_0$; in the second case we have $ww_0 < yw_0$ and $\mu^s_{ww_0,yw_0} \ne 0$ (see 11.6). In both cases, 6.6 shows that $D_{ww_0}(c_sc_{yw_0}) \ne 0$. Hence $ww_0 \le_{\mathcal{L}} yw_0$. Thus, the first equivalence in (a) is established. The second equivalence in (a) follows from the last sentence in 11.2.

Now (b) follows by applying (a) to y^{-1}, w^{-1} instead of y,w; (c) follows from (a) and (b); (d) follows from (a),(b),(c). The corollary is proved.

12. Examples of elements D_w

We preserve the setup of 3.1.

Proposition 12.1. *Assume that $L(s) > 0$ for all $s \in S$. For any $y \in W$ we have $D_1(T_y) = \text{sgn}(y)v^{-L(y)}$. Equivalently, with the identification in 10.6, we have*
$D_1 = \sum_{y \in W} \text{sgn}(y)v^{-L(y)}T_y$.

An equivalent statement is that $q'_{1,y} = \text{sgn}(y)v^{-L(y)}$. Since $q'_{1,y}$ are determined by the equations $\sum_y q'_{1,y}p_{y,w} = \delta_{1,w}$ (see 10.2(a)) it is enough to show that

$$\sum_y \text{sgn}(y)v^{-L(y)}p_{y,w} = \delta_{1,w}$$

for all $w \in W$. If $w = 1$ this is clear. Assume now that $w \ne 1$. We can find $s \in S$ such that $sw < w$. We must prove that

$$\sum_{y; y<sy} \text{sgn}(y)v^{-L(y)}(p_{y,w} - v_s^{-1}p_{sy,w}) = 0.$$

Each term of the last sum is 0, by 6.6(c). The proposition is proved.

Corollary 12.2. *Assume that W is finite and that $L(s) > 0$ for all $s \in S$. Then*
$c_{w_0} = \sum_{y \in W} v^{-L(yw_0)}T_y$.

This follows immediately from 12.1 and 11.5. Alternatively, we can argue as follows. We prove that $p_{y,w_0} = v^{-L(yw_0)}$ for all y, by descending induction on $l(y)$.

If $l(y)$ is maximal, that is $y = w_0$, then $p_{y,w_0} = 1$. Assume now that $l(y) < l(w_0)$. We can find $s \in S$ such that $l(sy) = l(y) + 1$. By the induction hypothesis we have $p_{sy,w_0} = v^{-L(syw_0)}$. By 6.6(c), we have

$$p_{y,w_0}$$
$$= v_s^{-1} p_{sy,w_0} = v^{-L(s)-L(syw_0)} = v^{-L(s)-L(w_0)+L(sy)} = v^{-L(w_0)+L(y)} = v^{-L(yw_0)}.$$

The corollary is proved.

12.3. From 11.5 we see that $D_{z^{-1}}$ can be explicitly computed when W is finite and c_{zw_0} is known. In particular, in the setup of 7.4 with $m = 2k+2 < \infty$, we can compute explicitly all $D_{z^{-1}}$ using 7.6(a). For example:

$$D_{s_1} = \sum_{s \in [0,k-1]} (1 - v^{2L_1} + v^{4L_1} - \dots + (-1)^s v^{2sL_1}) v^{-sL_1 - sL_2}$$
$$\times (T_{1_{2s+1}} - v^{-L_2} T_{2_{2s+2}} - v^{-L_2} T_{1_{2s+2}} + v^{-2L_2} T_{2_{2s+3}})$$
$$+ (1 - v^{2L_1} + v^{4L_1} - \dots + (-1)^k v^{2kL_1}) v^{-kL_1 - kL_2} (T_{1_{2k+1}} - v^{-L_2} T_{2_{2k+2}}).$$

Using this (for larger and larger m) one can deduce that an analogous formula holds in the setup of 7.4 with $m = \infty$:

$$D_{s_1} = \sum_{s \geq 0} (1 - v^{2L_1} + v^{4L_1} - \dots + (-1)^s v^{2sL_1}) v^{-sL_1 - sL_2}$$

(a) $$\times (T_{1_{2s+1}} - v^{-L_2} T_{2_{2s+2}} - v^{-L_2} T_{1_{2s+2}} + v^{-2L_2} T_{2_{2s+3}}) \in \mathcal{H}'.$$

(We use the identification in 10.6.)

13. The function a

13.1. We preserve the setup of 3.1.

In the remainder of these notes we assume that $L(s) > 0$ for all $s \in S$.

For x, y, z in W we define $f_{x,y,z} \in \mathcal{A}$, $f'_{x,y,z} \in \mathcal{A}$, $h_{x,y,z} \in \mathcal{A}$ by

$$T_x T_y = \sum_{z \in W} f_{x,y,z} T_z = \sum_{z \in W} f'_{x,y,z} c_z,$$

$$c_x c_y = \sum_{z \in W} h_{x,y,z} c_z.$$

We have

(a) $f_{x,y,z} = \sum_u p_{z,u} f'_{x,y,u}$

(b) $f'_{x,y,z} = \sum_u q'_{z,u} f_{x,y,u}$,

(c) $h_{x,y,z} = \sum_{x',y'} p_{x',x}p_{y',y}f'_{x',y',z}$.
All sums in (a)-(c) are finite. (a),(c) follow from the definitions; (b) follows from (a) using 10.2(a).

From 8.2, 5.6, we see that
(d) $h_{x,y,z} \neq 0 \implies z \leq_{\mathcal{R}} x, z \leq_{\mathcal{L}} y$,
(e) $h_{x,y,z} = h_{y^{-1},x^{-1},z^{-1}}$.
We show:
(f) $f_{x,y,z} \neq 0 \implies l(z) \leq l(x) + l(y)$;
(g) $f'_{x,y,z} \neq 0 \implies l(z) \leq l(x) + l(y)$;
(h) $h_{x,y,z} \neq 0 \implies l(z) \leq l(x) + l(y)$.
Now (f) follows from the definition, by induction on $l(x)$; (g) follows from (b) and (f) using that $q'_{z,z'} = 0$ unless $z \leq z'$; (h) follows from (c) and (g) using that $p_{x',x} = 0$ unless $x' \leq x$.

13.2. We say that $N \in \mathbf{N}$ is a *bound* for W, L if $v^{-N} f_{x,y,z} \in \mathcal{A}_{\leq 0}$ for all x, y, z in W. We say that W, L is *bounded* if there exists $N \in \mathbf{N}$ such that N is a bound for W, L.

Lemma 13.3. *If W is finite, then $N = L(w_0^S)$ is a bound for W, L.*

By 10.4(a) we have $f_{x,y,z} = \tau(T_xT_yT_{z^{-1}})$. By 10.4(c) we have $\tau(T_xT_yT_{z^{-1}}) \in v^{L(w_0^S)}\mathbf{Z}[v^{-1}]$. The lemma is proved.

13.4. *Conjecture. In the general case W, L admits a bound $N = \max_I L(w_0^I)$ where I runs over the subsets of S such that W_I is finite.*

For W is tame this is proved in [L7, 7.2] assuming that $L = l$, but the same proof remains valid without the assumption $L = l$.

We illustrate this in the setup of 7.1 with $m = \infty$. For $a, b \in \{1, 2\}$ and $k > 0, k' > 0$, we have
$T_{a_k}T_{b_{k'}} = T_{a_{k+k'}}$ if $b = a + k \mod 2$,
$T_{a_k}T_{b_{k'}} = T_{a_kb_{k'}} + \sum_{u \in [1,\min(k,k')]} \xi_{b+u-1}T_{a_{k+k'-2u+1}}$ if $b = a + k + 1 \mod 2$;
here, for $n \in \mathbf{Z}$ we set $\xi_n = v^{L_1} - v^{-L_1}$ if n is odd and $\xi_n = v^{L_2} - v^{-L_2}$ if n is even. We see that, in this case, $\max(L_1, L_2)$ is a bound for W, L.

Lemma 13.5. *Assume that W, L is bounded; let N be a bound for W, L. Then, for any x, y, z in W we have*
(a) $v^{-N}f'_{x,y,z} \in \mathcal{A}_{\leq 0}$,
(b) $v^{-N}h_{x,y,z} \in \mathcal{A}_{\leq 0}$.

(a) follows from 13.1(b) since $q'_{z,z'} \in \mathcal{A}_{\leq 0}$. (b) follows from (a) and 13.1(c) since $p_{x',x} \in \mathcal{A}_{\leq 0}$, $p_{y',y} \in \mathcal{A}_{\leq 0}$.

13.6. *In the remainder of this chapter we assume that W, L is bounded.* By 13.5(b), for any $z \in W$ there exists a unique integer $\mathbf{a}(z) \geq 0$ such that
(a) $h_{x,y,z} \in v^{\mathbf{a}(z)}\mathbf{Z}[v^{-1}]$ for all $x, y \in W$,

(b) $h_{x,y,z}\notin v^{\mathbf{a}(z)-1}\mathbf{Z}[v^{-1}]$ for some $x,y\in W$.
(We use that $h_{1,z,z}=1$.) We then have for any x,y,z:
(c) $h_{x,y,z}=\gamma_{x,y,z^{-1}}v^{\mathbf{a}(z)} \mod v^{\mathbf{a}(z)-1}\mathbf{Z}[v^{-1}]$
where $\gamma_{x,y,z^{-1}}\in\mathbf{Z}$ is well defined; moreover, for any $z\in W$ there exists x,y such that $\gamma_{x,y,z^{-1}}\neq 0$.

For any x,y,z we have
(d) $f'_{x,y,z}=\gamma_{x,y,z^{-1}}v^{\mathbf{a}(z)} \mod v^{\mathbf{a}(z)-1}\mathbf{Z}[v^{-1}]$.
This is proved (for fixed z) by induction on $l(x)+l(y)$ using (c) and 13.1(c). (Note that $p_{x',x}p_{y',y}$ is 1 if $x'=x,y'=y$ and is in $\mathcal{A}_{<0}$ otherwise.)

Proposition 13.7. *(a)* $\mathbf{a}(1)=0$.
(b) If $z\in W-\{1\}$*, then* $\mathbf{a}(z)\geq \min_{s\in S}L(s)>0$.

We prove (a). Let $x,y\in W$. Assume first that $y\neq 1$. We can find $s\in S$ such that $ys<y$. Then $c_y\in {}^s\mathcal{H}$. Since ${}^s\mathcal{H}$ is a left ideal (see 8.4) we have $c_xc_y\in {}^s\mathcal{H}$. Since $s1>1$, from the definition of ${}^s\mathcal{H}$ it then follows that $h_{x,y,1}=0$.

Similarly, if $x\neq 1$, then $h_{x,y,1}=0$. Since $h_{1,1,1}=1$, (a) follows.

In the setup of (b) we can find $s\in S$ such that $sz<z$. By 6.6(b) we have $h_{s,z,z}=v_s+v_s^{-1}$. This shows that $\mathbf{a}(z)\geq L(s)>0$. The proposition is proved.

Proposition 13.8. *Assume that W is finite.*
(a) We have $\mathbf{a}(w_0)=L(w_0)$.
(b) For any $w\in W-\{w_0\}$ *we have* $\mathbf{a}(w)<L(w_0)$.

From 13.5, 13.3, for any $w\in W$ we have $\mathbf{a}(w)\leq L(w_0)$.

We prove (a). From 6.6(b) we see that $T_sc_{w_0}=v_sc_{w_0}$ for any $s\in S$. Using this and 12.2, we see that
$c_{w_0}c_{w_0}=\sum_{y\in W}v^{-L(yw_0)}v^{L(y)}c_{w_0}$,
hence

$$h_{w_0,w_0,w_0}=\sum_{y\in W}v^{-L(w_0)}v^{2L(y)}\in v^{L(w_0)} \mod v^{L(w_0)-1}\mathbf{Z}[v^{-1}].$$

It follows that $\mathbf{a}(w_0)\geq L(w_0)$. Hence $\mathbf{a}(w_0)=L(w_0)$. This proves (a).

We prove (b). Let $z\in W$ be such that $\mathbf{a}(z)=L(w_0)$. We must prove that $z=w_0$. By 13.6(d), we can find x,y such that

$$f'_{x,y,z}=bv^{L(w_0)}+\text{strictly smaller powers of } v$$

where $b\in\mathbf{Z}-\{0\}$. For any $z'\neq z$ we have $f'_{x,y,z'}\in v^{L(w_0)}\mathbf{Z}[v^{-1}]$ (by 13.6 and the first sentence in the proof). Since $p_{z,z'}=1$ for $z=z'$ and $p_{z,z'}\in\mathcal{A}_{<0}$ for $z'<z$, we see that the equality $f_{x,y,z}=\sum_{z'}p_{z,z'}f'_{x,y,z'}$ (see 13.1(a)) implies that

$$f_{x,y,z}=bv^{L(w_0)}+\text{strictly smaller powers of } v$$

with $b\neq 0$. Now $f_{x,y,z}=\tau(T_xT_yT_{z^{-1}})$. Using now 10.4(c) we see that

$$\min(L(x),L(y),L(z^{-1}))=L(w_0).$$

It follows that $x=y=z^{-1}=w_0$. The proposition is proved.

Proposition 13.9. *(a) For any $z \in W$ we have $\mathbf{a}(z) = \mathbf{a}(z^{-1})$.*
(b) For any $x, y, z \in W$ we have $\gamma_{x,y,z} = \gamma_{y^{-1},x^{-1},z^{-1}}$.

(a),(b) follow from 13.1(e).

13.10. We show that, in the setup of 7.1 with $m = \infty$ and $L_2 \geq L_1$, the function $\mathbf{a} : W \to \mathbf{N}$ is given as follows:
(a) $\mathbf{a}(1) = 0$,
(b) $\mathbf{a}(1_1) = L_1, \mathbf{a}(2_1) = L_2$,
(c) $\mathbf{a}(1_k) = \mathbf{a}(2_k) = L_2$ if $k \geq 2$.
Now (a) is contained in 13.7(a). If $s_2 z < z$ then, by the proof of 13.7(b) we have $\mathbf{a}(z) \geq L_2$. By 13.4, L_2 is a bound for W, L hence $\mathbf{a}(z) \leq L_2$ so that $\mathbf{a}(z) = L_2$. If $zs_2 < z$ then the previous argument is applicable to z^{-1}. Using 13.9, we see that $\mathbf{a}(z) = \mathbf{a}(z^{-1}) = L_2$.

Assume next that $z = 1_{2k+1}$ where $k \geq 1$. By 7.5, 7.6, we have

$$c_{1_2} c_{2_{2k}} = c_1 c_2 c_{2_{2k}} = (v^{L_2} + v^{-L_2}) c_1 c_{2_{2k}} = (v^{L_2} + v^{-L_2}) c_{1_{2k+1}},$$

hence $h_{1_2, 2_{2k}, z} = v^{L_2} + v^{-L_2}$. Thus, $\mathbf{a}(z) \geq L_2$. By 13.4 we have $\mathbf{a}(z) \leq L_2$ hence $\mathbf{a}(z) = L_2$.

It remains to consider the case where $z = s_1$. Assume first that $L_1 = L_2$. Then $\mathbf{a}(s_1) \leq L_1$ by 13.4 and $\mathbf{a}(s_1) \geq L_1$ by 13.7(b). Hence $\mathbf{a}(s_1) = L_1$.

Assume next that $L_1 < L_2$. Then $\mathcal{I} = \sum_{w \in W - \{1, s_1\}} \mathcal{A} c_w$ is a two-sided ideal $\mathcal{I}$ of $\mathcal{H}$ (see 8.8). Hence if x or y is in $W - \{1, s_1\}$, then $c_x c_y \in \mathcal{I}$ and $h_{x,y,s_1} = 0$. Using

$$h_{1,1,s_1} = 0, h_{1,s_1,s_1} = h_{s_1,1,s_1} = 1, h_{s_1,s_1,s_1} = v^{L_1} + v^{-L_1}$$

we see that $\mathbf{a}(s_1) = L_1$. Thus, (a),(b),(c) are established.

13.11. In this subsection we assume that we are in the setup of 7.1 with $4 \leq m < \infty$ and $L_2 > L_1$. By 7.8, we have

$$h_{2_{m-1}, 2_{m-1}, 2_{m-1}} = (-1)^{(m-2)/2} v^{(mL_2 - (m-2)L_1)/2} + \text{ strictly smaller powers of } v.$$

Hence $\mathbf{a}(2_{m-1}) \geq (mL_2 - (m-2)L_1)/2$.

One can show that the function $\mathbf{a} : W \to \mathbf{N}$ is given as follows:
$\mathbf{a}(1) = 0$,
$\mathbf{a}(1_1) = L_1, \mathbf{a}(2_1) = L_2$,
$\mathbf{a}(1_{m-1}) = L_2, \mathbf{a}(2_{m-1}) = (mL_2 - (m-2)L_1)/2$,
$\mathbf{a}(2_m) = m(L_1 + L_2)/2$,
$\mathbf{a}(1_k) = \mathbf{a}(2_k) = L_2$ if $1 < k < m - 1$.
This remains true in the case where $L_1 = L_2$.

13.12. We conjecture that any two-sided cell of W would meet some finite W_I; this would imply that there are only finitely many two-sided cells in W. (Compare 18.2.)

On the other hand, the number of left cells in W can be infinite for some non-tame W with $L = l$ (Bédard [Be])

14. Conjectures

14.1. We preserve the setup of 3.1. *In this chapter we assume that W, L is bounded, see 13.2.*

For $n \in \mathbf{Z}$ define $\pi_n : \mathcal{A} \to \mathbf{Z}$ by $\pi_n(\sum_{k\in\mathbf{Z}} a_k v^k) = a_n$.

For $z \in W$ we define an integer $\Delta(z) \geq 0$ by

(a) $$p_{1,z} = n_z v^{-\Delta(z)} + \text{strictly smaller powers of } v, n_z \in \mathbf{Z} - \{0\}.$$

Note that $\Delta(z) = \Delta(z^{-1})$ and $\Delta(1) = 0$, $0 < \Delta(z) \leq L(z)$ for $z \neq 1$ (see 5.4). Let

$$\mathcal{D} = \{z \in W; \mathbf{a}(z) = \Delta(z)\}.$$

Clearly, $z \in \mathcal{D} \implies z^{-1} \in \mathcal{D}$.

Conjectures 14.2. *The following properties hold.*

P1. For any $z \in W$ we have $\mathbf{a}(z) \leq \Delta(z)$.

P2. If $d \in \mathcal{D}$ and $x, y \in W$ satisfy $\gamma_{x,y,d} \neq 0$, then $x = y^{-1}$.

P3. If $y \in W$, there exists a unique $d \in \mathcal{D}$ such that $\gamma_{y^{-1},y,d} \neq 0$.

P4. If $z' \leq_{\mathcal{LR}} z$ then $\mathbf{a}(z') \geq \mathbf{a}(z)$. Hence, if $z' \sim_{\mathcal{LR}} z$, then $\mathbf{a}(z') = \mathbf{a}(z)$.

P5. If $d \in \mathcal{D}, y \in W, \gamma_{y^{-1},y,d} \neq 0$, then $\gamma_{y^{-1},y,d} = n_d = \pm 1$.

P6. If $d \in \mathcal{D}$, then $d^2 = 1$.

P7. For any $x, y, z \in W$ we have $\gamma_{x,y,z} = \gamma_{y,z,x}$.

P8. Let $x, y, z \in W$ be such that $\gamma_{x,y,z} \neq 0$. Then $x \sim_{\mathcal{L}} y^{-1}$, $y \sim_{\mathcal{L}} z^{-1}$, $z \sim_{\mathcal{L}} x^{-1}$.

P9. If $z' \leq_{\mathcal{L}} z$ and $\mathbf{a}(z') = \mathbf{a}(z)$ then $z' \sim_{\mathcal{L}} z$.

P10. If $z' \leq_{\mathcal{R}} z$ and $\mathbf{a}(z') = \mathbf{a}(z)$ then $z' \sim_{\mathcal{R}} z$.

P11. If $z' \leq_{\mathcal{LR}} z$ and $\mathbf{a}(z') = \mathbf{a}(z)$ then $z' \sim_{\mathcal{LR}} z$.

P12. Let $I \subset S$. If $y \in W_I$, then $\mathbf{a}(y)$ computed in terms of W_I is equal to $\mathbf{a}(y)$ computed in terms of W.

P13. Any left cell Γ of W contains a unique element $d \in \mathcal{D}$. We have $\gamma_{x^{-1},x,d} \neq 0$ for all $x \in \Gamma$.

P14. For any $z \in W$ we have $z \sim_{\mathcal{LR}} z^{-1}$.

P15. Let v' be a second indeterminate and let $h'_{x,y,z} \in \mathbf{Z}[v', v'^{-1}]$ be obtained from $h_{x,y,z}$ by the substitution $v \mapsto v'$. If $x, x', y, w \in W$ satisfy $\mathbf{a}(w) = \mathbf{a}(y)$, then $\sum_{y'} h'_{w,x',y'} h_{x,y',y} = \sum_{y'} h_{x,w,y'} h'_{y',x',y}$.

In §15-§17 we will verify the conjectures above in a number of cases.

14.3. We consider the following auxiliary statement.

$\tilde{P}$. Let $x, y, z, z' \in W$ be such that $\gamma_{x,y,z^{-1}} \neq 0$, $z' \leftarrow_{\mathcal{L}} z$. Then there exists $x' \in W$ such that $\pi_{\mathbf{a}(z)}(h_{x',y,z'}) \neq 0$. In particular, $\mathbf{a}(z') \geq \mathbf{a}(z)$.

In this chapter we will show, that, if P1-P3 and $\tilde{P}$ are assumed to be true, then P4-P14 are automatically true. The arguments follow [L7],[L9].

14.4. $\tilde{P} \implies$ *P4.* Let z', z be as in P4. We can assume that $z \leftarrow_{\mathcal{L}} z'$ or $z \leftarrow_{\mathcal{R}} z'$. In the first case, from $\tilde{P}$ we get $\mathbf{a}(z') \geq \mathbf{a}(z)$. (We can find x, y such that $\gamma_{x,y,z^{-1}} \neq 0$.) In the second case, from $\tilde{P}$ we get $\mathbf{a}(z'^{-1}) \geq \mathbf{a}(z^{-1})$ hence $\mathbf{a}(z') \geq \mathbf{a}(z)$.

14.5. *P1,P3* $\implies$ *P5.* Let $x, y \in W$. Applying τ to $c_x c_y = \sum_{z\in W} h_{x,y,z} c_z$ gives

$$\sum_z h_{x,y,z} p_{1,z} = \sum_{x',y'} p_{x',x} p_{y',y} \tau(T_{x'} T_{y'}) = \sum_{x',y'} p_{x',x} p_{y',y} \delta_{x'y',1} = \sum_{x'} p_{x',x} p_{x'^{-1},y}$$

hence

(a) $$\sum_{z\in W} h_{x,y,z} p_{1,z} = \delta_{xy,1} \mod v^{-1}\mathbf{Z}[v^{-1}].$$

We take $x = y^{-1}$ and note that $h_{y^{-1},y,z} \in v^{\mathbf{a}(z)}\mathbf{Z}[v^{-1}]$, $p_{1,z} \in v^{-\Delta(z)}\mathbf{Z}[v^{-1}]$, hence

$$h_{y^{-1},y,z} p_{1,z} \in v^{\mathbf{a}(z)-\Delta(z)}\mathbf{Z}[v^{-1}].$$

The same argument shows that, if $z \in \mathcal{D}$, then

$$h_{y^{-1},y,z} p_{1,z} \in \gamma_{y^{-1},y,z^{-1}} n_z + \mathcal{A}_{<0}.$$

If $z \notin \mathcal{D}$ then, by P1, we have $\mathbf{a}(z) - \Delta(z) < 0$ so that $h_{y^{-1},y,z} p_{1,z} \in \mathcal{A}_{<0}$. We see that

$$\sum_{z\in W} h_{y^{-1},y,z} p_{1,z} = \sum_{z\in\mathcal{D}} \gamma_{y^{-1},y,z^{-1}} n_z \mod \mathcal{A}_{<0}.$$

Comparison with (a) gives $\sum_{z\in\mathcal{D}} \gamma_{y^{-1},y,z^{-1}} n_z = 1$. Equivalently,

$$\sum_{z\in\mathcal{D}} \gamma_{y^{-1},y,z} n_z = 1.$$

Using this and P3 we see that, in the setup of P5 we have $\gamma_{y^{-1},y,d} n_d = 1$. Since $\gamma_{y^{-1},y,d}, n_d$ are integers, we must have $\gamma_{y^{-1},y,d} = n_d = \pm 1$.

14.6. *P2,P3* $\implies$ *P6.* We can find x, y such that $\gamma_{x,y,d} \neq 0$. By P2, we have $x = y^{-1}$ so that $\gamma_{y^{-1},y,d} \neq 0$. This implies $\gamma_{y^{-1},y,d^{-1}} \neq 0$. (See 13.9(b)). We have $d^{-1} \in \mathcal{D}$. By the uniqueness in P3 we have $d = d^{-1}$.

14.7. *P2,P3,P4,P5* $\implies$ *P7.* We first prove the following statement.

(a) *Let $x, y, z \in W, d \in \mathcal{D}$ be such that $\gamma_{x,y,z} \neq 0$, $\gamma_{z^{-1},z,d} \neq 0$, $\mathbf{a}(d) = \mathbf{a}(z)$. Then $\gamma_{x,y,z} = \gamma_{y,z,x}$.*

Let $n = \mathbf{a}(d)$. From $\gamma_{x,y,z} \neq 0$ we deduce $h_{x,y,z^{-1}} \neq 0$ hence $z^{-1} \leq_{\mathcal{R}} x$, hence $n = \mathbf{a}(z) = \mathbf{a}(z^{-1}) \geq \mathbf{a}(x)$ (see P4). Computing the coefficient of c_d in two ways, we obtain

$$\sum_{z'} h_{x,y,z'} h_{z',z,d} = \sum_{x'} h_{x,x',d} h_{y,z,x'}.$$

Now $h_{z',z,d} \neq 0$ implies $d \leq_{\mathcal{R}} z'$ hence $\mathbf{a}(z') \leq \mathbf{a}(d) = n$ (see P4); similarly, $h_{x,x',d} \neq 0$ implies $d \leq_{\mathcal{L}} x'$ hence $\mathbf{a}(x') \leq \mathbf{a}(d) = n$. Thus we have

$$\sum_{z';\mathbf{a}(z')\leq n} h_{x,y,z'} h_{z',z,d} = \sum_{x';\mathbf{a}(x')\leq n} h_{x,x',d} h_{y,z,x'}.$$

By P2 and our assumptions, the left hand side is

$$\gamma_{x,y,z}\gamma_{z^{-1},z,d} v^{2n} + \text{strictly smaller powers of } v.$$

Similarly, the right hand side is

$$\gamma_{x,x^{-1},d}\pi_n(h_{y,z,x^{-1}})v^{2n} + \text{strictly smaller powers of } v.$$

Hence $\gamma_{x,x^{-1},d}\pi_n(h_{y,z,x^{-1}}) = \gamma_{x,y,z}\gamma_{z^{-1},z,d} \neq 0$. Thus,

$$\gamma_{x,x^{-1},d} \neq 0, \pi_n(h_{y,z,x^{-1}}) \neq 0.$$

We see that $\mathbf{a}(x^{-1}) \geq n$. But we have also $\mathbf{a}(x) \leq n$ hence $\mathbf{a}(x) = n$ and $\pi_n(h_{y,z,x^{-1}}) = \gamma_{y,z,x}$. Since $\gamma_{x,x^{-1},d} \neq 0$, we have (by P5) $\gamma_{x,x^{-1},d} = \gamma_{z^{-1},z,d}$. Using this and $\gamma_{x,x^{-1},d}\gamma_{y,z,x} = \gamma_{x,y,z}\gamma_{z^{-1},z,d}$ we deduce $\gamma_{y,z,x} = \gamma_{x,y,z}$, as required.

Next we prove the following statement.

(b) *Let $z \in W, d \in \mathcal{D}$ be such that $\gamma_{z^{-1},z,d} \neq 0$. Then* $\mathbf{a}(z) = \mathbf{a}(d)$.

We shall assume that (b) holds whenever $\mathbf{a}(z) > N_0$ and we shall deduce that it also holds when $\mathbf{a}(z) = N_0$. (This will prove (b) by descending induction on $\mathbf{a}(z)$ since $\mathbf{a}(z)$ is bounded above.) Assume that $\mathbf{a}(z) = N_0$. From $\gamma_{z^{-1},z,d} = \pm 1$ we deduce that $h_{z^{-1},z,d^{-1}} \neq 0$ hence $d^{-1} \leq_{\mathcal{L}} z^{-1}$ hence $\mathbf{a}(d^{-1}) \geq \mathbf{a}(z^{-1})$ (see P4) and $\mathbf{a}(d) \geq \mathbf{a}(z)$. Assume that $\mathbf{a}(d) > \mathbf{a}(z)$, that is, $\mathbf{a}(d) > N_0$. Let $d' \in \mathcal{D}$ be such that $\gamma_{d^{-1},d,d'} \neq 0$ (see P3). By the induction hypothesis applied to d, d' instead of z, d, we have $\mathbf{a}(d) = \mathbf{a}(d')$. From $\gamma_{z^{-1},z,d} \neq 0$,$\gamma_{d^{-1},d,d'} \neq 0$, $\mathbf{a}(d) = \mathbf{a}(d')$, we deduce (using (a)) that $\gamma_{z,d,z^{-1}} = \gamma_{z^{-1},z,d}$. Hence $\gamma_{z,d,z^{-1}} \neq 0$. It follows that $h_{z,d,z} \neq 0$, hence $z \leq_{\mathcal{L}} d$, hence $\mathbf{a}(z) \geq \mathbf{a}(d)$ (see P4). This contradicts the assumption $\mathbf{a}(d) > \mathbf{a}(z)$. Hence we must have $\mathbf{a}(z) = \mathbf{a}(d)$, as required.

We now prove P7. Assume first that $\gamma_{x,y,z} \neq 0$. Let $d \in \mathcal{D}$ be such that $\gamma_{z^{-1},z,d} \neq 0$ (see P3). By (b) we have $\mathbf{a}(z) = \mathbf{a}(d)$. Using (a) we then have $\gamma_{x,y,z} = \gamma_{y,z,x}$. Assume next that $\gamma_{x,y,z} = 0$; we must show that $\gamma_{y,z,x} = 0$. We assume that $\gamma_{y,z,x} \neq 0$. By the first part of the proof, we have

$$\gamma_{y,z,x} \neq 0 \implies \gamma_{y,z,x} = \gamma_{z,x,y} \neq 0 \implies \gamma_{z,x,y} = \gamma_{x,y,z} \neq 0,$$

a contradiction.

14.8. *P7 $\implies$ P8.* If $\gamma_{x,y,z} \neq 0$, then $h_{x,y,z^{-1}} \neq 0$, hence $z^{-1} \leq_{\mathcal{L}} y, z \leq_{\mathcal{L}} x^{-1}$. By P7 we also have $\gamma_{y,z,x} \neq 0$ (hence $x^{-1} \leq_{\mathcal{L}} z, x \leq_{\mathcal{L}} y^{-1}$) and $\gamma_{z,x,y} \neq 0$ (hence $y^{-1} \leq_{\mathcal{L}} x, y \leq_{\mathcal{L}} z^{-1}$). Thus, we have $x \sim_{\mathcal{L}} y^{-1}, y \sim_{\mathcal{L}} z^{-1}, z \sim_{\mathcal{L}} x^{-1}$.

14.9. *$\tilde{P}$,P4,P8 $\implies$ P9.* We can find a sequence $z'=z_0,z_1,\dots,z_n=z$ such that for any $j\in[1,n]$ we have $z_{j-1}\leftarrow_{\mathcal{L}} z_j$. By P4 we have $\mathbf{a}(z')=\mathbf{a}(z_0)\ge\mathbf{a}(z_1)\ge\dots\ge\mathbf{a}(z_n)=\mathbf{a}(z)$. Since $\mathbf{a}(z)=\mathbf{a}(z')$, we have $\mathbf{a}(z')=\mathbf{a}(z_0)=\mathbf{a}(z_1)=\dots=\mathbf{a}(z_n)=\mathbf{a}(z)$. Thus, it suffices to show that, if $z'\leftarrow_{\mathcal{L}} z$ and $\mathbf{a}(z')=\mathbf{a}(z)$, then $z'\sim_{\mathcal{L}} z$. Let $x,y\in W$ be such that $\gamma_{x,y,z^{-1}}\ne 0$. By $\tilde{P}$, there exists $x'\in W$ such that $\pi_{\mathbf{a}(z)}(h_{x',y,z'})\ne 0$. Since $\mathbf{a}(z')=\mathbf{a}(z)$, we have $\gamma_{x',y,z'^{-1}}\ne 0$. From $\gamma_{x,y,z^{-1}}\ne 0,\gamma_{x',y,z'^{-1}}\ne 0$ we deduce, using P8, that $y\sim_{\mathcal{L}} z, y\sim_{\mathcal{L}} z'$, hence $z\sim_{\mathcal{L}} z'$.

14.10. *P9 $\implies$ P10.* We apply P9 to z^{-1},z'^{-1}.

14.11. *P4,P9,P10 $\implies$ P11.* We can find a sequence $z'=z_0,z_1,\dots,z_n=z$ such that for any $j\in[1,n]$ we have $z_{j-1}\le_{\mathcal{L}} z_j$ or $z_{j-1}\le_{\mathcal{R}} z_j$. By P4, we have $\mathbf{a}(z')=\mathbf{a}(z_0)\ge\mathbf{a}(z_1)\ge\dots\ge\mathbf{a}(z_n)=\mathbf{a}(z)$. Since $\mathbf{a}(z)=\mathbf{a}(z')$, we have $\mathbf{a}(z')=\mathbf{a}(z_0)=\mathbf{a}(z_1)=\dots=\mathbf{a}(z_n)=\mathbf{a}(z)$. Applying P9 or P10 to z_{j-1},z_j we obtain $z_{j-1}\sim_{\mathcal{L}} z_j$ or $z_{j-1}\sim_{\mathcal{R}} z_j$. Hence $z'\sim_{\mathcal{LR}} z$.

14.12. *P3,P4,P8 for W and W_I $\implies$ P12.* We write $\mathbf{a}_I:W_I\to\mathbf{N}$ for the $\mathbf{a}$-function defined in terms of W_I. For $x,y,z\in W_I$, we write $h^I_{x,y,z},\gamma^I_{x,y,z}$ for the analogues of $h_{x,y,z},\gamma_{x,y,z}$ when W is replaced by W_I. Let $\mathcal{H}_I\subset\mathcal{H}$ be as in 9.9.

Let $d\in\mathcal{D}$ be such that $\gamma_{y^{-1},y,d}\ne 0$. (See P3.) Then $\pi_{\mathbf{a}(d)}(h_{y^{-1},y,d^{-1}})\ne 0$. Now $c_{y^{-1}}c_y\in\mathcal{H}_I$ hence $d\in W_I$ and $\pi_{\mathbf{a}(d)}(h^I_{y^{-1},y,d^{-1}})\ne 0$. Thus, $\mathbf{a}_I(d^{-1})\ge\mathbf{a}(d^{-1})$. The reverse inequality is obvious hence $\mathbf{a}_I(d)=\mathbf{a}(d)$. We see that $\gamma^I_{y^{-1},y,d}\ne 0$. From P8 we see that $y\sim_{\mathcal{L}} d$ (relative to W_I) and $y\sim_{\mathcal{L}} d$ (relative to W). From P4 we deduce that $\mathbf{a}_I(y)=\mathbf{a}_I(d)$ and $\mathbf{a}(y)=\mathbf{a}(d)$. It follows that $\mathbf{a}(y)=\mathbf{a}_I(y)$.

14.13. *$\tilde{P}$,P2,P3,P4,P6,P8 $\implies$ P13.* If $x\in\Gamma$ then, by P3, there exists $d\in\mathcal{D}$ such that $\gamma_{x^{-1},x,d}\ne 0$. By P8 we have $x\sim_{\mathcal{L}} d^{-1}$ hence $d^{-1}\in\Gamma$. By P6, we have $d=d^{-1}$ hence $d\in\Gamma$. It remains to prove the uniqueness of d. Let d',d'' be elements of $\mathcal{D}\cap\Gamma$. We must prove that $d'=d''$. We can find x',y',x'',y'' such that $\gamma_{x',y',d'}\ne 0$, $\gamma_{x'',y'',d''}\ne 0$. By P2, we have $x'=y'^{-1},x''=y''^{-1}$. By P8, we have $y'\sim_{\mathcal{L}} d'^{-1}=d'$ and $y''\sim_{\mathcal{L}} d''^{-1}=d''$, hence $y',y''\in\Gamma$. By the definition of left cells, we can find a sequence $y'=x_0,x_1,\dots,x_n=y''$ such that for any $j\in[1,n]$ we have $x_{j-1}\leftarrow_{\mathcal{L}} x_j$. Since $y'\sim_{\mathcal{L}} y''$, we have $x_j\in\Gamma$ for all j. For $j\in[1,n-1]$ let $d_j\in\mathcal{D}$ be such that $\gamma_{x_j^{-1},x_j,d_j}\ne 0$. Let $d_0=d',d_n=d''$. As in the beginning of the proof, we have $d_j\in\Gamma$ for each j. Let $j\in[1,n]$. Since $x_{j-1}\leftarrow_{\mathcal{L}} x_j$, we have (by P8) $\gamma_{x_j,d_j,x_j^{-1}}\ne 0$. Applying $\tilde{P}$ to x_j,d_j,x_j,x_{j-1} instead of x,y,z,z', we see that there exists u such that $\pi_{\mathbf{a}(x_j)}(h_{u,d_j,x_{j-1}})\ne 0$. Since $x_{j-1}\sim x_j$, we have $\mathbf{a}(x_{j-1})=\mathbf{a}(x_j)$ (see P4), hence $\pi_{\mathbf{a}(x_j)}(h_{u,d_j,x_{j-1}})=\gamma_{u,d_j,x_{j-1}^{-1}}\ne 0$. Using P8, we deduce $\gamma_{x_{j-1}^{-1},u,d_j}\ne 0$. Using P2 we see that $u=x_{j-1}$ and $\gamma_{x_{j-1}^{-1},x_{j-1},d_j}\ne 0$. We have also $\gamma_{x_{j-1}^{-1},x_{j-1},d_{j-1}}\ne 0$ and by the uniqueness in P3, it follows that $d_{j-1}=d_j$. It follows that $d'=d''$, as required.

14.14. *P6,P13 $\implies$ P14.* By P13, we can find $d\in\mathcal{D}$ such that $z\sim_{\mathcal{L}} d$. Since $d=d^{-1}$ (see P6), it follows that $z^{-1}\sim_{\mathcal{R}} d$. Thus, $z\sim_{\mathcal{LR}} z^{-1}$.

14.15. In this subsection we reformulate conjecture P15, assuming that P4,P9, P10 hold. Let $\tilde{\mathcal{A}} = \mathbf{Z}[v, v^{-1}, v', v'^{-1}]$ where v, v' are indeterminates. Let $\tilde{\mathcal{H}}$ be the free $\tilde{\mathcal{A}}$-module with basis $e_w (w \in W)$. Let $\mathcal{H}', c'_w, h'_{x,y,z}$ be obtained from $\mathcal{H}, c_w, h_{x,y,z}$ by changing the variable v to v'.

On $\tilde{\mathcal{H}}$ we have a left $\mathcal{H}$-module structure given by $v^n c_y e_w = \sum_x v^n h_{y,w,x} e_x$ and a right $\mathcal{H}'$-module structure given by $e_w(v'^n c'_y) = \sum_x v'^n h'_{w,y,x} e_x$. These module structures do not commute in general. For each $a \ge 0$ let $\tilde{\mathcal{H}}_{\ge a}$ be the $\tilde{\mathcal{A}}$-submodule of $\tilde{\mathcal{H}}$ spanned by $\{e_w; \mathbf{a}(w) \ge a\}$. By P4, this is a left $\mathcal{H}$-submodule and a right $\mathcal{H}'$-submodule of $\tilde{\mathcal{H}}$. We have

$$\dots \tilde{\mathcal{H}}_{\ge 2} \subset \tilde{\mathcal{H}}_{\ge 1} \subset \tilde{\mathcal{H}}_{\ge 0} = \tilde{\mathcal{H}}$$

and $gr\tilde{\mathcal{H}} = \oplus_{a \ge 0} \tilde{\mathcal{H}}_{\ge a}/\tilde{\mathcal{H}}_{\ge a+1}$ inherits a left $\mathcal{H}$-module structure and a right $\mathcal{H}'$-module structure from $\tilde{\mathcal{H}}$. Clearly, P15 is equivalent to the condition that these module structures on $gr\tilde{\mathcal{H}}$ commute. To check this last condition, it is enough to check that

the actions of $c_s, c'_{s'}$ commute on $gr\tilde{\mathcal{H}}$ for $s, s' \in S$.

Let $s, s' \in S$, $w \in W$. A computation using 6.6, 6.7, 8.2 shows that $(c_s e_w)c'_{s'} - c_s(e_w c'_{s'})$ is 0 if $sw < w$ or $ws' < w$, while if $sw > w, ws' > w$, it is

$$\sum_{y; sy<y, ys'<y} (h'_{w,s',y}(v_s + v_s^{-1}) - h_{s,w,y}(v'_{s'} + {v'_{s'}}^{-1}))e_y + \sum_{y; sy<y, ys'<y} \alpha_y e_y$$

where

$$\alpha_y = \sum_{y'; y's'<y'<sy'} h'_{w,s',y'} h_{s,y',y} - \sum_{y'; sy'<y'<y's'} h_{s,w,y'} h'_{y',s',y}.$$

If y satisfies $sy < y, ys' < y$ and either $h'_{w,s',y}$ or $h_{s,w,y}$ is $\neq 0$, then $\mathbf{a}(y) > \mathbf{a}(w)$. (We certainly have $\mathbf{a}(y) \ge \mathbf{a}(w)$ by P4. If we had $\mathbf{a}(y) = \mathbf{a}(w)$ and $h_{s,w,y} \neq 0$ then by P9 we would have $y \sim_{\mathcal{L}} w$ hence $\mathcal{R}(y) = \mathcal{R}(w)$ contradicting $ys' < y, ws' > w$. If we had $\mathbf{a}(y) = \mathbf{a}(w)$ and $h'_{s,w,y} \neq 0$ then by P10 we would have $y \sim_{\mathcal{R}} w$ hence $\mathcal{L}(y) = \mathcal{L}(w)$, contradicting $sy < y, sw > w$.) Hence, if $sw > w, ws' > w$, we have

$$(c_s e_w)c'_{s'} - c_s(e_w c'_{s'}) = \sum_{y; sy<y, ys'<y, \mathbf{a}(y)=\mathbf{a}(w)} \alpha_y e_y \mod \tilde{\mathcal{H}}_{\ge \mathbf{a}(w)+1}.$$

We see that P15 is equivalent to the following statement.

(a) *If $y, w \in W, s, s' \in S$ are such that $sw > w, ws' > w, sy < y, ys' < y, \mathbf{a}(y) = \mathbf{a}(w)$, then*

$\sum_{y'; y's'<y'<sy'} h'_{w,s',y'} h_{s,y',y} = \sum_{y'; sy'<y'<y's'} h_{s,w,y'} h'_{y',s',y}$.

15. Example: the split case

15.1. We preserve the setup of 3.1. We assume that $L = l$ that is, we are in the split case. From the results on Soergel bimodules in [EW], we see that

(a) $h_{x,y,z} \in \mathbf{N}[v,v^{-1}]$ for all x,y,z in W,
(b) $p_{y,w} \in \mathbf{N}[v^{-1}]$ for all y,w in W.
In this chapter we assume that W,l is bounded. We will show that $\tilde{P}$ and P1-P3 hold for W,l hence all of P1-P14 hold for W,l; moreover we show that P15 holds. (Note that for P1-P6, the boundedness of W,l will not be needed). The arguments in this chapter are based on [L9].

15.2. From 14.5(a) (which does not depend on the boundedness assumption) we see that for $x,y \in W$ we have
(a) $\sum_{z\in W} h_{x,y,z}p_{1,z} \in \mathbf{Z}[v^{-1}]$.
From 15.1(a),(b) we see that $h_{x,y,z}p_{1,z} \in \mathbf{N}[v,v^{-1}]$ for any $z \in W$. Hence in (a) there are no cancellations, so that
(b) $h_{x,y,z}p_{1,z} \in \mathbf{N}[v^{-1}]$ for any $z \in W$.
Let $z \in W$. By 5.4(a) we have $p_{1,z} = \sum_{j\ge 0} e_j v^{-l(z)+j}$ where $e_j \in \mathbf{Z}$ are zero for large j and $e_0 = 1$; moreover, by 15.1(b) we have $e_j \in \mathbf{N}$ for all j. Thus, $v^{-l(z)}h_{x,y,z} + \sum_{j>0} e_j v^{-l(z)+j}h_{x,y,z} \in \mathbf{N}[v^{-1}]$. Again in this sum there are no cancellations, hence

$$v^{-l(z)}h_{x,y,z} \in \mathbf{N}[v^{-1}].$$

(This was proved for finite W in [L7] and then for general W by Springer (unpublished).) This shows that the definition of $\mathbf{a}(z)$ in 13.6 can be given in our case without the boundedness assumption. Hence the definition of $\gamma_{x,y,z}$ in 13.6 can be given in our case without the boundedness assumption. Note that the definition of $\Delta(z)$ in 14.1 can be given without the boundedness assumption. Hence the definition of $\mathcal{D}$ in 14.1 can be given in our case without the boundedness assumption.

We show that *P1 holds for* (W,l).

We fix $z \in W$ and choose $x,y \in W$ so that $\gamma_{x,y,z^{-1}} \neq 0$. From the definitions,
(c) $h_{x,y,z}p_{1,z} \in \gamma_{x,y,z^{-1}}n_z v^{\mathbf{a}(z)-\Delta(z)} +$ strictly smaller powers of v
and the coefficient of $v^{\mathbf{a}(z)-\Delta(z)}$ is $\neq 0$. Comparison with (b) gives $\mathbf{a}(z)-\Delta(z) \le 0$.

15.3. *Proof of P2.* Assume that $x \neq y^{-1}$. From 14.5(a) we see that
(a) $\sum_{z\in W} h_{x,y,z}p_{1,z} \in v^{-1}\mathbf{Z}[v^{-1}]$.
As in 15.2, this implies (using 15.1(a),(b)) that
(b) $h_{x,y,z}p_{1,z} \in v^{-1}\mathbf{N}[v^{-1}]$ for any $z \in W$.
Assume now that $z = d^{-1} \in \mathcal{D}$. Then 15.2(c) becomes in our case

$$h_{x,y,z}p_{1,z} \in \gamma_{x,y,z^{-1}}n_z + v^{-1}\mathbf{Z}[v^{-1}].$$

Comparison with (b) gives $\gamma_{x,y,z^{-1}}n_z = 0$. Since $n_z \neq 0$, we have $\gamma_{x,y,z^{-1}} = 0$. This proves P2.

15.4. *Proof of P3.* From 14.5(a) we see that
(a) $\sum_{z\in W} h_{y^{-1},y,z}p_{1,z} \in 1 + v^{-1}\mathbf{Z}[v^{-1}]$.
As in 15.2, this implies (using 15.1(a),(b)) that there is a unique z, say $z = d^{-1}$ such that

(b) $h_{y^{-1},y,d^{-1}}p_{1,d^{-1}} \in 1 + v^{-1}\mathbf{N}[v^{-1}]$
and that
(c) $h_{y^{-1},y,z}p_{1,z} \in v^{-1}\mathbf{N}[v^{-1}]$
for all $z \neq d^{-1}$. For $z = d^{-1}$, 15.2(c) becomes

$$h_{y^{-1},y,d^{-1}}p_{1,d^{-1}} \in \gamma_{y^{-1},y,d}n_{d^{-1}}v^{\mathbf{a}(d)-\Delta(d)} + \text{strictly smaller powers of } v.$$

Here $\mathbf{a}(d) - \Delta(d) \leq 0$. Comparison with (b) gives $\mathbf{a}(d) - \Delta(d) = 0$ and $\gamma_{y^{-1},y,d}n_{d^{-1}} = 1$. Thus, $d \in \mathcal{D}$ and $\gamma_{y^{-1},y,d} \neq 0$. Thus, the existence part of P3 is established.

Assume that there exists $d' \neq d$ such that $d' \in \mathcal{D}$ and $\gamma_{y^{-1},y,d'} \neq 0$. For $z = d'^{-1}$, 15.2(c) becomes

$$h_{y^{-1},y,d'^{-1}}p_{1,d'^{-1}} \in \gamma_{y^{-1},y,d'}n_{d'^{-1}} + v^{-1}\mathbf{Z}[v^{-1}].$$

Comparison with (c) (with $z = d'^{-1}$) gives $\gamma_{y^{-1},y,d'}n_{d'^{-1}} = 0$ hence $\gamma_{y^{-1},y,d'} = 0$, a contradiction. This proves the uniqueness part of P3.

15.5. *Proof of $\tilde{P}$.* We may assume that $z' \neq z$. Then we can find $s \in S$ such that $sz' < z', sz > z$ and $h_{s,z,z'} \neq 0$. Since $h_{x,y,z} \neq 0$, we have (by 13.1(d)) $z \leq_{\mathcal{R}} x$ hence $\mathcal{L}(x) \subset \mathcal{L}(z)$ (by 8.6). Since $s \notin \mathcal{L}(z)$, we have $s \notin \mathcal{L}(x)$, that is, $sx > x$. We have $c_s c_x c_y = \sum_u p_u c_u$, where

$$p_u = \sum_w h_{x,y,w}h_{s,w,u} = \sum_{x'} h_{s,x,x'}h_{x',y,u}.$$

In particular,

$$p_{z'} = \sum_w h_{x,y,w}h_{s,w,z'} = h_{x,y,z}h_{s,z,z'} + \sum_{w;w\neq z} h_{x,y,w}h_{s,w,z'}.$$

By 6.5, we have $h_{s,z,z'} \in \mathbf{Z}$ hence

(a) $$\pi_n(p_{z'}) = \pi_n(h_{x,y,z})h_{s,z,z'} + \sum_{w;w\neq z} \pi_n(h_{x,y,w}h_{s,w,z'})$$

for any $n \in \mathbf{Z}$. In particular, this holds for $n = \mathbf{a}(z)$. By assumption, we have $\pi_n(h_{x,y,z}) \neq 0$ and $h_{s,z,z'} \neq 0$; hence, by 15.1(a), we have $\pi_n(h_{x,y,z}) > 0$ and $h_{s,z,z'} > 0$. Again, by 15.1(a) we have $\pi_n(h_{x,y,w}h_{s,w,z'}) \geq 0$ for any $w \neq z$. Hence from (a) we deduce $\pi_n(p_{z'}) > 0$. Since $p_{z'} = \sum_{x'} h_{s,x,x'}h_{x',y,z'}$, there exists x' such that $\pi_n(h_{s,x,x'}h_{x',y,z'}) \neq 0$. Since $sx > x$, we see from 6.5 that $h_{s,x,x'} \in \mathbf{Z}$ hence

$$\pi_n(h_{s,x,x'}h_{x',y,z'}) = h_{s,x,x'}\pi_n(h_{x',y,z'}).$$

Thus we have $\pi_n(h_{x',y,z'}) \neq 0$. This proves $\tilde{P}$ in our case.

15.6. Since $\tilde{P}$ and P1-P3 are known, we see that P1-P11 and P13,P14 hold in our case (see §14). The same arguments can be applied to W_I where $I \subset S$, hence P1-P11 and P13,P14 hold for W_I. By 14.12, P12 holds for W. Thus, P1-P14 hold for W.

15.7. *Proof of P15.* By 14.15, we see that it is enough to prove 14.15(a). Let y, w, s, s' be as in 14.15(a). In our case, by 6.5, the equation in 14.15(a) involves only integers, hence it is enough to prove it after specializing $v = v'$. If in 14.15 we specialize $v = v'$, then the left and right module structures in 14.15 clearly commute, since the left and right regular representations of $\mathcal{H}$ commute. Hence the coefficient of e_y in $((c_s e_w)c_{s'} - c_s(e_w c_{s'}))_{v=v'}$ is 0. By the computation in 14.15, this coefficient is

(a)

$$(h_{w,s',y} - h_{s,w,y})(v+v^{-1}) + \sum_{\substack{y' \\ y's'<y'<sy'}} h_{w,s',y'} h_{s,y',y} - \sum_{\substack{y' \\ sy'<y'<y's'}} h_{s,w,y'} h_{y',s',y} = 0.$$

By 6.5, $h_{s,w,y}$ is the coefficient of v^{-1} in $p_{y,w}$ and $h_{w,s',y} = h_{s',w^{-1},y^{-1}}$ is the coefficient of v^{-1} in $p_{y^{-1},w^{-1}} = p_{y,w}$. Thus, $h_{s,w,y} = h_{w,s',y}$ and (a) reduces to the equation in 14.15(a) (specialized at $v = v'$). This proves 14.15(a).

16. Example: the quasisplit case

16.1. In this subsection we review some results from [L12, §11].

Let k be an algebraically closed field of characteristic zero. Let $\mathfrak{C}$ be a k-linear category, that is a category in which the space of morphisms between any two objects has a given k-vector space structure such that composition of morphisms is bilinear and such that finite direct sums exist. A functor from one k-linear category to another is said to be k-linear if it respects the k-vector space structures.

Let $\mathcal{K}(\mathfrak{C})$ be the Grothendieck group of $\mathfrak{C}$ that is, the free abelian group generated by symbols $[A]$ for each $A \in \mathfrak{C}$ (up to isomorphism) with relations $[A \oplus B] = [A| + [B]$ for any $A, B \in \mathfrak{C}$. Let $\mathbf{n}$ be an integer ≥ 1. A k-linear functor $M \mapsto M^\sharp$, $\mathfrak{C} \to \mathfrak{C}$ is said to be $\mathbf{n}$-*periodic* if $(\sharp)^{\mathbf{n}} : \mathfrak{C} \to \mathfrak{C}$ is the identity functor. Assuming that such a functor is given we define a new k-linear category $\mathfrak{C}_\sharp$ as follows. The objects of $\mathfrak{C}_\sharp$ are pairs (A, ϕ) where $A \in \mathfrak{C}$ and $\phi : A^\sharp \to A$ is an isomorphism in $\mathfrak{C}$ such that the composition

$$A^{\sharp^{\mathbf{n}}} \xrightarrow{\phi^{\sharp^{\mathbf{n}-1}}} A^{\sharp^{\mathbf{n}-1}} \xrightarrow{\phi^{\sharp^{\mathbf{n}-2}}} \ldots \to A^\sharp \xrightarrow{\phi} A$$

is the identity map of A. Let (A, ϕ), (A', ϕ') be two objects of $\mathfrak{C}_\sharp$. We define a k-linear map $\mathrm{Hom}_{\mathfrak{C}}(A, A') \to \mathrm{Hom}_{\mathfrak{C}}(A, A')$ by $f \mapsto f^! := \phi' f^\sharp \phi^{-1}$. Note that the $\mathbf{n}$-th iteration of $^!$ applied to f is 1. By definition, $\mathrm{Hom}_{\mathfrak{C}_\sharp}((A, \phi), (A', \phi')) = \{f \in \mathrm{Hom}_{\mathfrak{C}}(A, A'); f = f^!\}$, a k-vector space. The direct sum of two objects (A, ϕ), (A', ϕ') is $(A \oplus A', \phi \oplus \phi')$. Clearly, if $(A, \phi) \in \mathfrak{C}_\sharp$, then $(A, \zeta\phi) \in \mathfrak{C}_\sharp$ for any $\zeta \in k$

such that $\zeta^{\mathbf{n}} = 1$. An object (A, ϕ) of $\mathfrak{C}_\sharp$ is said to be *traceless* if there exists an object B of $\mathfrak{C}$, an integer $t \ge 2$ dividing $\mathbf{n}$ and an isomorphism

$$A \cong B \oplus B^\sharp \oplus \ldots \oplus B^{\sharp^{t-1}}$$

under which ϕ corresponds to an isomorphism

$$B \oplus B^\sharp \oplus \ldots \oplus B^{\sharp^{t-1}} \xrightarrow{\sim} B \oplus B^\sharp \oplus \ldots \oplus B^{\sharp^{t-1}}$$

which carries the summand of $B^{\sharp^j}$ onto the summand $B^{\sharp^j}$ for $1 \le j \le t-1$) and the summand $B^{\sharp^t}$ onto the summand B.

Let $\mathcal{O}$ be the subring of k consisting of all $\mathbf{Z}$-linear combinations of $\mathbf{n}$-th roots of 1. We associate to $\mathfrak{C}$ and $\sharp$ an $\mathcal{O}$-module $\mathcal{K}_\sharp(\mathfrak{C})$. By definition $\mathcal{K}_\sharp(\mathfrak{C})$ is the $\mathcal{O}$-module generated by symbols $[B, \phi]$ one for each isomorphism class of objects (B, ϕ) of $\mathfrak{C}_\sharp$ subject to the following relations:
(a) $[B, \phi] + [B', \phi'] = [B \oplus B', \phi \oplus \phi']$;
(b) $[B, \phi] = 0$ if (B, ϕ) is traceless;
(c) $[B, \zeta\phi] = \zeta[B, \phi]$ if $\zeta \in k$ satisfies $\zeta^{\mathbf{n}} = 1$.

16.2. Let $\tilde{W}, \tilde{S}$ be a Coxeter group. (The set $\tilde{S}$ of simple reflections of $\tilde{W}$ is assumed to be finite.) We view $\tilde{S}$ as a subset of $\tilde{W}$. For any $I \subset \tilde{S}$ we denote by $\tilde{W}_I$ the subgroup of $\tilde{W}$ generated by I. Let $\tilde{l} : \tilde{W} \to \mathbf{N}$ be the length function of $\tilde{W}$. Let $\le$ be the standard partial order on $\tilde{W}$. Let $\iota : \tilde{W} \to \tilde{W}$ be an automorphism such that $\iota(\tilde{S}) = \tilde{S}$. We fix an integer $\mathbf{n} \ge 1$ such that $\iota^{\mathbf{n}} = 1$. Let $W = \{w \in \tilde{W}; \iota(w) = w\}$. Let S be the set of ι-orbits I on $\tilde{S}$ such that $\tilde{W}_I$ is finite; for such I let w_0^I be the longest element of $\tilde{W}_I$ (note that $w_0^I \in W$). According to Theorems A.8 and A.9 in the Appendix, W is a Coxeter group on the set of generators $\{w_0^I; I \in S\}$ and the restriction of $\tilde{l}$ to W is a weight function $L : W \to \mathbf{N}$. Let $l : W \to \mathbf{N}$ be the length function of W, S. We then say that W, L is in the quasisplit case. We define

$$\mathcal{H}, T_x, c_x, p_{y,x}, f_{x,y,z}, h_{x,y,z}$$

in terms of W, S, L as in 3.2, 5.2, 5.3 (here $x, y, z \in W$). Let

$$\tilde{\mathcal{H}}, \tilde{T}_x, \tilde{c}_x, \tilde{p}_{y,x}, \tilde{f}_{x,y,z}, \tilde{h}_{x,y,z}$$

be the analogous objects defined in terms of $\tilde{W}, \tilde{S}, \tilde{l}$ (here $x, y, z \in \tilde{W}$).

Let $\mathfrak{h}_{\mathbf{R}}$ be a reflection representation of $\tilde{W}$ over the real numbers $\mathbf{R}$, as in [EW, 1.1]; for any $s \in \tilde{S}$ we fix a linear form $\alpha_s : \mathfrak{h}_{\mathbf{R}} \to \mathbf{R}$ whose kernel is equal to the fixed point set of $s : \mathfrak{h}_{\mathbf{R}} \to \mathfrak{h}_{\mathbf{R}}$. Let $\mathfrak{h} = \mathbf{C} \otimes_{\mathbf{R}} \mathfrak{h}$; we extend α_s to a $\mathbf{C}$-linear function $\mathfrak{h} \to \mathbf{C}$ denoted again by α_s. Let R be the algebra of polynomial functions $\mathfrak{h} \to \mathbf{C}$ with the $\mathbf{Z}$-grading in which linear functions $\mathfrak{h} \to \mathbf{C}$ have degree 2. Note that $\tilde{W}$ acts naturally on R; we write this action as $w : r \mapsto {}^w r$ and for $s \in \tilde{S}$ we set

$R^s = \{r \in R; {}^s r = r\}$, a subalgebra of R. Let $R^{>0} = \{r \in R; r(0) = 0\}$. We can assume that there exists a $\mathbf{C}$-linear map $\xi \mapsto \iota(\xi)$ of the dual space of $\mathfrak{h}$ into itself whose $\mathbf{n}$-th power is 1 and is such that $\iota(w\xi) = \iota(w)(\iota(\xi))$ for $w \in W, x \in \mathfrak{h}$ and such that $\iota(\alpha_s) = \alpha_{\iota(s)}$ for $s \in \tilde{S}$. It induces an algebra automorphism $r \mapsto \iota(r)$ of R.

Let $\mathcal{R}$ be the category whose objects are $\mathbf{Z}$-graded (R, R)-bimodules in which for $M, M' \in \mathcal{R}$, $\mathrm{Hom}_{\mathcal{R}}(M, M')$ is the space of homomorphisms of (R, R)-bimodules $M \to M'$ compatible with the $\mathbf{Z}$-gradings. For $M \in \mathcal{R}$ and $n \in \mathbf{Z}$, the shift $M[n]$ is the object of $\mathcal{R}$ equal in degree i to M in degree $i + n$. For M, M' in $\mathcal{R}$ we set $MM' = M \otimes_R M'$; this is naturally an object of $\mathcal{R}$. For M, M' in $\mathcal{R}$ we set

$$M'^M = \oplus_{n\in\mathbf{Z}} \mathrm{Hom}_{\mathcal{R}}(M, M'[n]),$$

viewed as an object of $\mathcal{R}$ with $(rf)(m) = f(rm)$, $(fr)(m) = f(mr)$ for $m \in M, f \in M'^M, r \in R$. For any $M \in \mathcal{R}$ we set $\underline{M} = M/MR^{>0} = M \otimes_R \mathbf{C}$ where $\mathbf{C}$ is identified with $R/R^{>0}$. We view $\underline{M}$ as a $\mathbf{Z}$-graded $\mathbf{C}$-vector space.

For $s \in \tilde{S}$ let $B_s = R \otimes_{R^s} R[1] \in \mathcal{R}$. More generally, for any $x \in \tilde{W}$, Soergel [So, 6.16] shows that there is an object B_x of $\mathcal{R}$ (unique up to isomorphism) such that B_x is an indecomposable direct summand of $B_{s_1} B_{s_2} \dots B_{s_q}$ for some/any reduced expression $w = s_1 s_2 \dots s_q$ $(s_i \in \tilde{S})$ and such that B_x is not a direct summand of $B_{s'_1} B_{s'_2} \dots B_{s'_p}$ whenever $s'_1, \dots, s'_p \in \tilde{S}, p < q$.

Let $\tilde{C}$ be the full subcategory of $\mathcal{R}$ whose objects are isomorphic to finite direct sums of shifts of objects of the form B_x for various $x \in \tilde{W}$. Let C be the full subcategory of $\mathcal{R}$ whose objects are isomorphic to finite direct sums of objects of the form B_x for various $x \in \tilde{W}$.

Let $x \in \tilde{W}$. From [EW] it follows that $\mathrm{Hom}_{\mathcal{R}}(B_x, B_x) = \mathbf{C}$ and from [So, 6.16] it follows that $\dim \underline{R_x^{B_x}}_{l(x)} = 1$. Thus, as noted in [LV, 2.2], $\underline{R_x^{B_x}}_{l(x)} \otimes_{\mathbf{C}} B_x$ is an object of C isomorphic to B_x and defined up to unique isomorphism (even though B_x was defined only up to non-unique isomorphism). From now on we will use the notation B_x for this new object; this agrees with the earlier description of B_s.

From [So] it follows that for $M, M' \in \tilde{C}$ we have $MM' \in \tilde{C}$. For any $x \in \tilde{W}$ let R_x be the object of $\mathcal{R}$ such that $R_x = R$ as a left R-module and such that for $m \in R_x, r \in R$ we have $mr = ({}^x r)m$. The following result appears in [So, 6.15]:

(a) *For any $M \in \tilde{C}$, R_x^M is a finitely generated graded free right R-module; hence* $\dim_{\mathbf{C}} \underline{R_x^M} < \infty$.

We regard $\mathcal{K}(\tilde{C})$ as an $\mathcal{A}$-module by $v^n[M] = [M[-n]]$ for $M \in \tilde{C}, n \in \mathbf{Z}$. Note that $\mathcal{K}(\tilde{C})$ is an associative $\mathcal{A}$-algebra with product defined by $[M][M'] = [MM']$ for $M \in \tilde{C}, M' \in \tilde{C}$. From [So, 1.10, 5.3] we see that

(b) *the assignment* $M \mapsto \sum_{y\in W, i\in\mathbf{Z}} \dim \underline{R_y^M)}_i v^{-i+\tilde{l}(y)} T_y$ *defines an $\mathcal{A}$-algebra isomorphism* $\chi : \mathcal{K}(\tilde{C}) \xrightarrow{\sim} \tilde{\mathcal{H}}$.

From [EW, Theorem 1.1] it follows that for $x \in \tilde{W}$ we have

(c) $$\chi(B_x) = \tilde{c}_x.$$

16.3. For $M \in \mathcal{R}$ let $M^\sharp$ be the object of $\mathcal{R}$ which is equal to M as a graded $\mathbf{C}$-vector space, but left (resp. right) multiplication by $r \in R$ on $M^\sharp$ equals left (resp. right) multiplication by $\iota(r)$ on M. If $f : M_1 \to M_2$ is a morphism in $\mathcal{R}$ then f can be also viewed as a morphism $M_1^\sharp \to M_2^\sharp$ in $\mathcal{R}$. Clearly, $M \mapsto M^\sharp$ is a $\mathbf{C}$-linear $\mathbf{n}$-periodic functor $\mathcal{R} \to \mathcal{R}$. Hence $\mathcal{R}_\sharp$ is well defined, see 16.1. If $M_1, M_2 \in \mathcal{R}$ then the identity maps gives an identification $M_1^\sharp M_2^\sharp = (M_1 M_2)^\sharp$ as objects in $\mathcal{R}$.

Let $s \in \tilde{S}$. We define a $\mathbf{C}$-linear isomorphism $\omega_s : B_s[-1] \xrightarrow{\sim} B_{\iota(s)}[-1]$ given by $x \otimes_{R^s} y \mapsto \iota(x) \otimes_{R^{\iota(s)}} \iota(y)$ for $x, y \in R$. We have $\omega_s(rfr') = \iota(r)\omega_s(f)\iota(r')$ for $r, r' \in R, f \in B_s[-1]$. Hence ω_s can be viewed as an isomorphism $B_s[-1] \xrightarrow{\sim} B^\sharp_{\iota(s)}[-1]$ in $\mathcal{R}$ or as an isomorphism $B_s \xrightarrow{\sim} B^\sharp_{\iota(s)}$ in $\mathcal{R}$. Now let $x \in \tilde{W}$ and let $s_1 s_2 \dots s_n$ be a reduced expression for x. Since B_x is an indecomposable direct summand of $B_{s_1} B_{s_2} \dots B_{s_n}$ (and n is minimal with this property) we see that $B_x^\sharp$ is an indecomposable direct summand of

$$(B_{s_1} B_{s_2} \dots B_{s_n})^\sharp = B^\sharp_{s_1} B^\sharp_{s_2} \dots B^\sharp_{s_n} \cong B_{\iota^{-1}(s_1)} B_{\iota^{-1}(s_2)} \dots B_{\iota^{-1}(s_n)}$$

(and n is minimal with this property) hence by [So, 6.16] we have

(a) $$B_x^\sharp \cong B_{\iota^{-1}(x)}.$$

. In particular we have $B_x^\sharp \in \tilde{C}$. It follows that $M \in C \implies M^\sharp \in C$ and $M \in \tilde{C} \implies M^\sharp \in \tilde{C}$. Note that $M \mapsto M^\sharp$ are $\mathbf{C}$-linear, $\mathbf{n}$-periodic functors $C \to C$ and $\tilde{C} \to \tilde{C}$. Hence $C_\sharp, \tilde{C}_\sharp$ are defined as in 16.1 and $\mathcal{K}_\sharp(C)$, $\mathcal{K}_\sharp(\tilde{C})$ are well defined $\mathcal{O}$-modules.

Now if $x \in W$ then from (a) we have that there exists an isomorphism $\phi : B_x^\sharp \xrightarrow{\sim} B_x$ in C. Replacing ϕ by $c\phi$ for a suitable $c \in \mathbf{C}^*$ we can assume that $(B_x, \phi) \in C_\sharp$. (We use that $\mathrm{End}(B_x) = \mathbf{C}$, see [EW].)

16.4. For $x \in \tilde{W}$ we define $\mathfrak{f}_x : R_x^\sharp \to R_{\iota^{-1}(x)}$ by $r \mapsto \iota^{-1}(r)$. This is an isomorphism in $\mathcal{R}$. Now assume that $x \in W$; then $\mathfrak{f}_x : R_x^\sharp \to R_x$ and $(R_x, \mathfrak{f}_x) \in \mathcal{R}_\sharp$; thus, $(R_x[i], \mathfrak{f}_x[i]) \in \mathcal{R}_\sharp$ for any $i \in \mathbf{Z}$. Hence, if $(M, \phi) \in \tilde{C}_\sharp$ and $i \in \mathbf{Z}$, then $f \mapsto f^!$, $\mathrm{Hom}_{\mathcal{R}}(M, R_x[i]) \to \mathrm{Hom}_{\mathcal{R}}(M, R_x[i])$ is defined as in 16.1. Taking direct sum over $i \in \mathbf{Z}$ we obtain a map $f \mapsto f^!$, $R_x^M \to R_x^M$ whose $\mathbf{n}$-th iteration is 1. Since $R^{>0}$ is ι-stable we see that $f \mapsto f^!$ maps $R_x^M R^{>0}$ into iself hence it induces an $\mathbf{C}$-linear map $\underline{R_x^M} \to \underline{R_x^M}$ and (for any i) an $\mathbf{C}$-linear map $\underline{R_x^M}_i \to \underline{R_x^M}_i$ (whose $\mathbf{n}$-th power is 1), denoted by $\mathcal{Z}^M_{x,\phi,i}$. Let

$$\epsilon_i^x(M, \phi) = \mathrm{tr}_{\mathbf{C}}(\mathcal{Z}^M_{x,\phi,i}, \underline{R_x^M}_i) \in \mathcal{O}.$$

We now take $M = B_x$ (still assuming $x \in W$) so that $(B_x, \phi) \in C_\sharp$ for some ϕ. Then $\underline{R_x^{B_x}}_{\tilde{l}(x)} = \mathbf{C}$ hence $\epsilon^x_{\tilde{l}(x)}(B_x, \phi)$ is an $\mathbf{n}$-th root of 1 in $\mathbf{C}$. We can normalize

$\phi : B_x^\sharp \to B_x$ uniquely so that $\epsilon^x_{l(x)}(B_x, \phi) = 1$. We shall denote this normalized ϕ by ϕ_x.

Next we note that if $x, x' \in \tilde{W}$ then by [EW], $\mathrm{Hom}_{\mathcal{R}}(B_{x'}, B_x)$ is $\mathbf{C}$ if $x = x'$ and is 1 if $x \neq x'$. It follows that C is a semisimple abelian category and the B_x are its simple objects. Using this and [L12, 11.1.8] we deduce that

(a) $\mathcal{K}_\sharp(C)$ is the free $\mathcal{O}$-module with basis $\{[B_x, \phi_x]; x \in \tilde{W}\}$.

16.5. Let $\mathcal{O}' = \mathcal{O}[v, v^{-1}]$ where v is an indeterminate. We view $\mathcal{K}_\sigma(\tilde{C})$ as an $\mathcal{O}'$-module with $v^n[M, \phi] = [M[-n], \phi]$ for $(M, \phi) \in \tilde{C}_\sharp$, $n \in \mathbf{Z}$. We have the following result.

(a) *The $\mathcal{O}'$-linear map $q : \mathcal{O}' \otimes_{\mathcal{O}} \mathcal{K}_\sharp(C) \to \mathcal{K}_\sharp(\tilde{C})$ given by $v^n \otimes [M, \phi] \mapsto [M[-n], \phi]$ is an isomorphism.*

The map q is clearly well defined. To prove that it is surjective we shall use the functors $M \mapsto \tau_{\le i} M$ from $\tilde{C}$ to $\tilde{C}$ (resp. $M \mapsto \mathcal{H}^i M$ from $\tilde{C}$ to C) defined in [EW, 6.2]. (Here $i \in \mathbf{Z}$.) These define in an obvious way functors $\tilde{C}_\sharp \to \tilde{C}_\sharp$ (resp. $\tilde{C}_\sharp \to C_\sharp$) denoted again by $\tau_{\le i}$ (resp. $\mathcal{H}^i$). Let $(M, \phi) \in \tilde{C}_\phi$. From the definition we have an exact sequence in $\tilde{C}$ (with morphisms in $\tilde{C}_\sharp$)

$$0 \to \tau_{\le i-1} M \xrightarrow{e} \tau_{\le i} M \xrightarrow{e'} \mathcal{H}^i M[-i] \to 0$$

which is split but the splitting is not necessarily given by morphisms in $\tilde{C}_\sharp$. Thus there exist morphisms

$$\tau_{\le i-1} M \xleftarrow{f} \tau_{\le i} M \xleftarrow{f'} \mathcal{H}^i M[-i]$$

in $\tilde{C}$ such that $e'f' = 1$, $fe = 1$, $f'e' + ef = 1$. Now $f^!, f'^!$ are defined as in 1.1 and, since $e^! = e$, $e'^! = e'$ (notation of 16.1), we have $e'f'^! = 1$, $f^!e = 1$, $f'^!e' + ef^! = 1$ hence, setting $\tilde{f} = (f + f^! + (f^1)^! + \dots)/\mathbf{n}$, $\tilde{f}' = (f' + f'^! + (f'^!)^! + \dots)/\mathbf{n}$ (the last two sums have $\mathbf{n}$ terms) we have $e'\tilde{f}' = 1$, $\tilde{f}e = 1$, $\tilde{f}'e' + e\tilde{f} = 1$ and $\tilde{f}^! = \tilde{f}$, $\tilde{f}'^! = \tilde{f}'$. Thus we obtain a new splitting of the exact sequence above which is given by morphisms in $\tilde{C}_\sharp$. It follows that

$$(\tau_{\le i} M, \phi) \cong (\tau_{\le i-1} M, \phi) \oplus (\mathcal{H}^i M[-i], \phi)$$

in $\tilde{C}_\sharp$ (the maps ϕ are induced by $M^\sharp \to M$). Hence $[\tau_{\le i} M, \phi] = [\tau_{\le i-1} M, \phi] + [\mathcal{H}^i M[-i], \phi]$ in $\mathcal{K}_\sharp(\tilde{C})$. Since $[M, \phi] = [\tau_{\le i} M, \phi]$ for $i\mathbf{g}0$ and $0 = [\tau_{\le i} M, \phi]$ for $-i\mathbf{g}0$ we deduce that $[M, \phi] = \sum_i [\mathcal{H}^i M[-i], \phi]$. This proves the surjectivity of q.

We define $\mathcal{K}(\tilde{C}_\sharp) \to \mathcal{O}' \otimes \mathcal{K}(C_\sharp)$ by $[M, \phi] \mapsto \sum_{n \in \mathbf{Z}} v^{-n} [\mathcal{H}^n M, \phi_n]$ where ϕ_n is induced by ϕ. This clearly induces a homomorphism $q' : \mathcal{K}_\sharp(\tilde{C}) \to \mathcal{A} \otimes \mathcal{K}_\sharp(C)$ which satisfies $q'q = 1$. It follows that q is injective, completing the proof of (a).

Using (a) and 16.4(a) we see that

(b) $\mathcal{K}_\sharp(\tilde{C})$ is a free $\mathcal{O}'$-module with basis $\{[B_x, \phi_x]; x \in W\}$, (notation of 16.4).

16.6. Let $\mathcal{N}$ be the free $\mathcal{O}'$-module with basis $\{b_x; x \in W\}$. For any $(M,\phi) \in \tilde{C}_\sharp$ and any $y \in W$ we set

$$\epsilon^y(M,\phi) = \sum_{i\in\mathbf{Z}} \epsilon_i^y(M,\phi)v^{-i} \in \mathcal{O}'.$$

The homomorphism $\mathcal{K}(\tilde{C}_\sharp) \to \mathcal{N}$,

$$[M,\phi] \mapsto \sum_{y\in W} \epsilon^y(M,\phi)v^{\tilde{l}(y)}b_y,$$

clearly factors through an $\mathcal{O}'$-module homomorphism

(a) $$\chi' : \mathcal{K}_\sharp(\tilde{C}) \to \mathcal{N}.$$

We show:

(b) χ' is an isomorphism.

For $x \in W$ let $\tilde{b}_x = \chi'([B_x, \phi_x])$. We can write $\tilde{b}_x = \sum_{y\in W} f_{y,x}b_y$ where $f_{y,x} \in \mathcal{O}'$ are zero for all but finitely many y. In view of 16.5(b), to prove (b) it is enough to show:

(c) *Let $y \in W$. If $y \not\leq x$ then $f_{y,x} = 0$. If $y \leq x$ then $f_{y,x} = \dot{p}_{y,x}(v)$ where $\dot{p}_{x,x} = 1$ and $\dot{p}_{y,x} \in v^{-1}\mathcal{O}[v^{-1}]$ if $y < x$.*

Assume that $f_{y,x} \neq 0$. Then for some i we have $\epsilon_i^y(B_x,\phi_x) \neq 0$ hence $\underline{R_y^{B_x}} \neq 0$. Using 16.4(b),(c), we deduce that the coefficient of T_y in c_x is nonzero; thus we have $y \leq x$, as required. Next we assume that $y \leq x$. We have $f_{y,x} = \sum_i \epsilon_i^y(B_x,\phi_x)v^{-i+\tilde{l}(y)}$ hence it is enough to show that

$\epsilon_i^y(B_x,\phi_x) \neq 0$ implies $-i + \tilde{l}(y) \leq 0$, with strict inequality unless $x = y$.

Now $\epsilon_i^y(B_x,\phi_x) \neq 0$ implies $\underline{R_y^{B_x}}_i \neq 0$. Hence it is enough to show that

$\underline{R_y^{B_x}}_i \neq 0$ implies $-i + \tilde{l}(y) \leq 0$, with strict inequality unless $x = y$.

By 16.2(b),(c), we have

$$\sum_{j\in\mathbf{Z}} \dim \underline{R_y^{B_x}}_j u^{-j+\tilde{l}(y)} = \tilde{p}_{y,x}(u)$$

and it remains to use that $\tilde{p}_{x,x} = 1$ and $\tilde{p}_{y,x} \in v^{-1}\mathbf{Z}[v^{-1}]$ if $y < x$. This proves (c) hence also (b).

16.7. If $(M,\phi) \in \tilde{C}_\sharp$ and $(M',\phi') \in \tilde{C}_\sharp$ then $(MM', \phi\otimes\phi')$ is again an object of $\tilde{C}_\sharp$. Note that if (M,ϕ) or (M',ϕ') is traceless then $(MM', \phi\otimes\phi')$ is again traceless. It follows easily that $((M,\phi),(M',\phi')) \mapsto (MM', \phi\otimes\phi')$ defines an $\mathcal{O}'$-bilinear map $\mathcal{K}_\sharp(\tilde{C}) \times \mathcal{K}_\sharp(\tilde{C}) \to \mathcal{K}_\sharp(\tilde{C})$ which makes $\mathcal{K}_\sharp(\tilde{C})$ into an associative $\mathcal{O}'$-algebra with 1. (The unit element is $[B_1, \phi_1]$.) Via the isomorphism 16.6(a),(b), we obtain an associative $\mathcal{O}'$-algebra structure (with $1 = b_1$) on $\mathcal{N}$. One can show that the following identities hold in the algebra $\mathcal{N}$:

(a) $$b_w b'_w = b_{ww'} \text{ if } w, w' \in W, l(ww') = l(w) + l(w');$$

(b) $$(b_{w_0^I} - v^{\tilde{l}(w_0^I)})(b_{w_0^I} + v^{-\tilde{l}(w_0^I)}) = 0 \text{ for any } I \in S.$$

Thus $\mathcal{N}$ can be identified with the $\mathcal{O}'$-algebra $\mathcal{O}' \otimes_{\mathcal{A}} \mathcal{H}$ where $\mathcal{H}$ is as in 16.2 in such a way that b_w corresponds to $T_w \in \mathcal{H}$ for any $w \in W$.

16.8. For $M \in \tilde{C}$ let $D(M) \in \tilde{C}$ be the "dual" of M defined as in [So, 5.9]. Now $(M, \phi) \mapsto (D(M), D(\phi)^{-1})$ induces a ring homomorphism$^-$: $\mathcal{K}_\sharp(\tilde{C}) \to \mathcal{K}_\sharp(\tilde{C})$ which maps $[B_x, \phi_x]$ to itself for any $x \in W$ and is semilinear with respect to the ring involution$^-$: $\mathcal{O}' \to \mathcal{O}'$ given by $v^n \mapsto v^{-n}$ and $\zeta \mapsto \zeta^{-1}$ for any $\zeta \in \mathcal{O}$ such that $\zeta^{\mathbf{n}} = 1$. Via the isomorphism 16.6(a),(b), this becomes an $\mathcal{O}'$-semilinear ring homomorphism$^-$: $\mathcal{N} \to \mathcal{N}$ such that $\overline{\tilde{b}_x} = \tilde{b}_x$ for any $x \in W$. From the results in 16.6 for any $I \in S$ we have $\tilde{b}_{w_0^I} = b_{w_0^I} + z_I$ where $z_I \in v^{-1}\mathcal{O}[v^{-1}]$ satisfies $\overline{b_{w_0^I} + z_I} = b_{w_0^I} + z_I$ that is $\overline{b_{w_0^I}} = b_{w_0^I} + z_I - \overline{z_I}$. Applying$^-$to the equation $b_{w_0^I}^2 = (v^n - v^{-n})\beta_{w_0^I} + 1$ where $n = \tilde{l}(w_0^I)$ (see 16.7(b)) we obtain

$$(b_{w_0^I} + z_I - \overline{z_I})^2 = (v^{-n} - v^n)(b_{w_0^I} + z_I - \overline{z_I}) + 1$$

hence

$$(v^n - v^{-n})b_{w_0^I} + 1 + 2(z_I - \overline{z_I})b_{w_0^I} + (z_i - \overline{z_I})^2 = (v^{-n} - v^n)(b_{w_0^I} + z_I - \overline{z_I}) + 1.$$

We deduce that $z_I - \overline{z_I} = v^{-n} - v^n$ that is $\overline{z_I - v^{-n}} = z_I - v^{-n}$. Since $n > 0$ we have $z_I - v^{-n} \in \mathcal{O}[v^{-1}]$ hence $z_I - v^{-n} = 0$. Thus we have

$$\tilde{b}_{w_0^I} = b_{w_0^I} + v^{-\tilde{l}(w_0^I)}.$$

Since this holds for every $I \in S$ we see that under our identification $\mathcal{N} = \mathcal{O}' \otimes_{\mathcal{A}} \mathcal{H}$, $^-$: $\mathcal{N} \to \mathcal{N}$ corresponds to$^-$: $\mathcal{H} \to \mathcal{H}$ (as in 4.1) extended semilinearly to $\mathcal{O}' \otimes_{\mathcal{A}} \mathcal{H}$. For $w \in W$, both $\tilde{b}_w$ and c_w are fixed by$^-$and (by results in 16.6) their difference is in $\sum_{y \in W; y < w} v^{-1}\mathbf{Z}[v^{-1}]T_y$; it follows that $\tilde{b}_w = c_w$. In particular we have $\tilde{b}_w \in \mathcal{H}$.

16.9. Let $x, y \in W$ be such that $y \leq x$ and let $n \in \mathbf{Z}$. We show:

(a) If the coefficient of v^n in $p_{y,x}$ is $\neq 0$ then the coefficient of v^n in $\tilde{p}_{y,x}$ is $\neq 0$;

(b) if the coefficient of v^n in $\tilde{p}_{y,x}$ is 1 then the coefficient of v^n in $p_{y,x}$ is ± 1.

In the setup of (a), the coefficient of v^n in $f_{y,x}$ is $\neq 0$ (notation of 16.6). Hence $\epsilon^y_{\tilde{l}(y)-n}(B_x, \phi_x) \neq 0$ (see 16.6) so that

$$\mathrm{tr}_{\mathbf{C}}(\mathcal{Z}^{B_x}_{y,\phi_x,\tilde{l}(y)-n}, \underline{R^{B_x}_y}_{\tilde{l}(y)-n}) \neq 0$$

(notation of 16.4); in particular we have $\underline{R_y^M}_{\tilde{l}(y)-n} \neq 0$. From 16.2(b) we see that v^n appears in the coefficient of T_y in $\chi(B_x)$ with $\neq$ coefficient; from 16.2(c) we deduce that v^n appears in $\tilde{p}_{y,x}$ with $\neq$ coefficient. This proves (a).

In the setup of (b), using 16.2(b),(c) we see that $\dim \underline{R_y^M}_{\tilde{l}(y)-n} = 1$. Hence (with notation of 16.4), $\mathrm{tr}_{\mathbf{C}}(\mathcal{Z}^{B_x}_{y,\phi_x,\tilde{l}(y)-n}, \underline{R_y^{B_x}}_{\tilde{l}(y)-n})$ is the trace of a linear transformation of finite order of a one dimensional vector space so that it is a root of 1. Thus, $\epsilon^y_{\tilde{l}(y)-n}(B_x, \phi_x)0$ is a root of 1 so that by 16.6, the coefficient of v^n in $f_{x,y}$ is a root of 1 and the coefficient of v^n in $p_{y,x}$ is a root of 1. Since $p_{y,x}$ has integer coefficients, the coefficient of v^n in $p_{y,x}$ is ± 1. This proves (b).

16.10. Let x, y, z in W and let $n \in \mathbf{Z}$. We form $(B_xB_y, \phi_x \otimes \phi_y) \in \tilde{C}_\sharp$. Now $\phi_x \otimes \phi_y$ induces an isomorphism $\psi_n : (\mathcal{H}^n(B_xB_y))^\sharp \to \mathcal{H}^n(B_xB_y))$ so that $(\mathcal{H}^n(B_xB_y), \psi_n) \in \tilde{C}_\sharp$ (notation of 16.5). Let $V^n_{x,y,z} = \mathrm{Hom}_{\mathcal{R}}(B_z, \mathcal{H}^n(B_xB_y))$. We can find a linear isomorphism $\theta : V_n \to V_n$ of finite order such that under the obvious imbedding $V^n_{x,y,z} \otimes B_z \to \mathcal{H}^n(B_xB_y)$, $\theta \otimes \phi_z$ is compatible with ψ_n. From the definitions, the coefficient of v^n in $h_{x,y,z}$ is equal to $\mathrm{tr}(\theta, V^n_{x,y,z})$. We show:

(a) If the coefficient of v^n in $h_{x,y,z}$ is $\neq 0$ then the coefficient of v^n in $\tilde{h}_{x,y,z}$ is $\neq 0$;

(b) if the coefficient of v^n in $\tilde{h}_{x,y,z}$ is 1 then the coefficient of v^n in $h_{x,y,z}$ is ± 1.

In the setup of (a) we have $\mathrm{tr}(\theta, V^n_{x,y,z}) \neq 0$ hence $V^n_{x,y,z} \neq 0$. Thus B_z appears with nonzero multiplicity in $\mathcal{H}^n(B_xB_y)$. From the definitions we see that the coefficient of v^n in $\tilde{h}_{x,y,z}$ is $\neq 0$. Thus (a) holds.

In the setup of (b) we have $\dim V^n_{x,y,z} = 1$. Since $\theta : V^n_{x,y,z} \to V^n_{x,y,z}$ has finite order, it follows that $\mathrm{tr}(\theta, V^n_{x,y,z})$ is a root of 1. Hence the coefficient of v^n in $h_{x,y,z}$ is a root of 1; but that coefficient is an integer hence it is ± 1. Thus (b) holds.

16.11. We show that for x, y, z in W we have

(a) $$v^{-L(z)} h_{x,y,z} \in \mathbf{Z}[v^{-1}].$$

We must show that if $n \in \mathbf{Z}$ and the coefficient of v^n in $h_{x,y,z}$ is $\neq 0$ then $n \leq L(z)$. By 16.10(a), the coefficient of v^n in $\tilde{h}_{x,y,z}$ is $\neq 0$; hence by 15.2(b) (applied to $\tilde{W}, \tilde{l}$) we have $n \leq \tilde{l}(z)$ that is $n \leq L(z)$, as required. Using (a) we see that definition of $\mathbf{a}(z) \in \mathbf{N}$, $\Delta(z) \in \mathbf{N}$ (relative to L) and $\gamma_{x,y,z} \in \mathbf{Z}$ (for x, y, z in W) and $\mathcal{D}$ as in §13 makes sense for W, L even without the assumption that W, L is bounded. We shall denote by $\tilde{\mathbf{a}}(z)$, $\tilde{\Delta}(z)$, $\tilde{\gamma}_{x,y,z}$ (for x, y, z in $\tilde{W}$) and $\tilde{\mathcal{D}}$ the analogous objects defined in terms of $\tilde{W}, \tilde{l}$, see 15.2.

We now make the additional assumption that $\tilde{W}, \tilde{l}$ is bounded. We show:

(b) *W, L is bounded.*

By assumption there exists $N \geq 0$ such that for all x, y, z in $\tilde{W}$ we have $v^{-N}\tilde{h}_{x,y,z} \in \mathbf{Z}[v^{-1}]$. In particular, if $x, y, z \in W$ and n is such that the coefficient of v^n in $\tilde{h}_{x,y,z}$ is nonzero then $n \leq N$; using 16.10(a) we deduce that, if $x, y, z \in W$ and

n is such that the coefficient of v^n in $h_{x,y,z}$ is nonzero then $n \le N$, so that $v^{-N}h_{x,y,z} \in \mathbf{Z}[v^{-1}]$. This proves (b).

Under the assumption that $\tilde{W}, \tilde{l}$ is bounded, the results of §15 are applicable to $\tilde{W}, \tilde{l}$; in this chapter we will show that, under the same assumption, P1-P15 hold for W, L.

Lemma 16.12. *For $z \in W$ we have $\mathbf{a}(z) = \tilde{\mathbf{a}}(z)$ and $\tilde{\Delta}(z) \le \Delta(z)$.*

We can find $x, y \in W$ such that $\pi_{\mathbf{a}(z)}(h_{x,y,z}) \ne 0$. By 16.10(a) we have $\pi_{\mathbf{a}(z)}(\tilde{h}_{x,y,z}) \ne 0$. Hence $\mathbf{a}(z) \le \tilde{\mathbf{a}}(z)$. By P3,P5 for $\tilde{W}$, there is a unique $d \in \tilde{\mathcal{D}}$ such that $\tilde{\gamma}_{z^{-1},z,d} = \pm 1$. The uniqueness of d implies that d is fixed by u. Thus $d \in W$. By P7 for $\tilde{W}$, we have $\tilde{\gamma}_{z,d,z^{-1}} = \pm 1$. Hence $\pi_{\tilde{\mathbf{a}}(z)}(\tilde{h}_{z,d,z}) = \pm 1$. By 16.10(b), we have $\pi_{\tilde{\mathbf{a}}(z)}(h_{z,d,z}) = \pm 1$. Hence $\tilde{\mathbf{a}}(z) \le \mathbf{a}(z)$ so that $\tilde{\mathbf{a}}(z) = \mathbf{a}(z)$.

By definition, we have $\pi_{-\Delta(z)}(p_{1,z}) \ne 0$. Using 16.9(a), we deduce that $\pi_{-\Delta(z)}(\tilde{p}_{1,z}) \ne 0$. Hence $-\Delta(z) \le -\tilde{\Delta}(z)$. The lemma is proved.

Lemma 16.13. *We have $\mathcal{D} = \tilde{\mathcal{D}} \cap W$.*

Let $d \in \mathcal{D}$. We have $\mathbf{a}(d) = \Delta(d)$. Using 16.12, we deduce $\tilde{\mathbf{a}}(d) = \Delta(d)$. By P1 for $\tilde{W}$, we have $\tilde{\mathbf{a}}(d) \le \tilde{\Delta}(d)$. Hence $\Delta(d) \le \tilde{\Delta}(d)$. Using 16.12, we deduce $\Delta(d) = \tilde{\Delta}(d)$ so that $\tilde{\Delta}(d) = \tilde{\mathbf{a}}(d)$ and $d \in \tilde{\mathcal{D}}$.

Conversely, let $d \in \tilde{\mathcal{D}} \cap W$. We have $\tilde{\mathbf{a}}(d) = \tilde{\Delta}(d)$. Using 16.12 we deduce $\mathbf{a}(d) = \tilde{\Delta}(d)$. By P5 for $\tilde{W}$, we have $\pi_{-\tilde{\Delta}(d)}(\tilde{p}_{1,d}) = \pm 1$. Using 16.9(b) we deduce $\pi_{-\tilde{\Delta}(d)}(p_{1,d}) = \pm 1$. Hence $-\tilde{\Delta}(d) \le -\Delta(d)$. Using 16.12 we deduce $\Delta(d) = \tilde{\Delta}(d)$ so that $\Delta(d) = \mathbf{a}(d)$ and $d \in \mathcal{D}$. The lemma is proved.

Lemma 16.14. *(a) Let $x, y, z \in W$ be such that $\gamma_{x,y,z} \ne 0$. Then $\tilde{\gamma}_{x,y,z} \ne 0$.*

(b) Let $x, y, z \in W$ be such that $\tilde{\gamma}_{x,y,z} = \pm 1$. Then $\gamma_{x,y,z} = \pm 1$.

In the setup of (a) we have $\pi_{\mathbf{a}(z^{-1})}(h_{x,y,z^{-1}}) \ne 0$. Using 16.12 we deduce that $\pi_{\tilde{\mathbf{a}}(z^{-1})}(h_{x,y,z^{-1}}) \ne 0$. Using 16.10(a), we deduce that $\pi_{\tilde{\mathbf{a}}(z^{-1})}(\tilde{h}_{x,y,z^{-1}}) \ne 0$. Hence $\tilde{\gamma}_{x,y,z} \ne 0$.

In the setup of (b) we have $\pi_{\tilde{\mathbf{a}}(z^{-1})}(\tilde{h}_{x,y,z^{-1}}) = \pm 1$. Using 16.12, we deduce $\pi_{\mathbf{a}(z^{-1})}(\tilde{h}_{x,y,z^{-1}}) = \pm 1$. Using 16.10(b), we deduce $\pi_{\mathbf{a}(z^{-1})}(h_{x,y,z^{-1}}) = \pm 1$. Hence $\gamma_{x,y,z} = \pm 1$.

16.15. *Proof of P1.* By 16.12 and P1 for $\tilde{W}$, we have $\mathbf{a}(z) = \tilde{\mathbf{a}}(z) \le \tilde{\Delta}(z) \le \Delta(z)$, hence $\mathbf{a}(z) \le \Delta(z)$.

16.16. *Proof of P2.* In the setup of P2, we have (by 16.14) $\tilde{\gamma}_{x,y,d} \ne 0$ and $d \in \tilde{\mathcal{D}}$ (see 16.13). Using P2 for $\tilde{W}$, we deduce $x = y^{-1}$.

16.17. *Proof of P3.* Let $y \in W$. By P3 for $\tilde{W}$, there is a unique $d \in \tilde{\mathcal{D}}$ such that $\tilde{\gamma}_{y^{-1},y,d} \ne 0$. By the uniqueness of d, we have $u(d) = d$ hence $d \in W$. Using P5 for $\tilde{W}$, we see that $\tilde{\gamma}_{y^{-1},y,d} = \pm 1$. Using 16.14, we deduce $\gamma_{y^{-1},y,d} = \pm 1$. Since $d \in \mathcal{D}$ by 16.13, the existence part of P3 is established. Assume now that $d' \in \mathcal{D}$ satisfies

$\gamma_{y^{-1},y,d'} \neq 0$. Using 16.14, we deduce $\tilde{\gamma}_{y^{-1},y,d'} \neq 0$. Since $d' \in \tilde{\mathcal{D}}$ by 16.13, we can use the uniqueness in P3 for $\tilde{W}$ to deduce that $d = d'$. Thus P3 holds for W.

16.18. *Proof of P4.* We may assume that there exists $s \in S$ such that $h_{s,z,z'} \neq 0$ or $h_{z,s,z'} \neq 0$. In the first case, using 16.10(a), we deduce $\tilde{h}_{s,z,z'} \neq 0$. Hence $z' \leq_{\mathcal{L}} z$ (in $\tilde{W}$) and using P4 for $\tilde{W}$, we deduce that $\tilde{\mathbf{a}}(z') \geq \tilde{\mathbf{a}}(z)$. Using now 16.12, we see that $\mathbf{a}(z') \geq \mathbf{a}(z)$. The proof in the second case is entirely similar.

16.19. Now P5 is proved as in 14.5; P6 is proved as in 14.6; P7 is proved as in 14.7; P8 is proved as in 14.8; P12 is proved as in 14.12.

16.20. *Proof of P13.* If $z' \leftarrow_{\mathcal{L}} z$ in W, then there exists $s \in S$ such that $h_{s,z,z'} \neq 0$ hence, by 16.10(a), $\tilde{h}_{s,z,z'} \neq 0$, hence $z' \leq_{\mathcal{L}} z$ in $\tilde{W}$. It follows that

(a) $z' \leq_{\mathcal{L}} z$ (in W) implies $z' \leq_{\mathcal{L}} z$ (in $\tilde{W}$).

Hence

(b) $z' \sim_{\mathcal{L}} z$ (in W) implies $z' \sim_{\mathcal{L}} z$ (in $\tilde{W}$).

Thus any left cell of W is contained in a left cell of $\tilde{W}$.

In the setup of P13, let $\tilde{\Gamma}$ be the left cell of $\tilde{W}$ containing Γ. Let $x \in \Gamma$. By P3 for W, there exists $d \in \mathcal{D}$ such that $\gamma_{x^{-1},x,d} \neq 0$. By P8 for W, we have $x \sim_{\mathcal{L}} d^{-1}$ hence $d^{-1} \in \Gamma$. Using P6 we have $d = d^{-1}$, hence $d \in \Gamma$. It remains to prove the uniqueness of d. Let d', d'' be elements of $\mathcal{D} \cap \Gamma$. We must prove that $d' = d''$. Now d', d'' belong to $\tilde{\Gamma}$ and, by 16.13, are in $\tilde{\mathcal{D}}$. Using P13 for $\tilde{W}$, it follows that $d' = d''$. Thus P13 holds for W.

Lemma 16.21. *Let $x, y \in W$. We have $x \sim_{\mathcal{L}} y$ (in W) if and only if $x \sim_{\mathcal{L}} y$ (in $\tilde{W}$).*

If $x \sim_{\mathcal{L}} y$ (in W) then $x \sim_{\mathcal{L}} y$ (in $\tilde{W}$), by 16.20(b).

Assume now that $x \sim_{\mathcal{L}} y$ (in $\tilde{W}$). Let $d, d' \in \mathcal{D}$ be such that $x \sim_{\mathcal{L}} d$ (in W) and $y \sim_{\mathcal{L}} d'$ (in W); see P13. By the first line of the proof we have $x \sim_{\mathcal{L}} d$ (in $\tilde{W}$) and $y \sim_{\mathcal{L}} d'$ (in $\tilde{W}$). Hence $d \sim_{\mathcal{L}} d'$ (in $\tilde{W}$). Since $d, d' \in \tilde{\mathcal{D}}$, we deduce (using P13 for $\tilde{W}$) that $d = d'$. It follows that $x \sim_{\mathcal{L}} y$ (in W). The lemma is proved.

16.22. *Proof of P9.* We assume that $z' \leq_{\mathcal{L}} z$ (in W) and $\mathbf{a}(z') = \mathbf{a}(z)$. By 16.20(a), it follows that $z' \leq_{\mathcal{L}} z$ (in $\tilde{W}$) and, using 16.12, that $\tilde{\mathbf{a}}(z') = \tilde{\mathbf{a}}(z)$. Using now P9 in $\tilde{W}$, it follows that $z' \sim_{\mathcal{L}} z$ (in $\tilde{W}$). Using 16.21, we deduce that $z' \sim_{\mathcal{L}} z$ (in W).

16.23. Now P10 is proved as in 14.10; P11 is proved as in 14.11; P14 is proved as in 14.14.

16.24. We sketch a proof of P15 in our case.

A refinement of the proof of P15 given in 14.15, 15.7 provides, for any w, y, x, x' in $\tilde{W}$ and any k, an isomorphism of vector spaces

$$\oplus_{j+j'=k} \oplus_{y' \in \tilde{W}} V^{j'}_{w,x',y'} \otimes V^{j}_{x,y',y} \xrightarrow{\sim} \oplus_{j+j'=k} \oplus_{y' \in \tilde{W}} V^{j}_{x,w,y'} \otimes V^{j'}_{y',x',y}.$$

which (assuming that $\tilde{\mathbf{a}}(w) = \tilde{\mathbf{a}}(y)$) restricts to an isomorphism

$$\oplus_{y' \in \tilde{W}} V^{j'}_{w,x',y'} \otimes V^{j}_{x,y',y} \xrightarrow{\sim} \oplus_{y' \in \tilde{W}} V^{j}_{x,w,y'} \otimes V^{j'}_{y',x',y}$$

for any j, j' such that $j + j' = k$.

Assuming now that $w, y, x, x' \in W$, we can take traces of u in both sides; we deduce

$$\sum_{y' \in W} \pi_{j'}(h_{w,x',y'}) \pi_j(h_{x,y',y}) = \sum_{y' \in W} \pi_j(h_{x,w,y'}) \pi_{j'}(h_{y',x',y})$$

(the summands corresponding to $y' \in \tilde{W} - W$ do not contribute to the trace) or equivalently

$$\sum_{y' \in W} h'_{w,x',y'} h_{x,y',y} = \sum_{y' \in W} h_{x,w,y'} h'_{y',x',y},$$

as required.

17. Example: the infinite dihedral case

17.1. In this chapter we preserve the setup of 7.1. We assume that $m = \infty$ and that $L_2 > L_1$. We will show that P1-P15 hold in this case.

Let $\zeta = v^{L_2 - L_1} + v^{L_1 - L_2}$. For $a \in \{1, 2\}$, let $f_a = v^{L_a} + v^{-L_a}$. For $m, n \in \mathbf{Z}$ we define $\delta_{m<n}$ to be 1 if $m < n$ and to be 0 otherwise.

17.2. From 7.5, 7.6 we have for all $k' \in \mathbf{N}$:
$c_1 c_{2_{k'}} = c_{1_{k'+1}}$,
$c_2 c_{1_{k'}} = c_{2_{k'+1}} + \delta_{k'>1} \zeta c_{2_{k'-1}} + \delta_{k'>3} c_{2_{k'-3}}$.

Proposition 17.3. *For $k \geq 0, k' \geq 1$ we have*
(a) $c_{2_{2k+1}} c_{2_{k'}} = f_2 \sum_{u \in [0,k]; 2u \leq k'-1} c_{2_{2k+k'-4u}}$,
(b) $c_{1_{2k+2}} c_{2_{k'}} = f_2 \sum_{u \in [0,k]; 2u \leq k'-1} c_{1_{2k+k'+1-4u}}$.

Assume that $k = 0$. Using 17.2 we have $c_2 c_{2_{k'}} = f_2 c_{2_{k'}}$.

Assume now that $k = 1$. Using 17.2, we have $c_{2_3} = c_2 c_1 c_2 - \zeta c_2$. Using this and 17.2, we have

$$c_{2_3} c_{2_{k'}} = c_2 c_1 c_2 c_{2_{k'}} - \zeta c_2 c_{2_{k'}} = f_2 c_2 c_{1_{k'+1}} - f_2 \zeta c_{2_{k'}}$$
$$= f_2 c_{2_{k'+2}} + f_2 \zeta c_{2_{k'}} + \delta_{k'>2} f_2 c_{2_{k'-2}} - f_2 \zeta c_{2_{k'}} = f_2 c_{2_{k'+2}} + \delta_{k'>2} f_2 c_{2_{k'-2}},$$

as required. We prove the equality in (a) for fixed k', by induction on k. The cases $k = 0, 1$ are already known. If $k = 2$ then using 17.2, we have $c_{2_5} = c_2 c_1 c_{2_3} - \zeta c_{2_3} - c_{2_1}$. Using this, 17.2, and the induction hypothesis, we have

$$c_{2_5} c_{2_{k'}} = c_2 c_1 c_{2_3} c_{2_{k'}} - \zeta c_{2_3} c_{2_{k'}} - c_{2_1} c_{2_{k'}}$$
$$= f_2 c_2 c_1 c_{2_{k'+2}} + \delta_{k'>2} f_2 c_2 c_1 c_{2_{k'-2}} - \zeta f_2 c_{2_{k'+2}} - \delta_{k'>2} \zeta f_2 c_{2_{k'-2}} - f_2 c_{2_{k'}}$$
$$= f_2 c_2 c_{1_{k'+3}} + \delta_{k'>2} f_2 c_2 c_{1_{k'-1}} - \zeta f_2 c_{2_{k'+2}} - \delta_{k'>2} \zeta f_2 c_{2_{k'-2}} - f_2 c_{2_{k'}}$$
$$= f_2 c_{2_{k'+4}} + f_2 \zeta c_{2_{k'+2}} + f_2 c_{2_{k'}} + \delta_{k'>2} f_2 c_{2_{k'}} + \delta_{k'>2} f_2 \zeta c_{2_{k'-2}} + \delta_{k'>4} f_2 c_{2_{k'-4}}$$
$$- \zeta f_2 c_{2_{k'+2}} - \delta_{k'>2} \zeta f_2 c_{2_{k'-2}} - f_2 c_{2_{k'}} = f_2 c_{2_{k'+4}} + \delta_{k'>2} f_2 c_{2_{k'}} + \delta_{k'>4} f_2 c_{2_{k'-4}},$$

as required. A similar argument applies for $k\ge 3$. This proves (a).

(b) is obtained by multiplying both sides of (a) by c_1 on the left. The proposition is proved.

Proposition 17.4. *For $k\ge 0, k'\ge 1$, we have*

(a) $c_{2_{2k+1}}c_{1_{k'}}=\sum_{u\in[0,2k+2]}p_uc_{2_{k'+2k+1-2u}}$,

(b) $c_{1_{2k+2}}c_{1_{k'}}=\sum_{u\in[0,2k+2]}p_uc_{1_{k'+2k+2-2u}}$,

(c) $c_{1^{-1}_{k'}}c_{2_{2k+1}}=\sum_{u\in[0,2k+2]}p_uc_{2^{-1}_{k'+2k+1-2u}}$,

(d) $c_{1^{-1}_{k'}}c_{1^{-1}_{2k+2}}=\sum_{u\in[0,2k+2]}p_uc_{1^{-1}_{k'+2k+1-2u}}$,

(e) $c_{2_{2k+2}}c_{1_{k'}}=\sum_{u\in[0,2k+2]}f_1p_uc_{2_{k'+2k+1-2u}}$,

(f) $c_{1_{2k+3}}c_{1_{k'}}=\sum_{u\in[0,2k+2]}f_1p_uc_{1_{k'+2k+2-2u}}$,

(g) $c_{1_1}c_{1_{k'}}=f_1c_{1_{k'}}$,

where

$p_0=1$, $p_{2k+2}=\delta_{k'>2k+3}$,

$p_u=\delta_{k'>u}\zeta$ *for* $u=1,3,5,\dots,2k+1$,

$p_u=\delta_{k'>u-1}+\delta_{k'>u+1}$ *for* $u=2,4,6,\dots,2k$.

We prove (a). For $k=0$ the equality in (a) is $c_2c_{1_{k'}}=c_{2_{k'+1}}+\delta_{k'>1}\zeta c_{2_{k'-1}}+\delta_{k'>3}c_{2_{k'-3}}$ which is contained in 17.2. Assume now that $k=1$. Using $c_{2_3}=c_2c_1c_2-\zeta c_2$ and 17.2, we have

$$\begin{aligned}
&c_{2_3}c_{1_{k'}}=c_2c_1c_2c_{1_{k'}}-\zeta c_2c_{1_{k'}}=c_2c_1c_{2_{k'+1}}+\delta_{k'>1}\zeta c_2c_1c_{2_{k'-1}}+\delta_{k'>3}c_2c_1c_{2_{k'-3}}\\
&-\zeta c_{2_{k'+1}}-\delta_{k'>1}\zeta^2c_{2_{k'-1}}-\delta_{k'>3}\zeta c_{2_{k'-3}}=c_2c_{1_{k'+2}}+\delta_{k'>1}\zeta c_2c_{1_{k'}}\\
&+\delta_{k'>3}c_2c_{1_{k'-2}}-\zeta c_{2_{k'+1}}-\delta_{k'>1}\zeta^2c_{2_{k'-1}}-\delta_{k'>3}\zeta c_{2_{k'-3}}\\
&=c_{2_{k'+3}}+\zeta c_{2_{k'+1}}+\delta_{k'>1}c_{2_{k'-1}}+\delta_{k'>1}\zeta c_{2_{k'+1}}+\delta_{k'>1}\zeta^2c_{2_{k'-1}}+\delta_{k'>3}\zeta c_{2_{k'-3}}\\
&+\delta_{k'>3}c_{2_{k'-1}}+\delta_{k'>3}\zeta c_{2_{k'-3}}+\delta_{k'>5}c_{2_{k'-5}}-\zeta c_{2_{k'+1}}-\delta_{k'>1}\zeta^2c_{2_{k'-1}}-\delta_{k'>3}\zeta c_{2_{k'-3}}\\
&=c_{2_{k'+3}}+\delta_{k'>1}\zeta c_{2_{k'+1}}+(\delta_{k'>1}+\delta_{k'>3})c_{2_{k'-1}}+\delta_{k'>3}\zeta c_{2_{k'-3}}+\delta_{k'>5}c_{2_{k'-5}},\qquad\blacksquare
\end{aligned}$$

as required.

We prove the equality in (a) for fixed k', by induction on k. The cases $k=0,1$ are already known. Assume now that $k=2$. Using $c_{2_5}=c_2c_1c_{2_3}-\zeta c_{2_3}-c_{2_1}$, 17.2, and the case $k=1$, we have

$$
\begin{aligned}
&c_{2_5}c_{1_{k'}} = c_2c_1c_{2_3}c_{1_{k'}} - \zeta c_{2_3}c_{1_{k'}} - c_{2_1}c_{1_{k'}}\\
&= c_2c_1c_{2_{k'+3}} + \delta_{k'>1}\zeta c_2c_1c_{2_{k'+1}} + (\delta_{k'>1}+\delta_{k'>3})c_2c_1c_{2_{k'-1}} + \delta_{k'>3}\zeta c_2c_1c_{2_{k'-3}}\\
&+ \delta_{k'>5}c_2c_1c_{2_{k'-5}} - \zeta c_{2_{k'+3}} - \delta_{k'>1}\zeta^2 c_{2_{k'+1}} - (\delta_{k'>1}+\delta_{k'>3})\zeta c_{2_{k'-1}}\\
&- \delta_{k'>3}\zeta^2 c_{2_{k'-3}} - \delta_{k'>5}\zeta c_{2_{k'-5}} - c_{2_{k'+1}} - \delta_{k'>1}\zeta c_{2_{k'-1}} - \delta_{k'>3}c_{2_{k'-3}}\\
&= c_2c_{1_{k'+4}} + \delta_{k'>1}\zeta c_2c_{1_{k'+2}} + (\delta_{k'>1}+\delta_{k'>3})c_2c_{1_{k'}} + \delta_{k'>3}\zeta c_2c_{1_{k'-2}}\\
&+ \delta_{k'>5}c_2c_{1_{k'-4}} - \zeta c_{2_{k'+3}} - \delta_{k'>1}\zeta^2 c_{2_{k'+1}} - (\delta_{k'>1}+\delta_{k'>3})\zeta c_{2_{k'-1}}\\
&- \delta_{k'>3}\zeta^2 c_{2_{k'-3}} - \delta_{k'>5}\zeta c_{2_{k'-5}} - c_{2_{k'+1}} - \delta_{k'>1}\zeta c_{2_{k'-1}} - \delta_{k'>3}c_{2_{k'-3}}\\
&= c_{2_{k'+5}} + \zeta c_{2_{k'+3}} + c_{2_{k'+1}} + \delta_{k'>1}\zeta c_{2_{k'+3}} + \delta_{k'>1}\zeta^2 c_{2_{k'+1}} + \delta_{k'>1}\zeta c_{2_{k'-1}}\\
&+ (\delta_{k'>1}+\delta_{k'>3})c_{2_{k'+1}} + (\delta_{k'>1}+\delta_{k'>3})\zeta c_{2_{k'-1}} + 2\delta_{k'>3}c_{2_{k'-3}} + \delta_{k'>3}\zeta c_{2_{k'-1}}\\
&+ \delta_{k'>3}\zeta^2 c_{2_{k'-3}} + \delta_{k'>5}\zeta c_{2_{k'-5}} + \delta_{k'>5}c_{2_{k'-3}} + \delta_{k'>5}\zeta c_{2_{k'-5}} + \delta_{k'>7}c_{2_{k'-7}}\\
&- \zeta c_{2_{k'+3}} - \delta_{k'>1}\zeta^2 c_{2_{k'+1}} - (\delta_{k'>1}+\delta_{k'>3})\zeta c_{2_{k'-1}} - \delta_{k'>3}\zeta^2 c_{2_{k'-3}} - \delta_{k'>5}\zeta c_{2_{k'-5}}\\
&- c_{2_{k'+1}} - \delta_{k'>1}\zeta c_{2_{k'-1}} - \delta_{k'>3}c_{2_{k'-3}}\\
&= c_{2_{k'+5}} + \delta_{k'>1}\zeta c_{2_{k'+3}} + (\delta_{k'>1}+\delta_{k'>3})c_{2_{k'+1}} + \delta_{k'>3}c_{2_{k'-3}} + \delta_{k'>3}\zeta c_{2_{k'-1}}\\
&+ \delta_{k'>5}c_{2_{k'-3}} + \delta_{k'>5}\zeta c_{2_{k'-5}} + \delta_{k'>7}c_{2_{k'-7}}\\
&= c_{2_{k'+5}} + \delta_{k'>1}\zeta c_{2_{k'+3}} + (\delta_{k'>1}+\delta_{k'>3})c_{2_{k'+1}}\\
&+ \delta_{k'>3}\zeta c_{2_{k'-1}} + (\delta_{k'>3}+\delta_{k'>5})c_{2_{k'-3}} + \delta_{k'>5}\zeta c_{2_{k'-5}} + \delta_{k'>7}c_{2_{k'-7}}.
\end{aligned}
$$

A similar argument applies for $k \ge 4$. This proves (a).

(b) is obtained by multiplying both sides of (a) by c_1 on the left; (c),(d) are obtained by applying $h \mapsto h^\flat$ to both sides of (a),(b). We prove (e). We have

$$c_{2_{2k+2}}c_{1_{k'}} = c_{2_{2k+1}}c_1c_{1_{k'}} = f_1c_{2_{2k+1}}c_{1_{k'}}$$

and the last expression can be computed from (a). This proves (e). Similarly, (f) follows from (b); (g) is a special case of 6.6. The proposition is proved.

17.5. From 7.4,7.6 we see that the function $\Delta : W \to \mathbf{N}$ has the following values:
$\Delta(2_{2k}) = kL_1 + kL_2$,
$\Delta(2_{2k+1}) = -kL_1 + (k+1)L_2$,
$\Delta(1_1) = L_1$,
$\Delta(1_{2k+1}) = (k-1)L_1 + kL_2$, if $k \ge 1$,
$\Delta(1_{2k}) = kL_1 + kL_2$.

It follows that P1 holds and that $\mathcal{D}$ consists of $2_0 = 1_0, 2_1, 1_1, 1_3$. Thus, P6 holds.

The formulas in 17.3, 17.4 determine $h_{x,y,z}$ for all x, y, z except when $x = 1$ or $y = 1$, in which case $h_{1,y,z} = \delta_{y,z}$, $h_{x,1,z} = \delta_{x,z}$. From these formulas we see that the triples (x, y, d) with $d \in \mathcal{D}$, $\gamma_{x,y,d} \neq 0$ are:

$(2_{2k+1}, 2_{2k+1}, 2_1)$, $(1_{2k+2}, 2_{2k+2}, 1_3)$, $(1_1, 1_1, 1_1)$,
$(1, 1, 1)$, $(2_{2k+2}, 1_{2k+2}, 2_1)$,$(1_{2k+3}, 1_{2k+3}, 1_3)$,
where $k \geq 0$. This implies that P2,P3 hold. From the results in 8.8 we see that P4,P9,P13 hold. From 14.5 we see that P5 holds. From 14.7 we see that P7 holds. From 14.8 we see that P8 holds. From 14.10 we see that P10 holds. From 14.11 we see that P11 holds. From 14.12 we see that P12 holds. From 14.14 we see that P14 holds.

We now verify P15 in our case. With the notation in 14.15, it is enough to show that, if $a, b \in \{1, 2\}$, $w \in W$, $s_a w > w, w s_b > w$, then

$$(c_a e_w)c'_b - c_a(e_w c'_b) \in \tilde{\mathcal{H}}_{\geq \mathbf{a}(w)+1}.$$

Here $c_a = c_{s_a}, c'_b = c'_{s_b}$. If a or b is 1, then from 17.2 we have $(c_a e_w)c'_b - c_a(e_w c'_b) = 0$. Hence we may assume that $a = b = 2$ and $w = 1_{2k+1}$. Using 17.2 we have

$$\begin{aligned}
&c_2(e_{1_{2k+1}} c'_2) = c_2(e_{1_{2k+2}} + \delta_{k>0})\zeta' e_{1_{2k}} + \delta_{k>1} e_{1_{2k-2}} \\
&= e_{2_{2k+3}} + \zeta e_{2_{2k+1}} + \delta_{k>0} e_{2_{2k-1}} + \delta_{k>0}\zeta' e_{2_{2k+1}} + \delta_{k>0}\zeta\zeta' e_{2_{2k-1}} \\
&+ \delta_{k>1}\zeta' e_{2_{2k-3}} + \delta_{k>1} e_{2_{2k-1}} + \delta_{k>1}\zeta e_{2_{2k-3}} + \delta_{k>2} e_{2_{2k-5}} \\
&= e_{2_{2k+3}} + \zeta e_{2_{2k+1}} + \delta_{k>0}\zeta' e_{2_{2k+1}} + \delta_{k>0} e_{2_{2k-1}} + \delta_{k>1} e_{2_{2k-1}} \\
&+ \delta_{k>0}\zeta\zeta' e_{2_{2k-1}} + \delta_{k>1}(\zeta + \zeta') e_{2_{2k-3}} + \delta_{k>2} e_{2_{2k-5}}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&(c_2 e_{1_{2k+1}})c'_2 = e_{2_{2k+3}} + \zeta' e_{2_{2k+1}} + \delta_{k>0}\zeta e_{2_{2k+1}} + \delta_{k>0}) e_{2_{2k-1}} + \delta_{k>1} e_{2_{2k-1}} \\
&+ \delta_{k>0}\zeta\zeta' e_{2_{2k-1}} + \delta_{k>1}(\zeta + \zeta') e_{2_{2k-3}} + \delta_{k>2} e_{2_{2k-5}}.
\end{aligned}$$

Hence

$$c_2(e_{1_{2k+1}} c'_2) - (c_2 e_{1_{2k+1}})c'_2 = (\zeta - \zeta')(1 - \delta_{k>0}) e_{2_{2k+1}}.$$

If $k > 0$, the right hand side is zero. Thus we may assume that $k = 0$. In this case,

$$c_2(e_{1_1} c'_2) - (c_2 e_{1_1})c'_2 = (\zeta - \zeta') e_{2_1}.$$

We have $\mathbf{a}(1_1) = L_1 < L_2 = \mathbf{a}(2_1)$. This completes the verification of P15 in our case.

18. The ring J

18.1. *In this chapter we assume that W, L is bounded and that P1-P15 in §14 are valid.* In particular the results of this chapter are applicable if we are in the split case (see §15) or more generally in the quasisplit case (see §16) with W, L bounded.

Theorem 18.2. *Assume that W is tame.*
(a) W has only finitely many left cells.
(b) W has only finitely many right cells.
(c) W has only finitely many two-sided cells.
(d) $\mathcal{D}$ is a finite set.

We prove (a). Since $\mathbf{a}(w)$ is bounded above it is enough to show that, for any $a \in \mathbf{N}$, $\mathbf{a}^{-1}(a)$ is a union of finitely many left cells. By P4, $\mathbf{a}^{-1}(a)$ is a union of left cells. Let $\mathcal{H}^1$ be the $\mathbf{Z}$-algebra $\mathbf{Z} \otimes_{\mathcal{A}} \mathcal{H}$ where $\mathbf{Z}$ is regarded as an $\mathcal{A}$-algebra via $v \mapsto 1$. We write c_w instead of $1 \otimes c_w$. For any $a' \geq 0$ let $\mathcal{H}^1_{\geq a'}$ be the subgroup of $\mathcal{H}^1$ spanned by $\{c_w; \mathbf{a}(w) \geq a'\}$ (a two-sided ideal of $\mathcal{H}^1$, by P4). We have a direct sum decomposition

(e)
$$\mathcal{H}^1_{\geq a}/\mathcal{H}^1_{\geq a+1} = \oplus_\Gamma E_\Gamma$$

where Γ runs over the left cells contained in $\mathbf{a}^{-1}(a)$ and E_Γ is generated as a group by the images of $c_w, w \in \Gamma$; these images form a $\mathbf{Z}$-basis of E_Γ. Now $\mathcal{H}^1_{\geq a}/\mathcal{H}^1_{\geq a+1}$ inherits a left $\mathcal{H}^1$-module structure from $\mathcal{H}^1$ and (by P9) each E_Γ is a $\mathcal{H}^1$-submodule. Since W is tame, there exists a finitely generated abelian subgroup W_1 of finite index of W. Now $\mathcal{H}^1 = \mathbf{Z}[W]$ contains $\mathbf{Z}[W_1]$ as a subring. Since $\mathcal{H}^1_{\geq a}/\mathcal{H}^1_{\geq a+1}$ is a subquotient of $\mathcal{H}^1$ (a finitely generated $\mathbf{Z}[W_1]$-module) and $\mathbf{Z}[W_1]$ is a noetherian ring, it follows that $\mathcal{H}^1_{\geq a}/\mathcal{H}^1_{\geq a+1}$ is a finitely generated $\mathbf{Z}[W_1]$-module. Hence in the direct sum decomposition (e) with only non-zero summands, the number of summands must be finite. This proves (a).

Since any right cell is of the form Γ^{-1} where Γ is a left cell, we see that (b) follows from (a). Since any two-sided cell is a union of left cells, we see that (c) follows from (a). From P16 we see that (d) follows from (a). The theorem is proved.

18.3. Let J be the free abelian group with basis $(t_w)_{w \in W}$. We set

$$t_x t_y = \sum_{z \in W} \gamma_{x,y,z^{-1}} t_z.$$

The sum is finite since $\gamma_{x,y,z^{-1}} \neq 0 \implies h_{x,y,z} \neq 0$ and this implies that z runs through a finite set (for fixed x, y). We show that this defines an (associative) ring structure on J. We must check the identity

(a)
$$\sum_z \gamma_{x,y,z^{-1}} \gamma_{z,u,u'^{-1}} = \sum_w \gamma_{y,u,w^{-1}} \gamma_{x,w,u'^{-1}}$$

for any $x, y, u, u' \in W$. From P8,P4 we see that both sides of (a) are 0 unless

(b)
$$\mathbf{a}(x) = \mathbf{a}(y) = \mathbf{a}(u) = \mathbf{a}(u') = a$$

for some $a \in \mathbf{N}$. Hence we may assume that (b) holds. By P8,P4, in the first sum in (a) we may assume that $\mathbf{a}(z) = a$ and in the second sum in (a) we may assume that $\mathbf{a}(w) = a$. The equation $(c_x c_y)c_u = c_x(c_y c_u)$ in $\mathcal{H}$ implies

(c) $$\sum_z h_{x,y,z} h_{z,u,u'} = \sum_w h_{y,u,w} h_{x,w,u'}.$$

If $h_{x,y,z} h_{z,u,u'} \neq 0$ then $u' \leq_{\mathcal{R}} z \leq_{\mathcal{R}} x$ hence, by P4, $\mathbf{a}(u') \geq \mathbf{a}(z) \geq \mathbf{a}(x)$ and $\mathbf{a}(z) = a$. Hence in the first sum in (c) we may assume that $\mathbf{a}(z) = a$. Similarly in the second sum in (c) we may assume that $\mathbf{a}(w) = a$. Taking the coefficient of $v^{2\mathbf{a}(z)}$ in both sides of (c) we find (a).

For any commutative ring A with 1 we set $J_A = A \otimes J$; this is the free A-module with basis $\{t_x; x \in W\}$. It is naturally an A-algebra.

If $\mathcal{D}$ is finite, the algebra J_A has a unit element $\sum_{d \in \mathcal{D}} n_d t_d$. Here $n_d = \pm 1$ is as in 14.1(a), see P5. Let us check that $t_x \sum_d n_d t_d = t_x$ for $x \in W$. This is equivalent to the identity $\sum_d n_d \gamma_{x,d,z^{-1}} = \delta_{z,x}$. By P7 this is equivalent to $\sum_d n_d \gamma_{z^{-1},x,d} = \delta_{z,x}$. This follows from P2,P3,P5. The equality $(\sum_d n_d t_d) t_x = t_x$ is checked in a similar way.

If $\mathcal{D}$ is not necessarily finite, then J_A has only a generalized unit element in the sense the elements $t_d (d \in \mathcal{D})$ of J_A satisfy $t_d t_{d'} = \delta_{d,d'}$ for $d, d' \in \mathcal{D}$ and $\sum_{d,d' \in \mathcal{D}} t_d J_A t_{d'} = J_A$.

For any subset X of W, let J_A^X be the A-submodule of J_A generated by $\{t_x; x \in X\}$. (When $A = \mathbf{Z}$ we write J^X instead of $J_{\mathbf{Z}}^X$.) If $\mathbf{c}$ is a two-sided cell of W, L then, by P8, $J_A^{\mathbf{c}}$ is a subalgebra of J_A and $J_A = \oplus_{\mathbf{c}} J_A^{\mathbf{c}}$ is a direct sum decomposition of J_A as an algebra. If $\mathcal{D}$ is finite then $J_A^{\mathbf{c}}$ has a unit element $\sum_{d \in \mathcal{D} \cap \mathbf{c}} n_d t_d$. Similarly, if Γ is a left cell of W, L then $J_A^{\Gamma \cap \Gamma^{-1}}$ is a subalgebra of J_A with unit element $n_d t_d$ where $d \in \mathcal{D} \cap \Gamma$.

Proposition 18.4. *Assume that we are in the setup of 15.1. Let $x, y \in W$.*

(a) The condition $x \sim_{\mathcal{L}} y$ is equivalent to the condition that $t_x t_{y^{-1}} \neq 0$ and to the condition that, for some u, t_y appears with $\neq 0$ coefficient in $t_u t_x$.

(b) The condition $x \sim_{\mathcal{R}} y$ is equivalent to the condition that $t_{x^{-1}} t_y \neq 0$ and to the condition that, for some u, t_y appears with $\neq 0$ coefficient in $t_x t_u$.

(c) The condition $x \sim_{\mathcal{LR}} y$ is equivalent to the condition that $t_x t_u t_y \neq 0$ for some u and to the condition that, for some u, u', t_y appears with $\neq 0$ coefficient in $t_{u'} t_x t_u$.

Let $J^+ = \sum_z \mathbf{N} t_z$. By 15.1(a) we have $J^+ J^+ \subset J^+$.

We prove (a). The second condition is equivalent to $\gamma_{x,y^{-1},u} \neq 0$ for some u; the third condition is equivalent to $\gamma_{u,x,y^{-1}} \neq 0$ for some u. These conditions are equivalent by P7.

Assume that $\gamma_{x,y^{-1},u} \neq 0$ for some u. Using P8 we deduce that $x \sim_{\mathcal{L}} y$.

Assume now that $x \sim_{\mathcal{L}} y$. Let $d \in \mathcal{D}$ be such that $x \sim_{\mathcal{L}} d$. Then we have also $y \sim_{\mathcal{L}} d$. By P13 we have $\gamma_{x^{-1},x,d} \neq 0$, $\gamma_{y^{-1},y,d} \neq 0$. Hence $\gamma_{x^{-1},x,d} = 1, \gamma_{y^{-1},y,d} =$

1. Hence $t_{x^{-1}}t_x \in t_d + J^+$, $t_{y^{-1}}t_y \in t_d + J^+$. Since $t_d t_d = t_d$, it follows that $t_{x^{-1}}t_x t_{y^{-1}}t_y \in t_d t_d + J^+ = t_d + J^+$. In particular, $t_x t_{y^{-1}} \neq 0$. This proves (a).

The proof of (b) is entirely similar.

We prove (c). Using the associativity of J we see that the third condition on x, y is a transitive relation on W. Hence to prove that the first condition implies the third condition we may assume that either $x \sim_{\mathcal{L}} y$ or $x \sim_{\mathcal{R}} y$, in which case this follows from (a) or (b). The fact that the third condition implies the first condition also follows from (a),(b). Thus the first and third condition are equivalent.

Assume that $t_x t_u t_y \neq 0$ for some u. By (a),(b) we then have $x \sim_{\mathcal{L}} u^{-1}, u^{-1} \sim_{\mathcal{R}} y$. Hence $x \sim_{\mathcal{LR}} y$.

Conversely, assume that $x \sim_{\mathcal{LR}} y$. By P14 we have $x \sim_{\mathcal{LR}} y^{-1}$. By the earlier part of the proof, $t_{y^{-1}}$ appears with $\neq 0$ coefficient in $t_{u'}t_x t_u$ for some u, u'. We have $t_{u'}t_x t_u \in a t_{y^{-1}} + J^+$ where $a > 0$. Hence $t_{u'}t_x t_u t_y \in a t_{y^{-1}}t_y + J^+$. Since $t_{y^{-1}}t_y$ has a coefficient 1 and the other coefficients are ≥ 0, it folows that $t_{u'}t_x t_u t_y \neq 0$. Thus, $t_x t_u t_y \neq 0$. We see that the first and second conditions are equivalent. The proposition is proved.

18.5. Assume now that we are in the setup of 7.1 with $m = \infty$ and $L_2 > L_1$. From the formulas in 17.3,17.4 we can determine the multiplication table of J. We find

$$
\begin{aligned}
&t_{2_{2k+1}} t_{2_{2k'+1}} = \textstyle\sum_{u\in[0,\tilde{k}]} t_{2_{2k+2k'+1-4u}},\\
&t_{1_{2k+3}} t_{1_{2k'+3}} = \textstyle\sum_{u\in[0,\tilde{k}]} t_{1_{2k+2k'+3-4u}},\\
&t_{2_{2k+1}} t_{2_{2k'+2}} = \textstyle\sum_{u\in[0,\tilde{k}]} t_{2_{2k+2k'+2-4u}},\\
&t_{1_{2k+3}} t_{1_{2k'+2}} = \textstyle\sum_{u\in[0,\tilde{k}]} t_{1_{2k+2k'+2-4u}},\\
&t_{2_{2k+2}} t_{1_{2k'+3}} = \textstyle\sum_{u\in[0,\tilde{k}]} t_{2_{2k+2k'+2-4u}},\\
&t_{2_{2k+2}} t_{1_{2k'+2}} = \textstyle\sum_{u\in[0,\tilde{k}]} t_{2_{2k+2k'+1-4u}},\\
&t_{1_{2k+2}} t_{2_{2k'+1}} = \textstyle\sum_{u\in[0,\tilde{k}]} t_{1_{2k+2k'+2-4u}},\\
&t_{1_{2k+2}} t_{2_{2k'+2}} = \textstyle\sum_{u\in[0,\tilde{k}]} t_{1_{2k+2k'+3-4u}},\\
&t_{1_1} t_{1_1} = t_{1_1},\\
&t_1 t_1 = t_1;
\end{aligned}
$$

here $k, k' \geq 0$ and $\tilde{k} = \min(k, k')$. All other products are 0.

Let R be the free abelian group with basis $(b_k)_{k\in\mathbf{N}}$. We regard R as a commutative ring with multiplication

$$b_k b_{k'} = \sum_{u\in[0,\min(k,k')]} b_{k+k'-2u}.$$

Let $J_0 = \sum_{w\in W-\{1,1_1\}} \mathbf{Z}t_w$. The formulas above show that $J = J_0 \oplus \mathbf{Z}t_1 \oplus \mathbf{Z}t_{1_1}$ (direct sum of rings) and that the ring J_0 is isomorphic to the ring of 2×2 matrices with entries in R, via the isomorphism defined by:

$$t_{2_{2k+1}} \mapsto \begin{pmatrix} b_k & 0 \\ 0 & 0 \end{pmatrix}, \quad t_{1_{2k+3}} \mapsto \begin{pmatrix} 0 & 0 \\ 0 & b_k \end{pmatrix}, \quad t_{2_{2k+2}} \mapsto \begin{pmatrix} 0 & b_k \\ 0 & 0 \end{pmatrix}, \quad t_{1_{2k+2}} \mapsto \begin{pmatrix} 0 & 0 \\ b_k & 0 \end{pmatrix}.$$

Note that R is canonically isomorphic to the representation ring of $SL_2(\mathbf{C})$ with its canonical basis consisting of irreducible representations.

18.6. Assume that we are in the setup of 7.1 with $m = \infty$ and $L_2 = L_1$. By methods similar (but simpler) to those of §17 and 18.5, we find

$$t_{2_{2k+1}} t_{2_{2k'+1}} = \sum_{u \in [0, 2\min(k,k')]} t_{2_{2k+2k'+1-2u}}.$$

Let J^1 be the subring of J generated by $t_{2_{2k+1}}, k \in \mathbf{N}$. While, in 18.5, the analogue of J^1 was isomorphic to R as a ring with basis, in the present case, J^1 is canonically isomorphic to R', the subgroup of R generated by b_k with k even. (Note that R' is a subring of R, naturally isomorphic to the representation ring of $PGL_2(\mathbf{C})$.)

18.7. In the setup of 7.1 with $m = 4$ and $L_2 = 2, L_1 = 1$ (a special case of the situation in §16), we have

$$J = \mathbf{Z}t_1 \oplus \mathbf{Z}t_{1_1} \oplus J_0 \oplus \mathbf{Z}t_{2_3} \oplus \mathbf{Z}t_{2_4}$$

(direct sum of rings) where J_0 is the subgroup of J generated by $t_{2_1}, t_{2_2}, t_{1_2}, t_{1_3}$. The ring J_0 is isomorphic to the ring of 2×2 matrices with entries in $\mathbf{Z}$, via the isomorphism defined by:

$$t_{2_1} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad t_{1_3} \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad t_{2_2} \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad t_{1_2} \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Moreover, t_1, t_{1_1}, t_{2_4} are idempotent. On the other hand,

$$t_{2_3} t_{2_3} = -t_{2_3}.$$

Notice the minus sign! (It is a special case of the computation in 7.8.)

18.8. *Until the end of 18.12 we assume that $\mathcal{D}$ is finite.* In the following result, $n_d = \pm 1$ (for $d \in \mathcal{D}$) is as in 14.1(a) (see P.5).

Theorem 18.9. *The $\mathcal{A}$-linear maps $\phi : \mathcal{H} \to J_{\mathcal{A}}$, $\phi' : \mathcal{H} \to J_{\mathcal{A}}$ given by*

$$\phi(c_x^\dagger) = \sum_{z \in W, d \in \mathcal{D}; \mathbf{a}(d) = \mathbf{a}(z)} h_{x,d,z} n_d t_z \quad (x \in W),$$

$$\phi'(c_x^\dagger) = \sum_{z \in W, d \in \mathcal{D}; \mathbf{a}(d) = \mathbf{a}(z)} h_{d,x,z} n_d t_z \quad (x \in W),$$

are homomorphisms of $\mathcal{A}$-algebras with 1.

Note that ϕ' is the composition of the algebra isomorphism $\mathcal{H} \xrightarrow{\sim} \mathcal{H}^{opp}$ given by $c_w \mapsto c_{w^{-1}}$ (see 3.4) with $\phi : \mathcal{H}^{opp} \to (J_{\mathcal{A}})^{opp}$ and with $(J_{\mathcal{A}})^{opp} \xrightarrow{\sim} J_{\mathcal{A}}$ given by $t_w \mapsto t_{w^{-1}}$. Hence it is enough to prove the statement of the theorem concerning ϕ. Consider the equality

(a) $\sum_w h_{x_1,x_2,w}h'_{w,x_3,y} = \sum_w h_{x_1,w,y}h'_{x_2,x_3,w}$
(see P15) with $\mathbf{a}(x_2) = \mathbf{a}(y) = a$. In the left hand side we may assume that $y \le_{\mathcal{R}} w \le_{\mathcal{L}} x_2$ hence (by P4) $\mathbf{a}(y) \ge \mathbf{a}(w) \ge \mathbf{a}(x_2)$, hence $\mathbf{a}(w) = a$. Similarly in the right hand side we may assume that $\mathbf{a}(w) = a$. Picking the coefficient of v'^a in both sides of (a) gives
(b) $\sum_w h_{x_1,x_2,w}\gamma_{w,x_3,y^{-1}} = \sum_w h_{x_1,w,y}\gamma_{x_2,x_3,w^{-1}}$.
Let $x, x' \in W$. The desired identity $\phi(c_x^\dagger c_{x'}^\dagger) = \phi(c_x^\dagger)\phi(c_{x'}^\dagger)$ is equivalent to

$$\sum_{\substack{w\in W, d\in\mathcal{D}\\ \mathbf{a}(d)=a'}} h_{x,x',w}h_{w,d,u}n_d = \sum_{\substack{z,z'\in W, d,d'\in\mathcal{D}\\ \mathbf{a}(d)=\mathbf{a}(z)\\ \mathbf{a}(d')=\mathbf{a}(z')}} h_{x,d,z}h_{x',d',z'}\gamma_{z,z',u^{-1}}n_d n_{d'}$$

for any $u \in W$ such that $\mathbf{a}(u) = a'$. In the right hand we may assume that
$\mathbf{a}(d) = \mathbf{a}(z) = \mathbf{a}(d') = \mathbf{a}(z') = a'$
(by P8,P4). Hence the right hand side can be rewritten (using (b)):

$$\sum_{\substack{z'\in W, d,d'\in\mathcal{D}\\ \mathbf{a}(d)=\mathbf{a}(d')=\mathbf{a}(z')=a'}} h_{x',d',z'} \sum_{z;\mathbf{a}(z)=a'} h_{x,d,z}\gamma_{z,z',u^{-1}}n_d n_{d'}$$
$$= \sum_{\substack{z'\in W, d,d'\in\mathcal{D}\\ \mathbf{a}(d)=\mathbf{a}(d')=\mathbf{a}(z')=a'}} h_{x',d',z'} \sum_{w;\mathbf{a}(w)=a'} h_{x,w,u}\gamma_{d,z',w^{-1}}n_d n_{d'}.$$

By P2,P3,P5, this equals

$$\sum_{z'\in W, d'\in\mathcal{D};\mathbf{a}(d')=\mathbf{a}(z')=a'} h_{x',d',z'}h_{x,z',u}n_{d'}$$

which by the identity $(c_x c_{x'})c_{d'} = c_x(c_{x'}c_{d'})$ equals

$$\sum_{w\in W, d'\in\mathcal{D};\mathbf{a}(d')=a'} h_{x,x',w}h_{w,d',u}n_{d'}.$$

Thus ϕ is compatible with multiplication.

Next we show that ϕ is compatible with the unit elements of the two algebras. An equivalent statement is that for any $z \in W$ such that $\mathbf{a}(z) = a$, the sum $\sum_{d\in\mathcal{D};\mathbf{a}(d)=a} h_{1,d,z}n_d$ equals n_z if $z \in \mathcal{D}$ and is 0 if $z \notin \mathcal{D}$. This is clear since $h_{1,d,z} = \delta_{z,d}$.

18.10. If we identify the $\mathcal{A}$-modules $\mathcal{H}$ and $J_{\mathcal{A}}$ via $c_w^\dagger \mapsto t_w$, the obvious left $J_{\mathcal{A}}$-module structure on $J_{\mathcal{A}}$ becomes the left $J_{\mathcal{A}}$-module structure on $\mathcal{H}$ given by

$$t_x * c_w^\dagger = \sum_{z\in W} \gamma_{x,w,z^{-1}}c_z^\dagger$$

Let $\mathcal{H}_a = \oplus_{w;\mathbf{a}(w)=a}\mathcal{A}c^\dagger_w, \mathcal{H}_{\geq a} = \oplus_{w;\mathbf{a}(w)\geq a}\mathcal{A}c^\dagger_w$. We have $t_x * c^\dagger_w \in \mathcal{H}_{\mathbf{a}(w)}$ for all x, w. We show that for any $h \in \mathcal{H}, w \in W$ we have

(a) $$hc^\dagger_w = \phi(h) * c^\dagger_w \mod \mathcal{H}_{\geq \mathbf{a}(w)+1}.$$

Indeed, we may assume that $h = c^\dagger_x$. Using 18.9(b), we have

$$\begin{aligned}\phi(c^\dagger_x) * c^\dagger_w &= \sum_{\substack{d\in\mathcal{D},z\\ \mathbf{a}(d)=\mathbf{a}(z)}} h_{x,d,z} n_d t_z * c^\dagger_w \\ &= \sum_{\substack{d\in\mathcal{D},z,u\\ \mathbf{a}(d)=\mathbf{a}(z)}} h_{x,d,z}\gamma_{z,w,u^{-1}}\hat{n}_d c^\dagger_u \\ &= \sum_{\substack{d\in\mathcal{D},t,u\\ \mathbf{a}(d)=\mathbf{a}(w)=\mathbf{a}(u)}} h_{x,t,u}\gamma_{d,w,t^{-1}} n_d c^\dagger_u \\ &= \sum_{\substack{u\\ \mathbf{a}(w)=\mathbf{a}(u)}} h_{x,w,u} c^\dagger_u = c^\dagger_x c^\dagger_w \mod \mathcal{H}_{\geq \mathbf{a}(w)+1},\end{aligned}$$

as required.

18.11. Let $\mathcal{A} \to R$ be a ring homomorphism of $\mathcal{A}$ into a commutative ring R with 1. Let $\mathcal{H}_R = R \otimes_{\mathcal{A}} \mathcal{H}$, $\mathcal{H}_{R,\geq a} = R \otimes_{\mathcal{A}} \mathcal{H}_{\geq a}$. Then ϕ extends to a homomorphism of R-algebras $\phi_R : \mathcal{H}_R \to J_R$. The $J_{\mathcal{A}}$-module in 18.10 extends to a J_A-module structure on $\mathcal{H}_R$ denoted again by $*$. From 18.10(a) we deduce

(a) $hc^\dagger_w = \phi_R(h) * c^\dagger_w \mod \mathcal{H}_{R,\geq \mathbf{a}(w)+1}$ for any $h \in \mathcal{H}_R, w \in W$.

Proposition 18.12. *(a) If N is a bound for W, L, then* $(\ker \phi_R)^{N+1} = 0$.

(b) If $R = R_0[v, v^{-1}]$ where R_0 is a commutative ring with 1, v is an indeterminate and $\mathcal{A} \to R$ is the obvious ring homomorphism, then $\ker \phi_R = 0$.

We prove (a). If $h \in \ker \phi_R$ then by 18.11(a), we have $h\mathcal{H}_{R,\geq a} \subset \mathcal{H}_{R,\geq a+1}$ for any $a \geq 0$. Applying this repeatedly, we see that, if $h_1, h_2, \ldots, h_{N+1} \in \mathcal{H}$, we have $h_1 h_2 \ldots h_{N+1} \in \mathcal{H}_{R,\geq N+1} = 0$. This proves (a).

We prove (b). Let $h = \sum_x p_x c^\dagger_x \in \ker \phi_R$ where $p_x \in R$. Assume that $h \neq 0$. Then $p_x \neq 0$ for some x. We can find $a \geq 0$ such that $p_x \neq 0 \implies \mathbf{a}(x) \geq a$ and $X = \{x \in W; p_x \neq 0, \mathbf{a}(x) = a\}$ is non-empty. We can find $b \in \mathbf{Z}$ such that $p_x \in v^b\mathbf{Z}[v^{-1}]$ for all $x \in X$ and such that $X' = \{x \in X; \pi_b(p_x) \neq 0\}$ is non-empty. Let $x_0 \in X'$. We can find $d \in \mathcal{D}$ such that $\gamma_{x_0,d,x_0^{-1}} = \gamma_{x_0^{-1},x_0,d} \neq 0$. We have $hc^\dagger_d = \sum_x p_x c^\dagger_x c^\dagger_d$. If $\mathbf{a}(x) > a$, then $c^\dagger_x c^\dagger_d \in \mathcal{H}_{R,\geq a+1}$. Hence $hc^\dagger_d = \sum_{x\in X} p_x c^\dagger_x c^\dagger_d \mod \mathcal{H}_{R,\geq a+1}$. Since $\phi_R(h) = 0$, from 18.11(a) we have $hc^\dagger_d \in \mod \mathcal{H}_{R,\geq a+1}$. It follows that $\sum_{x\in X} p_x c^\dagger_x c^\dagger_d \in \mathcal{H}_{R,\geq a+1}$. In particular the coefficient of $c^\dagger_{x_0}$ in $\sum_{x\in X} p_x c^\dagger_x c^\dagger_d$ is 0. In other words, $\sum_{x\in X} p_x h_{x,d,x_0} = 0$. The coefficient of v^{a+b} in the last sum is

$$\sum_{x\in X} \pi_b(p_x)\gamma_{x,d,x_0^{-1}} = \pi_b(p_{x_0})\gamma_{x_0,d,x_0^{-1}}$$

and this is on the one hand 0 and on the other hand is non-zero since $\pi_b(p_{x_0}) \neq 0$ and $\gamma_{x_0,d,x_0^{-1}} \neq 0$, by the choice of x_0, d. This contradiction completes the proof.

18.13. We fix a commutative ring A with 1. We will show that, without assuming that $\mathcal{D}$ is finite, J_A can be imbedded naturally in a larger A-algebra which has a unit element.

Let $\tilde{J}_A$ be the set of formal sums $\sum_{w\in W} f(w)t_w$ where $f : W \to A$ is any function. We regard $\tilde{J}_A$ as an A-module in an obvious way. For a function $f : W \to A$, the support of f is $\text{supp}(f) = \{w \in W; f(w) \neq 0\}$. Note that J_A may be identified with the A-submodule of $\tilde{J}_A$ consisting of all $\sum_{w\in W} f(w)t_w$ such that $f : W \to A$ has finite support.

We say that $f : W \to A$ is *left (resp. right) admissible* if $\text{supp}(f)$ has finite intersection with any left (resp. right) cell in W. If $\tilde{f} : W \to A$ is given by $\tilde{f}(z) = f(z^{-1})$ then clearly f is left admissible if and only if $\tilde{f}$ is right admissible.

For two functions $f, f' : W \to A$ we try to define $f'' : W \to A$ by $f''(z) = \sum_{x,y\in W} f(x)f'(y)\gamma_{x,y,z^{-1}}$; the sum may be infinite in general hence may not make sense. We show:

(a) *If both f, f' are left admissible then f'' is well defined and left admissible. If both f, f' are right admissible then f'' is well defined and right admissible.*

Assume first that f, f' are left admissible. To prove the first sentence of (a) it is enough to show that for any $d \in \mathcal{D}$, the set

$$\{(x, y, z) \in W^3; f(x) \neq 0, f'(y) \neq 0, z \sim_{\mathcal{L}} d, \gamma_{x,y,z^{-1}} \neq 0\}$$

is finite. Using P8 we see that this set is contained in

$$U = \{(x, y, z) \in W^3; f(x) \neq 0, f'(y) \neq 0, z \sim_{\mathcal{L}} d, x \sim_{\mathcal{L}} y^{-1}, y \sim_{\mathcal{L}} z, l(z) \leq l(x) + l(y)\}.$$

(Note that if $\gamma_{x,y,z^{-1}} \neq 0$ then $h_{x,y,z} \neq 0$ hence $l(z) \leq l(x)+l(y)$, see 13.1.) Hence it is enough to show that U is finite. Let

$F = \{y \in W; f'(y) \neq 0, y \sim_{\mathcal{L}} d\}$,

$F' = \{x \in W; x \sim_{\mathcal{L}} d', f(x) \neq 0, x \sim_{\mathcal{L}} h^{-1}$ for some $h \in F\}$,

$F'' = \{z \in W; l(z) \leq l(x) + l(y)$ for some $x \in F', y \in F\}$.

Now F is finite since f' is left admissible. Hence F' is finite. It follows that F'' is finite (we use that f is left admissible). Then F'' must be also finite since there are only finitely many elements of fixed length in W. If $(x, y, z) \in U$ then $y \in F$ hence $x \in F'$. We have clearly $z \in F''$. Thus, $U \subset F'' \times F \times F'''$ so that U is finite. This proves the first sentence in (a).

Next we assume that f, f' are right admissible. To prove the second sentence of (a) it is enough to show that for any $d \in \mathcal{D}$, the set

$$\{(x, y, z) \in W^3; f(x) \neq 0, f'(y) \neq 0, z \sim_{\mathcal{R}} d, \gamma_{x,y,z^{-1}} \neq 0\}$$
$$= \{(x, y, z) \in W^3; \tilde{f}(x^{-1}) \neq 0, \tilde{f}'(y^{-1}) \neq 0, z^{-1} \sim_{\mathcal{L}} d, \gamma_{x,y,z^{-1}} \neq 0\}$$

is finite. Since $\tilde{f}, \tilde{f}'$ are left admissible, it is enough to show, by the first part of the proof applied to $\tilde{f}', \tilde{f}$ instead of f, f', that $\gamma_{x,y,z^{-1}} \neq 0$ implies $\gamma_{y^{-1},x^{-1},z} \neq 0$. But the identity $h_{x,y,z} = h_{y^{-1},x^{-1},z^{-1}}$ implies $\gamma_{x,y,z^{-1}} = \gamma_{y^{-1},x^{-1},z}$; this completes the proof of (a).

Let $\grave{J}_A$ (resp. $\acute{J}_A$) be the set of formal sums $\sum_{w\in W} f(w)t_w \in \tilde{J}_A$ where $f : W \to A$ is a left (resp. right) admissible function. Note that $\grave{J}_A$ and $\acute{J}_A$ are A-submodules of $\tilde{A}_J$. We define A-algebra structures on $\grave{J}_A$ and on $\acute{J}_A$ by

$$(\sum_{x\in W} f(x)t_x)(\sum_{y\in W} f'(y)t_y) = \sum_{z\in W} f''(z)t_z$$

with f, f', f'' as in (a). We show:

(b) *These algebra structures are associative.*

We must show that if either each of $f_1, f_2, f_3 : W \to A$ is left admissible or each of $f_1, f_2, f_3 : W \to A$ is right admissible, then for any $u_1 \in W$ we have

$$\sum_{x,y,z,u\in W} f_1(x)f_2(y)f_3(z)\gamma_{x,y,u^{-1}}\gamma_{u,z,u_1^{-1}}$$
$$= \sum_{x,y,z,u'\in W} f_1(x)f_2(y)f_3(z)\gamma_{y,z,u'^{-1}}\gamma_{x,u',u_1^{-1}}.$$

(Both sums are finite by repeated application of (a).) It is enough to show that for any x, y, z, u_1 in W we have

$$\sum_u \gamma_{x,y,u^{-1}}\gamma_{u,z,u_1^{-1}} = \sum_{u'} \gamma_{y,z,u'^{-1}}\gamma_{x,u',u_1^{-1}}$$

This follows from the associativity of J_A.

Note that both algebras $\grave{J}_A$, $\acute{J}_A$ have a unit element, namely $1 = \sum_{d\in\mathcal{D}} n_d t_d$; this is checked, using P2,P3,P5,P7, in the same way as in 18.3.

Let $\check{J}_A = \grave{J}_A \cap \acute{J}_A$. Thus $\check{J}_A$ consists of all formal sums $\sum_{w\in W} f(w)t_w \in \tilde{J}_A$ such that $f : W \to A$ is both left admissible and right admissible. Note that $\check{J}_A$ is an A-subalgebra of both $\grave{J}_A$ and of $\acute{J}_A$ with unit element $1 = \sum_{d\in\mathcal{D}\cap c} n_d t_d$. Note also that J_A is a subalgebra of $\check{J}_A$. Now the map $\sum_{x\in W} f(x)t_x \mapsto \sum_{x\in W} f(x^{-1})t_x$ defines algebra isomorphisms $\grave{J}_A \xrightarrow{\sim} (\acute{J}_A)^{opp}$, $\check{J}_A \xrightarrow{\sim} (\check{J}_A)^{opp}$, $J_A \xrightarrow{\sim} (J_A)^{opp}$ where the upperscript ^{opp} denotes the opposed algebra.

If $\mathcal{D}$ is finite (which is the case if W is tame, see 18.2), we have $\grave{J}_A = \acute{J}_A = \check{J}_A = J_A$.

18.14. We show:

(a) *Let $w \in W$. The function $W \to \mathcal{A}$, $z \mapsto \sum_{d\in\mathcal{D};\mathbf{a}(d)=\mathbf{a}(z)} h_{w,d,z}n_d$ is left admissible. The function $W \to \mathcal{A}$, $z \mapsto \sum_{d\in\mathcal{D};\mathbf{a}(d)=\mathbf{a}(z)} h_{d,w,z}n_d$ is right admissible.*

To prove the first assertion of (a) it is enough to show that for any $d \in \mathcal{D}$, the set $\{z \in W; z \sim_{\mathcal{L}} d, h_{w,d,z} \neq 0\}$ is finite. This set is contained in $\{z \in W; l(z) \leq l(w) + l(d)\}$ (see 13.1) which is clearly finite. This proves the first assertion of (a). The second assertion of (a) is proved in a similar way.

From (a) we see that the $\mathcal{A}$-linear maps $\phi : \mathcal{H} \to {}^{\check{}}J_{\mathcal{A}}$, $\phi' : \mathcal{H} \to J^{\check{}}_{\mathcal{A}}$ given by

(b)
$$\phi(c_x^\dagger) = \sum_{d\in\mathcal{D}, z\in W; \mathbf{a}(z)=\mathbf{a}(d)} h_{x,d,z} n_d t_z,$$

(c)
$$\phi'(c_x^\dagger) = \sum_{z\in c, d\in\mathcal{D}; \mathbf{a}(d)=\mathbf{a}(z)} h_{d,x,z} n_d t_z$$

are well defined. The proof of the following result is essentially the same as that of Theorem 18.9.

(d) *$\phi : \mathcal{H} \to {}^{\check{}}J_{\mathcal{A}}$ and $\phi' : \mathcal{H} \to J^{\check{}}_{\mathcal{A}}$ are homomorphisms of $\mathcal{A}$-algebras with* 1.

Now let $R, \mathcal{A} \to R$ be as in 18.2(b), let $\mathcal{H}_R$ be as in 18.11; we define an R-algebra homomorphism $\phi_R : \mathcal{H}_R \to {}^{\check{}}J_R$ by the same formula as $\phi : \mathcal{H} \to {}^{\check{}}J_R$ in which $h_{x,d,z}$ is viewed as an element of R. A proof similar to that of 18.12(b) shows that

(e) $\ker \phi_R = 0$.

18.15. In the remainder of this chapter we assume that $L = l$ and we fix a two-sided cell $\mathbf{c}$ of W. We shall study a categorical version $C^{\mathbf{c}}$ of the ring $J^{\mathbf{c}}$. To do this, we shall use the theory of Soergel modules as in 16.2. We shall use the notation of 16.2 with $\tilde{W} = W$. Thus, $R, R^{>0}, \mathcal{R}, \tilde{C}, C, B_x (x \in W)$ are defined as in 16.2. For $M, M' \in \mathcal{R}$, $MM' \in \mathcal{R}$, $M^{M'} \in \mathcal{R}$ are defined as in 16.2. If $L \in \tilde{C}$ and $j \in \mathbf{Z}$ we write $L^j \in C$ for what in [EW, 6.2] is denoted by $\mathcal{H}^j(L)$. (The fact that L^j is well defined follows from results of [So] and [EW].) For any subset X of $\mathbf{c}$, let C^X be the full subcategory of C whose objects are isomorphic to finite direct sums of objects of the form $B_x (x \in X)$. For any $L \in C$ there is a unique direct sum decomposition $L = \underline{L} \oplus L'$ where $\underline{L} \in C^{\mathbf{c}}$ and L' is a direct sum of objects of the form $B_x (x \notin \mathbf{c})$. (The uniquenes of this direct sum decomposition follows from the results of [So] and [EW].) For $M \in C^{\mathbf{c}}$ we have $M = \oplus_{z\in\mathbf{c}} E_z^M \otimes B_z$ where E_z^M are well defined finite dimensional $\mathbf{C}$-vector spaces which are 0 for all but finitely many z.

Let a be the value of the $\mathbf{a}$-function on $\mathbf{c}$. By arguments similar to those in [L14] and making use of the results in [EW] we see that for $L, L' \in C^{\mathbf{c}}$ we have $\underline{(LL')^j} = 0$ if $j > a$ and $L, L' \mapsto L\underline{\bullet}L' := \underline{(LL')^a}$ defines a monoidal structure on $C^{\mathbf{c}}$. (For three objects L, L', L'' of $C^{\mathbf{c}}$ we have $(L\underline{\bullet}L')\underline{\bullet}L'' = L\underline{\bullet}(L'\underline{\bullet}L'') = \underline{(LL'L'')^{2a}}$.)

For $x, y \in\in \mathbf{c}$ we have

$$B_x \underline{\bullet} B_y = \oplus_{z\in\mathbf{c}} V_{x,y,z^{-1}} \otimes B_z$$

where $V_{x,y,z^{-1}} = E_z^{B_x\underline{\bullet}B_y}$ are canonically defined $\mathbf{C}$-vector spaces which are 0 for all but finitely many z. Note that

(a)
$$\dim V_{x,y,z^{-1}} = \gamma_{x,y,z^{-1}}.$$

For $x\in\mathbf{c}$ let d_x be the unique element of $\mathcal{D}\cap\mathbf{c}$ such that $x\sim_{\mathcal{L}} d_x$.

If $x\in\mathbf{c}$ and $d=d_{x^{-1}}$, we have canonically

$$(\check{V}_{d,d,d}\otimes B_d)\underline{\bullet}B_x=B_x.$$

(We shall denote the dual space of a $\mathbf{C}$-vector space V by $\check{V}$.) Indeed, by (a), it is enough to show that $\check{V}_{d,d,d}\otimes V_{d,x,x^{-1}}=\mathbf{C}$. From $(B_d\underline{\bullet}B_d)\underline{\bullet}B_x=B_d\underline{\bullet}(B_d\underline{\bullet}B_x)$ we deduce using (a) that $V_{d,d,d}\otimes V_{d,x,x^{-1}}=V_{d,x,x^{-1}}\otimes V_{d,x,x^{-1}}$ where $V_{d,x,x^{-1}},V_{d,d,d}$ are 1-dimensional hence $V_{d,d,d}=V_{d,x,x^{-1}}$ and the desired equality follows. Note also that if $d'\in\mathcal{D}\cap\mathbf{c},d'\neq d_{x^{-1}}$, then $(\check{V}_{d',d',d'}\otimes B_{d'})\underline{\bullet}B_x=0$. Similarly, if $x\in\mathbf{c}$ and $d=d_x$, we have canonically

$$B_x\underline{\bullet}(\check{V}_{d,d,d}\otimes B_d)=B_x.$$

(We use the equality $\check{V}_{d,d,d}\otimes V_{x,d,x^{-1}}=\mathbf{C}$ which follows from $B_x\underline{\bullet}(B_d\underline{\bullet}B_d)=(B_x\underline{\bullet}B_d)\underline{\bullet}B_d$.) Moreover, if $d'\in\mathcal{D}\cap\mathbf{c}$ and $d'\neq d_x$, then $B_x\underline{\bullet}(\check{V}_{d',d',d'}\otimes B_{d'})=0$. Thus, $\oplus_{d\in\mathcal{D}\mathbf{c}}(\check{V}_{d,d,d}\otimes B_d)$ plays the role of a unit object for the monoidal category $C^{\mathbf{c}}$, although it does not belong to $C^{\mathbf{c}}$ (unless $\mathcal{D}\cap\mathbf{c}$ is finite).

From (a) we see that if Γ,Γ',Γ'' are left cells contained in $\mathbf{c}$ and $L\in C^{\Gamma'^{-1}\cap\Gamma}$, $L'\in C^{\Gamma^{-1}\cap\Gamma''}$, then $L\underline{\bullet}L'\in C^{\Gamma'^{-1}\cap\Gamma''}$. In particular $L,L'\mapsto L\underline{\bullet}L'$ defines a monoidal structure on $C^{\Gamma^{-1}\cap\Gamma}$. This monoidal structure admits a unit object, namely $\check{V}_{d,d,d}\otimes B_d$, where $d\in\mathcal{D}\cap\Gamma$.

18.16. We show that for $x,y,z\in\mathbf{c}$ we have canonically

(a) $$V_{x,y,z}=V_{y^{-1},x^{-1},z^{-1}}.$$

For any $M\in\mathcal{R}$ let $M^\sharp$ be the object of $\mathcal{R}$ which is equal to M as a graded $\mathbf{C}$-vector space, but left (resp. right) multiplication by $r\in R$ on $M^\sharp$ equals right (resp. left) multiplication by r on M. In [LV, 3.1] it is shown that $M\in C$ implies $M^\sharp\in C$ and $M\in\tilde{C}$ implies $M^\sharp\in\tilde{C}$; it follows that for $M\in\tilde{C}$ and $j\in\mathbf{Z}$ we have canonically $(M^\sharp)^j=(M^j)^\sharp$. More precisely in *loc.cit.* it is shown that, if $x\in W$, then we have $B_x^\sharp\cong B_{x^{-1}}$. Let $\phi:B_x^\sharp\to B_{x^{-1}}$ be an isomorphism. It is well defined up to multiplication by a number in $\mathbf{C}^*$. We show that there is a canonical choice for it. For any $M\in\mathcal{R}$ we have an obvious isomorphism $(R_x)^M\to(R_x^\sharp)^{M^\sharp}$ (identity map) of $\mathbf{C}$-vector spaces. Despite the fact that this is not necessarily an isomorphism in $\mathcal{R}$, it induces for any i an isomorphism $\underline{(R_x)^M}_i\to\underline{(R_x^\sharp)^{M^\sharp}}_i$ of $\mathbf{C}$-vector spaces. (We use that $R^{>0}(R_x^M)=(R_x^M)R^{>0}$, see [LV, 3.2].) Taking $M=B_x$, $i=l(x)$, we may thus identify $\underline{(R_x)^{B_x}}_{l(x)}\to\underline{(R_x^\sharp)^{B_x^\sharp}}_{l(x)}$ as $\mathbf{C}$-vector spaces. We now identify $R_x^\sharp=R_{x^{-1}}$ as in [LV, 3.2] and we identify $B_x^\sharp$ with $B_{x^{-1}}$ via ϕ. We obtain an identification of $\underline{(R_x)^{B_x}}_{l(x)}$ with $\underline{(R_{x^{-1}})^{B_{x^{-1}}}}_{l(x)}$ that is of $\mathbf{C}$ with $\mathbf{C}$. This is multiplication by some $\lambda\in\mathbf{C}-\{0\}$. By replacing ϕ be a nonzero scalar multiple we can achieve that $\lambda = 1$. This gives a canonical choice for ϕ; we denote it by ϕ_x. We can now identify $B_x^\sharp = B_{x^{-1}}$ via ϕ_x.

For $M, M' \in \tilde{C}$ we have canonically $(MM')^\sharp = (M')^s h M^\sharp$. Hence for $x, y \in \mathbf{c}$ we have canonically

$$\oplus_{z\in\mathbf{c}} V_{y^{-1},x^{-1},z^{-1}} B_z = \underline{(B_{y^{-1}}B_{x^{-1}})^a}$$
$$= \underline{(B_y^\sharp B_x^\sharp)^a} = \underline{((B_x B_y)^\sharp)^a} = \underline{((B_x B_y)^a)^\sharp}$$
$$= \oplus_{z\in\mathbf{c}}(V_{x,y,z}B_{z^{-1}})\sharp = \oplus_{z\in\mathbf{c}} V_{x,y,z}B_z.$$

Now (a) follows.

18.17. In the remainder of this chapter we assume that W, S is an affine Weyl group, see 1.15 and that I is a subset of S such that the group W_I generated by I is finite of maximum possible order. We have $W = W_I\mathcal{T}$, $W_I \cap \mathcal{T} = \{1\}$, where $\mathcal{T}$ is the normal subgroup of W defined in 1.16. Let w_0^I be the longest element of W_I. We assume that $\mathbf{c}$ is the two-sided cell of W containing w_0^I. There is a unique automorphism $w \mapsto w^*$ of W such that $x^* = w_0^I x w_0^I$ for all $x \in W_I$ and $y^* = w_0^I y^{-1} w_0^I$ for all $y \in \mathcal{T}$. This automorphism maps S onto S, I onto I and $\mathbf{c}$ onto $\mathbf{c}$. We shall assume, as we may, that (with notation in [LV, 2.1]), there is an involutive automorphism $e \mapsto e^*$ of the dual space $\check{\mathfrak{h}}$ of the reflection representation $\mathfrak{h}$ of W such that $(we)^* = w^*e^*$ for all $w \in W, e \in \check{\mathfrak{h}}$ and $(\alpha_s)^* = \alpha_{s^*}$ for all $s \in S$.

From the definitions we see that for any $x, y, z \in \mathbf{c}$ we have canonically

(a) $$V_{x^*,y^*,z^*} = V_{x,y,z}.$$

Let $\Gamma = \{w \in W; w$ has maximal length in wW_I. According to [L7, 8.5], Γ is a left cell of W. It is clearly contained in $\mathbf{c}$.

The set $\Gamma^{-1} \cap \Gamma$ is the set of all $w \in W$ such that w has maximal length in $W_I w W_I$. Hence each W_I, W_I double coset in W contains a unique element of $\Gamma^{-1} \cap \Gamma$, see 9.15(e).

By [L17, 8.2], if $w \in W$ has maximal length in its W_I, W_I double coset then $w^* = w^{-1}$. In particular, we have

(b) $$w^* = w^{-1} \text{ for all } w \in \Gamma^{-1} \cap \Gamma.$$

18.18. We show that for x, y, z in $\Gamma^{-1} \cap \Gamma$ we have canonically

(a) $$V_{x,y,z} = V_{y,x,z}.$$

The method of proof has some common features with one in [LX, 3.5]

Using 18.17(a), 18.17(b) and 18.16(a) we see that we have canonically

$$V_{x,y,z} = V_{x^*,y^*,z^*} = V_{x^{-1},y^{-1},z^{-1}} = V_{y,x,z}.$$

This proves (a). From (a) we deduce that for x, y in $\Gamma^{-1} \cap \Gamma$ we have canonically

(b) $$B_x \underline{\bullet} B_y = B_y \underline{\bullet} B_x$$

in $C^{\Gamma^{-1} \cap \Gamma}$. Now let M, M' be two objects of $C^{\Gamma^{-1} \cap \Gamma}$. We show that we have canonically

(c) $$M \underline{\bullet} M' = M' \underline{\bullet} M$$

in $C^{\Gamma^{-1} \cap \Gamma}$. We have canonically

$$M = \oplus_{x \in \Gamma^{-1} \cap \Gamma} E_{M,x} \otimes B_x, \quad M' = \oplus_{y \in \Gamma^{-1} \cap \Gamma} E_{M',y} \otimes B_y.$$

Hence, using (b), we have

$$\begin{aligned} M \underline{\bullet} M' &= \oplus_{x,y \in \Gamma^{-1} \cap \Gamma} (E_{M,x} \otimes E_{M',y}) \otimes B_x \underline{\bullet} B_y \\ &= \oplus_{x,y \in \Gamma^{-1} \cap \Gamma} (E_{M',y} \otimes E_{M,x}) \otimes B_y \underline{\bullet} B_x = M' \underline{\bullet} M. \end{aligned}$$

We see that the monoidal category $C^{\Gamma^{-1} \cap \Gamma}$ has a natural commutativity constraint.

18.19. Let d be the unique element of $\mathcal{D} \cap \Gamma$. We define a contravariant functor $D : C^{\Gamma^{-1} \cap \Gamma} \to C^{\Gamma^{-1} \cap \Gamma}$ by

$$M \mapsto DM = \oplus_{z \in \Gamma^{-1} \cap \Gamma} \check{E}_z^M \otimes \check{V}_{z,z^{-1},d} \otimes \check{V}_{d,d,d} \otimes B_{z^{-1}} \in C^{\Gamma^{-1} \cap \Gamma}.$$

For $M \in C^{\Gamma^{-1} \cap \Gamma}$ we have

$$DDM = \oplus_{z \in \Gamma^{-1} \cap \Gamma} E_z^M \otimes V_{z,z^{-1},d} \otimes V_{d,d,d} \otimes \check{V}_{z^{-1},z,d} \otimes \check{V}_{d,d,d} \otimes B_z = M.$$

Here we have used that $V_{z,z^{-1},d} \otimes V_{d,d,d} \otimes \check{V}_{z^{-1},z,d} \otimes \check{V}_{d,d,d} = \mathbf{C}$ since, by 18.15(a), $V_{d,d,d}$ and $V_{z,z^{-1},d} = V_{z^{-1},z,d}$ are 1-dimensional (note that by 18.17(a) and 18.17(b) we have $V_{z,z^{-1},d} = V_{z,z^*,d} = V_{z^*,z,d^*} = V_{z^{-1},z,d}$).

For $x \in \Gamma^{-1} \cap \Gamma$ we have

$$\begin{aligned} B_x \underline{\bullet} D(B_x) &= \check{V}_{x,x^{-1},d} \otimes \check{V}_{d,d,d} \otimes B_x \underline{\bullet} B_{x^{-1}} \\ &= \oplus_{z \in \Gamma^{-1} \cap \Gamma} \check{V}_{x,x^{-1},d} \otimes \check{V}_{d,d,d} \otimes V_{x,x^{-1},z^{-1}} \otimes B_z \\ &= \check{V}_{x,x^{-1},d} \otimes \check{V}_{d,d,d} \otimes V_{x,x^{-1},d} \otimes B_d \oplus M_1 = M_0 \oplus M_1 \end{aligned}$$

where

$$M_0 = \check{V}_{d,d,d} \otimes B_d,$$

$$M_1 = \oplus_{z \in \Gamma^{-1} \cap \Gamma; z \neq d} \check{V}_{x,x^{-1},d} \otimes \check{V}_{d,d,d} \otimes V_{x,x^{-1},z^{-1}} B_z.$$

(We have again used that $V_{x,x^{-1},d}$ is 1-dimensional.) Thus we have obvious morphisms $M_0 \xrightarrow{j_x} B_x \underline{\bullet} D(B_x) \xrightarrow{j'_x} M_0$ where M_0 is the unit object of $C^{\Gamma^{-1}\cap\Gamma}$. Now let $M \in C^{\Gamma^{-1}\cap\Gamma}$. We have

$$M \underline{\bullet} DM = \oplus_{x,x' \in \Gamma^{-1}\cap\Gamma} (E_x^M \otimes \check{E}_{x'}^M) \otimes B_x \underline{\bullet} DB_{x'}$$

Let $j : M_0 \to M \underline{\bullet} DM$ be the morphism whose x, x' component is 0 if $x \neq x'$ and is $a_x \otimes j_x$ when $x = x'$ (here $a_x : \mathbf{C} \to E_x^M \otimes \check{E}_x^M$ is the obvious imbedding). Let $j' : M \underline{\bullet} DM \to M_0$ be the morphism whose x, x' component is 0 if $x \neq x'$ and is $a'_x \otimes j'_x$ when $x = x'$ (here $a'_x : E_x^M \otimes \check{E}_x^M \to \mathbf{C}$ is the obvious projection). The morphisms $M_0 \xrightarrow{j} M \underline{\bullet} DM \xrightarrow{j'} M_0$ provide a rigid structure for the monoidal category $C^{\Gamma^{-1}\cap\Gamma}$.

18.20. Let G be the simple adjoint group over $\mathbf{C}$ of type dual to that of W, S. Let *RepG* be the tensor category of finite dimensional rational representations of G. Note that the simple objects of *RepG* are naturally indexed by the elements of W which have maximal length in their W_I, W_I double coset, hence they are indexed by $\Gamma^{-1} \cap \Gamma$. For $x \in \Gamma^{-1} \cap \Gamma$ let $\mathcal{V}_x$ be the corresponding simple object of *RepG*. Now [L5, Cor.8.7] can be interpreted as follows:

For any $x, y \in \Gamma^{-1} \cap \Gamma$ *we have* $\mathcal{V}_x \otimes \mathcal{V}_{x'} \cong \oplus_{z \in \Gamma^{-1}\cap\Gamma} \mathcal{V}_z^{\oplus \gamma_{x,y,z^{-1}}}$.

Using 1.1(a) this can be restated as follows:

(a) *For any* $x, y \in \Gamma^{-1} \cap \Gamma$ *we have* $\mathcal{V}_x \otimes \mathcal{V}_{x'} \cong \oplus_{z \in \Gamma^{-1}\cap\Gamma} V_{x,y,z^{-1}} \otimes \mathcal{V}_z$.

Using (a) we see that the rigid symmetric monoidal category $C^{\Gamma^{-1}\cap\Gamma}$ satisfies the assumptions in Deligne's theorem [De, 0.6]: the finite $\otimes$-generation and the property [De, 0.5(i)(b)] follow from the analogous statements for the category *RepG* where they are obvious. We deduce that $C^{\Gamma^{-1}\cap\Gamma}$ is equivalent as a tensor category to the category of representations of some supergroup. One can show that this supergroup is in fact a group isomorphic to G (we omit the proof).

18.21. In this subsection we make no assumption on $\mathbf{c}$. Let Γ be the set of all $x \in \mathbf{c}$ such that x has minimal length in xW_I. According to [LX], Γ is exactly one left cell of W. By methods similar to those in 18.18, 18.19 we see that the monoidal category $C^{\Gamma^{-1}\cap\Gamma}$ is rigid and has a natural commutativity constraint.

19. Algebras with trace form

19.1. Let R be a field and let A be an associative R-algebra with 1 of finite dimension over R. We assume that A is semisimple and split over R and that we are given a *trace form* on A that is, an R-linear map $\tau : A \to R$ such that $(a, a') = \tau(aa') = \tau(a'a)$ is a non-degenerate (symmetric) R-bilinear form $(,) : A \times A \to R$. Note that $(aa', a'') = (a, a'a'')$ for all a, a', a'' in A. Let Mod$A$ be the category whose objects are left A-modules of finite dimension over R. We write $E \in$ Irr$A$ for "E is a simple object of Mod$A$".

Let $(a_i)_{i\in I}$ be an R-basis of A. Define an R-basis $(a'_i)_{i\in I}$ of A by $(a_i, a'_j) = \delta_{ij}$. Then

(a) $\sum_i a_i \otimes a'_i \in A \otimes A$ is independent of the choice of (a_i).

Proposition 19.2. *(a) We have* $\sum_i \tau(a_i)a'_i = 1$.

(b) If $E \in \mathrm{Irr}A$*, then* $\sum_i \mathrm{tr}(a_i, E)a'_i$ *is in the centre of* A*. It acts on* E *as a scalar* $f_E \in R$ *times the identity and on* $E' \in \mathrm{Irr}A$*, not isomorphic to* E*, as zero. Moreover,* f_E *does not depend on the choice of* (a_i)*.*

(c) One can attach uniquely to each $E \in \mathrm{Irr}A$ *a scalar* $g_E \in R$ *(depending only on the isomorphism class of* E*), so that*

$\sum_E g_E \mathrm{tr}(a, E) = \tau(a)$ *for all* $a \in A$*,*

where the sum is taken over all $E \in \mathrm{Irr}A$ *up to isomorphism.*

(d) For any $E \in \mathrm{Irr}A$ *we have* $f_E g_E = 1$*. In particular,* $f_E \neq 0, g_E \neq 0$*.*

(e) If $E, E' \in \mathrm{Irr}A$*, then* $\sum_i \mathrm{tr}(a_i, E)\mathrm{tr}(a'_i, E')$ *is* $f_E \dim E$ *if* E, E' *are isomorphic and is* 0*, otherwise.*

Let $A = \oplus_{n=1}^t A_n$ be the decomposition of A as a sum of simple algebras. Let $\tau_n : A_n \to R$ be the restriction of τ. Then τ_n is a trace form for A_n, whose associated form is the restriction of $(,)$ and $(A_n, A_{n'}) = 0$ for $n \neq n'$. Hence we can choose (a_i) so that each a_i is contained in some A_n and then a'_i will be contained in the same A_n as a_i.

We prove (a). From 19.1(a) we see that $\sum_i \tau(a_i)a'_i$ is independent of the choice of (a_i). Hence we may choose (a_i) as in the first paragraph of the proof. We are thus reduced to the case where A is simple. In that case the assertion is easily verified.

We prove (b). From 19.1(a) we see that $\sum_i \mathrm{tr}(a_i, E)a'_i$ is independent of the choice of (a_i). Hence we may choose (a_i) as in the first paragraph of the proof. We are thus reduced to the case where A is simple. In that case the assertion is easily verified.

We prove (c). It is enough to note that $a \mapsto \mathrm{tr}(a, E)$ form a basis of the space of R-linear functions $A \to R$ which vanish on all $aa' - a'a$ and τ is such a function.

We prove (d). We consider the equation in (c) for $a = a_i$, we multiply both sides by a'_i and sum over i. Using (a), we obtain

$$\sum_i \sum_E g_E \mathrm{tr}(a_i, E)a'_i = \sum_i \tau(a_i)a'_i = 1.$$

Hence $\sum_E g_E \sum_i \mathrm{tr}(a_i, E)a'_i = 1$. By (b), the left hand side acts on a $E' \in \mathrm{Irr}A$ as a scalar $g_{E'} f_{E'}$ times the identity. This proves (d).

(e) follows immediately from (b). The proposition is proved.

19.3. Now let A' be a semisimple subalgebra of A such that τ', the restriction of τ to A' is a trace form of A'. (We do not assume that the unit element $1_{A'}$ of A' coincides to the unit element 1 of A.) If $E \in \mathrm{Mod}A$ then $1_{A'}E$ is naturally an object of $\mathrm{Mod}A'$. Hence if $E' \in \mathrm{Irr}A'$, then the multiplicity $[E' : 1_{A'}E]$ of E' in $1_{A'}E'$ is well defined.

Note that, if $a' \in A'$, then $\mathrm{tr}(a', 1_{A'}E) = \mathrm{tr}(a', E)$.

Lemma 19.4. *Let $E' \in \mathrm{Irr}A'$. We have $g_{E'} = \sum_E [E' : 1_{A'}E] g_E$, sum over all $E \in \mathrm{Irr}A$ (up to isomorphism).*

By the definition of $g_{E'}$, it is enough to show that

(a) $\sum_{E'} \sum_E [E' : 1'E] g_E \mathrm{tr}(a', E') = \tau(a')$

for any $a' \in A'$. Here E' (resp. E) runs over the isomorphism classes of simple objects of $\mathrm{Mod}A'$ (resp. $\mathrm{Mod}A$). The left hand of (a) is

$$\sum_E g_E \sum_{E'} [E' : 1'E] \mathrm{tr}(a', E') = \sum_E g_E \mathrm{tr}(a', 1'E) = \sum_E g_E \mathrm{tr}(a', E) = \tau(a').$$

This completes the proof.

20. The function $\mathbf{a}_E$

20.1. In this chapter we assume that W is finite (hence W, L is automatically bounded) and that P1-P15 are satisfied.

The results of §19 will be applied in the following cases.

(a) $A = \mathcal{H}_{\mathbf{C}}, R = \mathbf{C}$. Here $\mathcal{A} \to \mathbf{C}$ takes v to 1. We identify $\mathcal{H}_{\mathbf{C}}$ with the group algebra $\mathbf{C}[W]$ by $w \mapsto T_w$ for all w. It is well known that $\mathbf{C}[W]$ is a semisimple split algebra. We take τ so that $\tau(x) = \delta_{x,1}$ for $x \in W$. Then the bases (x) and (x^{-1}) are dual with respect to $(,)$.

We will say "W-module" instead of "$\mathbf{C}[W]$-module". We will write $\mathrm{Mod}W, \mathrm{Irr}W$ instead of $\mathrm{Mod}\mathbf{C}[W], \mathrm{Irr}\mathbf{C}[W]$.

(b) $A = J_{\mathbf{C}}$, $R = \mathbf{C}$. Since $\mathbf{C}[W]$ is semisimple, we see from 18.12(a) that the kernel of $\phi_{\mathbf{C}} : \mathbf{C}[W] \to J_{\mathbf{C}}$ is 0 so that $\phi_{\mathbf{C}}$ is injective. Since $\dim \mathbf{C}[W] = \dim J_{\mathbf{C}} = \sharp W$ it follows that $\phi_{\mathbf{C}}$ is an isomorphism. In particular $J_{\mathbf{C}}$ is a semisimple split algebra. We take $\tau : J_{\mathbf{C}} \to \mathbf{C}$ so that $\tau(t_z)$ is n_z if $z \in \mathcal{D}$ and 0, otherwise. Then $(t_x, t_y) = \delta_{xy,1}$. The bases (t_x) and $(t_{x^{-1}})$ are dual with respect to $(,)$.

(c) $A = \mathcal{H}_{\mathbf{C}(v)}, R = \mathbf{C}(v)$. Here $\mathcal{A} \to \mathbf{C}$ takes v to v. The homomorphism $\phi_{\mathbf{C}(v)} : \mathcal{H}_{\mathbf{C}(v)} \to J_{\mathbf{C}(v)}$ is injective. This follows from 18.12(b), using the fact that injectivity is preserved by tensoring with a field of fractions. Since $\mathcal{H}_{\mathbf{C}(v)}, J_{\mathbf{C}(v)}$ have the same dimension, it follows that $\phi_{\mathbf{C}(v)}$ is an isomorphism. Since $J_{\mathbf{C}(v)} = \mathbf{C}(v) \otimes J_{\mathbf{C}}$, and $J_{\mathbf{C}}$ is semisimple, split, it follows that $J_{\mathbf{C}(v)}$ is semisimple, split, hence $\mathcal{H}_{\mathbf{C}(v)}$ is semisimple, split. We take $\tau : \mathcal{H}_{\mathbf{C}(v)} \to \mathbf{C}(v)$ so that $\tau(T_w) = \delta_{w,1}$. The bases (T_x) and $(T_{x^{-1}})$ are dual with respect to $(,)$.

Remark. The argument above shows also that,

(d) if $R = R_0(v)$, with R_0 an arbitrary field and $\mathcal{A} \to R$ carries v to v, then $\phi_R : \mathcal{H}_R \to J_R$ is an isomorphism;

(e) if R in 18.11 is a field of characteristic 0 then $\phi_R : \mathcal{H}_R \to J_R$ is an isomorphism if and only if $\mathcal{H}_R$ is a semisimple R-algebra.

20.2. For any $E \in \mathrm{Mod}W$ we denote by $E_\spadesuit$ the corresponding $J_{\mathbf{C}}$-module. Thus, $E_\spadesuit$ coincides with E as a $\mathbf{C}$-vector space and the action of $j \in J_{\mathbf{C}}$ on $E_\spadesuit$ is the same as the action of $\phi_{\mathbf{C}}^{-1}(j)$ on E. The $J_{\mathbf{C}}$-module structure on $E_\spadesuit$ extends in a natural way to a $J_{\mathbf{C}(v)}$-module structure on $E_v = \mathbf{C}(v)\otimes_{\mathbf{C}} E_\spadesuit$. We will also regard E_v as an $\mathcal{H}_{\mathbf{C}(v)}$-module via the algebra isomorphism $\phi_{\mathbf{C}(v)} : \mathcal{H}_{\mathbf{C}(v)} \xrightarrow{\sim} J_{\mathbf{C}(v)}$. If E is simple, then $E_\spadesuit$ and E_v are simple.

Let $E \in \mathrm{Irr}W$. Then $E_\spadesuit$ is a simple $J^{\mathbf{c}}_{\mathbf{C}}$-module for a unique two-sided cell $\mathbf{c}$ of W. Then for any $x \in \mathbf{c}$, we write $E \sim_{\mathcal{LR}} x$. If $E, E' \in \mathrm{Irr}W$, we write $E \sim_{\mathcal{LR}} E'$ if for some $x \in W$ we have $E \sim_{\mathcal{LR}} x$, $E' \sim_{\mathcal{LR}} x$.

20.3. There is the following direct relationship between E and E_v (without going through J):

$$\mathrm{tr}(x, E) = \mathrm{tr}(T_x, E_v)|_{v=1} \text{ for all } x \in W.$$

Indeed, it is enough to show that $\mathrm{tr}(c_x, E) = \mathrm{tr}(c_x, E_v)|_{v=1}$. Both sides are equal to $\sum_{z\in W, d\in\mathcal{D}} \gamma_{x,d,z^{-1}} \hat{n}_z \mathrm{tr}(t_z, E_\spadesuit)$.

20.4. Assume that $E \in \mathrm{Irr}W$. We have

(a) $(f_{E_v})_{v=1} \dim(E) = \sharp W$.

Indeed, setting $v = 1$ in $\sum_{x\in W} \mathrm{tr}(T_x, E_v)\mathrm{tr}(T_{x^{-1}}, E_v) = f_{E_v} \dim(E)$ gives

$$\sum_{x\in W} \mathrm{tr}(x, E)\mathrm{tr}(x^{-1}, E) = (f_{E_v})_{v=1} \dim(E).$$

The left hand side equals $\sharp W$; (a) follows.

20.5. Let $I \subset S$, let $E' \in \mathrm{Irr}W_I$ and let $E \in \mathrm{Irr}W$. We have

(a) $[E'_v : E_v] = [E' : E]$.

The right hand side is $\sharp(W_I)^{-1} \sum_{x\in W_I} \mathrm{tr}(x, E')\mathrm{tr}(x^{-1}, E)$. The left hand side is

$$f^{-1}_{E'_v} \dim(E')^{-1} \sum_{x\in W_I} \mathrm{tr}(T_x, E'_v)\mathrm{tr}(T_{x^{-1}}, E_v).$$

Since this is a constant, it is equal to its value for $v = 1$. Hence it is equal to

$$(f^{-1}_{E'_v})_{v=1} \dim(E')^{-1} \sum_{x\in W_I} \mathrm{tr}(x, E'_v)\mathrm{tr}(x^{-1}, E_v).$$

Thus it is enough to show that $(f_{E'_v})_{v=1} \dim(E') = \sharp(W_I)$. But this is a special case of 20.4(a).

Proposition 20.6. *Let $E \in \mathrm{Irr}W$.*

(a) There exists a unique integer $\mathbf{a}_E \geq 0$ such that $\mathrm{tr}(T_x, E_v) \in v^{-\mathbf{a}_E}\mathbf{C}[v]$ for all $x \in W$ and $\mathrm{tr}(T_x, E_v) \notin v^{-\mathbf{a}_E+1}\mathbf{C}[v]$ for some $x \in W$.

(b) For $x \in W$ we have $\mathrm{tr}(T_x, E_v) = \mathrm{sgn}(x)v^{-\mathbf{a}_E}\mathrm{tr}(t_x, E_\spadesuit) \mod v^{-\mathbf{a}_E+1}\mathbf{C}[v]$.

(c) Let $\mathbf{c}$ be the two-sided cell such that $E_\spadesuit \in \mathrm{Irr}J^{\mathbf{c}}_{\mathbf{C}}$. Then $\mathbf{a}_E = \mathbf{a}(z)$ for any $z \in \mathbf{c}$.

Let $a = \mathbf{a}(z)$ for any $z \in \mathbf{c}$. Let $x \in W$. By definition,

$$\mathrm{tr}(c^\dagger_x, E_v) = \sum_{z\in W, d\in\mathcal{D}; \mathbf{a}(d)=\mathbf{a}(z)} h_{x,d,z} n_d \mathrm{tr}(t_z, E_\spadesuit).$$

In the last sum we have $\mathrm{tr}(t_z, E_\spadesuit) = 0$ unless $z \in \mathbf{c}$ in which case $\mathbf{a}(z) = a$ and $h_{x,d,z} = \overline{h}_{x,d,z} = \gamma_{x,d,z^{-1}} v^{-a} \mod v^{-a+1}\mathbf{Z}[v]$. Thus we have

$$\mathrm{tr}(c_x^\dagger, E_v) = \sum_{z \in W, d \in \mathcal{D}} \gamma_{x,d,z^{-1}} n_d \mathrm{tr}(t_z, E_\spadesuit) v^{-a} \mod v^{-a+1}\mathbf{C}[v].$$

For each z in the last sum we have $\sum_{d \in \mathcal{D}} \gamma_{x,d,z^{-1}} n_d = \delta_{x,z}$. This gives

(d) $$\mathrm{tr}(c_x^\dagger, E_v) = \mathrm{tr}(t_x, E_\spadesuit) v^{-a} \mod v^{-a+1}\mathbf{C}[v].$$

We have $T_x = \sum_{y; y \le x} q'_{y,x} c_y$ with $q'_{y,x}$ as in 10.1. Hence $\mathrm{sgn}(x)\overline{T}_x = T_x^\dagger = \sum_{y; y \le x} q'_{y,x} c_y^\dagger$. Applying $^-$ gives $\mathrm{sgn}(x) T_x = \sum_{y; y \le x} \overline{q}'_{y,x} c_y^\dagger$. Hence

$$\mathrm{tr}(T_x, E_v) = \mathrm{sgn}(x) \sum_{y; y \le x} \overline{q}'_{y,x} \mathrm{tr}(c_y^\dagger, E_v).$$

Using (d) together with $\overline{q}'_{x,x} = 1$, $\overline{q}'_{y,x} \in v\mathbf{Z}[v]$ for $y < x$ (see 10.1), we deduce

$$\mathrm{tr}(T_x, E_v) = \mathrm{sgn}(x) \mathrm{tr}(t_x, E_\spadesuit) v^{-a} \mod v^{-a+1}\mathbf{C}[v].$$

Since $E_\spadesuit \in \mathrm{Irr} J_{\mathbf{C}}$, we have $\mathrm{tr}(t_x, E_\spadesuit) \ne 0$ for some $x \in W$. The proposition follows.

Corollary 20.7. *$f_{E_v} = f_{E_\spadesuit} v^{-2\mathbf{a}_E}$ + strictly higher powers of v.*

Using 19.2(e) for $\mathcal{H}_{\mathbf{C}(v)}$ and $J_{\mathbf{C}}$, we obtain

$$\begin{aligned} f_{E_v} \dim E &= \sum_x \mathrm{tr}(T_x, E_v) \mathrm{tr}(T_{x^{-1}}, E_v) \\ &\in \sum_x \mathrm{tr}(t_x, E_\spadesuit) \mathrm{tr}(t_{x^{-1}}, E_\spadesuit) v^{-2\mathbf{a}_E} + v^{-2\mathbf{a}_E+1}\mathbf{C}[v] \\ &= f_{E_\spadesuit} \dim E v^{-2\mathbf{a}_E} + v^{-2\mathbf{a}_E+1}\mathbf{C}[v]. \end{aligned}$$

The corollary follows.

Let $^-: \mathbf{C}[v, v^{-1}] \to \mathbf{C}[v, v^{-1}]$ be the $\mathbf{C}$-algebra homomorphism given by $v^n \mapsto v^{-n}$ for all n.

Corollary 20.8. *For any $h \in \mathcal{H}$ we have $\mathrm{tr}(\bar{h}, E_v) = \overline{\mathrm{tr}(h, E_v)}$.*

We can assume that $h = c_x^\dagger$ where $x \in W$. As in the proof of 20.6 we have

$$\mathrm{tr}(c_x^\dagger, E_v) = \sum_{z \in W, d \in \mathcal{D}; \mathbf{a}(d) = \mathbf{a}(z)} h_{x,d,z} n_d \mathrm{tr}(t_z, E_\spadesuit).$$

Hence it suffices to note that $\overline{h_{x,d,z}} = h_{x,d,z}$ for all d, z in the last sum.

For $E\in \text{Mod}W$ we write $E^\dagger$,$E^\dagger_v$, $E^\dagger_\spadesuit$ instead of $E\otimes \text{sgn}$, $(E\otimes\text{sgn})_v$, $(E\otimes\text{sgn})_\spadesuit$.

Lemma 20.9. *Let $E\in \text{Irr}W$. For any $x\in W$ we have*

$$\text{tr}(T_x,E^\dagger_v)=(-1)^{l(x)}\overline{\text{tr}(T_x,E_v)}.$$

There is a unique a $\mathbf{C}(v)$-algebra involution $^\dagger:\mathcal{H}_{\mathbf{C}(v)}\to\mathcal{H}_{\mathbf{C}(v)}$ extending $^\dagger:\mathcal{H}\to\mathcal{H}$ (see 3.5). Let $(E_v)^\dagger$ be the $\mathcal{H}_{\mathbf{C}(v)}$-module with underlying vector space E_v such that the action of $h\in\mathcal{V}_{\mathbf{C}(v)}$ on $(E_v)^\dagger$ is the same as the action of $h^\dagger$ on E_v. Clearly, $(E_v)^\dagger\in\text{Irr}\mathcal{H}_{\mathbf{C}(v)}$. For $x\in W$ we have

$$\text{tr}(T_x,(E_v)^\dagger)=(-1)^{l(x)}\text{tr}(T^{-1}_{x^{-1}},E_v)=(-1)^{l(x)}\overline{\text{tr}(T_x,E_v)}.$$

(The last equation follows from 20.8.) Setting $v=1$ we obtain

$$\text{tr}(T_x,(E_v)^\dagger)|_{v=1}=(-1)^{l(x)}\text{tr}(x,E)=\text{tr}(x,E^\dagger).$$

Using 20.3, we deduce that $(E_v)^\dagger\cong E^\dagger_v$ in $\text{Mod}\mathcal{H}_{\mathbf{C}(v)}$. The lemma follows.

Proposition 20.10. *For any $x\in W$ we have*

$$\text{tr}(T_x,E_v)=\text{tr}(t_x,E^\dagger_\spadesuit)v^{\mathbf{a}_{E^\dagger}}+\textit{ strictly lower powers of } v.$$

By 20.9 and 20.6 we have

$$\text{tr}(T_x,E_v)=\text{sgn}(x)\overline{\text{tr}(T_x,E^\dagger_v)}=\overline{\text{tr}(t_x,E^\dagger_\spadesuit)v^{-\mathbf{a}_{E^\dagger}}+\text{ strictly higher powers of } v}$$
$$=\text{tr}(t_x,E^\dagger_\spadesuit)v^{\mathbf{a}_{E^\dagger}}+\text{ strictly lower powers of } v.$$

The proposition is proved.

Corollary 20.11. $f_{E_v}=f_{E^\dagger_\spadesuit}v^{2\mathbf{a}_{E^\dagger}}+$ *strictly lower powers of v.*

Using 20.10 we have

$$f_{E_v}\dim E=\sum_x\text{tr}(T_x,E_v)\text{tr}(T_{x^{-1}},E_v)$$
$$\in\sum_x\text{tr}(t_x,E^\dagger_\spadesuit)\text{tr}(t_{x^{-1}},E^\dagger_\spadesuit)v^{2\mathbf{a}_{E^\dagger}}+v^{2\mathbf{a}_{E^\dagger}-1}\mathbf{C}[v^{-1}]$$
$$=f_{E^\dagger_\spadesuit}\dim E v^{2\mathbf{a}_{E^\dagger}}+v^{2\mathbf{a}_{E^\dagger}-1}\mathbf{C}[v^{-1}].$$

Lemma 20.12. *Let $E' \in \mathrm{Irr} W_I$. With notation of 19.2, we have*

$$g_{E'_v} = \sum_{E; E \in \mathrm{Irr} W} [E' : E] g_{E_v}.$$

We apply 19.4 with $A = \mathcal{H}_{\mathbf{C}(v)}$ and A' the analogous algebra for W_I instead of W, identified naturally with a subspace of A. (In this case the unit elements of the two algebras are compatible hence $1_{A'} E_v = E_v$.) It remains to use 20.5(a).

Lemma 20.13. *Let $E \in \mathrm{Irr} W$.*
(a) For any $x \in W$, $\mathrm{tr}(t_{x^{-1}}, E_\spadesuit)$ is the complex conjugate of $\mathrm{tr}(t_x, E_\spadesuit)$.
(b) $f_{E_\spadesuit}$ is a strictly positive real number.

We prove (a). Let $\langle , \rangle : E \times E \to \mathbf{C}$ be a positive definite hermitian form. We define $\langle , \rangle' : E_\spadesuit \times E_\spadesuit \to \mathbf{C}$ by $\langle e, e' \rangle' = \sum_{z \in W} \langle t_z e, t_z e' \rangle$. This is again a positive definite hermitian form on $E_\spadesuit$. We show that

$$\langle t_x e, e' \rangle' = \langle e, t_{x^{-1}} e' \rangle'$$

for all e, e'. This is equivalent to

$$\sum_{y,z} \gamma_{z,x,y^{-1}} \langle t_y e, t_z e' \rangle = \sum_{y,z} \gamma_{y,x^{-1},z^{-1}} \langle t_y e, t_z e' \rangle$$

which follows from $\gamma_{z,x,y^{-1}} = \gamma_{y,x^{-1},z^{-1}}$. We see that $t_{x^{-1}}$ is the adjoint of t_x with respect to a positive definite hermitian form. (a) follows.

We prove (b). By 19.2(e) we have $f_{E_\spadesuit} \dim(E) = \sum_x \mathrm{tr}(t_x, E_\spadesuit) \mathrm{tr}(t_{x^{-1}}, E_\spadesuit)$. The right hand side of this equality is a real number ≥ 0, by (a). Hence so is the left hand side. Now $f_{E_\spadesuit} \neq 0$ by 19.2(d) and (b) follows.

Proposition 20.14. *Let $E' \in \mathrm{Irr} W_I$.*
(a) For any $E \in \mathrm{Irr} W$ such that $[E' : E] \neq 0$ we have $\mathbf{a}_{E'} \leq \mathbf{a}_E$.
(b) We have $g_{E'_\spadesuit} = \sum [E' : E] g_{E_\spadesuit}$, sum over all $E \in \mathrm{Irr} W$ (up to isomorphism) such that $\mathbf{a}_E = \mathbf{a}_{E'}$.

Let X be the set of all E (up to isomorphism) such that $[E' : E] \neq 0$ and such that $\mathbf{a}_E$ is minimum, say equal to a. Assume first that $a < \mathbf{a}_{E'}$. Using 19.2(d) we rewrite 20.12 in the form

(c) $v^{-2a} f_{E'_v}^{-1} = \sum_E [E' : E] v^{-2a} f_{E_v}^{-1}$.

By 20.7, we have

(d) $(v^{-2\mathbf{a}_E} f_{E_v}^{-1})|_{v=0} = f_{E_\spadesuit}^{-1}$, $(v^{-2\mathbf{a}_{E'}} f_{E'_v}^{-1})|_{v=0} = f_{E'_\spadesuit}^{-1}$,

hence by setting $v = 0$ in (c) we obtain

$$0 = \sum_{E \in X} [E' : E] f_{E_\spadesuit}^{-1}.$$

The right hand side is a real number > 0 by 20.8(b). This is a contradiction. Thus we must have $a \ge \mathbf{a}_{E'}$ and (a) is proved.

We now rewrite (c) in the form

(e) $v^{-2\mathbf{a}_{E'}} f_{E'_v}^{-1} = \sum_E [E' : E] v^{-2\mathbf{a}_{E'}} f_{E_v}^{-1}$.

Using (d) and (a) we see that, setting $v = 0$ in (e) gives

$$f_{E'_\spadesuit}^{-1} = \sum_{E; \mathbf{a}_E = \mathbf{a}_{E'}} [E' : E] f_{E_\spadesuit}^{-1}.$$

This proves (b).

20.15. Let $K(W)$ be the $\mathbf{C}$-vector space with basis indexed by the $E \in \mathrm{Irr}W$ (up to isomorphism). If $\tilde{E} \in \mathrm{Mod}W$ we identify $\tilde{E}$ with the element $\sum_E [E : \tilde{E}] E \in K(W)$ (E as above). We define a $\mathbf{C}$-linear map $\mathbf{j}_{W_I}^W : K(W_I) \to K(W)$ by

$\mathbf{j}_{W_I}^W(E') = \sum_E [E' : E] E$,

sum over all $E \in \mathrm{Irr}W$ (up to isomorphism) such that $\mathbf{a}_E = \mathbf{a}_{E'}$; here $E' \in \mathrm{Irr}W_I$. We call this *truncated induction*.

Let $I'' \subset I' \subset S$. We show that the following transitivity formula holds:

(a) $$\mathbf{j}_{W_{I'}}^W \mathbf{j}_{W_{I''}}^{W_{I'}} = \mathbf{j}_{W_{I''}}^W : K(W_{I''}) \to K(W).$$

Let $E'' \in \mathrm{Irr}W_{I''}$. We must show that

$$[E'' : E] = \sum_{E'; \mathbf{a}_{E'} = \mathbf{a}_{E''}} [E'' : E'][E' : E]$$

for any $E'' \in \mathrm{Irr}W_{I''}, E \in \mathrm{Irr}W$ such that $\mathbf{a}_{E''} = \mathbf{a}_E$; in the sum we have $E' \in \mathrm{Irr}W_{I'}$. Clearly,

$$[E'' : E] = \sum_{E'} [E'' : E'][E' : E].$$

Hence it is enough to show that, if $[E'' : E'][E' : E] \ne 0$, then we automatically have $\mathbf{a}_{E'} = \mathbf{a}_{E''}$. By 2.10(a) we have $\mathbf{a}_{E''} \le \mathbf{a}_{E'} \le \mathbf{a}_E$. Since $\mathbf{a}_{E''} = \mathbf{a}_E$, the desired conclusion follows.

20.16. For any $x \in W$ we set

$$\gamma_x = \sum_{E; E \in \mathrm{Irr}W} \mathrm{tr}(t_x, E_\spadesuit) E \in K(W).$$

We sometimes write γ_x^W instead of γ_x, to emphasize dependence on W. Note that γ_x is a $\mathbf{C}$-linear combination of E such that $E \sim_{\mathcal{LR}} x$. Hence, if E, E' appear with $\ne 0$ coefficient in γ_x then $E \sim_{\mathcal{LR}} E'$.

Proposition 20.17. *If $x \in W_I$, then $\gamma_x^W = \mathbf{j}_{W_I}^W(\gamma_x^{W_I})$.*

An equivalent statement is

(a) $$\mathrm{tr}(t_x, E_\spadesuit) = \sum_{E';\mathbf{a}_E=\mathbf{a}_{E'}} \mathrm{tr}(t_x, E'_\spadesuit)[E' : E]$$

for any $E \in \mathrm{Irr}W$; in the sum we have $E' \in \mathrm{Irr}W_I$. Clearly, we have

(b) $$v^{\mathbf{a}_E}\mathrm{tr}(T_x, E_v) = \sum_{E';E'\in\mathrm{Irr}W_I} v^{\mathbf{a}_E}\mathrm{tr}(T_x, E'_v)[E' : E].$$

In the right hand side we may assume that $\mathbf{a}_{E'} \le \mathbf{a}_E$. Using this and 20.6, we see that setting $v = 0$ in (b) gives (a). The proposition is proved.

Lemma 20.18. *(a) We have $\mathbf{a}_{\mathrm{sgn}} = L(w_0)$.*
(b) We have $f_{\mathrm{sgn}_\spadesuit} = 1$.
(c) We have $\gamma_{w_0} = \mathrm{sgn}$.

sgn_v is the one dimensional $\mathcal{H}_{\mathbf{C}(v)}$-module on which T_x acts as $\mathrm{sgn}(x)v^{-L(x)}$. (This follows from 20.3.) From 20.6(b) we see that $\mathbf{a}_{\mathrm{sgn}} = L(w_0)$ and that $\mathrm{tr}(t_{w_0}, \mathrm{sgn}_\spadesuit) = 1$. This proves (a). To prove (c) it remains to show that, if $\mathrm{tr}(t_{w_0}, E_\spadesuit) \ne 0$ (E simple) then $E \cong \mathrm{sgn}$. This assumption shows, by 20.6(c), that $E_\spadesuit \in \mathrm{Irr}J^{\mathbf{c}}_{\mathbf{C}}$ where $\mathbf{c}$ is the two-sided cell such that $\mathrm{sgn}_\spadesuit \in \mathrm{Irr}J^{\mathbf{c}}_{\mathbf{C}}$. Since $\mathrm{tr}(t_{w_0}, \mathrm{sgn}_\spadesuit) = 1$, we have $w_0 \in \mathbf{c}$. From 13.8 it follows that $\{w_0\}$ is a two-sided cell. Thus $\mathbf{c} = \{w_0\}$ and $J^{\mathbf{c}}_{\mathbf{C}}$ is one dimensional. Hence it cannot have more than one simple module. Thus, $E \cong \mathrm{sgn}$. This yields (c) and also (b). The lemma is proved.

20.19. Assume that I, I' form a partition of S such that $W = W_I \times W_{I'}$. If $E \in \mathrm{Irr}W_I$ and $E' \in \mathrm{Irr}W_{I'}$, then $E \boxtimes E' \in \mathrm{Irr}W$. From the definitions,

$$\mathbf{a}_{E\boxtimes E'} = \mathbf{a}_E + \mathbf{a}_{E'}, f_{(E\boxtimes E')_\spadesuit} = f_{E_\spadesuit} f_{E'_\spadesuit}.$$

Moreover, if $x \in W_I, x' \in W_{I'}$, then

$$\gamma^W_{xx'} = \gamma^{W_I}_x \boxtimes \gamma^{W_{I'}}_{x'}.$$

20.20. Until the end of 20.23 we assume that w_0 is in the centre of W. Then, for any $E \in \mathrm{Irr}W$, w_0 acts on E as ϵ_E times identity where $\epsilon_E = \pm 1$. Now $E \mapsto \epsilon_E E$ extends to a $\mathbf{C}$-linear involution $\zeta : K(W) \to K(W)$.

Lemma 20.21. *Let $E \in \mathrm{Irr}W$. For any $x \in W$ we have*
$\mathrm{tr}(T_{w_0 x}, E_v) = \epsilon_E v^{-\mathbf{a}_E + \mathbf{a}_{E^\dagger}} \overline{\mathrm{tr}(T_x, E_v)}$.

Since w_0 is in the centre of W, T_{w_0} is in the centre of $\mathcal{H}_{\mathbf{C}(v)}$ hence it acts on E_v as a scalar $\lambda \in \mathbf{C}(v)$ times the identity. Now $\mathrm{tr}(T_x, E_v) \in \mathbf{C}[v, v^{-1}]$ and

$\mathrm{tr}(T_x^{-1}, E_v) \in \mathbf{C}[v, v^{-1}]$. In particular, $\lambda \in \mathbf{C}[v, v^{-1}]$ and $\lambda^{-1} \in \mathbf{C}[v, v^{-1}]$. This implies $\lambda = cv^n$ where $c \in \mathbf{C}$. For $v = 1$, λ becomes ϵ_E. Hence $\lambda = \epsilon_E v^n$ for some n. We have

$$\mathrm{tr}(T_{w_0 x}, E_v) = \mathrm{tr}(T_{w_0} T_{x^{-1}}^{-1}, E_v) = \lambda \mathrm{tr}(T_{x^{-1}}^{-1}, E_v) = \lambda \overline{\mathrm{tr}(T_x, E_v)}.$$

We have

$$\sum_x \mathrm{tr}(T_{w_0 x}, E_v)\mathrm{tr}(T_{x^{-1} w_0}, E_v) = \lambda^2 \sum_x \overline{\mathrm{tr}(T_x, E_v)\mathrm{tr}(T_{x^{-1}}, E_v)}$$

hence $f_{E_v} \dim(E) = \lambda^2 \overline{f_{E_v}} \dim(E)$ so that $f_{E_v} = v^{2n} \overline{f_{E_v}}$. By 20.9, we have

$$\sum_x \overline{\mathrm{tr}(T_x, E_v)\mathrm{tr}(T_{x^{-1}}, E_v)} = \sum_x \mathrm{tr}(T_x, E_v^\dagger)\mathrm{tr}(T_{x^{-1}}, E_v^\dagger)$$

hence $\overline{f_{E_v}} = f_{E_v^\dagger}$. We see that $f_{E_v} = v^{2n} f_{(E^\dagger)_v}$. Comparing the lowest terms we see that $-2\mathbf{a}_E = 2n - 2\mathbf{a}_{E^\dagger}$ hence $n = -\mathbf{a}_E + \mathbf{a}_{E^\dagger}$ and that

(a) $f_{E_\spadesuit} = f_{E_\spadesuit^\dagger}$.

Lemma 20.22. $v^{\mathbf{a}_E} \mathrm{tr}(T_{w_0 x}, E_v) = \epsilon_E (-1)^{l(x)} v^{\mathbf{a}_{E^\dagger}} \mathrm{tr}(T_x, E_v^\dagger)$.

We combine 20.8, 20.21.

Lemma 20.23. *For any $x \in W$ we have $\gamma_{x w_0} = \mathrm{sgn}(x)\zeta(\gamma_x) \otimes \mathrm{sgn}$.*

An equivalent statement is

$$\mathrm{tr}(t_{x w_0}, E_\spadesuit) = \mathrm{sgn}(x)\mathrm{tr}(t_x, E_\spadesuit^\dagger)\epsilon_{E^\dagger}$$

for any $E \in \mathrm{Irr}W$. Setting $v = 0$ in the identity in 20.22 gives

$$\mathrm{sgn}(x w_0)\mathrm{tr}(t_{w_0 x}, E_\spadesuit) = \epsilon_E \mathrm{tr}(t_x, E_\spadesuit^\dagger).$$

It remains to show that $\epsilon_{E^\dagger} = \epsilon_E \mathrm{sgn}(w_0)$. This is clear.

20.24. By the Cayley-Hamilton theorem, any element $r \in J$ satisfies an equation of the form $r^n + a_1 r^{n-1} + \cdots + a_n = 0$ where $a_i \in \mathbf{Z}$. (We use that the structure constants of J are integers.) This holds in particular for $r = t_x$ where $x \in W$. Hence for any $\mathcal{E} \in \mathrm{Irr}J_{\mathbf{C}}$, $\mathrm{tr}(t_x, \mathcal{E})$ is an algebraic integer. If R is a subfield of $\mathbf{C}$ such that the group algebra $R[W]$ is split over R, then J_R is split over R and it follows that for $x, \mathcal{E}$ as above, $\mathrm{tr}(t_x, \mathcal{E})$ is an algebraic integer in R. In particular, if we can take $R = \mathbf{Q}$, then $\mathrm{tr}(t_x, \mathcal{E}) \in \mathbf{Z}$.

21. Study of a left cell

21.1. In this chapter we preserve the setup of 20.1. Let Γ be a left cell of W, L. Let d be the unique element in $\Gamma \cap \mathcal{D}$. The $\mathcal{A}$-submodule $\sum_{y\in\Gamma} \mathcal{A}c_y^\dagger$ of $\mathcal{H}$ can be regarded as an $\mathcal{H}$-module by the rule $c_x^\dagger \cdot c_w^\dagger = \sum_{z\in\Gamma} h_{x,y,z}c_z^\dagger$ with $x \in W, y \in W$. By change of scalars ($v \mapsto 1$) this gives rise to an $\mathcal{H}_{\mathbf{C}} = \mathbf{C}[W]$-module $[\Gamma]$. On the other hand, $J_{\mathbf{C}}^\Gamma = \oplus_{y\in\Gamma}\mathbf{C}t_y$ is a left ideal in $J_{\mathbf{C}}$ by 14.2(P8).

Lemma 21.2. *The* $\mathbf{C}$*-linear isomorphism* $t_y \mapsto c_y^\dagger$ *for* $y \in \Gamma$ *is an isomorphism of* $J_{\mathbf{C}}$*-modules* $J_{\mathbf{C}}^\Gamma \xrightarrow{\sim} [\Gamma]_\spadesuit$.

We have $\Gamma \subset X_a = \{w \in W; \mathbf{a}(x) = a\}$ for some $a \in \mathbf{N}$. The $\mathcal{A}$-submodule $\sum_{y\in X_a} \mathcal{A}c_y^\dagger$ of $\mathcal{H}$ can be regarded as an $\mathcal{H}$-module by the rule

$$c_x^\dagger \cdot c_w^\dagger = \sum_{z\in X_a} h_{x,y,z}c_z^\dagger$$

with $x \in W, y \in W$. By change of scalars ($v \mapsto 1$) this gives rise to an $\mathcal{H}_{\mathbf{C}} = \mathbf{C}[W]$-module $[X_a]$. On the other hand, $J_{\mathbf{C}}^{X_a} = \oplus_{y\in X_a}\mathbf{C}t_y$ is a left (even two-sided) ideal in $J_{\mathbf{C}}$. The $\mathbf{C}$-linear map in the lemma extends by the same formula to a $\mathbf{C}$-linear isomorphism $J_{\mathbf{C}}^{X_a} \xrightarrow{\sim} [X_a]_\spadesuit$. It is enough to show that this is $J_{\mathbf{C}}$-linear. This follows from the computation in 18.10. The lemma is proved.

Lemma 21.3. *Let* $\mathcal{E} \in \mathrm{Irr}J_{\mathbf{C}}$*. The* $\mathbf{C}$*-linear map* $u : \mathrm{Hom}_{J_{\mathbf{C}}}(J_{\mathbf{C}}^\Gamma, \mathcal{E}) \to t_d\mathcal{E}$ *given by* $\xi \mapsto \xi(n_dt_d)$ *is an isomorphism.*

u is well defined since $\xi(n_dt_d) = t_d\xi(t_d) \in t_d\mathcal{E}$. We define a linear map in the opposite direction by $e \mapsto [j \mapsto je]$. It is clear that this is the inverse of u. (We use that $jn_dt_d = j$ for $j \in J_{\mathbf{C}}^\Gamma$.) The lemma is proved.

Proposition 21.4. *We have* $\gamma_d = n_d\sum_E[E : [\Gamma]]E$ *(sum over all* $E \in \mathrm{Irr}W$ *up to isomorphism).*

An equivalent statement is that $\mathrm{tr}(n_dt_d, E_\spadesuit) = [E : [\Gamma]]$, for E as above. By 21.2, we have $[E : [\Gamma]] = [E_\spadesuit : J_{\mathbf{C}}^\Gamma]$. Hence it remains to show that $\mathrm{tr}(n_dt_d, \mathcal{E}) = [\mathcal{E} : J_{\mathbf{C}}^\Gamma]$ for any $\mathcal{E} \in \mathrm{Irr}J_{\mathbf{C}}$. Since $\mathcal{E} = \oplus_{d'\in\mathcal{D}}n_{d'}t_{d'}\mathcal{E}$ and $n_dt_d : \mathcal{E} \to \mathcal{E}$ is the projection to the summand $n_dt_d\mathcal{E}$, we see that $\mathrm{tr}(n_dt_d, \mathcal{E}) = \dim(t_d\mathcal{E})$. It remains to show that $\dim(t_d\mathcal{E}) = [\mathcal{E} : J_{\mathbf{C}}^\Gamma]$. This follows from 21.3.

Proposition 21.5. $[\Gamma]^\dagger, [\Gamma w_0]$ *are isomorphic in* $\mathrm{Mod}W$.

We may identify $[\Gamma]^\dagger$ with the W-module with $\mathbf{C}$-basis $(e_y)_{y\in\Gamma}$ where $s \in S$ acts by $e_y \mapsto -e_y + \sum_{z\in\Gamma} h_{s,y,z}e_z$.

On the other hand we may identify $[\Gamma w_0]$ with the W-module with $\mathbf{C}$-basis $(e'_{yw_0})_{y\in\Gamma}$ where $s \in S$ acts by $e'_{yw_0} \mapsto e'_{yw_0} - \sum_{z\in\Gamma} h_{s,yw_0,zw_0}e'_{zw_0}$.

The W-module dual to $[\Gamma]^\dagger$ has a $\mathbf{C}$-basis $(e''_y)_{y\in\Gamma}$ (dual to (e_y)) in which the action of $s \in S$ is given by $e''_y \mapsto -e''_y + \sum_{z\in\Gamma} h_{s,z,y}e''_z$. We define a $\mathbf{C}$-isomorphism between this last space and $[\Gamma w_0]$ by $e''_y \mapsto \text{sgn}(y)e'_{yw_0}$ for all y. We show that this comutes with the action of W. It suffices to show that for any $s \in S$, we have

(a) $-h_{s,z,y} = \text{sgn}(y)\text{sgn}(z)h_{s,yw_0,zw_0}$ for all $z \neq y$ and

(b) $1 - h_{s,y,y} = -1 + h_{s,yw_0,yw_0}$ for all y.

We use 6.6. Assume first that $sz > z$. If $sy > y$ and $y \neq z$, both sides of (a) are 0. If $sy < y < z$ then (a) follows from 11.6. If $y = sz$ then both sides of (a) are -1. If $sy < y$ but $y \not< z$ or $y \neq sz$ then both sides of (a) are 0.

Assume next that $sz < z$. If $z \neq y$ then both sides of (a) are 0.

If $sy > y$, both sides of (b) are 1. If $sy < y$, both sides of (b) are -1. Thus (a),(b) are verified. Since $[\Gamma]^\dagger$ and its dual are isomorphic in Mod$W$ (they are defined over $\mathbf{Q}$), the lemma follows.

Corollary 21.6. *Let $E \in \text{Irr}W$ and let $\mathbf{c}$ be the two-sided cell of W such that $E_\spadesuit \in \text{Irr}J^{\mathbf{c}}_{\mathbf{C}}$. Then $E^\dagger_\spadesuit \in \text{Irr}J^{\mathbf{c}w_0}_{\mathbf{C}}$.*

Replacing Γ by $\mathbf{c}$ in the definition of $[\Gamma]$ we obtain a W-module $[\mathbf{c}]$. Then 21.2, 21.5 hold with Γ replaced by $\mathbf{c}$ with the same proof. Our assumption implies (by 21.2 for $\mathbf{c}$) that E appears in the W-module $[\mathbf{c}]$. Using 21.5 for $\mathbf{c}$ we deduce that $E^\dagger$ appears in the W-module $[\mathbf{c}w_0]$. Using 21.2 for $\mathbf{c}w_0$, we deduce that $E^\dagger_\spadesuit$ appears in the $J_{\mathbf{C}}$-module $J^{\mathbf{c}w_0}_{\mathbf{C}}$. The corollary follows.

Corollary 21.7. *Let $E, E' \in \text{Irr}W$ be such that $E \sim_{\mathcal{LR}} E'$. Then $E^\dagger \sim_{\mathcal{LR}} E'^\dagger$.*

By assumption there exists a two-sided cell $\mathbf{c}$ such that $E_\spadesuit, E'_\spadesuit \in \text{Irr}J^{\mathbf{c}}_{\mathbf{C}}$. By 21.6, $E^\dagger_\spadesuit, E'^\dagger_\spadesuit \in \text{Irr}J^{\mathbf{c}w_0}_{\mathbf{C}}$. The corollary follows.

21.8. The results of §19 are applicable to A, the $\mathbf{C}$-subspace $J^{\Gamma\cap\Gamma^{-1}}_{\mathbf{C}}$ of $J_{\mathbf{C}}$ spanned by $\Gamma \cap \Gamma^{-1}$ and $R = \mathbf{C}$. This is a $\mathbf{C}$-subalgebra of $J_{\mathbf{C}}$ with unit element $n_d t_d$. In 21.9 we will show that $J^{\Gamma\cap\Gamma^{-1}}_{\mathbf{C}}$ is semisimple. It is then clearly split. We take $\tau : J^{\Gamma\cap\Gamma^{-1}}_{\mathbf{C}} \to \mathbf{C}$ so that $\tau(t_x) = n_d\delta_{x,d}$. (This is the restriction of $\tau : J_{\mathbf{C}} \to \mathbf{C}$.) We have $(t_x, t_y) = \delta_{xy,1}$. The bases (t_x) and $(t_{x^{-1}})$ (where x runs through $\Gamma\cap\Gamma^{-1}$) are dual with respect to $(,)$.

21.9. We show that the $\mathbf{C}$-algebra $J^{\Gamma\cap\Gamma^{-1}}_{\mathbf{C}}$ is semisimple. It is enough to prove the analogous statement for the $\mathbf{Q}$-algebra A', the $\mathbf{Q}$-span of $\Gamma \cap \Gamma^{-1}$ in $J_{\mathbf{Q}}$. We define a $\mathbf{Q}$-bilinear pairing $(|) : A' \times A' \to \mathbf{Q}$ by $(t_x|t_y) = \delta_{x,y}$ for $x, y \in \Gamma \cap \Gamma^{-1}$. Let $j \mapsto \tilde{j}$ be the $\mathbf{Q}$-linear map $A' \to A'$ given by $\tilde{t}_x = t_{x^{-1}}$ for all x. We show that

(a) $$(j_1 j_2|j_3) = (j_2|\tilde{j}_1 j_3)$$

for all j_1, j_2, j_3 in our ring. We may assume that $j_1 = t_x, j_2 = t_y, j_3 = t_z$. Then (a) follows from

$$\gamma_{x,y,z^{-1}} = \gamma_{x^{-1},z,y^{-1}}.$$

Now let I be a left ideal of A'. Let $I^\perp = \{a \in A'; (a|I) = 0\}$. Since $(|)$ is positive definite, we have $A' = I \oplus I^\perp$. From (c) we see that $I^\perp$ is a left ideal. This proves that A' is semisimple.

The same proof could be used to show directly that $J_{\mathbf{C}}$ is semisimple.

Proposition 21.10. *Let $E, E' \in \mathrm{Irr}W$, $N = \sum_{x\in\Gamma\cap\Gamma^{-1}} \mathrm{tr}(t_x, E_\spadesuit)\mathrm{tr}(t_{x^{-1}}, E'_\spadesuit)$. Then $N = f_{E_\spadesuit}[E : [\Gamma]]$ if E, E' are isomorphic and $N = 0$, otherwise.*

If $\mathcal{E} \in \mathrm{Irr}J_{\mathbf{C}}$, then $t_d\mathcal{E}$ is either 0 or in $\mathrm{Irr}J_{\mathbf{C}}^{\Gamma\cap\Gamma^{-1}}$. Moreover, $\mathcal{E} \mapsto t_d\mathcal{E}$ defines a bijection between the set of simple $J_{\mathbf{C}}$-modules (up to isomorphism) which appear in the $J_{\mathbf{C}}$-module $J_{\mathbf{C}}^{\Gamma}$ and the set of simple $J_{\mathbf{C}}^{\Gamma\cap\Gamma^{-1}}$-modules (up to isomorphism). We then have $\dim(t_d\mathcal{E}) = [\mathcal{E} : J_{\mathbf{C}}^{\Gamma}]$. For $j \in J_{\mathbf{C}}^{\Gamma\cap\Gamma^{-1}}$ we have $\mathrm{tr}(j, \mathcal{E}) = \mathrm{tr}(j, t_d\mathcal{E})$. If $t_dE_\spadesuit = 0$ or $t_dE'_\spadesuit = 0$, then $N = 0$ and the result is clear. If $t_dE_\spadesuit \neq 0$ and $t'_dE_\spadesuit \neq 0$ then, by 19.2(e), we see that $N = f_{t_dE_\spadesuit}[E_\spadesuit : J_{\mathbf{C}}^{\Gamma}]$ if E, E' are isomorphic and to 0, otherwise. It remains to show that $f_{t_dE_\spadesuit} = f_{E_\spadesuit}$, $[E : [\Gamma]] = [E_\spadesuit : J_{\mathbf{C}}^{\Gamma}]$ and the analogous equalities for E'. Now $f_{t_dE_\spadesuit} = f_{E_\spadesuit}$ follows from 19.4 applied to $(A', A) = (J_{\mathbf{C}}^{\Gamma\cap\Gamma^{-1}}, J_{\mathbf{C}})$; the equality $[E : [\Gamma]] = [E_\spadesuit : J_{\mathbf{C}}^{\Gamma}]$ follows from 21.2. The proposition is proved.

22. Constructible representations

22.1. In this chapter we preserve the setup of 20.1.

We define a class $\mathrm{Con}(W)$ of W-modules (relative to our fixed $L : W \to \mathbf{N}$) by induction on $\sharp S$. If $\sharp S = 0$ so that $W = \{1\}$, $\mathrm{Con}(W)$ consists of the unit representation. Assume now that $\sharp S > 0$. Then $\mathrm{Con}(W)$ consists of the W-modules of the form $\mathbf{j}_{W_I}^W(E')$ or $\mathbf{j}_{W_I}^W(E') \otimes \mathrm{sgn}$ for various subsets $I \subset S, I \neq S$ and various $E' \in \mathrm{Con}(W_I)$. (If we restrict ourselves to I such that $\sharp(S - I) = 1$ we get the same class of W-modules, by the transitivity of truncated induction.) The W-modules in $\mathrm{Con}(W)$ are said to be the *constructible representations* of W.

Now the unit representation of W is constructible (it is obtained by truncated induction from the unit representation of the subgroup with one element). Hence $\mathrm{sgn} \in \mathrm{Con}(W)$.

Lemma 22.2. *If $E \in \mathrm{Con}(W)$, then there exists a left cell Γ of W such that $E = [\Gamma]$.*

We argue by induction on $\sharp S$. If $\sharp S = 0$ the result is obvious. Assume now that $\sharp S > 0$. Let $E \in \mathrm{Con}(W)$.

Case 1. $E = \mathbf{j}_{W_I}^W(E')$ where $I \subset S, I \neq S$ and $E' \in \mathrm{Con}(W_I)$. By the induction hypothesis there exists a left cell Γ' of W_I such that $E' = [\Gamma']$. Let $d \in \Gamma' \cap \mathcal{D}$. By 21.4 we have $\gamma_d^{W_I} = [\Gamma'] = E'$. By 20.17 we have $E = \mathbf{j}_{W_I}^W(E') = \mathbf{j}_{W_I}^W(\gamma_d^{W_I}) = \gamma_d^W$. Let Γ be the left cell of W that contains d. By 21.4 we have $\gamma_d^W = [\Gamma]$. Hence $E = [\Gamma]$.

Case 2. $E = \mathbf{j}_{W_I}^W(E') \otimes \mathrm{sgn}$ where $I \subset S, I \neq S$ and $E' \in \mathrm{Con}(W_I)$. Then by Case 1, $E \otimes \mathrm{sgn} = [\Gamma]$ for some left cell Γ of W. By 21.5 we have $E = [\Gamma w_0]$. The lemma is proved.

Proposition 22.3. *For any $E \in \mathrm{Irr}W$ there exists a constructible representation of W which contains a simple component isomorphic to E.*

The general case can be easily reduced to the case where W is irreducible. Assume now that W is irreducible. If $L = al$ for some $a > 0$, the constructible representations of W are listed in [L8] and the proposition is easily checked. (See also the discussion of types A, D in 22.5, 22.26.) In the cases where W is irreducible but L is not of the form al, the constructible representations are described later in this chapter and this yields the proposition in all cases.

22.4. Let $W = \mathfrak{S}_n$ be the group of permutations of $1, 2, \dots, n$. We regard W as a Coxeter group with generators

$$s_1 = (1,2), s_2 = (2,3), \dots, s_{n-1} = (n-1, n),$$

(transpositions). We take $L = al$ where $a > 0$.

The simple W-modules (up to isomorphism) are in 1-1 correspondence with the partitions $\alpha = (\alpha_1 \ge \alpha_2 \ge \dots)$ such that $\alpha_N = 0$ for large N and $\sum_i \alpha_i = n$. The correspondence (denoted by $\alpha \mapsto \pi_\alpha$) is defined as follows. Let α be as above, let $(\alpha'_1 \ge \alpha'_2 \ge \dots)$ be the partition dual to α. Let π_α be the simple W-module whose restriction to $\mathfrak{S}_{\alpha_1} \times \mathfrak{S}_{\alpha_2} \dots$ contains 1 and whose restriction to $\mathfrak{S}_{\alpha'_1} \times \mathfrak{S}_{\alpha'_2} \dots$ contains the sign representation. We have (a consequence of results of Steinberg):

$$f_{(\pi_\alpha)_v} = v^{-\sum_i 2\binom{\alpha'_i}{2}} + \text{strictly higher powers of } v.$$

It follows that

(a) $\mathbf{a}_{\pi_\alpha} = \sum_i a\binom{\alpha'_i}{2}$ and $f_{(\pi_\alpha)_\spadesuit} = 1$.

Lemma 22.5. *In the setup of 22.4, a W-module is constructible if and only if it is simple.*

For any sequence $\beta = (\beta_1, \beta_2, \dots)$ in $\mathbf{N}$ such that $\beta_N = 0$ for large N and $\sum_i \beta_i = n$, we set

$$I_\beta = \{s_i; i \in [1, n-1], i \ne \beta_1, i \ne \beta_1 + \beta_2, \dots\}.$$

From 22.4(a) we see easily that, if β is the same as α' up to order, then

(a) $\mathbf{j}_{W_{I_\beta}}^W(\mathrm{sgn}) = \pi_\alpha$.

Since the sgn $\in \mathrm{Con}(W_{I_\beta})$, it follows that $\pi_\alpha \in \mathrm{Con}(W)$. Thus any simple W-module is constructible.

We now show that any constructible representation E of $W = \mathfrak{S}_n$ is simple. We may assume that $n \ge 1$ and that the analogous result is true for any $W_{I'} \ne W$. We may assume that $E = \mathbf{j}_{W_{I_\beta}}^W(C)$ where β is as above, $W_{I_\beta} \ne W$ and $C \in \mathrm{Con}(W_{I_\beta})$. By the induction hypothesis, C is simple. Since the analogue of (a) holds for W_{I_β} (instead of W) we have $C = \mathbf{j}_{W_{I_{\beta'}}}^{W_{I_\beta}}(\mathrm{sgn})$ for some β' such that $W_{I_{\beta'}} \subset W_{I_\beta}$. By the transitivity of truncated induction we have $E = \mathbf{j}_{W_{I_{\beta'}}}^W(\mathrm{sgn})$. Hence, by (a), for β' instead of β, E is simple. The lemma is proved.

22.6. We now develop some combinatorics which is useful for the verification of 22.3 for W of classical type.

Let $a > 0, b \ge 0$ be integers. We can write uniquely $b = ar + b'$ where $r, b' \in \mathbf{N}$ and $b' < a$. Let $N \in \mathbf{N}$. Let $\mathcal{M}_{a,b}^N$ be the set of multisets $\tilde{Z} = \{\tilde{z}_1 \le \tilde{z}_2 \le \dots \le \tilde{z}_{2N+r}\}$ of integers ≥ 0 such that

(a) if $b' = 0$, there are at least $N + r$ distinct entries in $\tilde{Z}$, no entry is repeated more than twice and all entries of $\tilde{Z}$ are divisible by a;

(b) if $b' > 0$, all inequalities in $\tilde{Z}$ are strict and N entries of $\tilde{Z}$ are divisible by a and $N + r$ entries of $\tilde{Z}$ are equal to b' modulo a.

The entries which appear in $\tilde{Z}$ exactly once are called the *singles* of $\tilde{Z}$; they form a set Z. The other entries of $\tilde{Z}$ are called the *doubles* of $\tilde{Z}$.

For example, the multiset $\tilde{Z}^0$ whose entries are (up to order)

$$0, a, 2a, \dots, (N-1)a, b', a + b', 2a + b', \dots, (N + r - 1)a + b'$$

belongs to $\mathcal{M}_{a,b}^N$. Clearly, the sum of entries of $\tilde{Z}$ minus the sum of entries of $\tilde{Z}^0$ is ≥ 0 and divisible by a, hence it is equal to an for a well defined $n \in \mathbf{N}$ said to be the *rank* of $\tilde{Z}$. We have

$$\sum_{k=1}^{2N+r} \tilde{z}_k = an + aN^2 + N(b-a) + a\binom{r}{2} + b'r.$$

Note that $\tilde{Z}^0$ has rank 0. Let $\mathcal{M}_{a,b;n}^N$ be the set of multisets of rank n in $\mathcal{M}_{a,b}^N$. We define an (injective) map $\mathcal{M}_{a,b}^N \to \mathcal{M}_{a,b}^{N+1}$ by

$$\{\tilde{z}_1 \le \tilde{z}_2 \le \dots \le \tilde{z}_{2N+r}\} \mapsto \{0, b', \tilde{z}_1 + a \le \tilde{z}_2 + a \le \dots \le \tilde{z}_{2N+r} + a\}.$$

This restricts for any $n \in \mathbf{N}$ to an (injective) map

(c) $\mathcal{M}_{a,b;n}^N \to \mathcal{M}_{a,b;n}^{N+1}$.

It is easy to see that, for fixed n, $\sharp(\mathcal{M}_{a,b;n}^N)$ is bounded as $N \to \infty$, hence the maps (c) are bijections for large N. Let $\mathcal{M}_{a,b;n}$ be the inductive limit of $\mathcal{M}_{a,b;n}^N$ as $N \to \infty$ (with respect to the maps (c)).

22.7. Let $\mathrm{Sy}_{a,b;n}^N$ be the set consisting of all tableaux (or *symbols*)

(a)
$$\begin{matrix} \lambda_1, \lambda_2, \dots, \lambda_{N+r} \\ \mu_1, \mu_2, \dots, \mu_N \end{matrix}$$

where $\lambda_1 < \lambda_2 < \dots < \lambda_{N+r}$ are integers ≥ 0, congruent to b' modulo a, $\mu_1, \mu_2, \dots, \mu_N$ are integers ≥ 0, divisible by a and

$$\sum_i \lambda_i + \sum_j \mu_j = an + aN^2 + N(b-a) + a\binom{r}{2} + b'r.$$

If we arrange the entries of Λ in a single row, we obtain a multiset $\tilde{Z} \in \mathcal{M}^N_{a,b;n}$. This defines a (surjective) map $\pi_N : \mathrm{Sy}^N_{a,b;n} \to \mathcal{M}^N_{a,b;n}$. We define an (injective) map

(b) $\mathrm{Sy}^N_{a,b;n} \to \mathrm{Sy}^{N+1}_{a,b;n}$

by associating to (a) the symbol

$$\begin{matrix} b', \lambda_1 + a, \lambda_2 + a, \dots, \lambda_{N+r} + a \\ \qquad 0, \mu_1 + a, \mu_2 + a, \dots, \mu_N + a. \end{matrix}$$

This is compatible with the map $\mathcal{M}^N_{a,b} \to \mathcal{M}^{N+1}_{a,b}$ in 22.6 (via π_N, π_{N+1}).

Since for fixed n, $\sharp(\mathrm{Sy}^N_{a,b;n})$ is bounded as $N \to \infty$, the maps (b) are bijections for large N. Let $\mathrm{Sy}_{a,b;n}$ be the inductive limit of $\mathrm{Sy}^N_{a,b;n}$ as $N \to \infty$ (with respect to the maps (b)).

22.8. Let $\tilde{Z} = \{\tilde{z}_1 \le \tilde{z}_2 \le \dots \le \tilde{z}_{2N+r}\} \in \mathcal{M}^N_{a,b;n}$. Let t be an integer which is large enough so that the multiset

(a) $\{at + b' - \tilde{z}_1, at + b' - \tilde{z}_2, \dots, at + b' - \tilde{z}_{2N+r}\}$

is contained in the multiset

(b) $\{0, a, 2a, \dots, ta, b', a + b', 2a + b', \dots, ta + b'\}$

and let $\bar{\tilde{Z}}$ be the complement of (a) in (b). Then $\bar{\tilde{Z}} \in \mathcal{M}^{t+1-N-r}_{a,b}$. The sum of entries of $\bar{\tilde{Z}}$ is

$$\sum_{k \in [0,t]} (2ka + b') - (at + b')(2N + r) + \sum_h \tilde{z}_h$$

$$= at(t+1) + (t+1)b' - (at + b')(2N + r) + an + aN^2 + N(b - a) + a\binom{r}{2} + b'r$$

$$= an + a(t + 1 - N - r)^2 + (t + 1 - N - r)(b - a) + a\binom{r}{2} + b'r.$$

Thus, $\bar{\tilde{Z}}$ has rank n.

We define a bijection $\pi_N^{-1}(\tilde{Z}) \xrightarrow{\sim} \pi^{-1}_{t+1-N-r}(\bar{\tilde{Z}})$ by $\Lambda \mapsto \overline{\Lambda}$ where Λ is as in 22.7(a) and $\overline{\Lambda}$ is

$$\begin{aligned} &\{b', a + b', 2a + b', 3a + b', \dots, ta + b'\} \\ &- \{at + b' - \mu_1, at + b' - \mu_2, \dots, at + b' - \mu_N\} \\ &\{0, a, 2a, 3a, \dots, ta\} - \{at + b' - \lambda_1, at + b' - \lambda_2, \dots, at + b' - \lambda_{N+r}\}. \end{aligned}$$

22.9. Let $W = W_n$ be the group of permutations of $1, 2, \dots, n, n', \dots, 2', 1'$ which commute with the involution $i \mapsto i', i' \mapsto i$. We regard W_n as a Coxeter group with generators $s_1, s_2, \dots, s_n$ given as products of transpositions by

$s_1 = (1, 2)(1', 2'), s_2 = (2, 3)(2', 3'), \dots, s_{n-1} = (n - 1, n)((n - 1)', n')$,

$s_n = (n, n')$.

22.10. A permutation in W defines a permutation of the n element set consisting of the pairs $(1,1'),(2,2'),\dots,(n,n')$. Thus we have a natural homomorphism of W_n onto $\mathfrak{S}_n$, the symmetric group in n letters. Define a homomorphism $\chi_n : W_n \to \pm 1$ by

$\chi_n(\sigma)=1$ if $\{\sigma(1),\sigma(2),\dots,\sigma(n)\}\cap\{1',2',\dots,n'\}$ has even cardinal,

$\chi_n(\sigma)=-1$, otherwise.

The simple W-modules (up to isomorphism) are in 1-1 correspondence with the ordered pairs α,β where $\alpha=(\alpha_1\ge\alpha_2\ge\dots)$ and $\beta=(\beta_1\ge\beta_2\ge\dots)$ are partitions such that $\alpha_N=\beta_N=0$ for large N and $\sum_i\alpha_i+\sum_j\beta_j=n$. The correspondence (denoted by $\alpha,\beta\mapsto E^{\alpha,\beta}$) is defined as follows. Let α,β be as above, let $(\alpha'_1\ge\alpha'_2\ge\dots)$ be the partition dual to α and let $(\beta'_1\ge\beta'_2\ge\dots)$ be the partition dual to β. Let $k=\sum_i\alpha_i, l=\sum_j\beta_j$. Let π_α be the simple $\mathfrak{S}_k$-module defined as in 22.4 and let π_β be the analogously defined simple $\mathfrak{S}_l$-module. We regard π_α,π_β as simple modules of W_k,W_l via the natural homomorphisms $W_k\to\mathfrak{S}_k, W_l\to\mathfrak{S}_l$ as above. We identify $W_k\times W_l$ with the subgroup of W consisting of all permutations in W which map $1,2,\dots,k,k',\dots,2',1'$ into itself hence also map $k+1,k+2,\dots,n,n',\dots,(k+2)',(k+1)'$ into itself. Consider the representation $\pi_\alpha\otimes(\pi_\beta\otimes\chi_l)$ of $W_k\times W_l$. We induce it to W; the resulting representation of W is irreducible; we denote it by $E^{\alpha,\beta}$.

We fix $a>0, b\ge 0$ and we write $b=ar+b'$ as in 22.6.

Let α,β be as above. Let N be an integer such that $\alpha_{N+r+1}=0$, $\beta_{N+1}=0$. (Any large enough integer satisfies these conditions.) We set

$$\lambda_i=a(\alpha_{N+r-i+1}+i-1)+b', (i\in[1,N+r]),\quad \mu_j=a(\beta_{N-j+1}+j-1), (j\in[1,N]).$$

We have $0\le\lambda_1<\lambda_2<\dots<\lambda_{N+r}$, $0\le\mu_1<\mu_2<\dots<\mu_N$. Let Λ denote the tableau 22.7(a). It is easy to see that $\Lambda\in\mathrm{Sy}^N_{a,b;n}$. Moreover, if N is replaced by $N+1$, then Λ is replaced by its image under $\mathrm{Sy}^N_{a,b;n}\to\mathrm{Sy}^{N+1}_{a,b;n}$ (see 22.7). Let $[\Lambda]=E^{\alpha,\beta}$. Note that $[\Lambda]$ depends only on the image of Λ under the canonical map $\mathrm{Sy}^N_{a,b;n}\to\mathrm{Sy}_{a,b;n}$. In this way, we see that

the simple W-modules are naturally in bijection with the set $\mathrm{Sy}_{a,b;n}$.

For $i\in[1,N]$ we have $a(\alpha_{N-i+1}+i-1)+b=a(\alpha_{N+r-i-r+1}+i+r-1)+b'=\lambda_{i+r}$.

If N is large we have $\lambda_i=a(i-1)+b'$ for $i\in[1,r]$ and $\mu_j=a(j-1)$ for $j\in[1,r]$.

22.11. Let q,y be indeterminates. With the notation in 22.10, let

$$H_\alpha(q)=q^{-\sum_i\binom{\alpha'_i}{2}}\prod_{i,j}\frac{q^{\alpha_i+\alpha'_j-i-j+1}-1}{q-1},$$

$$G_{\alpha,\beta}(q,y)=q^{-\sum_i\alpha'_i\beta'_i/2}\prod_{i,j}(q^{\alpha_i+\beta'_j-i-j+1}y+1);$$

both products are taken over all $i\ge 1, j\ge 1$ such that $\alpha_i\ge j, \alpha'_j\ge i$.

Define a weight function $L : W \to \mathbf{N}$ by $L(s_1) = L(s_2) = \cdots = L(s_{n-1}) = a$, $L(s_n) = b$. We now assume that both a, b are > 0. We also assume that a, b are such that W, L satisfies the assumptions of 18.1. Then $f_{E_v^{\alpha,\beta}}$ is defined in terms of this L.

Lemma 22.12 (Hoefsmit [H]). *We have*

$$f_{E_v^{\alpha,\beta}} = H_\alpha(v^{2a})H_\beta(v^{2a})G_{\alpha,\beta}(v^{2a}, v^{2b})G_{\beta,\alpha}(v^{2a}, v^{-2b}).$$

We will rewrite the expression above using the following result.

Lemma 22.13. *Let N be a large integer. We have*

$$H_\alpha(q) = q^{\sum_{i\in[1,N-1]}\binom{i}{2}} \frac{\prod_{i=1}^N \prod_{h\in[1,\alpha_{N-i+1}+i-1]} \frac{q^h-1}{q-1}}{\prod_{1\le i<j\le N} \frac{q^{\alpha_{N-j+1}+j-1}-q^{\alpha_{N-i+1}+i-1}}{q-1}},$$

$$G_{\alpha,\beta}(q,y)G_{\beta,\alpha}(q,y^{-1}) = q^{\sum_{i\in[1,N-1]} i^2}(\sqrt{y}+\sqrt{y}^{-1})^N$$
$$\times \frac{\prod_{i=1}^N \prod_{h\in[1,\alpha_{N-i+1}+i-1]}(q^h y+1)\prod_{j=1}^N \prod_{h\in[1,\beta_{N-j+1}+j-1]}(q^h y^{-1}+1)}{\prod_{i,j\in[1,N]}(q^{\alpha_{N-i+1}+i-1}\sqrt{y}+q^{\beta_{N-j+1}+j-1}\sqrt{y}^{-1})}.$$

The proof is by induction on n. We omit it.

Proposition 22.14. *(a) If $b' = 0$ then $f_{[\Lambda]^\spadesuit}$ is equal to 2^d where $2d+r$ is the number of singles in Λ. If $b' > 0$ then $f_{[\Lambda]^\spadesuit} = 1$.*

(b) We have $\mathbf{a}_{[\Lambda]} = A_N - B_N$ where

$$A_N = \sum_{i\in[1,N+r],j\in[1,N]} \min(\lambda_i,\mu_j) + \sum_{1\le i<j\le N+r} \min(\lambda_i,\lambda_j) + \sum_{1\le i<j\le N} \min(\mu_i,\mu_j),$$

$$B_N = \sum_{i\in[1,N+r],j\in[1,N]} \min(a(i-1)+b', a(j-1))$$
$$+ \sum_{1\le i<j\le N+r} \min(a(i-1)+b', a(j-1)+b') + \sum_{1\le i<j\le N} \min(a(i-1), a(j-1)).$$

It is enough to prove (a) assuming that N is large. Since

$$A_{N+1} - A_N = a(N+r)N + a\binom{N}{2} + a\binom{N+r}{2} + b'N + b'(N+r) = B_{N+1} - B_N,$$

we have $A_{N+1} - B_{N+1} = A_N - B_N$ hence it is enough to prove (b) assuming that N is large. In the remainder of the proof we assume that N is large.

For $f, f' \in \mathbf{C}(v)$ we write $f \cong f'$ if $f' = fg$ with $g \in \mathbf{C}(v)$, $g|_{v=0} = 1$. Using 22.12, 22.13, we see that

$$f_{[\Lambda]_v} \cong \prod_{i\in[1,N]} (v^{2a-2b}+1)(v^{4a-2b}+1)\dots(v^{2\mu_i-2b}+1)$$
$$(v^b+v^{-b})^N v^{2a\sum_{i=1}^{N-1}(2i^2-i)} \prod_{i,j\in[1,N]} (v^{2\lambda_{i+r}-b}+v^{2\mu_j-b})^{-1}$$
$$\prod_{1\le i<j\le N} (v^{2\lambda_{j+r}-2b} - v^{2\lambda_{i+r}-2b})^{-1} \prod_{1\le i<j\le N} (v^{2\mu_j} - v^{2\mu_i})^{-1}$$

hence

$$f_{[\Lambda]_v} = 2^d v^{-K} + \text{strictly higher powers of } v$$

where $d = 0$ if $b' > 0$,

$$\begin{aligned} d &= \sharp(j\in[1,N] : b \le \mu_j) - \sharp(i,j\in[1,N] : \lambda_{i+r} = \mu_j) \\ &= N - \sharp(i\in[1,r], j\in[1,N] : (i-1)a = \mu_j) - \sharp(i,j\in[1,N] : \lambda_{i+r} = \mu_j) \\ &= N - \sharp(i\in[1,r], j\in[1,N] : \lambda_i = \mu_j) - \sharp(i,j\in[1,N] : \lambda_{i+r} = \mu_j) \\ &= N - \sharp(i\in[1,N+r], j\in[1,N] : \lambda_i = \mu_j) = (\sharp \text{ singles} - r)/2, \end{aligned}$$

if $b' = 0$,

$$\begin{aligned} -K &= -bN + 2a\sum_{i\in[1,N-1]}(2i^2-i) + \sum_{j\in[1,N]}\sum_{\substack{k\in[1,r]\\ ak\le\mu_j}}(2ak-2b) \\ &- \sum_{i,j\in[1,N]}(-b+2\min(\lambda_{i+r},\mu_j)) - \sum_{1\le i<j\le N}(-2b+2\min(\lambda_{i+r},\lambda_{j+r})) \\ &- \sum_{1\le i<j\le N} 2\min(\mu_i,\mu_j) = -bN + 2a\sum_{i\in[1,N-1]}(2i^2-i) + 2bN^2 - bN \\ &+ \sum_{j\in[1,N]}\sum_{k\in[1,r],ak\le\mu_j}(2ak-2b) - \sum_{i,j\in[1,N]} 2\min(\lambda_{i+r},\mu_j) \\ &- \sum_{1\le i<j\le N} 2\min(\lambda_{i+r},\lambda_{j+r}) - \sum_{1\le i<j\le N} 2\min(\mu_i,\mu_j) \\ &= \sum_{j\in[1,N]}\sum_{k\in[1,r],ak\le\mu_j}(2ak-2b) - \sum_{i,j\in[1,N]} 2\min(\lambda_{i+r},\mu_j) \\ &- \sum_{1\le i<j\le N} 2\min(\lambda_{i+r},\lambda_{j+r}) - \sum_{1\le i<j\le N} 2\min(\mu_i,\mu_j) + \star. \end{aligned}$$

(We will generally write $^\bigstar$ for an expression which depends only on a, b, N.) We have

$$\sum_{\substack{j\in[1,N]\\ k\in[1,r]\\ ak\le\mu_j}}(2ak-2b)=\sum_{\substack{j\in[1,r]\\ k\in[1,r]\\ ak\le\mu_j}}(2ak-2b)+\sum_{\substack{j\in[r+1,N]\\ k\in[1,r]\\ ak\le\mu_j}}(2ak-2b)$$
$$=\sum_{\substack{j\in[1,r]\\ k\in[1,r]\\ ak\le a(j-1)}}(2ak-2b)+\sum_{\substack{j\in[r+1,N]\\ k\in[1,r]}}(2ak-2b)={}^\bigstar,$$

hence

$$-K$$
$$=-2\Big(\sum_{i,j\in[1,N]}\min(\lambda_{i+r},\mu_j)-\sum_{1\le i<j\le N}\min(\lambda_{i+r},\lambda_{j+r})-\sum_{1\le i<j\le N}\min(\mu_i,\mu_j)\Big)$$
$$+{}^\bigstar.$$

We have

$$\sum_{i\in[1,r],j\in[1,N]}\min(\lambda_i,\mu_j)=\sum_{i\in[1,r],j\in[1,N]}\min(a(i-1)+b',\mu_j)$$
$$=\sum_{i\in[1,r],j\in[1,r]}\min(a(i-1)+b',a(j-1))+\sum_{i\in[1,r],j\in[r+1,N]}\min(a(i-1)+b',\mu_j)$$
$$=\sum_{i\in[1,r],j\in[1,r]}\min(a(i-1)+b',a(j-1))+\sum_{i\in[1,r],j\in[r+1,N]}(a(i-1)+b')={}^\bigstar,$$

hence

$$\sum_{i,j\in[1,N]}\min(\lambda_{i+r},\mu_j)=\sum_{i\in[1,N+r],j\in[1,N]}\min(\lambda_i,\mu_j)+{}^\bigstar.$$

We have

$$\sum_{1\le i<j\le N+r}\min(\lambda_i,\lambda_j)=\sum_{1\le i<j\le N}\min(\lambda_{i+r},\lambda_{j+r})+\sum_{i\in[1,r]}\lambda_i(N+r-i)$$
$$=\sum_{1\le i<j\le N}\min(\lambda_{i+r},\lambda_{j+r})+\sum_{i\in[1,r]}(a(i-1)+b')(N+r-i)$$
$$=\sum_{1\le i<j\le N}\min(\lambda_{i+r},\lambda_{j+r})+{}^\bigstar.$$

We see that

(c) $$-K=-2A_N+{}^\bigstar.$$

In the special case where $\alpha=\beta=(0\ge 0\ge\dots)$ we have $K=0$. On the other hand, by (c), we have $0=-2B_N+{}^\bigstar$ where $^\bigstar$ is as in (c). Hence in general we have $-K=-2A_N+2B_N$. This proves the proposition, in view of 20.11 and 20.21(a).

22.15. We identify $\mathfrak{S}_k \times W_l$ $(k + l = n)$ with the subgroup of W consisting of all permutations in W which map $\{1, 2, \ldots, k\}$ into itself (hence also map $\{1', 2', \ldots, k'\}$ and $\{k+1, \ldots, n, n', \ldots, (k+1)'\}$ into themselves. This is a standard parabolic subgroup of W. We consider an irreducible representation of $\mathfrak{S}_k \times W_l$ of the form $\mathrm{sgn}_k \boxtimes [\Lambda']$ where sgn_k is the sign representation of $\mathfrak{S}_k$ and $\Lambda' \in \mathrm{Sy}^N_{a,b;l}$. We may assume that Λ' has at least k entries. We want to associate to Λ' a symbol in $\mathrm{Sy}^N_{a,b;n}$ by increasing each of the k largest entries in Λ' by a. It may happen that the set of r largest entries of Λ' is not uniquely defined but there are two choices for it. (This can only happen if $b' = 0$.) Then the same procedure gives rise to two distinct symbols Λ^I, Λ^{II} in $\mathrm{Sy}^N_{a,b;n}$.

Lemma 22.16. *(a)* $g_{(\mathrm{sgn}_k \otimes [\Lambda'])_\spadesuit} = g_{[\Lambda']_\spadesuit}$ *is equal to* $g_{[\Lambda]_\spadesuit}$ *or to* $g_{[\Lambda^I]_\spadesuit} + g_{[\Lambda^{II}]_\spadesuit}$.
(b) $\mathbf{a}_{\mathrm{sgn}_k \otimes [\Lambda']} = a\binom{k}{2} + \mathbf{a}_{[\Lambda']}$ *is equal to* $\mathbf{a}_{[\Lambda]}$ *or to* $\mathbf{a}_{[\Lambda^I]} = \mathbf{a}_{[\Lambda^{II}]}$.

Λ, if defined, has the same number of singles as Λ'. Moreover, Λ^I (and Λ^{II}), if defined, has one more single than Λ'. Hence (a) follows from 22.14(a) using 20.18, 20.19.

By 22.14(b), the difference $\mathbf{a}_{[\tilde{\Lambda}]} - \mathbf{a}_{[\Lambda]}$ (where $\tilde{\Lambda}$ is either Λ or Λ^I or Λ^{II}) is a times the number of $i < j$ in $[1, k]$. Thus, it is $a\binom{k}{2}$. Hence (a) follows from 20.18, 20.19. The lemma is proved.

Lemma 22.17. $\mathbf{j}^W_{\mathfrak{S}_k \times W_l}(\mathrm{sgn}_k \otimes [\Lambda'])$ *equals* $[\Lambda]$ *or* $[\Lambda^I] + [\Lambda^{II}]$.

By a direct computation (involving representations of symmetric groups) we see that:
(a) if Λ is defined then $[[\Lambda'] : [\Lambda]] \ge 1$;
(b) if Λ^I, Λ^{II} are defined then $[[\Lambda'] : [\Lambda^I]] \ge 1$ and $[[\Lambda'] : [\Lambda^{II}]] \ge 1$.
In the setup of (a) we have (by 20.14(b)):
$g_{[\Lambda']_\spadesuit} = \sum_{E; \mathbf{a}_E = \mathbf{a}_{E'}} [[\Lambda'] : E] g_{E_\spadesuit}$ hence using 22.16(a) we have

$$g_{[\Lambda]_\spadesuit} = \sum_{E; \mathbf{a}_E = \mathbf{a}_{E'}} [[\Lambda'] : E] g_{E_\spadesuit}. \tag{c}$$

By 22.16(b), $E = [\Lambda]$ enters in the last sum and its contribution is $\ge g_{[\Lambda]_\spadesuit}$; the contribution of the other E is ≥ 0 (see 20.13(b)). Hence (c) forces $[[\Lambda'] : [\Lambda]] = 1$ and $[[\Lambda'] : E] = 0$ for all other E in the sum. In this case the lemma follows.

In the setup of (b) we have (by 20.14(b)) $g_{[\Lambda']_\spadesuit} = \sum_{E; \mathbf{a}_E = \mathbf{a}_{E'}} [[\Lambda'] : E] g_{E_\spadesuit}$ hence, by 22.16(a),

$$g_{[\Lambda^I]_\spadesuit} + g_{[\Lambda^{II}]_\spadesuit} = \sum_{E; \mathbf{a}_E = \mathbf{a}_{E'}} [[\Lambda'] : E] g_{E_\spadesuit}. \tag{d}$$

By 22.16(b), $E = [\Lambda^I]$ and $E = [\Lambda^{II}]$ enter in the last sum and their contribution is $\ge g_{[\Lambda^I]_\spadesuit} + g_{[\Lambda^{II}]_\spadesuit}$; the contribution of the other E is ≥ 0 (see 20.13(b)). Hence (d) forces $[[\Lambda'] : [\Lambda^I]] = [[\Lambda'] : [\Lambda^{II}]] = 1$ and $[[\Lambda'] : E] = 0$ for all other E in the sum. The lemma follows.

Lemma 22.18. $[\Lambda] \otimes \text{sgn} = [\overline{\Lambda}]$. *(Notation of 22.14.)*

This can be reduced to a known statement about the symmetric group. We omit the details.

22.19. Let Z be a totally ordered finite set $z_1 < z_2 < \cdots < z_M$. For any $r \in [0, M]$ such that $r = M \mod 2$ let $\underline{Z}_r$ be the set of subsets of Z of cardinal $(M - r)/2$. An involution $\iota : Z \to Z$ is said to be r-admissible if the following hold:

(a) ι has exactly r fixed points;

(b) if $M = r$, there is no further condition; if $M > r$, there exist two consecutive elements z, z' of Z such that $\iota(z) = z', \iota(z') = z$ and the induced involution of $Z - \{z, z'\}$ is r-admissible.

Let $\text{Inv}_r(Z)$ be the set of r-admissible involutions of Z. To $\iota \in \text{Inv}_r(Z)$ we associate a subset $\mathcal{S}_\iota$ of $\underline{Z}_r$ as follows: a subset $Y \subset Z$ is in $\mathcal{S}_\iota$ if it contains exactly one element in each non-trivial ι-orbit. Clearly, $\sharp(\mathcal{S}_\iota) = 2^{p_0}$ where $p_0 = (M - r)/2$. (In fact, $\mathcal{S}_\iota$ is naturally an affine space over the field $\mathbf{F}_2$.)

Lemma 22.20. *Assume that $p_0 > 0$. Let $Y \in \underline{Z}_r$.*

(a) We can find two consecutive elements z, z' of Z such that exactly one of z, z' is in Y.

(b) There exists $\iota \in \text{Inv}_r(Z)$ such that $Y \in \mathcal{S}_\iota$.

(c) Assume that for some $k \in [0, p_0 - 1]$, $z_1, z_2, \ldots, z_k$ belong to Y but $z_{k+1} \notin Y$. Let l be the smallest number such that $l > k$ and $z_l \in Y$. There exists $\iota \in \text{Inv}_r(Z)$ such that $Y \in \mathcal{S}_\iota$ and $\iota(z_l) = z_{l-1}$.

We prove (a). Let z_k be the smallest element of Y. If $k > 1$ then we can take $(z, z') = (z_{k-1}, z_k)$. Hence we may assume that $z_1 \in Y$. Let $z_{k'}$ be the next smallest element of Y. If $k' > 2$ then we can take $(z, z') = (z_{k'-1}, z_{k'})$. Continuing like this we see that we may assume that $Y = \{z_1, z_2, \ldots, z_{p_0}\}$. Since $p_0 < M$, we may take $(z, z') = (z_{p_0}, z_{p_0+1})$.

We prove (b). Let z, z' be as in (a). Let $Z' = Z - \{z, z'\}$ with the induced order. Let $Y' = Y \cap Z'$. If $p_0 \geq 2$ then by induction on p_0 we may assume that there exists $\iota' \in \text{Inv}_r(Z')$ such that $Y' \in \mathcal{S}_{\iota'}$. Extend ι' to an involution ι of Z by $z \mapsto z', z' \mapsto z$. Then $\iota \in \text{Inv}_r(Z)$ and $Y \in \mathcal{S}_\iota$. If $p_0 = 1$, define $\iota : Z \to Z$ so that $z \mapsto z', z' \mapsto z$ and $\iota = 1$ on $Z - \{z, z'\}$. Then $\iota \in \text{Inv}_r(Z)$ and $Y \in \mathcal{S}_\iota$.

We prove (c). We have $l \geq k + 2$. Hence $z_{l-1} \notin Y$. Let $(z, z') = (z_{l-1}, z_l)$. We continue as in the proof of (b), except that instead of invoking an induction hypothesis, we invoke (b) itself.

22.21. Assume that $M > r$. We consider the graph whose set of vertices is $\underline{Z}_r$ and in which two vertices $Y \neq Y'$ are joined if there exists $\iota \in \text{Inv}_r(Z)$ such that $Y \in \mathcal{S}_\iota, Y' \in \mathcal{S}_\iota$.

Lemma 22.22. *This graph is connected.*

We show that any vertex $Y = \{z_{i_1}, z_{i_2}, \ldots, z_{i_{p_0}}\}$ is in the same connected component as $Y_0 = \{z_1, z_2, \ldots, z_{p_0}\}$. We argue by induction on $m_Y = i_1 + i_2 +$

$\dots + i_{p_0}$. If $m_Y = 1 + 2 + \dots + p_0$ then $Y = Y_0$ and there is nothing to prove. Assume now that $m > 1 + 2 + \dots + p_0$ so that $Y \ne Y_0$. Then the assumption of Lemma 22.20(c) is satisfied. Hence we can find l such that $z_l \in Y, z_{l-1} \notin Y$ and $\iota \in \mathrm{Inv}_r(Z)$ such that $Y \in \mathcal{S}_\iota$ and $\iota(z_l) = z_{l-1}$. Let $Y' = (Y - \{z_l\}) \cup \{z_{l-1}\}$. Then $Y' \in \mathcal{S}_\iota$ hence Y, Y' are joined in our graph. We have $m_{Y'} = m_Y - 1$ hence by the induction hypothesis Y', Y_0 are in the same connected component. It follows that Y, Y_0 are in the same connected component. The lemma is proved.

22.23. Assume that $b' = 0$. Let $\tilde{Z} \in \mathcal{M}^N_{a,b;n}$. Let Z be the set of singles of $\tilde{Z}$. Each set $Y \in \underline{Z}_r$ gives rise to a symbol Λ_Y in $\pi_N^{-1}(\tilde{Z})$: the first row of Λ_Y consists of $Z - Y$ and one element in each double of $\tilde{Z}$; the second row consists of Y and one element in each double of $\tilde{Z}$. For any $\iota \in \mathrm{Inv}_r(Z)$ we set

$$c(\tilde{Z}, \iota) = \oplus_{Y \in \mathcal{S}_\iota} [\Lambda_Y] \in \mathrm{Mod} W.$$

Proposition 22.24. *(a) In the setup of 22.23, let $\iota \in \mathrm{Inv}_r(Z)$. Then $c(\tilde{Z}, \iota) \in \mathrm{Con}(W)$.*

(b) All constructible representations of W are obtained as in (a).

We prove (a) by induction on n. If $n = 0$ the result is clear. Assume that $n \ge 1$. We may assume that 0 is not a double of $\tilde{Z}$. Let at be the largest entry of $\tilde{Z}$.

(A) Assume that there exists $i, 0 \le i < t$, such that ai does not appear in $\tilde{Z}$. Then $\tilde{Z}$ is obtained from $\tilde{Z}' \in \mathcal{M}^N_{a,b;n-k}$ with $n - k < n$ by increasing each of the k largest entries by a and this set of largest entries is unambiguously defined. The set Z' of singles of $\tilde{Z}'$ is naturally in order preserving bijection with Z. Let ι' correspond to ι under this bijection. By the induction hypothesis, $c(\tilde{Z}', \iota') \in \mathrm{Con}(W_{n-k})$. Since, by 22.5, the sign representation sgn_k of $\mathfrak{S}_k$ is constructible, it follows that $\mathrm{sgn}_k \boxtimes c(\tilde{Z}', \iota') \in \mathrm{Con}(\mathfrak{S}_k \times W_{n-k})$. Using 22.17, we have

$$\mathbf{j}^W_{\mathfrak{S}_k \times W_{n-k}}(\mathrm{sgn}_k \boxtimes c(\tilde{Z}', \iota')) = c(\tilde{Z}, \iota)$$

hence $c(\tilde{Z}, \iota) \in \mathrm{Con}(W)$.

(B) Assume that there exists $i, 0 < i \le t$ such that ai is a double of $\tilde{Z}$. Let $\bar{\tilde{Z}}$ be as in 22.8 (with respect to our t). Then 0 is not a double of $\bar{\tilde{Z}}$ and the largest entry of $\bar{\tilde{Z}}$ is at. Let $\bar{Z}$ be the set of singles of $\bar{\tilde{Z}}$. We have $\bar{Z} = \{at - z; z \in Z\}$. Thus $\bar{Z}, Z$ are naturally in (order reversing) bijection under $j \mapsto at - j$. Let $\iota' \in \mathrm{Inv}_r(\bar{Z})$ correspond to ι under this bijection. Since $at - ai$ does not appear in $\bar{\tilde{Z}}$, (A) is applicable to $\bar{\tilde{Z}}$. Hence $c(\bar{\tilde{Z}}, \iota') \in \mathrm{Con}(W)$. By 22.18 we have $c(\bar{\tilde{Z}}, \iota') \otimes \mathrm{sgn} = c(\tilde{Z}, \iota)$ hence $c(\tilde{Z}, \iota) \in \mathrm{Con}(W)$.

(C) Assume that we are not in case (A) and not in case (B). Then $\tilde{Z} = \{0, a, 2a, \dots, ta\} = Z$. We can find $ia, (i+1)a$ in Z such that ι interchanges $ia, (i+1)a$ and induces on $Z - \{ia, (i+1)a\}$ an r-admissible involution ι_1. We have

$$\tilde{Z}' = \{0, a, 2a, \dots, ia, ia, (i+1)a, (i+2)a, \dots, (t-1)a\} \in \mathcal{M}^N_{a,b;n-k}$$

with $n-k<n$. The set of singles of $\tilde{Z}'$ is

$$Z'=\{0,a,2a,\dots,(i-1)a,(i+1)a,\dots,(t-1)a\}.$$

It is in natural (order preserving) bijection with $Z-\{ia,(i+1)a\}$. Hence ι_1 induces $\iota'\in \mathrm{Inv}_r(Z')$. By the induction hypothesis we have $c(\tilde{Z}',\iota')\in \mathrm{Con}(W_{n-k})$. Hence $\mathrm{sgn}_k\boxtimes c(\tilde{Z}',\iota')\in \mathrm{Con}(\mathfrak{S}_k\times W_{n-k})$ where sgn_k is as in (A). By 22.17 we have

$$\mathbf{j}_{\mathfrak{S}_k\times W_{n-k}}^{W}(\mathrm{sgn}_k\boxtimes c(\tilde{Z}',\iota'))=c(\tilde{Z},\iota)$$

hence $c(\tilde{Z},\iota)\in \mathrm{Con}(W)$. This proves (a).

We prove (b) by induction on n. If $n=0$ the result is clear. Assume now that $n\ge 1$. By an argument like the ones used in (B) we see that the class of representations of W obtained in (a) is closed under $\otimes$sgn. Therefore, to show that $C\in \mathrm{Con}(W)$ is obtained in (a), we may assume that $C=\mathbf{j}_{\mathfrak{S}_k\times W_{n-k}}^{W}(C')$ for some $k>0$ and some $C'\in \mathrm{Con}(\mathfrak{S}_k\times W_{n-k})$. By 22.5 we have $C'=E\boxtimes C''$ where E is a simple $\mathfrak{S}_k$-module and $C''\in \mathrm{Con}(W_{n-k})$. Using 22.5(a) we have $E=\mathbf{j}_{\mathfrak{S}_{k'}\times\mathfrak{S}_{k''}}^{\mathfrak{S}_k}(\mathrm{sgn}\boxtimes E')$ where $k'+k''=k, k'>0$ and E' is a simple $\mathfrak{S}_{k''}$-module. Let $\tilde{C}=\mathbf{j}_{\mathfrak{S}_{k''}\times W_{n-k}}^{W_{n-k'}}(E'\otimes C')\in \mathrm{Con}(W_{n-k'})$. Then $C=\mathbf{j}_{\mathfrak{S}_{k'}\times W_{n-k'}}^{W}(\mathrm{sgn}_{k'}\otimes\tilde{C})$. By the induction hypothesis, $\tilde{C}$ is of the form described in (a). Using an argument as in (A) or (C) we deduce that C is of the form described in (a). The proposition is proved.

Proposition 22.25. *Assume that $b'>0$.*

(a) Let $E\in \mathrm{Irr}W$. Then $E\in \mathrm{Con}(W)$.

(b) All constructible representations of W are obtained as in (a).

We prove (a). We may assume that $E=[\Lambda]$ where $\Lambda\in \mathrm{Sy}_{a,b;n}^{N}$ does not contain both 0 and b'. We argue by induction on n. If $n=0$ the result is clear. Assume now that $n\ge 1$.

(A) Assume that either (1) there exist two entries z,z' of Λ such that $z'-z>a$ and there is no entry z'' of Λ such that $z<z''<z'$, or (2) there exists an entry z' of Λ such that $z'\ge a$ and there is no entry z'' of Λ such that $z''<z'$. Let Λ' be the symbol obtained from Λ by substracting a from each entry $\tilde{z}$ of Λ such that $\tilde{z}\ge z'$ and leaving the other entries of Λ unchanged. Then $\Lambda'\in \mathrm{Sy}_{a,b;n-k}^{N}$ with $n-k<n$. By the induction hypothesis, $[\Lambda']\in \mathrm{Con}(W_{n-k})$. Since, by 22.5, the sign representation sgn_k of $\mathfrak{S}_k$ is constructible, it follows that $\mathrm{sgn}_k\boxtimes[\Lambda']\in \mathrm{Con}(\mathfrak{S}_k\times W_{n-k})$. Using 22.17, we have $\mathbf{j}_{\mathfrak{S}_k\times W_{n-k}}^{W}(\mathrm{sgn}_k\boxtimes[\Lambda'])=[\Lambda]$ hence $[\Lambda]\in \mathrm{Con}(W)$.

(B) Assume that there exist two entries z,z' of Λ such that $0<z'-z<a$. Let t be the smallest integer such that $at+b'\ge\lambda_i$ for all $i\in[1,N+r]$ and $at\ge\mu_j$ for all $j\in[1,N]$. Let $\overline{\Lambda}\in \mathrm{Sy}_{a,b;n}^{t+1-N-r}$ be as in 22.8 with respect to this t. Then $\overline{\Lambda}$ does not contain both 0 and b'. Now (A) is applicable to $\overline{\Lambda}$. Hence $[\overline{\Lambda}]\in \mathrm{Con}(W)$. By 22.18 we have $[\overline{\Lambda}]\otimes\mathrm{sgn}=[\Lambda]$ hence $[\Lambda]\in \mathrm{Con}(W)$.

(C) Assume that we are not in case (A) and not in case (B). Then the entries of Λ are either $0, a, 2a, \dots, ta$ or $b', a+b', 2a+b', \dots, ta+b'$. This cannot happen for $n \ge 1$. This proves (a).

The proof of (b) is entirely similar to that of 22.24(b). The proposition is proved.

22.26. We now assume that $n \ge 2$ and that $W' = W'_n$ is the kernel of $\chi_n : W_n \to \pm 1$ in 22.10. We regard W'_n as a Coxeter group with generators $s_1, s_2, \dots, s_{n-1}$ as in 22.9 and $s'_n = (n-1, n')((n-1)', n)$ (product of transpositions). Let $L : W' \to \mathbf{N}$ be the weight function given by $L(w) = al(w)$ for all w. Here $a > 0$.

For $\Lambda \in \mathrm{Sy}^N_{a,0}$ we denote by Λ^{tr} the symbol whose first (resp. second) row is the second (resp. first) row of Λ. We then have $\Lambda^{tr} \in \mathrm{Sy}^N_{a,0}$. From the definitions we see that the simple W_n-modules $[\Lambda], [\Lambda^{tr}]$ have the same restriction to W'; this restriction is a simple W'-module $[\underline{\Lambda}]$ if $\Lambda \ne \Lambda^{\mathrm{tr}}$ and is a direct sum of two non-isomorphic simple W'-modules $[{}^I\underline{\Lambda}]$,$[{}^{II}\underline{\Lambda}]$ if $\Lambda = \Lambda^{\mathrm{tr}}$. In this way we see that

the simple W'-modules are naturally in bijection with the set of orbits of the involution of $\mathrm{Sy}_{a,0;n}$ induced by $\Lambda \mapsto \Lambda^{tr}$ except that each fixed point of this involution corresponds to two simple W'-modules.

Let $\tilde{Z} \in \mathcal{M}^N_{a,0;n}$. Let Z be the set of singles of $\tilde{Z}$. Assume first that $Z \ne \emptyset$. Each set $Y \in \underline{Z}_0$ gives rise to a symbol Λ_Y in $\mathrm{Sy}^N_{a,0;n}$: the first row of Λ_Y consists of $Z - Y$ and one element in each double of $\tilde{Z}$; the second row consists of Y and one element in each double of $\tilde{Z}$. For any $\iota \in \mathrm{Inv}_0(Z)$ we define $c(\tilde{Z}, \iota) \in \mathrm{Mod}W$ by

$$c(\tilde{Z}, \iota) \oplus c(\tilde{Z}, \iota) = \oplus_{Y \in \mathcal{S}_\iota} [\Lambda_Y] \in \mathrm{Mod}W.$$

Note that Y and $Z - Y$ have the same contribution to the sum. A proof entirely similar to that of 22.24 shows that $c(\tilde{Z}, \iota) \in \mathrm{Con}(W)$. Moreover, if $Z = \emptyset$ and $\Lambda = \Lambda^{tr} \in \mathrm{Sy}^N_{a,1;n}$ is defined by $\pi_N(\Lambda) = \tilde{Z}$, then $[{}^I\underline{\Lambda}] \in \mathrm{Con}(W)$ and $[{}^{II}\underline{\Lambda}] \in \mathrm{Con}(W)$. All constructible representations of W are obtained in this way.

22.27. Assume that W is of type F_4 and that the values of $L : W \to \mathbf{N}$ on S are a, a, b, b where $a > b > 0$.

Case 1. Assume that $a = 2b$. There are four simple W-modules $\rho_1, \rho_2, \rho_8, \rho_9$ (subscript equals dimension) with $\mathbf{a} = 3b$. Then

$$\rho_1 \oplus \rho_8, \rho_2 \oplus \rho_9, \rho_8 \oplus \rho_9 \in \mathrm{Con}(W).$$

(They are obtained by $\mathbf{j}$ from the W_I of type B_3 with parameters a, b, b.)

The simple W-modules $\rho_1^\dagger, \rho_2^\dagger, \rho_8^\dagger, \rho_9^\dagger$ have $\mathbf{a} = 15b$ and

$$\rho_1^\dagger \oplus \rho_8^\dagger, \rho_2^\dagger \oplus \rho_9^\dagger, \rho_8^\dagger \oplus \rho_9^\dagger \in \mathrm{Con}(W).$$

There are five simple W-modules $\rho_{12}, \rho_{16}, \rho_6, \rho'_6, \rho_4$ (subscript equals dimension) with $\mathbf{a} = 7b$. Then

$$\rho_4 \oplus \rho_{16}, \rho_{12} \oplus \rho_{16} \oplus \rho_6, \rho_{12} \oplus \rho_{16} \oplus \rho'_6 \in \mathrm{Con}(W).$$

All 12 simple W-modules other than the 13 listed above, are constructible. All constructible representations of W are thus obtained.

Case 2. Assume that $a \notin \{b, 2b\}$. The simple W-modules $\rho_{12}, \rho_{16}, \rho_6, \rho'_6, \rho_4$ in Case 1 now have $\mathbf{a} = 3a + b$ and

$$\rho_4 \oplus \rho_{16}, \rho_{12} \oplus \rho_{16} \oplus \rho_6, \rho_{12} \oplus \rho_{16} \oplus \rho'_6 \in \mathrm{Con}(W).$$

All 20 simple W-modules other than the 5 listed above, are constructible. All constructible representations of W are thus obtained.

22.28. Assume that W is of type G_2 and that the values of $L : W \to \mathbf{N}$ on S are a, b where $a > b > 0$. Let ρ_2, ρ'_2 be the two 2-dimensional simple W-modules. They have $\mathbf{a} = a$ and $\rho_2 \oplus \rho'_2$ is constructible. All 4 simple W-modules other than the 2 listed above, are constructible. All constructible representations of W are thus obtained.

22.29. Let $\mathcal{L}$ be the set of all weight functions $L : W \to \mathbf{N}$ such that $L(s) > 0$ for all $s \in S$. We assume that $P1 - P15$ in §14 hold for any $L \in \mathcal{L}$. For $L, L' \in \mathcal{L}$ we write $L \sim L'$ if the constructible representations of W with respect to L are the same as those with respect to L'. This is an equivalence relation on $\mathcal{L}$. From the results in this chapter we see that any equivalence class for $\sim$ contains some L which is attached to some $(G, F, \mathcal{P}, \mathbf{E})$ as in 0.3.

We expect that the constructible representations of W are exactly the representations of W carried by the left cells of W (for fixed $L \in \mathcal{L}$). (For $L = l$ this holds by [L8]. For W of type F_4 and general L this holds by [G].) This would imply that for $L, L' \in \mathcal{L}$ we have $L \sim L'$ if and only if the representations of W carried by the left cells of W with respect to L are the same as those with respect to L'.

23. Two-sided cells

23.1. We preserve the setup of 20.1. We define a graph $\mathcal{G}_W$ as follows. The vertices of $\mathcal{G}_W$ are the simple W-modules up to isomorphism. Two non-isomorphic simple W-modules are joined in $\mathcal{G}_W$ if they both appear as components of some constructible representation of W. Let $\frac{\mathcal{G}_W}{\sim}$ be the set of connected components of $\mathcal{G}_W$. The connected components of $\mathcal{G}_W$ are determined explicitly by the results in §22 for W irreducible.

For example, in the setup of 22.4,22.5 we have $\mathcal{G}_W = \frac{\mathcal{G}_W}{\sim}$. In the setup of 22.24, $\frac{\mathcal{G}_W}{\sim}$ is naturally in bijection with $\mathcal{M}_{a,b;n}$. (Here, 22.22 is used). In the setup of 22.25, we have $\mathcal{G}_W = \frac{\mathcal{G}_W}{\sim}$.

We show that:

(a) *if E, E' are in the same connected component of $\mathcal{G}_W$ then $E \sim_{\mathcal{LR}} E'$.*

We may assume that both E, E' appear in some constructible representation of W. By 22.2, there exists a left cell Γ such that $[E : [\Gamma]] \neq 0$, $[E' : [\Gamma]] \neq 0$. By 21.2, we have $[E_\spadesuit : J^\Gamma_{\mathbf{C}}] \neq 0$, $[E'_\spadesuit : J^\Gamma_{\mathbf{C}}] \neq 0$. Hence $E \sim_{\mathcal{LR}} E'$, as desired.

23.2. Let c_W be the set of two-sided cells of W, L. Consider the (surjective) map $\mathrm{Irr}W \to c_W$ which to E associates the two-sided cell $\mathbf{c}$ such that $E \sim_{\mathcal{LR}} x$ for $x \in \mathbf{c}$. By 23.1 this induces a (surjective) map

(a) $\omega_W : \frac{\mathcal{G}_W}{\sim} \to c_W$.

We conjecture that ω_W is a bijection. This is made plausible by:

Proposition 23.3. *Assume that W, L is split. Then ω_W is a bijection.*

Let $E, E' \in \mathrm{Irr}W$ be such that $E \sim_{\mathcal{LR}} E'$. By 22.3, we can find constructible representations C, C' such that $[E : C] \neq 0, [E' : C'] \neq 0$. By 22.2, we can find left cells Γ, Γ' such that $C = [\Gamma], C' = [\Gamma']$. Then $[E : [\Gamma]] \neq 0$, $[E' : [\Gamma']] \neq 0$. Let $d \in \mathcal{D} \cap \Gamma, d' \in \mathcal{D} \cap \Gamma'$. Since $\gamma_d = [\Gamma]$ and $[E : [\Gamma] \neq 0$, we have $E \sim_{\mathcal{LR}} d$. Similarly, $E' \sim_{\mathcal{LR}} d'$. Hence $d' \sim_{\mathcal{LR}} d'$. By 18.4(c), there exists $u \in W$ such that $t_d t_u t_{d'} \neq 0$. (Here we use the splitness assumption.) Note that $j \mapsto j t_u t_{d'}$ is a $J_{\mathbf{C}}$-linear map $J^{\Gamma}_{\mathbf{C}} \to J^{\Gamma'}_{\mathbf{C}}$. This map is non-zero since it takes t_d to $t_d t_u t_{d'} \neq 0$. Thus, $\mathrm{Hom}_{J_{\mathbf{C}}}(J^{\Gamma}_{\mathbf{C}}, J^{\Gamma'}_{\mathbf{C}}) \neq 0$. Using 21.2, we deduce that $\mathrm{Hom}_W([\Gamma], [\Gamma']) \neq 0$. Hence there exists $\tilde{E} \in \mathrm{Irr}W$ such that $\tilde{E}$ is a component of both $[\Gamma] = C$ and $[\Gamma'] = C'$. Thus, both $E, \tilde{E}$ appear in C and both $\tilde{E}, E'$ appear in C'. Hence E, E' are in the same connected component of $\mathcal{G}_W$. The proposition is proved.

23.4. Assume now that $W, S, L, \tilde{W}, \iota$ are as in 16.2 and $\tilde{W}$ is an irreducible Weyl group.

Let $c^!_{\tilde{W}}$ be the set of all ι-stable two-sided cells of $\tilde{W}$. Let $c^\star_{\tilde{W}}$ be the set of all two-sided cells of $\tilde{W}$ which meet W. We have $c^\star_{\tilde{W}} \subset c^!_{\tilde{W}} \subset c_{\tilde{W}}$. Let $f : c_W \to c^\star_{\tilde{W}}$ be the map which attaches to a two-sided cell of W the unique two-sided cell of $\tilde{W}$ containing it; this map is well defined by 16.20(b) and is obviously surjective.

Proposition 23.5. *In the setup of 23.4, ω_W is a bijection and $f : c_W \to c^\star_{\tilde{W}}$ is a bijection.*

Since ω_W, f are surjective, the composition $f\omega_W : \frac{\mathcal{G}_W}{\sim} \to c^\star_{\tilde{W}}$ is surjective. Hence it is enough to show that this composition is injective. For this it suffices to check one of the two statements below:

(a) $\sharp(\frac{\mathcal{G}_W}{\sim}) = \sharp(c^\star_{\tilde{W}})$;

(b) the composition $\frac{\mathcal{G}_W}{\sim} \xrightarrow{f\omega_W} c^\star_{\tilde{W}} \subset c_{\tilde{W}} \xrightarrow{f'} \mathbf{N}\oplus\mathbf{N}$ (where $f'(\mathbf{c}) = (\mathbf{a}(x), \mathbf{a}(xw_0))$ for $x \in \mathbf{c}$) is injective.

Note that the value of the composition (b) at E is $(\mathbf{a}_E, \mathbf{a}_{E^\dagger})$.

Case 1. W is of type G_2 and $\tilde{W}$ is of type D_4. Then (b) holds: the composition (b) takes distinct values $(0, 12), (1, 7), (3, 3), (7, 1), (12, 0)$ on the 5 elements of $\frac{\mathcal{G}_W}{\sim}$.

Case 2. W is of type F_4 and $\tilde{W}$ is of type E_6. Then again (b) holds.

Case 3. W is of type B_n with $n \geq 2$ and $\tilde{W}$ is of type A_{2n} or A_{2n+1}. Then ι is conjugation by the longest element $\tilde{w}_0$ of $\tilde{W}$. We show that (a) holds.

Let Y be the set of all $E \in \mathrm{Irr}\tilde{W}$ (up to isomorphism) such that $\mathrm{tr}(\tilde{w}_0, E) \neq 0$. Let Y' be the set of all $E' \in \mathrm{Irr}W$ (up to isomorphism). By 23.4 and 23.1 we have a natural bijection between $c_{\tilde{W}}$ and the set of isomorphism classes of $E \in \mathrm{Irr}\tilde{W}$. If $\mathbf{c} \in c_{\tilde{W}}$ corresponds to E, then the number of fixed points of ι on $\mathbf{c}$ is clearly $\pm\dim(E)\mathrm{tr}(\tilde{w}_0, E)$. Hence $\sharp(c^\star_{\tilde{W}}) = \sharp Y$. From 23.1 we have $\sharp(\frac{\mathcal{G}_W}{\sim}) = \sharp Y$. Hence to show (a) it suffices to show that $\sharp Y = \sharp Y'$. But this is shown in [L3].

Case 4. Assume that $\tilde{W}$ is of type D_n and W is of type B_{n-1} with $n \geq 3$. We will show that (a) holds. We change notation and write W' instead of $\tilde{W}$, W'^ι instead of W. Then W' is as in 22.26 and we may assume that $\iota : W' \to W'$ is conjugation by s_n (as in 22.26). Let $\mathcal{M}^{N,!}_{1,0;n}$ be the set of all elements in $\mathcal{M}^N_{1,0;n}$ whose set of singles is non-empty. Let

$$\mathcal{M}^!_{1,0;n} = \lim_{N\to\infty} \mathcal{M}^{N,!}_{1,0;n}.$$

By 22.26 and 23.3, $c^!_{W'}$ is naturally in bijection with $\mathcal{M}^!_{1,0;n}$. By 23.1, $\frac{\mathcal{G}_{W'^\iota}}{\sim}$ is naturally in bijection with $\mathcal{M}_{1,2;n-1}$. The identity map is clearly a bijection $\mathcal{M}^N_{1,2;n-1} \xrightarrow{\sim} \mathcal{M}^{N+1,!}_{1,0;n}$. It induces a bijection $\mathcal{M}_{1,2;n-1} \xrightarrow{\sim} \mathcal{M}^!_{1,0;n}$. Hence to prove that $\sharp(\frac{\mathcal{G}_{W'^\iota}}{\sim}) \leq \sharp(c^\star_{W'})$ it suffices to prove that $\sharp(\mathcal{M}^!_{1,0;n}) = \sharp(\mathcal{M}^\star_{1,0;n})$. In other words, we must show that

(c) *any ι-stable two-sided cell of W' meets W'^ι.*

Now 22.26 and 23.3 provide an inductive procedure to obtain any ι-stable two-sided cell of W'. Namely such a cell is obtained by one of two procedures:

(i) we consider a ι-stable two-sided cell in a parabolic subgroup of type $\mathfrak{S}_k \times D_{n-k}$ (where $n-k \in [2, n-1]$) and we attach to it the unique two-sided cell of W' that contains it;

(ii) we take a two-sided cell obtained in (i) and multiply it on the right by the longest element of W'.

Since we may assume that (c) holds when n is replaced by $n-k \in [2, n-1]$, we see that the procedures (i) and (ii) yield only two-sided cells that contain ι-fixed elements. This proves (c). The proposition is proved.

24. Virtual cells

24.1. In this chapter we preserve the setup of 20.1.

A *virtual cell* of W (with respect to $L : W \to \mathbf{N}$) is an element of $K(W)$ of the form γ_x (see 20.16) for some $x \in W$.

Lemma 24.2. *Let $x \in W$ and let Γ be the left cell containing x.*

(a) If $\gamma_x \neq 0$ then $x \in \Gamma \cap \Gamma^{-1}$.

(b) γ_x is a $\mathbf{C}$-linear combination of $E \in \mathrm{Irr}W$ such that $[E : [\Gamma]] \neq 0$.

Assume that $\gamma_x \neq 0$. Then there exists $\mathcal{E} \in \mathrm{Irr}J_{\mathbf{C}}$ such that $\mathrm{tr}(t_x, \mathcal{E}) \neq 0$. We have $\mathcal{E} = \oplus_{d\in\mathcal{D}} t_d\mathcal{E}$ and $t_x : \mathcal{E} \to \mathcal{E}$ maps the summand $t_d\mathcal{E}$ (where $x \sim_{\mathcal{L}} d$) into $t_{d'}\mathcal{E}$, where $d' \sim_{\mathcal{L}} x^{-1}$ and all other summands to 0. Since $\mathrm{tr}(t_x, \mathcal{E}) \neq 0$, we must have $t_d\mathcal{E} = t_{d'}\mathcal{E} \neq 0$ hence $d = d'$ and $x \sim_{\mathcal{L}} x^{-1}$. This proves (a).

We prove (b). Let $d \in \mathcal{D} \cap \Gamma$. Assume that $E \in \mathrm{Irr}W$ appears with $\neq 0$ coefficient in γ_x. Then $\mathrm{tr}(t_x, E_\spadesuit) \neq 0$. As we have seen in the proof of (a), we have $t_d E_\spadesuit \neq 0$. Using 21.3,21.2, we deduce $[E_\spadesuit : J^\Gamma_{\mathbf{C}}] \neq 0$ and $[E_\spadesuit : [\Gamma]_\spadesuit] \neq 0$. Hence $[E : [\Gamma]] \neq 0$. The lemma is proved.

24.3. In the remainder of this chapter we will give a number of explicit computations of virtual cells.

Lemma 24.4. *In the setup of 22.10, w_0 acts on $[\Lambda]$ as multiplication by* $\epsilon_{[\Lambda]} = (-1)^{\sum_j (a^{-1}\mu_j - j+1)}$.

Using the definitions we are reduced to the case where $k = n$ or $l = n$. If $k = n$ we have $\epsilon_{[\Lambda]} = 1$ since $[\Lambda]$ factors through $\mathfrak{S}_n$ and the longest element of W_n is in the kernel of $W_n \to \mathfrak{S}_n$. Similarly, if $l = n$ we have $\epsilon_{[\Lambda]} = \epsilon_{\chi_n} = (-1)^n$. The lemma is proved.

Proposition 24.5. *Assume that we are in the setup of 22.23. Let $\iota \in \mathrm{Inv}_r(Z)$ and let $\kappa : \mathcal{S}_\iota \to \mathbf{F}_2$ be an affine-linear function. Let* $c(\tilde{Z}, \iota, \kappa) = \sum_{Y \in \mathcal{S}_\iota} (-1)^{\kappa(Y)} [\Lambda_Y] \in K(W)$.
There exists $x \in W$ such that $\gamma_x = \pm c(\tilde{Z}, \iota, \kappa)$.

To some extent the proof is a repetition of the proof of 22.24(a), but we have to keep track of κ, a complicating factor.

We argue by induction on the rank n of $\tilde{Z}$. If $n = 0$ the result is clear. Assume now that $n \geq 1$. We may assume that 0 is not a double of $\tilde{Z}$. Let at be the largest entry of $\tilde{Z}$.

(A) Assume that there exists $i, 0 \leq i < t$, such that ai does not appear in $\tilde{Z}$. Then $\tilde{Z}$ is obtained from a multiset $\tilde{Z}'$ of rank $n-k < n$ by increasing each of the k largest entries by a and this set of largest entries is unambiguously defined. The set Z' of singles of $\tilde{Z}'$ is naturally in bijection with Z.

Let ι', κ' correspond to ι, κ under this bijection. By the induction hypothesis, there exists $x' \in W_{n-k}$ such that $\gamma_{x'}^{W_{n-k}} = \pm c(\tilde{Z}', \iota', \kappa') \in K(W_{n-k})$. Let $w_{0,k}$ be the longest element of $\mathfrak{S}_k$. Then

(a) $$\gamma_{w_{0,k}x'}^{\mathfrak{S}_k \times W_{n-k}} = \gamma_{w_{0,k}}^{\mathfrak{S}_k} \boxtimes \gamma_{x'}^{W_{n-k}} = \mathrm{sgn}_k \boxtimes \gamma_{x'}^{W_{n-k}}$$

and

(b) $$\gamma_{w_{0,k}x'}^{W} = \mathbf{j}_{\mathfrak{S}_k \times W_{n-k}}^{W} (\gamma_{w_{0,k}x'}^{\mathfrak{S}_k \times W_{n-k}})$$
$$= \mathbf{j}_{\mathfrak{S}_k \times W_{n-k}}^{W} (\mathrm{sgn}_k \boxtimes \gamma_{x'}^{W_{n-k}}) = \pm \mathbf{j}_{\mathfrak{S}_k \times W_{n-k}}^{W} (\mathrm{sgn}_k \boxtimes c(\tilde{Z}', \iota', \kappa')) = \pm c(\tilde{Z}, \iota, \kappa),$$

as required.

(B) Assume that there exists $i, 0 < i \leq t$ such that ai is a double of $\tilde{Z}$. Let $\bar{\tilde{Z}}$ be as in 22.8 (with respect to our t). Then 0 is not a double of $\bar{\tilde{Z}}$ and the largest entry of $\bar{\tilde{Z}}$ is at. Let $\bar{Z}$ be the set of singles of $\bar{\tilde{Z}}$. We have $\bar{Z} = \{at - z; z \in Z\}$. Thus $\bar{Z}, Z$ are naturally in (order reversing) bijection under $j \mapsto at - j$. Let $\iota' \in \mathrm{Inv}_r(\bar{Z})$ correspond to ι under this bijection and let $\kappa' : \mathcal{S}_{\iota'} \to \mathbf{F}_2$ correspond to κ under this bijection. Define $\kappa'' : \mathcal{S}_{\iota'} \to \mathbf{F}_2$ by $\kappa''(Y) = \kappa'(Y) + \sum_{y \in Y} a^{-1}y$ (an affine-linear function). Since $at - ai$ does not appear in $\bar{\tilde{Z}}$, (A) is applicable to $\bar{\tilde{Z}}$. Hence there exists $x' \in W$ such that $\gamma_{x'} = \pm c(\bar{\tilde{Z}}, \iota', \kappa'')$. By 20.23, 22.18, 24.4, we have

$$\begin{aligned}\gamma_{x'w_0} \otimes \mathrm{sgn} &= (-1)^{l(x')}\zeta(\gamma_{x'}) = \pm\zeta(c(\bar{\tilde{Z}}, \iota', \kappa'')) \otimes \mathrm{sgn} \\ &= \pm c(\bar{\tilde{Z}}, \iota', \kappa') \otimes \mathrm{sgn} = \pm c(\tilde{Z}, \iota, \kappa),\end{aligned}$$

as desired.

(C) Assume that we are not in case (A) and not in case (B). Then $\tilde{Z} = \{0, a, 2a, \dots, ta\} = Z$. We can find $ia, (i+1)a$ in Z such that ι interchanges $ia, (i+1)a$ and induces on $Z - \{ia, (i+1)a\}$ an r-admissible involution ι_1.

(C1) Assume first that $\kappa(Y) = \kappa(Y * \{ia, (i+1)a\})$ for any $Y \in \mathcal{S}_\iota$. ($*$ is symmetric difference.) Let

$$\tilde{Z}' = \{0, a, 2a, \dots, ia, ia, (i+1)a, (i+2)a, \dots, (t-1)a\}.$$

This has rank $n - k < n$. The set of singles of $\tilde{Z}'$ is

$$Z' = \{0, a, 2a, \dots, (i-1)a, (i+1)a, \dots, (t-1)a\}.$$

It is in natural (order preserving) bijection with $Z - \{ia, (i+1)a\}$. Hence ι_1 induces $\iota' \in \mathrm{Inv}_r(Z')$. We have an obvious surjective map of affine spaces $\pi : \mathcal{S}_\iota \to \mathcal{S}_{\iota'}$ and κ is constant on the fibres of this map. Hence there is an affine-linear map $\kappa' : \mathcal{S}_{\iota'} \to \mathbf{F}_2$ such that $\kappa = \kappa'\pi$. By the induction hypothesis, there exists $x' \in W_{n-k}$ such that $\gamma_{x'}^{W_{n-k}} = \pm c(\tilde{Z}', \iota', \kappa') \in K(W_{n-k})$. Let $w_{0,k}$ be the longest element of $\mathfrak{S}_k$. Then (a), (b) hold and we are done.

(C2) Assume next that $\kappa(Y) \neq \kappa(Y * \{ia, (i+1)a\})$ for some (or equivalently any) $Y \in \mathcal{S}_\iota$. We have

$$\bar{\tilde{Z}} = \{0, 0, a, a, 2a, 2a, \dots, ta, ta\} - \{at - 0, at - a, \dots, at - at\} = \tilde{Z} = Z.$$

Let $\iota' \in \mathrm{Inv}_r(Z)$ correspond to ι under the bijection $z \mapsto ta - z$ of Z with itself; let $\kappa' : \mathcal{S}_{\iota'} \to \mathbf{F}_2$ correspond to κ under this bijection. Let $\kappa'' : \mathcal{S}_{\iota'} \to \mathbf{F}_2$ be given by $\kappa''(Y) = \kappa'(Y) + \sum_{y \in Y} a^{-1}y$ (an affine-linear function). Note that ι' interchanges $(t-i-1)a, (t-i)a$ and induces on $Z - \{(t-i-1)a, (t-i)a\}$ an r-admissible involution. We show that for any $Y \in \mathcal{S}_{\iota'}$ we have

$\kappa''(Y) = \kappa''(Y * \{(t-i-1)a, (t-i)a\})$,

or equivalently

$\kappa'(Y) = \kappa'(Y * \{(t-i-1)a, (t-i)a\}) + 1$.
This follows from our assumption $\kappa(Y) = \kappa(Y * \{ia, (i+1)a\}) + 1$ for any $Y \in \mathcal{S}_\iota$. We see that case (C1) applies to ι', κ'' so that there exists $x' \in W$ with $\gamma_{x'} = \pm c(\tilde{Z}, \iota', \kappa'')$. By 20.23, 22.18, 24.4, we have

$$\gamma_{x'w_0} = (-1)^{l(x')}\zeta(\gamma_{x'}) \otimes \mathrm{sgn} = \pm\zeta(c(\bar{\tilde{Z}}, \iota', \kappa'')) \otimes \mathrm{sgn}$$
$$= \pm c(\bar{\tilde{Z}}, \iota', \kappa') \otimes \mathrm{sgn} = \pm c(\tilde{Z}, \iota, \kappa),$$

as desired. The proposition is proved.

24.6. Assume that we are in the setup of 22.27. By 22.27,

$$\rho_4 + \rho_{16}, \rho_{12} + \rho_{16} + \rho_6, \rho_{12} + \rho_{16} + \rho_6'$$

are constructible, hence (by 22.2, 21.4) are of the form $n_d\gamma_d$ for suitable $d \in \mathcal{D}$, hence are $\pm$ virtual cells.

Let $d \in \mathcal{D}$ be such that $n_d\gamma_d = \rho_{12} + \rho_{16} + \rho_6$. Let Γ be the left cell that contains d. Recall (21.4) that $[\Gamma] = A \oplus B \oplus C$ where $A = \rho_{12}, B = \rho_{16}, C = \rho_6$. By the discussion in 21.10 we see that $J_{\mathbf{C}}^{\Gamma\cap\Gamma^{-1}}$ has exactly three simple modules (up to isomorphism), namely $t_dA_\spadesuit, t_dB_\spadesuit, t_dC_\spadesuit$, and these are 1-dimensional. Since $J^{\Gamma\cap\Gamma^{-1}}$ is a semisimple algebra (21.9), it follows that it is commutative of dimension 3. Hence $\Gamma \cap \Gamma^{-1}$ consists of three elements d, x, y. Let p_A, p_B, p_C denote the traces of t_x on $A_\spadesuit, B_\spadesuit, C_\spadesuit$ respectively. Let q_A, q_B, q_C denote the traces of t_y on $A_\spadesuit, B_\spadesuit, C_\spadesuit$ respectively. By 20.26, $p_A, p_B, p_C, q_A, q_B, q_C$ are integers. Recall that the traces of n_dt_d on $A_\spadesuit, B_\spadesuit, C_\spadesuit$ are $1, 1, 1$ respectively. Since $f_{A_\spadesuit}, f_{B_\spadesuit}, f_{C_\spadesuit}$ are $6, 2, 3$ we see that the orthogonality formula 21.10 gives

$$1 + p_A^2 + q_A^2 = 6, 1 + p_B^2 + q_B^2 = 2, 1 + p_C^2 + q_C^2 = 3,$$
$$1 + p_Ap_B + q_Aq_B = 0, 1 + p_Ap_C + q_Aq_C = 0, 1 + p_Bp_C + q_Bq_C = 0.$$

Solving these equations with integer unknowns we see that there exist $\epsilon, \epsilon' \in \{1, -1\}$ so that (up to interchanging x, y) we have

$$(p_A, q_A) = (2\epsilon, \epsilon'), (p_B, q_B) = (0, -\epsilon'), (p_C, q_C) = (-\epsilon, \epsilon').$$

Then $\epsilon\gamma_x = 2\rho_{12} - \rho_6$, $\epsilon'\gamma_y = \rho_{12} - \rho_{16} + \rho_6$. Hence
$2\rho_{12} - \rho_6, \rho_{12} - \rho_{16} + \rho_6$ are $\pm$ virtual cells.
The same argument shows that $2\rho_{12} - \rho_{6'}, \rho_{12} - \rho_{16} + \rho_{6'}$ are $\pm$ virtual cells. A similar (but simpler) argument shows that $\rho_4 - \rho_{16}$ is $\pm$ a virtual cell.

Assume now that we are in the setup of 22.27 (Case 1). By 22.27,

$$\rho_1 + \rho_2, \rho_1 + \rho_8, \rho_2 + \rho_9, \rho_8 + \rho_9, \rho_1^\dagger + \rho_2^\dagger, \rho_1^\dagger + \rho_8^\dagger, \rho_2^\dagger + \rho_9^\dagger, \rho_8^\dagger + \rho_9^\dagger,$$

are constructible, hence by 22.2, 21.4 are of the form $n_d\gamma_d$ for suitable $d \in \mathcal{D}$, hence are $\pm$ virtual cells. By an argument similar to that above (but simpler) we see that

$$\rho_1 - \rho_2, \rho_1 - \rho_8, \rho_2 - \rho_9, \rho_8 - \rho_9, \rho_1^\dagger - \rho_2^\dagger, \rho_1^\dagger - \rho_8^\dagger, \rho_2^\dagger - \rho_9^\dagger, \rho_8^\dagger - \rho_9^\dagger,$$

are $\pm$ virtual cells.

24.7. Assume that we are in the setup of 22.29. By 22.29, $\rho_2+\rho'_2$ is constructible, hence by 22.2, 21.4, is of the form $n_d\gamma_d$ for some $d\in\mathcal{D}$, hence is $\pm$ a virtual cell. As in 24.6, we see that $\rho_2-\rho'_2$ is $\pm$ a virtual cell.

25. Relative Coxeter groups

25.1. Let W,S be a Coxeter group and let $u\in A_W$ (see 1.17). We assume that W is a Weyl group or an affine Weyl group. Let J be a u-stable subset of S such that W_J is finite (that is, $J\ne S$ when W is infinite). Let $U:W\to\{\text{permutations of } R\}$ be as in 1.5. Let $\mathcal{W}$ be the set of all $w\in W$ such that $U(w)$ carries $\{(1,s);s\in J\}$ onto itself. (A subgroup of W.) Alternatively,

$$\mathcal{W}=\{w\in W;wW_J=W_Jw, w \text{ has minimal length in } wW_J=W_Jw\}.$$

Let K be the set of all u-orbits k on $S-J$ such that $W_{J\cup k}$ is finite. (In the case where W is infinite, K consists of all u-orbits on $S-J$ if $\sharp(u\backslash(S-J))\ge 2$ and $K=\emptyset$ if $\sharp(u\backslash(S-J))=1$.) We assume that J is u-*excellent* in the following sense: for any $k\in K$ we have $w_0^{J\cup k}Jw_0^{J\cup k}=J$.

For $k\in K$ we have $w_0^{J\cup k}w_0^Jw_0^{J\cup k}=w_0^J$ hence

$$\tau_k:=w_0^{J\cup k}w_0^J=w_0^Jw_0^{J\cup k}$$

satisfies $\tau_k^2=1$.

If $k\in K$ then $U(w_0^{J\cup k})$ maps $\{(1,s);s\in J\cup k\}$ onto $\{(-1,s);s\in J\cup k\}$. It also maps $\{(\pm 1,s);s\in J\}$ onto $\{(\pm 1,s);s\in J\}$. Hence it maps $\{(1,s);s\in J\}$ onto $\{(-1,s);s\in J\}$. Similarly, $U(w_0^J)$ maps $\{(-1,s);s\in J\}$ onto $\{(1,s);s\in J\}$. Hence $U(\tau_k)=U(w_0^J)U(w_0^{J\cup k})$ maps $\{(1,s);s\in J\}$ onto $\{(1,s);s\in J\}$. Thus, $\tau_k\in\mathcal{W}$. More precisely, $\tau_k\in\mathcal{W}^u$, the fixed point set of $u:\mathcal{W}\to\mathcal{W}$.

The following result is proved in [L1] assuming that W is a Weyl group (see [L13] for the case where W is an affine Weyl group).

(a) $\mathcal{W}^u$ *is a Coxeter group on the generators* $\{\tau_k;k\in K\}$. *Moreover, if* W *is a Weyl group then* $\mathcal{W}^u$ *is a Weyl group; if* W *is an affine Weyl group and* $\sharp(u\backslash(S-J))\ge 2$ *then* $\mathcal{W}^u$ *is an affine Weyl group; if* W *is an affine Weyl group and* $\sharp(u\backslash(S-J))=1$ *then* $\mathcal{W}^u=\{1\}$.

25.2. We now strengthen our assumption on J by assuming that there exists an adjoint reductive group G_J defined over $\mathbf{F}_q$ whose Coxeter graph is J (a full subgraph of the Coxeter graph of W), such that $u:J\to J$ is induced by the Frobenius map of G_J and that $G_J(\mathbf{F}_q)$ admits a unipotent cuspidal representation E; let $\mathbf{c}_0$ be the two-sided cell of W_J (with the weight function given by length) corresponding to this unipotent representation in the classification [L6]. The function $\mathbf{a}:W\to\mathbf{N}$ (see 13.6) (defined in terms of the weight function given by the length) takes a constant value a on $\mathbf{c}_0$ and a constant value a_k on $\mathbf{c}_0\tau_k$ for $k\in K$ (see 9.13, P.11, 15.6). The function $\{\tau_k;k\in K\}\to\mathbf{Z}$ given by $\tau_k\mapsto a_k-a$ takes equal values at two elements $\tau_k,\tau_{k'}$ that are conjugate in $\mathcal{W}^u$ (case by case check) hence it is the restriction of a weight function $L : \mathcal{W}^u \to \mathbf{Z}$. This weight function takes > 0 values on $\{\tau_k; k \in K\}$. Let $\mathbf{a}_L : \mathcal{W}^u \to \mathbf{N}$ be the function defined like $\mathbf{a} : W \to \mathbf{N}$ (see 13.6) in terms of $\mathcal{W}^u$ (instead of W) and the weight function just defined. Define $\mathbf{a}' : \mathcal{W}^u \to \mathbf{N}$ by $\mathbf{a}'(x) = \mathbf{a}(yx)$ where y is any element of $\mathbf{c}_0$. This is independent of the choice of y, by 9.13, P.11, 15.6.

Conjecture 25.3. *(a)* $\mathbf{a}_L = \mathbf{a}'$.

(b) Let $\mathbf{c}$ *be a two-sided cell of* $\mathcal{W}^u$ *(relative to the weight function* L*) as in 25.2. There exists a (necessarily unique) two-sided cell* $\tilde{\mathbf{c}}$ *of* W *(relative to the weight function given by length) such that* $yx \in \tilde{\mathbf{c}}$ *for any* $y \in \mathbf{c}_0, x \in \mathbf{c}$. *Moreover the map* $\mathbf{c} \mapsto \tilde{\mathbf{c}}$ *is injective.*

This would reduce the problem of computing the two-sided cells of $\mathcal{W}^u$ (relative to the weight function L) to the analogous problem for W (relative to the weight function given by length).

26. Representations

26.1. Let W, S be an affine Weyl group and let $u \in A_W$ (see 1.17). Let J be a u-stable subset of S with $J \neq S$. Let $\mathcal{U}(J)$ be the set of isomorphism classes of unipotent cuspidal representations of $G_J(\mathbf{F}_q)$ (as in 25.2). Note that $\mathcal{U}(J)$ is independent of the choice of G_J. Let $E \in \mathcal{U}(J)$. Let $\mathcal{H}(W, J, E)$ be the Iwahori-Hecke algebra attached to $\mathcal{W}^u$ (defined as in 25.1 in terms of W, S, J) and to the weight function $L : \mathcal{W}^u \to \mathbf{N}$ (defined as in 25.2). Let Ω be as in 1.18. Let

$$\Omega^u = \{a \in \Omega; ua = au\}, \Omega^{u,J} = \{a \in \Omega^u; a(J) = J\}.$$

If $a \in \Omega^{u,J}$ then $a : W \to W$ restricts to an automorphism of $\mathcal{W}^u$ as a Coxeter group; this automorphism is compatible with the weight function $L : \mathcal{W}^u \to \mathbf{N}$ hence it induces an automorphism of the algebra $\mathcal{H}(W, J, E)$. Hence we may form a semidirect product algebra $\mathcal{H}(W, J, E) \otimes_{\mathcal{A}} \mathcal{A}[\Omega^{u,J}]$ where $\mathcal{A}[\Omega^{u,J}]$ is the group algebra of $\Omega^{u,J}$ over $\mathcal{A}$.

Let $v_0 \in \mathbf{C}^*$ be such that $v_0 = 1$ or v_0 is not a root of 1. Let

$$(\mathcal{H}(W, J, E) \otimes_{\mathcal{A}} \mathcal{A}[\Omega^{u,J}])_{v_0}$$

be the $\mathbf{C}$-algebra obtained from $\mathcal{H}(W, J, E) \otimes_{\mathcal{A}} \mathcal{A}[\Omega^{u,J}]$ by the change of scalars $\mathcal{A} \to \mathbf{C}, v \mapsto v_0$. Let

$$\mathcal{I} = \sqcup \mathrm{Irr}(\mathcal{H}(W, J, E) \otimes_{\mathcal{A}} \mathcal{A}[\Omega^{u,J}])_{v_0}$$

where Irr stands for the set of isomorphism classes of simple modules of an algebra and the disjoint union is taken over all (J, E) as above modulo the action of Ω^u.

On the other hand, let $\mathcal{G}$ be a connected, simply connected almost simple reductive group over $\mathbf{C}$, of type "dual" to that of W. Let $A(\mathcal{G})$ be the group of automorphisms of $\mathcal{G}$ modulo the group of inner automorphisms of $\mathcal{G}$. There is a natural action of $A(\mathcal{G})$ on $\mathcal{G}$ (well defined up to conjugacy) and we form the semidirect product $\tilde{\mathcal{G}}$ of $\mathcal{G}$ and $A(\mathcal{G})$ via this action. Note that $\mathcal{G}$ may be identified with the identity component of $\tilde{\mathcal{G}}$. Let $\mathcal{J}$ be the set of all pairs $(C, \mathcal{E})$ where C is a $\mathcal{G}$-conjugacy class in $\tilde{\mathcal{G}}$ and $\mathcal{E}$ is an irreducible $\mathcal{G}$-equivariant local system on C.

Theorem 26.2. *There is a natural bijection $\mathcal{I} \leftrightarrow \mathcal{J}$.*

This is shown in [L13],[L16]. Using this bijection we may transfer the partition of $\mathcal{I}$ into pieces indexed by the various (J, E) into a partition of $\mathcal{J}$ into pieces again indexed by the various (J, E). This partition can be described purely in terms of the geometry of $\tilde{\mathcal{G}}$ (see [L16]).

27. A new realization of Hecke algebras

27.1. Let $G, F, \mathcal{P}, \mathbf{E}, W, S, J, \mathcal{W}, u, \dots$ be as in 0.3. Let $H = H(G^F, \mathcal{P}^F, \mathbf{E})$. In this chapter we give a new realization of the Hecke algebra H as a function space. We will identify $\bar{\mathbf{Q}}_l = \mathbf{C}$ (where l is a prime number invertible in $\mathbf{F}_q$) via some field isomorphism.

Let $P_0 \in \mathcal{P}^F$. Let L be an F-stable Levi subgroup of P_0, NL the normalizer of L in G, Z_L the centre of L. Let $M = NL/Z_L$. We have canonically $NL/L = \mathcal{W}$ hence $M = \sqcup_{w \in \mathcal{W}} M_w$ where M_w is the inverse image of w under the obvious map $NL/Z_L \to NL/L$. We have $M_1 = L/Z_L = L_{ad}$. The conjugation action defines an (injective) homomorphism $M \to Aut(L)$ which restricts, for any $w \in \mathcal{W}$, to an

(a) isomorphism of M_w onto an L_{ad}-coset $Aut(L)_w$ in $Aut(L)$.

By known properties of unipotent representations, there is a unique L_{ad}^F-module structure on $\mathbf{E}_{P_0}$ that extends the given P_0^F-module structure on $\mathbf{E}_{P_0}$ via the obvious homomorphism $P_0^F \to L_{ad}^F$. We choose an M^F-module structure $\iota : M^F \to GL(\mathbf{E}_{P_0})$ on $\mathbf{E}_{P_0}$ that extends this L_{ad}^F-module structure. (This exists by known properties of unipotent cuspidal representations.)

Let $w \in \mathcal{W}$ and let $\mathcal{O}_w$ be the corresponding good G-orbit on $\mathcal{P} \times \mathcal{P}$. For $(P_1, P_2) \in \mathcal{O}_w$ let $\bar{P}_2 \xrightarrow{\psi_{P_1}^{P_2}} \bar{P}_1$ be the unique isomorphism which takes the image of any $x \in P_1 \cap P_2$ under $P_1 \cap P_2 \to \bar{P}_2$ to the image of $x \in P_1 \cap P_2$ under $P_1 \cap P_2 \to \bar{P}_1$. Then the composition

$$\bar{P}_0 \xrightarrow{\mathrm{Ad}(g_2)} \bar{P}_2 \xrightarrow{\psi_{P_1}^{P_2}} \bar{P}_1 \xrightarrow{\mathrm{Ad}(g_1^{-1})} \bar{P}_0$$

where $g_1, g_2 \in G$, $g_1 P_0 g_1^{-1} = P_1, g_2 P_0 g_2^{-1} = P_2$, may be regarded as an element of $Aut(L)_w$ (we identify $\bar{P}_0 = L$). This corresponds under (a) to an element $\alpha_{g_1, g_2} \in M_w$.

27.2. Assume now that $F(w) = w$. Define ${}^w\phi \in H$ as follows: if $(P_1, P_2) \in \mathcal{O}_w$ then $({}^w\phi)_{P_1}^{P_2} : \mathbf{E}_{P_2} \to \mathbf{E}_{P_1}$ is the composition

$$\mathbf{E}_{P_2} \xrightarrow{g_2^{-1}} \mathbf{E}_{P_0} \xrightarrow{\iota(\alpha_{g_1, g_2})} \mathbf{E}_{P_0} \xrightarrow{g_1} \mathbf{E}_{P_1}$$

where $g_1, g_2 \in G^F$, $g_2 P_0 g_2^{-1} = P_2, g_1 P_0 g_1^{-1} = P_1$; if $(P_1, P_2) \notin \mathcal{O}_w$ then $({}^w\phi)_{P_1}^{P_2} : \mathbf{E}_{P_2} \to \mathbf{E}_{P_1}$ is 0. ($({}^w\phi)_{P_1}^{P_2}$ is independent of the choices of g_1, g_2.)

27.3. For w as in 27.2 we have $\mathcal{O}_{w^{-1}} = \{(P_2, P_1) \in \mathcal{P} \times \mathcal{P}; (P_1, P_2) \in \mathcal{O}_w\}$. Let $\mathcal{U} = \{P_1 \in \mathcal{P}^F; (P_0, P_1) \in \mathcal{O}_w\}$.
Then $\sharp\mathcal{U} = q^{l(w)}$ where l is length in W. The composition $({}^w\phi)({}^{w^{-1}}\phi)$ has as (P_0, P_0)-component the sum over all $P_1 \in \mathcal{U}$ of the compositions

$$\mathbf{E}_{P_0} \xrightarrow{\iota(\alpha_{g_1,1})} \mathbf{E}_{P_0} \xrightarrow{g_1} \mathbf{E}_{P_1} \xrightarrow{g_1^{-1}} \mathbf{E}_{P_0} \xrightarrow{\iota(\alpha_{1,g_1})} \mathbf{E}_{P_0}$$

where $g_1 \in G^F$, $g_1 P_0 g_1^{-1} = P_1$, that is $q^{l(w)}$ times the identity map of $\mathbf{E}_{P_0}$. Thus,
(a) $({}^w\phi)({}^{w^{-1}}\phi) = q^{l(w)}({}^1\phi) +$ linear combination of ${}^{w'}\phi$ with $w' \neq 1$.

27.4. Let $w, w' \in \mathcal{W}^u$ be such that $l(ww') = l(w) + l(w')$. (Here l is length in W.) Then
(a) $(P_1, P_2) \in \mathcal{O}_w, (P_2, P_3) \in \mathcal{O}_{w'} \implies (P_1, P_3) \in \mathcal{O}_{ww'}$,
(b) if $(P_1, P_3) \in \mathcal{O}_{ww'}$ then there is a unique $P_2 \in \mathcal{P}$ such that $(P_1, P_2) \in \mathcal{O}_w, (P_2, P_3) \in \mathcal{O}_{w'}$.
If P_1, P_2, P_3 are as in (a) we have $\psi_{P_1}^{P_3} = \psi_{P_1}^{P_2}\psi_{P_2}^{P_3} : \mathbf{E}_{P_3} \to \mathbf{E}_{P_1}$. From the definitions we see that
(c) $({}^w\phi)({}^{w'}\phi) = {}^{ww'}\phi$.

27.5. For $w \in \mathcal{W}^u$, ${}^w\phi$ is a basis element of $H_{\mathcal{O}_w}$. If $w = t_{k_1}t_{k_2}\dots t_{k_r}$ is a reduced expression in $\mathcal{W}^u$ (see 0.3) then $l(w) = l(t_{k_1}) + l(t_{k_2}) + \dots + l(t_{k_r})$ (where l is as in 27.4) and $T_w = T_{\tau_{k_1}}T_{\tau_{k_2}}\dots T_{\tau_{k_r}}$ (notation of 0.3) is a well defined basis element of $H_{\mathcal{O}_w}$ independent of the reduced expression. Hence ${}^w\phi = x_w T_w$ where
(a) $x_w = x_{\tau_{k_1}}x_{\tau_{k_2}}\dots x_{\tau_{k_r}}$
with $x_w \in \mathbf{C}^*$ for all $w \in \mathcal{W}^u$. From 27.4(c) we see that ${}^1\phi$ is the unit element of H. Hence ${}^1\phi = T_1$. By 27.3(a), we have $({}^{\tau_k}\phi)({}^{\tau_k}\phi) = q^{l(\tau_k)}({}^1\phi) + \dots$ hence
(b) $x_{\tau_k}^2 T_{\tau_k}^2 = q^{l(\tau_k)}T_1 +$ linear combination of $T_{w'}$ with $w' \neq 1$.
On the other hand, by 0.3(d) we have $T_{\tau_k}^2 = (q^{N_k/2} - q^{-N_k/2})T_{\tau_k} + T_1$. Comparing with (b) we see that $x_{\tau_k}^2 = q^{l(\tau_k)}$ hence $x_{\tau_k} = \epsilon_k q^{l(\tau_k)/2}$ where $\epsilon_k \in \{1, -1\}$. From (a) we see that for $w \in \mathcal{W}^u$ we have $x_w = \epsilon_w q^{l(w)/2}$ where $w \mapsto \epsilon_w$ is a function $\mathcal{W}^u \to \{1, -1\}$ satisfying $\epsilon_k\epsilon_{k'}\epsilon_k \dots = \epsilon_{k'}\epsilon_k\epsilon_{k'}\dots$ for $k \neq k'$ (both products have a number of terms equal to the order of $\tau_k\tau_{k'}$ in $\mathcal{W}^u$). It follows that $w \mapsto \epsilon_w$ is a group homomorphism $\mathcal{W}^u \to \{1, -1\}$. Since $M^F/L_{ad}^F = \mathcal{W}^u$, we may regard ϵ as a homomorphism $M^F \to \mathbf{C}^*$ which is trivial on L_{ad}^F.

27.6. Replacing $\iota : M^F \to GL(\mathbf{E}_{P_0})$ by its tensor product with $\epsilon : M^F \to \mathbf{C}^*$ we obtain a new homomorphism $\iota_0 : M^F \to GL(\mathbf{E}_{P_0})$. If we now redefine ${}^w\phi$ in terms of ι_0 rather than ι, then the ϵ-factors disappear and we have
(a) ${}^w\phi = q^{l(w)/2}T_w, w \in \mathcal{W}^u$.
Note that ι_0 is uniquely determined by property (a) and by its restriction to L_{ad}^F.

27.7. Let $D = \dim \mathbf{E}_{P_0}$. Let Y be the set of all triples (P, P', gU_P) where $P, P' \in \mathcal{P}$ and $gU_P \in G/U_P$ is such that $gPg^{-1} = P'$ (hence $gU_P = U_{P'}g$). Now Y is naturally defined over $\mathbf{F}_q$, with Frobenius map

$F:(P,P',gU_P)\to(F(P),F(P'),F(g)U_{F(P)})$.
Let Y_0 be the set of all triples (P,P',gU_P^F) where $P,P'\in\mathcal{P}^F$ and $gU_P^F\in G^F/U_P^F$ is such that $gPg^{-1}=P'$ (hence $gU_P^F=U_{P'}^Fg$). We have a bijection $Y_0\xrightarrow{\sim}Y^F$ given by $(P,P',gU_P^F)\mapsto(P,P',gU_P)$.

Let $\mathfrak{B}$ be the vector space of all functions $f:Y_0\to\mathbf{C}$. We define a multiplication $\mathfrak{B}\times\mathfrak{B}\to\mathfrak{B}$, $f',f''\mapsto f'*f''$ by

$$\text{(a)}\quad (f'*f'')(P,P',gU_P^F)=\frac{D}{\sharp\bar{P}^F}\sum_{\tilde{P},g'U_P^F,g''U_{\tilde{P}}^F}f'(P,\tilde{P},g'U_P^F)f''(\tilde{P},P',g''U_{\tilde{P}}^F)$$

where the sum is taken over all

$$\tilde{P}\in\mathcal{P}^F,g'U_P^F\in G^F/U_P^F,g''U_{\tilde{P}}^F\in G^F/U_{\tilde{P}}^F$$

such that

$$g'Pg'^{-1}=\tilde{P},g''\tilde{P}g''^{-1}=P',g''g'\in U_{P'}^Fg=gU_P^F.$$

Equivalently,

$$\text{(b)}\quad (f'*f'')(P,P',gU_P^F)=\frac{D}{\sharp\bar{P}^F}\sharp(U_P^F)^{-1}\sum_{\tilde{P},g'}f'(P,\tilde{P},g'U_P^F)f''(\tilde{P},P',gg'^{-1}U_{\tilde{P}}^F)$$

where the sum is taken over all $\tilde{P}\in\mathcal{P}^F,g'\in G^F$ such that $g'Pg'^{-1}=\tilde{P}$. With this multiplication, $\mathfrak{B}$ becomes an associative algebra.

Define $\kappa:H\to\mathfrak{B}$ by $\phi\mapsto\kappa(\phi)$ where $\kappa(\phi)(P,P',gU_P^F)$ is the trace of the composition

$$\mathbf{E}_P\xrightarrow{g}\mathbf{E}_{P'}\xrightarrow{\phi_P^{P'}}\mathbf{E}_P.$$

(This is independent of the choice of g in its U_P^F-coset; $\phi_P^{P'}$ is as in 0.1.)

Lemma 27.8. *$\kappa:H\to\mathfrak{B}$ is an algebra homomorphism.*

Let $\phi,\phi'\in H$ and let $(P,P',gU_P^F)\in Y_0$. Then

$$\kappa(\phi\phi')(P,P',gU_P^F)=\operatorname{tr}(\mathbf{E}_P\xrightarrow{g}\mathbf{E}_{P'}\xrightarrow{(\phi\phi')_P^{P'}}\mathbf{E}_P)$$
$$=\sum_{\tilde{P}}\operatorname{tr}(\mathbf{E}_P\xrightarrow{g}\mathbf{E}_{P'}\xrightarrow{\phi'^{P'}_{\tilde{P}}}\mathbf{E}_{\tilde{P}}\xrightarrow{\phi_P^{\tilde{P}}}\mathbf{E}_P).$$

On the other hand,

$$\begin{aligned}
&\kappa(\phi) * \kappa(\phi')(P, P', gU_P^F)\\
&= \sharp(U_P^F)^{-1} \frac{D}{\sharp \bar{P}^F} \sum_{\tilde{P}, g'; g'Pg'^{-1} = \tilde{P}} \kappa(\phi)(P, \tilde{P}, g'U_P^F)\kappa(\phi')(\tilde{P}, P', gg'^{-1}U_{\tilde{P}}^F)\\
&= \frac{D}{\sharp P^F} \sum_{\tilde{P}, g'; g'Pg'^{-1} = \tilde{P}} \mathrm{tr}(\mathbf{E}_P \xrightarrow{g'} \mathbf{E}_{\tilde{P}} \xrightarrow{\phi_P^{\tilde{P}}} \mathbf{E}_P)\mathrm{tr}(\mathbf{E}_{\tilde{P}} \xrightarrow{gg'^{-1}} \mathbf{E}_{P'} \xrightarrow{\phi_{\tilde{P}}^{P'}} \mathbf{E}_{\tilde{P}})\\
&= \frac{D}{\sharp P^F} \sum_{\tilde{P}, g'; g'Pg'^{-1} = \tilde{P}} \mathrm{tr}(\mathbf{E}_P \xrightarrow{g'} \mathbf{E}_{\tilde{P}} \xrightarrow{\phi_P^{\tilde{P}}} \mathbf{E}_P)\mathrm{tr}(\mathbf{E}_P \xrightarrow{g} \mathbf{E}_{P'} \xrightarrow{\phi_{\tilde{P}}^{P'}} \mathbf{E}_{\tilde{P}} \xrightarrow{g'^{-1}} \mathbf{E}_P).
\end{aligned}$$

It is then enough to show that for any $\tilde{P} \in \mathcal{P}^F$, we have

$$\begin{aligned}
&\mathrm{tr}(\mathbf{E}_P \xrightarrow{g} \mathbf{E}_{P'} \xrightarrow{\phi'^{P'}_{\tilde{P}}} \mathbf{E}_{\tilde{P}} \xrightarrow{\phi_P^{\tilde{P}}} \mathbf{E}_P)\\
&= \frac{D}{\sharp P^F} \sum_{\substack{g' \in G^F\\ g'Pg'^{-1} = \tilde{P}}} \mathrm{tr}(\mathbf{E}_P \xrightarrow{g'} \mathbf{E}_{\tilde{P}} \xrightarrow{\phi_P^{\tilde{P}}} \mathbf{E}_P)\mathrm{tr}(\mathbf{E}_P \xrightarrow{g} \mathbf{E}_{P'} \xrightarrow{\phi_{\tilde{P}}^{P'}} \mathbf{E}_{\tilde{P}} \xrightarrow{g'^{-1}} \mathbf{E}_P).
\end{aligned}$$

Let $\gamma \in G^F$ be such that $\gamma P \gamma^{-1} = \tilde{P}$. We rewrite the equality to be proved using the substitution $g' = \gamma h$:

(a) $$\mathrm{tr}(\mathbf{E}_P \xrightarrow{AB} \mathbf{E}_P) = \frac{D}{\sharp(P^F)} \sum_{h \in P^F} \mathrm{tr}(\mathbf{E}_P \xrightarrow{Ah} \mathbf{E}_P)\mathrm{tr}(\mathbf{E}_P \xrightarrow{h^{-1}B} \mathbf{E}_P)$$

where A is the composition $\mathbf{E}_P \xrightarrow{\gamma} \mathbf{E}_{\tilde{P}} \xrightarrow{\phi_P^{\tilde{P}}} \mathbf{E}_P$ and B is the composition $\mathbf{E}_P \xrightarrow{g} \mathbf{E}_{P'} \xrightarrow{\phi_{\tilde{P}}^{P'}} \mathbf{E}_{\tilde{P}} \xrightarrow{\gamma^{-1}} \mathbf{E}_P$. (Then AB is the composition $\mathbf{E}_P \xrightarrow{g} \mathbf{E}_{P'} \xrightarrow{\phi_{\tilde{P}}^{P'}} \mathbf{E}_{\tilde{P}} \xrightarrow{\phi_P^{\tilde{P}}} \mathbf{E}_P$.) Now (a) follows immediately from the Schur orthogonality relations for the matrix coefficients of the irreducible representation of P^F on $\mathbf{E}_P$. The lemma is proved.

27.9. Let $w \in \mathcal{W}^u$. Let $f_w : Y_0 \to \mathbf{C}$ be the image of $q^{-l(w)/2}({}^w\phi)$ (defined as in 27.2 in terms of ι_0) under $\kappa : H \to \mathfrak{B}$.

If $(P_1, P_2, gU_{P_1}^F) \in Y_0, (P_1, P_2) \notin \mathcal{O}_w$ then $f_w(P_1, P_2, gU_{P_1}^F) = 0$.

If $(P_1, P_2, gU_{P_1}^F) \in Y_0, (P_1, P_2) \in \mathcal{O}_w$ then $f_w(P_1, P_2, gU_{P_1}^F)$ is $q^{-l(w)/2}$ times the trace of the composition

$$\mathbf{E}_{P_1} \xrightarrow{g_2^{-1}g} \mathbf{E}_{P_0} \xrightarrow{\iota_0(\alpha_{g_1, g_2})} \mathbf{E}_{P_0} \xrightarrow{g_1} \mathbf{E}_{P_1}$$

where $g_1, g_2 \in G^F$, $g_2 P_0 g_2^{-1} = P_2, g_1 P_0 g_1^{-1} = P_1$; here we may assume that $g_1 = g^{-1} g_2$ hence

$$f_w(P_1, P_2, gU_{P_1}^F) = q^{-l(w)/2} \mathrm{tr}(\mathbf{E}_{P_0} \xrightarrow{\iota_0(\alpha_{g^{-1}g_2, g_2})} \mathbf{E}_{P_0})$$

where $g_2 \in G^F$, $g_2 P_0 g_2^{-1} = P_2$ (with $\alpha_{g^{-1}g_2, g_2}$ as in 27.1).

In particular, if $(P_1, P_2, gU_{P_1}^F) \in Y_0, P_1 \neq P_2$ then $f_1(P_1, P_2, gU_{P_1}^F) = 0$; if $P_1 \in \mathcal{P}^F, g \in P_1^F$ then

$$f_1(P_1, P_1, gU_{P_1}^F) = \mathrm{tr}(\mathbf{E}_{P_1} \xrightarrow{g} \mathbf{E}_{P_1}).$$

Thus, f_1 is not identically zero.

Here are some properties of the functions f_w which follow immediately from the corresponding properties of the functions ${}^w\phi$ using 27.8.

(a) $(f_{\tau_k} - q^{-N_k/2} f_1)(f_{\tau_k} + q^{N_k/2} f_1) = 0$ for all k,

(b) $f_w f_{w'} = f_{ww'}, w, w' \in \mathcal{W}^u, l(ww') = l(w) + l(w')$.

Let $\bar{H} = \kappa(H)$. This is a subalgebra of $\mathfrak{B}$ generated as a vector space by $\{f_w; w \in \mathcal{W}^u\}$. From (b) we see that $f_1 f_w = f_w f_1 = f_w$ for all $w \in \mathcal{W}^u$, hence f_1 is the unit element of the algebra $\bar{H}$. From (a) we see that f_{τ_k} is invertible in this algebra for any k and then from (b) we see that f_w is invertible in this algebra for any $w \in W$. Since $f_1 \neq 0$ we have $f_w \neq 0$ for any $w \in \mathcal{W}^u$. Now the f_w have disjoint supports. (The support of f_w is contained in Y_w^F where $Y_w = \{(P_1, P_2, gU_{P_1}^F) \in Y; (P_1, P_2) \in \mathcal{O}_w\}$.) It follows that the elements $f_w (w \in \mathcal{W}^u)$ are linearly independent in the vector space $\bar{H}$. Hence $\kappa : H \to \bar{H}$ is an isomorphism of algebras.

We have thus obtained a new model $\bar{H}$ for the Hecke algebra H as the vector space of functions $f : Y_0 \to \mathbf{C}$ spanned by the functions $f_w (w \in \mathcal{W}^u)$ with multiplication $*$ as in 27.7.

27.10. Let

$Z = \{(P', gU_{P_0}); P' \in \mathcal{P}, g \in G/U_{P_0}; gP_0 g^{-1} = P'\}$.

Let $w \in \mathcal{W}^u$. Let

$Z_w = \{(P', gU_{P_0}) \in Z, (P_0, P') \in \mathcal{O}_w\}$.

We have a morphism $\lambda : Z_w \to Aut(L)_w = M_w$ where $\lambda(P', gU_{P_0})$ is the composition

$$L = \bar{P}_0 \xrightarrow{\mathrm{Ad}(g)} \bar{P}' \xrightarrow{\psi_{P_0}^{P'}} \bar{P}_0 = L.$$

Note that λ is a smooth morphism with connected fibres.

Now Z_w, M_w are naturally defined over $\mathbf{F}_q$ with Frobenius maps F and λ commutes with F. Hence λ restricts to a map

(a) $$Z_w^F \xrightarrow{\lambda} M_w^F.$$

Define $f_w^0 : Z_w^F \to \mathbf{C}$ by

$$f_w^0(P', gU_{P_0}) = f_w(P_0, P', gU_{P_0}).$$

(Here $g \in G^F/U_{P_0}^F$.) We have

$$f_w^0 = q^{-l(w)/2}\lambda^*(\chi(\iota_0)_w)$$

where $\chi(\iota_0)_w : M_w^F \to \mathbf{C}$ is the character of $\iota_0 : M^F \to GL(\mathbf{E}_{P_0})$ restricted to M_w^F.

27.11. The obvious homomorphism $Aut(L) \to Aut(L_{ad})$ defines for any $w \in \mathcal{W}^u$ an isomorphism of $M_w = Aut(L)_w$ with a connected component of the reductive algebraic group $Aut(L_{ad})$ with identity component L_{ad}. Hence we have the notion of character sheaf on M_w (see [L10]). Let $\hat{M}_w$ be the set of isomorphism classes of character sheaves on M_w. Let $A \in \hat{M}_w$. Then A is L_{ad}-equivariant for the conjugation action of M_w. Since λ is smooth with connected fibres of fixed dimension, a suitable shift of $\lambda^*(A)$ is a simple perverse sheaf $\tilde{A}$ on Z_w. Let $\tilde{A}^\sharp$ be the unique simple perverse sheaf on Z, whose support is the closure in Z of the support of $\tilde{A}$ and which satisfies $\tilde{A}^\sharp|_{Z_w} = \tilde{A}$.

Let $\hat{M}_w^F$ be the set of all $A \in \hat{M}_w$ such that $F^*A \cong A$. For any $A \in \hat{M}_w^F$ we choose an isomorphism $\phi : F^*A \xrightarrow{\sim} A$. There are induced isomorphisms $\phi : F^*\tilde{A} \xrightarrow{\sim} \tilde{A}, \phi : F^*\tilde{A}^\sharp \xrightarrow{\sim} \tilde{A}^\sharp$. Let

$$\chi_{A,\phi} : M_w^F \to \bar{\mathbf{Q}}_l, \chi_{\tilde{A},\phi} : {Z_w}^F \to \bar{\mathbf{Q}}_l, \chi_{\tilde{A}^\sharp,\phi} : Z^F \to \bar{\mathbf{Q}}_l$$

be the corresponding characteristic functions (alternating sums of traces of Frobenius at stalks of cohomology sheaves at various F-fixed points). We have $c_{\tilde{A},\phi} = (-1)^N\lambda^*(c_{A,\phi})$ where $N = \dim Z_w - \dim M_w$ and $c_{\tilde{A},\phi} = c_{\tilde{A}^\sharp,\phi}|_{Z_w^F}$.

It is known [L10] that the functions $\chi_{A,\phi}$ (where A runs through $\hat{M}_w^F$) form a basis for the vector space of functions $M_w^F \to \mathbf{C}$ that are constant on the orbits of M^{0F} (acting on M_w^F by conjugation). Hence

$$\chi(\iota_0)_w = \sum_{A \in \hat{M}_w^F} \xi_A \chi_{A,\phi}$$

where $\xi_A \in \mathbf{C}$ are uniquely determined. Applying λ^* to both sides we deduce

$$q^{l(w)/2} f_w^0 = \sum_A \xi_A (-1)^N \chi_{\tilde{A},\phi}.$$

Hence

$$f_w^0 = q^{-l(w)/2}(-1)^N \sum_A \xi_A \chi_{\tilde{A}^\sharp,\phi}|_{Z_w^F}.$$

The following conjecture provides a geometric interpretation of the polynomials $p_{y,w}$ (see 5.3) attached to the Coxeter group $\mathcal{W}^u$ with its weight function $L : \mathcal{W}^u \to \mathbf{N}$.

Conjecture 27.12. *Assume that $y \in \mathcal{W}^u$. We have*

$$q^{-l(w)/2}(-1)^N \sum_A \xi_A \chi_{\tilde{A}^\sharp,\phi}|_{Z_y^F} = p_{y,w}|_{v=\sqrt{q}} f_y^0,$$

$$\sum_A \xi_A \chi_{\tilde{A}^\sharp,\phi}|_{Z^F - \cup_{y\in\mathcal{W}^u} Z_y^F} = 0.$$

27.13. We now consider the special case where $\mathcal{P}$ is the set of Borel subgroups of G and $\mathbf{E}$ is the trivial vector bundle $\mathbf{C}$. Then $\mathcal{W} = W$. In this case the homomorphism ι_0 is trivial. For $w \in \mathcal{W}^u = W^u$ and $(P_1, P_2, gU_{P_1}^F) \in Y_0$ we have

$f_w(P_1, P_2, gU_{P_1}^F) = 0$ if $(P_1, P_2) \notin \mathcal{O}_w$,

$f_w(P_1, P_2, gU_{P_1}^F) = q^{-l(w)/2}$, if $(P_1, P_2) \in \mathcal{O}_w$.

In particular, the functions in $\kappa(H)$ do not depend on the third coordinate $gU_{P_1}^F$ which can therefore be omitted. For f', f'' in $\kappa(H)$ we have

$(f' * f'')(P, P') = \sum_{\tilde{P}\in\mathcal{P}^F} f'(P, \tilde{P}) f''(\tilde{P}, P')$.

In the present case, Conjecture 27.12 states that $p_{y,w}|_{v=\sqrt{q}}$ is (up to normalization) the restriction to Z_y^F of the characteristic function of the intersection cohomology sheaf of the closure of Z_w in Z. Equivalently, if for $w \in W^u$ we set

$\mathcal{P}_w = \{P' \in \mathcal{P}; (P_0, P') \in \mathcal{O}_w),$

then $p_{y,w}|_{v=\sqrt{q}}$ is (up to normalization) the restriction to $\mathcal{P}_y^F$ of the characteristic function of the intersection cohomology sheaf of the closure of $\mathcal{P}_w$ in $\mathcal{P}$.

This property is known to be true; it is proved in [KL2] in the case where $u = 1$ on W and is stated in the general case in [L3].

27.14. Let $\mathbf{G}, F, \mathcal{P}, \mathbf{E}, W, S, J, \mathcal{W}, u, \dots$ be as in 0.6. Let $H = H(\mathbf{G}^F, \mathcal{P}^F, \mathbf{E})$. Everything in 27.1-27.13 extends to this case (we replace G by $\mathbf{G}$ throughout) with the following modifications. In the definition of $\mathfrak{B}$ (see 27.7) we must now restrict ourselves to functions $f : Y_0 \to \mathbf{C}$ such that

$$\{(P, P') \in \mathcal{P}^F \times \mathcal{P}^F; f(P, P', gU_P^F) \neq 0 \text{ for some } g \in \mathbf{G}^F\}$$

is contained in the union of finitely many $\mathbf{G}$-orbits on $\mathcal{P} \times \mathcal{P}$. Also, when defining the multiplication $*$ in 27.7 only the definition 27.7(a) makes now sense (in 27.7(b) the quantity $\sharp(U_P^F)$ is infinite hence does not make sense). In 27.13 one should use Iwahori subgroups instead of Borel subgroups.

Appendix

A.1. Let $\tilde{W}$ be the Coxeter group associated to a finite set $\tilde{S}$ and to the Coxeter matrix $(m_{s,s'})_{s,s'\in\tilde{S}}$. We view $\tilde{S}$ as a subset of $\tilde{W}$; for any $I \subset \tilde{S}$ we denote by $\tilde{W}_I$ the subgroup of $\tilde{W}$ generated by I. For any $I \subset \tilde{S}$ let $[I]$ the full subgraph of the Coxeter graph of $\tilde{W}$ with set of vertices I. Let $\tilde{l} : \tilde{W} \to \mathbf{N}$ be the length function of $\tilde{W}$. Let $\leq$ be the standard partial order on $\tilde{W}$.

For any $m \in \mathbf{Z}_{\geq 1} \cup \{\infty\}$ we set $\kappa_m = 2\cos(\pi/m) \in \mathbf{R}$.

Let $\mathbf{E}$ be the $\mathbf{R}$-vector space with basis $\{e_s; s \in \tilde{S}\}$. Following [Bo,Ch.V, 4.1], for any $s \in \tilde{S}$ we define a linear map $\sigma_s : \mathbf{E} \to \mathbf{E}$ by $\sigma_s(e_{s'}) = e_{s'} + \kappa_{m_{s,s'}} e_s$ for all $s' \in \tilde{S}$. According to [Bo, Ch.V, 4.3] there is a unique group homomorphism $\sigma : \tilde{W} \to GL(\mathbf{E})$ ("reflection representation" of $\tilde{W}$) such that $\sigma(s) = \sigma_s$ for all $s \in \tilde{S}$. Let $(,) : \mathbf{E} \times \mathbf{E} \to \mathbf{R}$ be the symmetric bilinear form given by $(e_s, e_{s'}) = -\kappa_{m_{s,s'}}/2$ for $s, s' \in \tilde{S}$. Note that for any $w \in \tilde{W}$, $\sigma(w)$ is an isometry of this form.

Assume that we are given a group automorphism $\tau : \tilde{W} \to \tilde{W}$ such that $\tau(\tilde{S}) = \tilde{S}$; we have necessarily $m_{\tau(s),\tau(s')} = m_{s,s'}$ for any s, s' in $\tilde{S}$. We define a vector space isomorphism $\tau : \mathbf{E} \xrightarrow{\sim} \mathbf{E}$ by $\tau(e_s) = e_{\tau(s)}$ for any $s \in \tilde{S}$. For any $s, s' \in \tilde{S}$ we have $\tau(\sigma_s(e_{s'}) = \sigma_{\tau(s)}(\tau(e_{s'}))$. Hence for any $w \in \tilde{W}$ we have $\tau\sigma(w)) = \sigma(\tau(w))\tau : \mathbf{E} \to \mathbf{E}$.

Let S be the set of τ-orbits I on $\tilde{S}$ such that $\tilde{W}_I$ is finite; for any $I \in S$ let w_0^I be the longest element of the finite Coxeter group $\tilde{W}_I$. Let $'\tilde{W}$ be the subgroup of $\tilde{W}$ generated by $\{w_0^I; I \in S\}$. Let $W = \{w \in \tilde{W}; \tau(w) = w\}$. We show:

(a) $W = {}'\tilde{W}$.

In the proof we shall make use of the following fact.

(b) If $I \subset \tilde{S}$ and $w \in \tilde{W}$ satisfies $sw < w$ for any $s \in I$ then $\tilde{l}(yw) = \tilde{l}(w) - \tilde{l}(y)$ for any $y \in \tilde{W}_I$. In particular, $y \mapsto \tilde{l}(y)$ is bounded on $\tilde{W}_I$ so that $\tilde{W}_I$ is finite.

The proof of the first sentence in (b) is identical to the proof of 9.8(d) (with the assumption that w has maximal length in its $\tilde{W}_I$ coset replaced by $sw < w$ for any $s \in I$). The second sentence in (b) follows from the first since $\tilde{l}(y) \leq \tilde{l}(w)$ for any $y \in \tilde{W}_I$.

We prove (a). The inclusion $'\tilde{W} \subset W$ is obvious. We now prove the reverse inclusion. Let $w \in W$. We show that $w \in {}'\tilde{W}$ by induction on $\tilde{l}(w)$. If $\tilde{l}(w) = 0$ then $w = 1$ and the result is obvious. Assume now that $\tilde{l}(w) \geq 1$. We can find $s \in \tilde{S}$ such that $sw < w$. Then for any i we have $\tau^i(sw) < \tau^i(w)$ that is $\tau^i(s)w < w$. Thus $s'w < w$ for any $s' \in I$ where I is the τ-orbit of s. Using (b) we see that $\tilde{W}_I$ is finite and $\tilde{l}(w_0^I w) = \tilde{l}(w) - \tilde{l}(w_0^I)$. The induction hypothesis is applicable to $w_0^I w$ instead of w and yields $w_0^I w \in {}'\tilde{W}$. It follows that $w \in {}'\tilde{W}$. This proves (a).

Next we show:

(c) Let $I \neq I'$ in S be such that $\tilde{W}_{I \cup I'}$ is infinite. Then the subgroup $\mathcal{W}$ of $\tilde{W}$ generated by w_0^I and $w_0^{I'}$ is infinite.

Note that each element of $\mathcal{W}$ is fixed by τ. Assume that $\mathcal{W}$ is finite. Then we can find $w \in \mathcal{W}$ of maximal length among the elements of $\mathcal{W}$. If $sw > w$ for some $s \in I$ then for any i we have $\tau^i(sw) > \tau^i(w)$ hence $\tau^i(s)w > w$; thus $s'w > w$ for any $s' \in I$. Using 9.7 we deduce that $\tilde{l}(w_0^I w) = \tilde{l}(w_0^I) + \tilde{l}(w) > \tilde{l}(w)$ contradicting the maximality of $\tilde{l}(w)$. Thus we have $sw < w$ for any $s \in I$. Similarly we have $sw < w$ for any $s \in I'$. Using (b) we see that $\tilde{W}_{I \cup I'}$ is finite. This contradiction proves (c).

Let $I \in S$. We set $m = \max\{m_{s,s'}; s \in I, s' \in I\}$. We show that if $m \ge 3$ then the following holds.

(d) There exists $i \in \mathbf{Z}$ such that $\tau^i : I \to I$ is a fixed point free involution and $m_{s,s'} = m$ if $s, s' \in I, s \ne s'$ are in the same τ^i-orbit and $m_{s,s'} = 2$ if $s, s' \in I$ are not in the same τ^i-orbit.

We can find $s_0 \in I, s'_0 \in I$ such that $m_{s_0,s'_0} = m \ge 3$. Let $K \subset I$ be set of vertices of the connected component $[K]$ of the Coxeter graph of $\tilde{W}_I$ that contains s_0 and s'_0. If $s, s' \in K$ then $s' = \tau^i(s)$ for some i and we have necessarily $\tau^i(K) = K$ (indeed, $\tau^i(K), K$ are sets of vertices of connected components of the Coxeter graph of $\tilde{W}_I$ containing s'). Thus the group of automorphisms of $[K]$ acts transitively on K. Using the known classification of finite Coxeter groups we see that $\sharp(K) = 2$ that is, $K = \{s_0, s'_0\}$. We also see that for some $i \in \mathbf{Z}$ we have $\tau^i(s_0) = s'_0$, $\tau^i(s'_0) = s_0$. Since I is a τ-orbit, we deduce that I is a disjoint union of τ^i-orbits of size 2 and $\tau^i : I \to I$ is an involution; moreover for each τ^i-orbit $\{s, s'\}$ on I we have $m_{s,s'} = m$. We also see that if s, s' are not in the same τ^i-orbit, then $m_{s,s'} = 2$. This proves (d).

Now let $I \ne I'$ in S be such that $\tilde{W}_{I \cup I'}$ is finite. Let

$m = \max\{m_{s,s'}; s \in I, s' \in I\}$, $m' = \max\{m_{s,s'}; s \in I', s' \in I'\}$, $\mu = \max\{m_{s,s'}; s \in I, s' \in I'\}$.

Note that $m < \infty$, $m' < \infty$, $\mu < \infty$. We show that if $\mu \ge 3$ then (after possibly interchanging I, I'), (e),(f) below hold.

(e) There exists $p \in \{1, 2, 3\}$ (with $p = 1$ if $\mu > 3$) and a p-fold covering $u : I \to I'$ which commutes with the action of τ, such that for any $s \in I, s' \in I'$ we have $m_{s,s'} = \mu$ if $s' = u(s)$ and $m_{s,s'} = 2$ if $s' \ne u(s)$.

(f) If $p \in \{2, 3\}$ then $m \le 2, m' \le 2$. If $p = 1$ and $\mu \ge 4$ then $m \le 2, m' \le 2$. If $p = 1$ and $\mu = 3$ then (m, m') is $(4, 2)$ or $(3, 2)$ or $(2, 2)$ or $(1, 1)$.

We can find $s_0 \in I, s'_0 \in I'$ such that $m_{s_0,s'_0} = \mu \ge 3$. Let $K \subset I$ be the set of vertices of the connected component $[K]$ of the Coxeter graph of $\tilde{W}_{I \cup I'}$ that contains s_0 and s'_0. Let $Aut[K]$ be the group of automorphisms of the Coxeter graph $[K]$.

If $s_1, s_2 \in I \cap J$ then we can find $i \in \mathbf{Z}$ such that $\tau^i(s_1) = s_2$. Then K, $t^i(K)$ are sets of vertices of connected components of the Coxeter graph of $\tilde{W}_{I \cup I'}$ and both contain s_2; hence $K = t^i(K)$. We see that

(g) for any s_1, s_2 in $I \cap K$ there exists an element of $Aut[K]$ which carries s_1 to s_2. Similarly, for any s'_1, s'_2 in $I' \cap K$ there exists an element of $Aut[K]$ which carries s'_1 to s'_2. In particular, $Aut[K]$ acts on K with at most two orbits.

For any $s \in I$ we set $\mathcal{K}'_s = \{s' \in I'; m_{s,s'} \ge 3\}$; for any $s' \in I'$ we set $\mathcal{K}_{s'} = \{s \in I; m_{s,s'} \ge 3\}$. Let $\mathcal{K}' = \mathcal{K}'_{s_0}$, $\mathcal{K} = \mathcal{K}_{s'_0}$. We have $\mathcal{K}' \subset I' \cap K$, $\mathcal{K} \subset I \cap K$.

Assume first that $\mu \ge 4$. If there exists $s' \in \mathcal{K}'_{s_0} - \{s'_0\}$, then using the known classification of finite Coxeter groups we see that $\mu \in \{4, 5\}$. Since $s', s'_0 \in I' \cap K$ we can find i such that $\tau^i \in Aut[K]$ and τ^i carries s'_0 to s' hence is nontrivial in $\mathrm{Aut}[K]$. By the classification of finite Coxeter groups we deduce that $\mu = 4$ and τ^i

interchanges s_0, s'_0. This contradicts the fact that s_0, s'_0 are in different τ-orbits. Thus $\sharp(\mathcal{K}'_{s_0}) = 1$. Using the fact that I is a τ-orbit, we deduce that $\sharp(\mathcal{K}'_s) = 1$ for any $s \in I$. Similarly, we have $\sharp(\mathcal{K}_{s'}) = 1$ for any $s' \in I'$. Hence we have a bijection $u : I \to I'$ given by $s \mapsto s'$ where $\mathcal{K}'_s = \{s'\}, \mathcal{K}_{s'} = \{s\}$ and (e) is verified in this case. Assume now that $s \in I$ satisfies $m_{s,s_0} \geq 3$. Then $s, s_0 \in I \cap K$ hence we can find j such that $\tau^j \in Aut[K]$ and τ^j carries s_0 to s. Thus $[K]$ has at least three distinct vertices s, s_0, s'_0 with $m_{s_0,s'_0} = 4$ and has an automorphism which carries s_0 to s. This is impossible, by the classification of finite Coxeter groups. We see that $m \leq 2$. Similarly we have $m' \leq 2$. Thus (f) holds in this case.

Next we assume that $\mu = 3$. If $m \geq 3$ then by (d) we have $m_{s_0,s_1} = m$ for some $s_1 \in I$ and by (g) some automorphism r of $[K]$ carries s_0 to s_1. Also, using (g) and the classification of finite Coxeter groups we see that $m \in \{3, 4\}$ and there exists $s'_1 \in I' - \{s'_0\}$ such that $K = \{s'_0, s_0, s_1, s'_1\}$ and the edges of $[K]$ are $(s'_0, s_0), (s_0, s_1), (s_1, s'_1)$. Note that r interchanges s_0 with s_1 and s'_0 with s'_1. In particular s'_0 is not connected with any $s' \in I' - \{s'_0\}$ in the Coxeter graph of $\tilde{W}_{I \cup I'}$ so that $m' \leq 2$. We also see that in this case $\mathcal{K} = \{s_0\}$, $\mathcal{K}' = \{s'_0\}$. Similarly, if $m' \geq 3$, then $m' \in \{3, 4\}$, $m \leq 2$ and $\mathcal{K} = \{s_0\}$, $\mathcal{K}' = \{s'_0\}$. Assume now that $\sharp(\mathcal{K}') \geq 3$. Since $\mathcal{K}' \subset I' \cap K$, $Aut[K]$ has at least three distinct elements (see (g)). Using the classification of finite Coxeter groups, we deduce that $\sharp(\mathcal{K}') = 3$, $K = \mathcal{K}'$, hence $\mathcal{K} = \{s_0\}$; using the fact that I and I' are τ-orbits, we see that we have $\sharp(\mathcal{K}'_s) = 3$ for any $s \in I$ and $\sharp(\mathcal{K}_{s'}) = 1$ for any $s' \in I'$, so that if we define $u : I' \to I$ by $u(s') = s$ where s is such that $s \in \mathcal{K}_{s'}$, then u satisfies (e) with I, I' interchanged and (f) holds as well.

Thus we can assume that $\sharp(\mathcal{K}') \leq 2$; similarly we can assume that $\sharp(\mathcal{K}) \leq 2$. Assume now that $\sharp(\mathcal{K}') = 2$, $\sharp(\mathcal{K}) \leq 2$; we write $\mathcal{K}' = \{s'_0, s'_1\}$. This is not compatible with the inequality $m \geq 3$, by a previous argument. Thus $m \leq 2$ and similarly, $m' \leq 2$. Hence $[K]$ is a graph of type A_n, $n \geq 2$. From (g) we see that $n \leq 3$. Since $\{s_0, s'_0, s'_1\} \subset K$ it follows that $\{s_0, s'_0, s'_1\} = K$ that is $\mathcal{K}' = K$; hence $\mathcal{K} = \{s_0\}$. Using the fact that I and I' are τ-orbits, we see that we have $\sharp(\mathcal{K}'_s) = 2$ for any $s \in I$ and $\sharp(\mathcal{K}_{s'}) = 1$ for any $s' \in I'$ so that if we define $u : I' \to I$ by $u(s') = s$ where s is such that $s \in \mathcal{K}_{s'}$, then u satisfies (e) with I, I' interchanged and (f) holds as well.

Thus (e),(f) hold if $\sharp(\mathcal{K}') \geq 2$; similarly they hold if $\sharp(\mathcal{K}) \geq 2$. We may assume therefore that $\sharp(\mathcal{K}') = \sharp(\mathcal{K}) = 1$. Using the fact that I and I' are τ-orbits, we see that we have $\sharp(\mathcal{K}'_s) = 1$ for any $s \in I$ and $\sharp(\mathcal{K}_{s'}) = 1$ for any $s' \in I'$ so that if we define $u : I' \to I$ by $u(s') = s$ where s is such that $s \in \mathcal{K}_{s'}$ then u satisfies (e) with I, I' interchanged. Thus (e) is proved. It remains to prove (f) in the case where $p = 1$ and $\mu = 3$. If $m \geq 3$ then by an earlier argument we have $m \in \{3, 4\}$ and $m' \leq 2$. If $m' \geq 3$ then again by an earlier argument we have $m' \in \{3, 4\}$, $m \leq 2$. Interchanging I, I' we have again $m \in \{3, 4\}$ and $m' \leq 2$. Thus we can assume that $m \leq 2$, $m' \leq 2$. Then (f) is clear. This completes the proof of (f).

A.2. Let E be an $\mathbf{R}$-vector space with basis $e_1, e_2, \dots, e_k, f_1, f_2, \dots, f_k$. Let $\mu \in \mathbf{Z}_{\geq 3}$. Let $e = (e_1 + \cdots + e_k)/\sqrt{k}$, $f = (f_1 + \cdots + f_k)/\sqrt{k}$.

For $i\in[1,k]$ we define a linear map $s_i:E\to E$ by
$s_i(e_i)=-e_i$, $s_i(e_j)=e_j$ for $j\ne i$, $s_i(f_i)=f_i+\kappa_\mu e_i$, $s_i(f_j)=f_j$ for $j\ne i$.
For $i\in[1,k]$ we define a linear map $t_i:E\to E$ by
$t_i(f_i)=-f_i$, $t_i(f_j)=f_j$ for $j\ne i$, $t_i(e_i)=e_i+\kappa_\mu f_i$, $t_i(e_j)=e_j$ for $j\ne i$.
Note that $s_1,\dots,s_k$ commute and $t_1,\dots,t_k$ commute. We set $\sigma=s_1s_2\dots s_k$, $\tilde\sigma=t_1t_2\dots t_k$. For all j we have
$\sigma(e_j)=-e_j$, $\sigma(f_j)=f_j+\kappa_\mu e_j$, $\tilde\sigma(e_j)=e_j+\kappa_\mu f_j$, $\tilde\sigma(f_j)=-f_j$.
Hence
(a) $\sigma\tilde\sigma(e_j)=(\kappa_\mu^2-1)e_j+\kappa_\mu f_j$, $\sigma\tilde\sigma(f_j)=-\kappa_\mu e_j-f_j$ for all j.
We see that
(b) $\sigma(e)=-e$, $\sigma(f)=f+\kappa_\mu e$, $\tilde\sigma(e)=e+\kappa_\mu f$, $\tilde\sigma(f)=-f$.
Note that $E_j=\mathbf{R}e_j+\mathbf{R}f_j$ is $\sigma\tilde\sigma$-stable. From (a) we see that the characteristic polynomial of $\sigma\tilde\sigma$ on E_j is $X^2-(\kappa_\mu^2-2)X+1$. Hence
(c) $(\sigma\tilde\sigma)^m=1$ on E.

A.3. Let E be an $\mathbf{R}$-vector space with basis $e_1,e_2,\dots,e_k,e'_1,e'_2,\dots,e'_k,f_1,\dots,f_k$. Let $e=(e_1+\dots+e_k+e'_1+\dots+e'_k)/\sqrt{2k}$, $f=(f_1+\dots+f_k)/\sqrt{k}$. For $i\in[1,k]$ we define a linear map $s_i:E\to E$ by
$s_i(e_i)=-e_i$, $s_i(e_j)=e_j$ for $j\ne i$, $s_i(e'_j)=e'_j$ for all j,
$s_i(f_i)=f_i+e_i$, $s_i(f_j)=f_j$ for $j\ne i$.
For $i\in[1,k]$ we define a linear map $s'_i:E\to E$ by
$s'_i(e_j)=e_j$ for all j, $s'_i(e'_i)=-e'_i$, $s'_i(e'_j)=e'_j$ for $j\ne i$,
$s'_i(f_i)=f_i+e'_i$, $s'_i(f_j)=f_j$ for $j\ne i$.
For $i\in[1,k]$ we define a linear map $t_i:E\to E$ by
$t_i(e_i)=e_i+f_i$, $t_i(e_j)=e_j$ for $j\ne i$,
$t_i(e'_i)=e'_i+f_i$, $t_i(e'_j)=e'_j$ for $j\ne i$,
$t_i(f_i)=-f_i$, $t_i(f_j)=f_j$ for $j\ne i$.
Note that $s_1,\dots,s_k,s'_1,\dots,s'_k$ commute and $t_1,\dots,t_k$ commute. We set $\sigma=s_1s_2\dots s_ks'_1s'_2\dots s'_k$, $\tilde\sigma=t_1t_2\dots t_k$. For all j we have
$\sigma(e_j)=-e_j$, $\sigma(e'_j)=-e'_j$, $\sigma(f_j)=f_j+e_j+e'_j$, $\tilde\sigma(e_j)=e_j+f_j$, $\tilde\sigma(e'_j)=e'_j+f_j$, $\tilde\sigma(f_j)=-f_j$.
Hence
(a) $\sigma\tilde\sigma(e_j)=f_j+e'_j$, $\sigma\tilde\sigma(e'_j)=f_j+e_j$, $\sigma\tilde\sigma(f_j)=-f_j-e_j-e'_j$.
We see that
(b) $\sigma(e)=-e$, $\sigma(f)=f+\sqrt{2}e$, $\tilde\sigma(e)=e+\sqrt{2}f$, $\tilde\sigma(f)=-f$.
Note that $E_j=\mathbf{R}e_j+\mathbf{R}e'_j+\mathbf{R}f_j$ is $\sigma\tilde\sigma$-stable. The characteristic polynomial of $\sigma\tilde\sigma$ on E_j is $(X^2+1)(X+1)$. Hence
(c) $(\sigma\tilde\sigma)^4=1$ on E.

A.4. Let E be an $\mathbf{R}$-vector space with basis
$e_1,e_2,\dots,e_k,e'_1,e'_2,\dots,e'_k,e''_1,\dots,e''_k,f_1,\dots,f_k$.
Let $e=(e_1+\dots+e_k+e'_1+\dots+e'_k+e''_1+\dots+e''_k)/\sqrt{3k}$, $f=(f_1+\dots+f_k)/\sqrt{k}$.
For $i\in[1,k]$ we define a linear map $s_i:E\to E$ by
$s_i(e_i)=-e_i$, $s_i(e_j)=e_j$ for $j\ne i$, $s_i(e'_j)=e'_j$ for all j,

$s_i(e''_j)=e''_j$ for all j, $s_i(f_i)=f_i+e_i$, $s_i(f_j)=f_j$ for $j\ne i$.
For $i\in[1,k]$ we define a linear map $s'_i:E\to E$ by
$s'_i(e_j)=e_j$ for all j, $s'_i(e'_i)=-e'_i$, $s'_i(e'_j)=e'_j$ for $j\ne i$,
$s'_i(e''_j)=e''_j$, for all j, $s'_i(f_i)=f_i+e'_i$, $s'_i(f_j)=f_j$ for $j\ne i$.
For $i\in[1,k]$ we define a linear map $s''_i:E\to E$ by
$s''_i(e_j)=e_j$ for all j, $s''_i(e'_j)=e'_j$ for all j, $s''_i(e''_i)=-e''_i$,
$s''_i(e''_j)=e''_j$ for $j\ne i$, $s''_i(f_i)=f_i+e''_i$, $s''_i(f_j)=f_j$ for $j\ne i$.
For $i\in[1,k]$ we define a linear map $t_i:E\to E$ by
$t_i(e_i)=e_i+f_i$, $t_i(e_j)=e_j$ for $j\ne i$,
$t_i(e'_i)=e'_i+f_i$, $t_i(e'_j)=e'_j$ for $j\ne i$,
$t_i(e''_i)=e''_i+f_i$, $t_i(e''_j)=e''_j$ for $j\ne i$,
$t_i(f_i)=-f_i$, $t_i(f_j)=f_j$ for $j\ne i$.
Note that $s_1,\dots,s_k,s'_1,\dots,s'_k,s''_1,\dots,s''_k$ commute and $t_1,\dots,t_k$ commute. We set
$\sigma=s_1s_2\dots s_ks'_1s'_2\dots s'_ks''_1\dots s''_k$, $\tilde\sigma=t_1t_2\dots t_k$.
For all j we have
$\sigma(e_j)=-e_j$, $\sigma(e'_j)=-e'_j$, $\sigma(e''_j)=-e''_j$, $\sigma(f_j)=f_j+e_j+e'_j+e''_j$,
$\tilde\sigma(e_j)=e_j+f_j$, $\tilde\sigma(e'_j)=e'_j+f_j$, $\tilde\sigma(e''_j)=e''_j+f_j$, $\tilde\sigma(f_j)=-f_j$.
Hence
(a) $\sigma\tilde\sigma(e_j)=e'_j+e''_j+f_j$, $\sigma\tilde\sigma(e'_j)=e_j+e''_j+f_j$, $\sigma\tilde\sigma(e''_j)=e_j+e'_j+f_j$, $\sigma\tilde\sigma(f_j)=-f_j-e_j-e'_j-e''_j$.
We see that
(b) $\sigma(e)=-e$, $\sigma(f)=f+\sqrt{3}e$, $\tilde\sigma(e)=e+\sqrt{3}f$, $\tilde\sigma(f)=-f$.

Note that $E_j=\mathbf{R}e_j+\mathbf{R}e'_j+\mathbf{R}e''_j+\mathbf{R}f_j$ is $\sigma\tilde\sigma$-stable. From (a) we see that $(\sigma\tilde\sigma)^6=1$ on E_j. (Its characteristic polynomial is $(X^2-X+1)(X+1)^2$.) Hence
(c) $(\sigma\tilde\sigma)^6=1$ on E.

A.5. Let E be an $\mathbf{R}$-vector space with basis
$e_1,e_2,\dots,e_k,e'_1,e'_2,\dots,e'_k,f_1,\dots,f_k,f'_1,\dots,f'_k$.
Let $e=(e_1+\dots+e_k+e'_1+\dots+e'_k)\sqrt{2}/\sqrt{2k}$, $f=(f_1+\dots+f_k+f'_1+\dots+f'_k)/\sqrt{2k}$.

For $i\in[1,k]$ we define a linear map $s_i:E\to E$ by
$s_i(e_i)=-e_i$, $s_i(e_j)=e_j$ for $j\ne i$, $s_i(e'_i)=e'_i+e_i$,
$s_i(e'_j)=e'_j$ for $j\ne i$, $s_i(f_i)=f_i+e_i$, $s_i(f_j)=f_j$ for $j\ne i$,
$s_i(f'_j)=f'_j$ for all j.
For $i\in[1,k]$ we define a linear map $s'_i:E\to E$ by
$s'_i(e_i)=e_i+e'_i$, $s'_i(e_j)=e_j$ for $j\ne i$,
$s'_i(e'_i)=-e'_i$, $s'_i(e'_j)=e'_j$ for $j\ne i$,
$s'_i(f_j)=f_j$ for all j, $s'_i(f'_i)=f'_i+e'_i$, $s'_i(f'_j)=f'_j$ for $j\ne i$.

For $i\in[1,k]$ we define a linear map $t_i:E\to E$ by
$t_i(e_i)=e_i+f_i$, $t_i(e_j)=e_j$ for $j\ne i$, $t_i(e'_j)=e'_j$ for all j,
$t_i(f_i)=-f_i$, $t_i(f_j)=f_j$ for $j\ne i$, $t_i(f'_j)=f'_j$ for all j.
For $i\in[1,k]$ we define a linear map $t'_i:E\to E$ by
$t'_i(e_i)=e_i+f_i$, $t'_i(e_j)=e_j$ for $j\ne i$, $t'_i(e'_j)=e'_j$ for all j,

$t'_i(f_i)=-f_i$, $t'_i(f_j)=f_j$ for $j\ne i$, $t'_i(f'_j)=f'_j$ for all j.
We set
$\sigma=s_1s'_1s_1s_2s'_2s_2\dots s_ks'_ks_k$, $\tilde\sigma=t_1t_2\dots t_kt'_1t'_2\dots t'_k$.
For all j we have
$\sigma(e_j)=-e'_j$, $\sigma(e'_j)=-e_j$, $\sigma(f_j)=f_j+e_j+e'_j$, $\sigma(f'_j)=f'_j+e_j+e'_j$,
$\tilde\sigma(e_j)=e_j+f_j$, $\tilde\sigma(e'_j)=e'_j+f'_j$, $\tilde\sigma(f_j)=-f_j$, $\tilde\sigma(f'_j)=-f'_j$.
Hence
(a) $\sigma\tilde\sigma(e_j)=e_j+f_j$, $\sigma\tilde\sigma(e'_j)=e'_j+f'_j$, $\sigma\tilde\sigma(f_j)=-e_j-e'_j-f_j$, $\sigma\tilde\sigma(f'_j)=-e_j-e'_j-f'_j$.
We see that
(b) $\sigma(e)=-e$, $\sigma(f)=f+\sqrt{2}e$, $\tilde\sigma(e)=e+\sqrt{2}f$, $\tilde\sigma(f)=-f$.
Note that $E_j=\mathbf{R}e_j+\mathbf{R}e'_j+\mathbf{R}f_j+\mathbf{R}f'_j$ is $\sigma\tilde\sigma$-stable. From (a) we see that the characteristic polynomial of $\sigma\tilde\sigma$ on E_j is $(X^2+1)(X^2-1)$. Hence
(c) $(\sigma\tilde\sigma)^4=1$ on E.

A.6. Let E be an $\mathbf{R}$-vector space with basis
$e_1,e_2,\dots,e_k,e'_1,e'_2,\dots,e'_k,f_1,\dots,f_k,f'_1,\dots,f'_k$.
Let $e=(e_1+\dots+e_k+e'_1+\dots+e'_k)\sqrt{2+\sqrt{2}}/\sqrt{2k}$, $f=(f_1+\dots+f_k+f'_1+\dots+f'_k)/\sqrt{2k}$.

For $i\in[1,k]$ we define a linear map $s_i:E\to E$ by
$s_i(e_i)=-e_i$, $s_i(e_j)=e_j$ for $j\ne i$, $s_i(e'_i)=e'_i+\sqrt{2}e_i$, $s_i(e'_j)=e'_j$ for $j\ne i$,
$s_i(f_i)=f_i+e_i$, $s_i(f_j)=f_j$ for $j\ne i$, $s_i(f'_j)=f'_j$ for all j.
For $i\in[1,k]$ we define a linear map $s'_i:E\to E$ by
$s'_i(e_i)=e_i+\sqrt{2}e'_i$, $s'_i(e_j)=e_j$ for $j\ne i$,
$s'_i(e'_i)=-e'_i$, $s'_i(e'_j)=e'_j$ for $j\ne i$,
$s'_i(f_j)=f_j$ for all j, $s'_i(f'_i)=f'_i+e'_i$, $s'_i(f'_j)=f'_j$ for $j\ne i$.
For $i\in[1,k]$ we define a linear map $t_i:E\to E$ by
$t_i(e_i)=e_i+f_i$, $t_i(e_j)=e_j$ for $j\ne i$, $t_i(e'_j)=e'_j$ for all j,
$t_i(f_i)=-f_i$, $t_i(f_j)=f_j$ for $j\ne i$, $t_i(f'_j)=f'_j$ for all j.
For $i\in[1,k]$ we define a linear map $t'_i:E\to E$ by
$t'_i(e_i)=e_i+f_i$, $t'_i(e_j)=e_j$ for $j\ne i$, $t'_i(e'_j)=e'_j$ for all j,
$t'_i(f_i)=-f_i$, $t'_i(f_j)=f_j$ for $j\ne i$, $t'_i(f'_j)=f'_j$ for all j.
We set $\sigma=s_1s'_1s_1s'_1s_2s'_2s_2s'_2\dots s_ks'_ks_ks'_k$, $\tilde\sigma=t_1t_2\dots t_kt'_1t'_2\dots t'_k$. For all j we have
$\sigma(e_j)=-e_j$, $\sigma(e'_j)=-e'_j$, $\sigma(f_j)=f_j+2e_j+\sqrt{2}e'_j$,
$\sigma(f'_j)=f'_j+\sqrt{2}e_j+2e'_j$, $\tilde\sigma(e_j)=e_j+f_j$, $\tilde\sigma(e'_j)=e'_j+f'_j$,
$\tilde\sigma(f_j)=-f_j$, $\tilde\sigma(f'_j)=-f'_j$.
Hence
(a) $\sigma\tilde\sigma(e_j)=e_j+\sqrt{2}e'_j+f_j$, $\sigma\tilde\sigma(e'_j)=\sqrt{2}e_j+e'_j+f'_j$,
$\sigma\tilde\sigma(f_j)=-2e_j-\sqrt{2}e'_j-f_j$, $\sigma\tilde\sigma(f'_j)=-\sqrt{2}e_j-2e'_j-f'_j$.
We see that

(b) $\sigma(e)=-e$, $\sigma(f)=f+\sqrt{2+\sqrt{2}}e$, $\tilde\sigma(e)=e+\sqrt{2+\sqrt{2}}f$, $\tilde\sigma(f)=-f$.
Note that $E_j=\mathbf{R}e_j+\mathbf{R}e'_j+\mathbf{R}f_j+\mathbf{R}f'_j$ is $\sigma\tilde\sigma$-stable. From (a) we see that the characteristic polynomial of $\sigma\tilde\sigma$ on E_j is $(X^2-\sqrt{2}X+1)(X^2+\sqrt{2}X+1)$. Hence

(c) $(\sigma\tilde\sigma)^8=1$ on E.

A.7. For any I,I' in S we define $M_{I,I'}\in\mathbf{Z}_{>0}\cup\{\infty\}$ as follows. If $I=I'$ we set $M_{I,I'}=1$. If $\tilde W_{I\cup I'}$ is infinite we set $M_{I,I'}=\infty$. Now assume that $I\ne I'$ and $\tilde W_{I\cup I'}$ is finite. Let

(a) $m=\max\{m_{s,s'};s\in I,s'\in I\}$, $m'=\max\{m_{s,s'};s\in I',s'\in I'\}$, $\mu=\max\{m_{s,s'};s\in I,s'\in I'\}$.

If $\mu=2$ we set $M_{I,I'}=2$.

If $\mu=3$ and p in A.1(e) is 2 or 3 we set $M_{I,I'}=2p$.

If $\mu=3$, p in A.1(e) is 1 and (m,m') is $(4,2)$ or $(2,4)$, we set $M_{I,I'}=8$.

If $\mu=3$, p in A.1(e) is 1 and (m,m') is $(3,2)$ or $(2,3)$, we set $M_{I,I'}=4$.

If $\mu\ge 3$ and $m\le 2,m'\le 2$ we set $M_{I,I'}=\mu$.

Let W' be the Coxeter group with generators g_I $(I\in S)$ with relations $(g_Ig_{I'})^{M_{I,I'}}=1$ for any $I,I'\in S$ such that $M_{I\cup I'}<\infty$. We have the following result.

Theorem A.8. *The map $g_I\mapsto w_0^I$ $(I\in S)$ extends uniquely to a group homomorphism $\lambda:W'\to\tilde W$ which is an isomorphism of W' onto the subgroup $W=\{w\in\tilde W;\tau(w)=w\}$ of $\tilde W$.*

We show that for any $I,I'\in S$ such that $\tilde W_{I\cup I'}$ is finite we have

(a) $(w_0^Iw_0^{I'})^{M_{I,I'}}=1$ in $\tilde W$.

If $I=I'$ this is a well known property of the longest element in a finite Coxeter group. Now assume that $I\ne I'$. By the injectivity of σ, see [Bo, Ch.V,4.4], it is enough to show that $(\sigma(w_0^Iw_0^{I'}))^{M_{I,I'}}=1$ in $GL(\mathbf{E})$. Let m,m',μ be as in A.7(a). Let $\mathbf{E}_{I\cup I'}$ be the subspace of $\mathbf{E}$ spanned by $\{e_s;s\in I\cup I'\}$. Note that $\sigma(w_0^Iw_0^{I'})$ leaves stable this subspace and induces the identity map on $\mathbf{E}/\mathbf{E}_{I\cup I'}$. Moreover, the bilinear form $(,)$ is positive definite on $\mathbf{E}_{I\cup I'}$ (see [Bo, Ch.5,4.8]) hence $\mathbf{E}$ is the direct sum of $\mathbf{E}_{I\cup I'}$ and its perpendicular in $\mathbf{E}$ on which the isometry $\sigma(w_0^Iw_0^{I'})$ must act as the identity. Thus it is enough to show that $(\sigma(w_0^Iw_0^{I'}))^{M_{I,I'}}=1$ on the subspace $\mathbf{E}_{I\cup I'}$.

If $\mu=2$ then $w_0^I,w_0^{I'}$ commute hence

$(w_0^Iw_0^{I'})^{M_{I,I'}}=(w_0^Iw_0^{I'})^2=(w_0^I)^2(w_0^{I'})^2=1$,

as required.

If $\mu=3$ and p in A.1(e) is 2 (resp. 3) then from A.3(c) (resp. A.4(c)) we see that the 4-th power (resp. 6-th power) of $\sigma(w_0^Iw_0^{I'})|\mathbf{E}_{I\cup I'}$ is 1, as required.

If $\mu=3$ and p in A.1(e) is 1 and (m,m') is $(4,2)$ or $(2,4)$ (resp. (m,m') is $(3,2)$ or $(2,3)$) then from A.6(c) (resp. A.5(c)) we see that the 8-th power (resp. 4-th power) of $\sigma(w_0^Iw_0^{I'})|\mathbf{E}_{I\cup I'}$ is 1, as required.

If $\mu=3$ and p in A.1(e) is 1 or if $\mu>3$ then from A.2(c) we see that the μ-th power of $\sigma(w_0^Iw_0^{I'})|\mathbf{E}_{I\cup I'}$ is 1, as required. This proves (a).

From (a) we see that the map $g_I \mapsto w_0^I$ ($I \in S$) extends (uniquely) to a group homomorphism $\lambda : W' \to \tilde{W}$. From A.1(a) we see that the image of λ is exactly W. It remains to show that λ is injective.

For any $I \in S$ we set $\tilde{e}_I = \sum_{s\in I} e_s$, $\psi_I = sqrt(\tilde{e}_I, \tilde{e}_I) \in \mathbf{R}_{>0}$ (we have $(\tilde{e}_I, \tilde{e}_I) \in \mathbf{R}_{>0}$ by [Bo, Ch.V,4.8,Thm.2]); we also set

$$e_I = \tilde{e}_I/\psi_I.$$

Note that $(e_I, e_I) = 1$. Setting $m = \max\{m_{s,s'}; s, s' \in I\}$, we have $\psi_I = 1$ if $m = 1$ and

$$\psi_I = \sqrt{\sharp(I)(1 - \kappa_m/2)}$$

if $m \geq 2$. (If $m \leq 2$ this is obvious; if $m \geq 3$, this follows from A.1(d).) For example, if $m \leq 2$ we have $\psi_I = \sqrt{\sharp(I)}$; if $m = 3$ we have $\psi_I = \sqrt{\sharp(I)\kappa_4^{-1}}$; if $m = 4$ we have $\psi_I = \sqrt{\sharp(I)\kappa_8^{-1}}$.

We show that for $I, I' \in S$ such that $\tilde{W}_{I\cup I'}$ is finite we have

(b) $\sigma(w_0^I)(e_{I'}) = e_{I'} + \kappa_{M_{I,I'}} e_I$.

When $I = I'$ this reduces to $\sigma(w_0^I)(e_I) = -e_I$ which follows easily from the definitions.

We now assume that $I \neq I'$. Let m, m', μ be as in A.7(a).

If $\mu = 2$ we have from the definitions $\sigma(w_0^I)(e_{I'}) = e_{I'}$, as required (since $\kappa_{M_{I,I'}} = 0$).

If $\mu = 3$ and p in A.1(e) is 2 (resp. 3) then from A.3(b) (resp. A.4(b)) we see that (b) holds. (We use that $\kappa_4 = \sqrt{2}$, $\kappa_6 = \sqrt{3}$.)

If $\mu = 3$ and p in A.1(e) is 1 and (m, m') is $(4, 2)$ or $(2, 4)$ (resp. (m, m') is $(3, 2)$ or $(2, 3)$) then from A.6(b) (resp. A.5(b)) we see that (b) holds. (We use that $\kappa_4 = \sqrt{2}$, $\kappa_8 = \sqrt{2 + \sqrt{2}}$.)

If $\mu = 3$ and p in A.1(e) is 1 or if $\mu > 3$ then from A.2(b) we see that (b) holds. This proves (b).

We show that for $I, I' \in S$ such that $\tilde{W}_{I\cup I'}$ is infinite we have

(c) $\sigma(w_0^I)(e_{I'}) = e_{I'} + x e_I$, $\sigma(w_0^{I'})(e_I) = e_I + x e_{I'}$ where $x \in \mathbf{R}_{\geq 2}$; moreover, we have $x = -2(e_I, e_{I'})$.

By A.1(c), the product $w_0^I w_0^{I'}$ has infinite order in W.

Next we remark that, by a standard argument, if $s_{i_1} s_{i_2} \dots s_{i_q}$ is a reduced expression in $\tilde{W}$ ($q \geq 1$) then $\sigma(s_{i_1} s_{i_2} \dots s_{i_{q-1}}) e_{s_{i_q}}$ is an $\mathbf{R}_{\geq 0}$-linear combination of elements $e_{s_{i_r}}$, $r \in [1, q]$ and at least one coefficient is > 0; moreover if s_{i_q} is different from $s_{i_1}, s_{i_2}, \dots, s_{i_{q-1}}$ then the coefficient of $e_{s_{i_q}}$ is 1.

Now let $s' \in I'$. We have $m_{s,s'} \geq 3$ for some $s \in I$. (Otherwise, $w_0^I, w_0^{I'}$ would commute and $w_0^I w_0^{I'}$ would have order 2, which is not the case, since it has infinite order). Hence we can find a reduced expression $s_{i_1} s_{i_2} \dots s_{i_q}$ for w_0^I with $s_{i_r} \in I$

and s_{i_q} such that $m_{s_{i_q},s'} \ge 3$. We have $\sigma(s_{i_q})e_{s'} = e_{s'} + ce_{s_{i_q}}$ where $c \in \mathbf{R}_{>0}$ and

$$\begin{aligned}\sigma(w_0^I)e_{s'} &= \sigma(s_{i_1}s_{i_2}\dots s_{i_q})e_{s'}\\ &= \sigma(s_{i_1}s_{i_2}\dots s_{i_{q-1}})(e_{s'} + ce_{s_{i_q}})\\ &= \sigma(s_{i_1}s_{i_2}\dots s_{i_{q-1}})e_{s'} + c\sigma(s_{i_1}s_{i_2}\dots s_{i_{q-1}})e_{s_{i_q}}.\end{aligned}$$

By the remark above this is equals $e_{s'}$ plus an $\mathbf{R}_{\ge 0}$-linear combination of elements $e_{s_{i_r}}$, $r \in [1,q]$ and at least one of the $e_{s_{i_r}}$ appears with coefficient > 0. Taking sum over all $s' \in I'$ we see that $\sigma(w_0^I)e_{I'}$ is equal to $e_{I'}$ plus an $\mathbf{R}_{\ge 0}$-linear combination of elements e_s, $s \in I$, and at least one of the e_s, $s \in I$, appears with coefficient > 0. Since $\tau : \mathbf{E} \to \mathbf{E}$ commutes with the action of $\sigma(w_0^I)$ and it keeps fixed $e_{I'}$ it follows that $\sigma(w_0^I)e_{I'} - e_{I'}$ is fixed by τ hence is of the form xe_I for a well defined $x \in \mathbf{R}_{>0}$. Thus we have $\sigma(w_0^I)e_{I'} = e_{I'} + xe_I$ with $x \in \mathbf{R}_{>0}$. Similarly we have $\sigma(w_0^{I'})(e_I) = e_I + ye_{I'}$ with $y \in \mathbf{R}_{>0}$. Since $\sigma(w_0^I)e_I = -e_I$, $\sigma(w_0^{I'})e_{I'} = -e_{I'}$, we see that the 2-dimensional subspace $\mathcal{E} = \mathbf{R}e_I + \mathbf{R}e_{I'}$ is stable under $\sigma(w_0^I)$ and $\sigma(w_0^{I'})$.

We have $\sigma(w_0^I)\sigma(w_0^{I'})(e_I) = (xy-1) + ye_{I'}$, $\sigma(w_0^I)\sigma(w_0^{I'})(e_{I'}) = -xe_I - e_{I'}$. We see that $\sigma(w_0^I)\sigma(w_0^{I'}) : \mathbf{E}_{I,I'} \to \mathbf{E}_{I,I'}$ has determinant 1 and trace $xy - 2$.

Assume that the bilinear form $(,)$ is positive definite on $\mathcal{E}$. Then $\mathbf{E}$ is the direct sum of $\mathcal{E}$ and $\mathcal{E}^\perp$, the perpendicular to χ with respect to $(,)$, and both $\sigma(w_0^I)$ and $\sigma(w_0^{I'})$ are contained in the group $\mathcal{G}$ of isometries of $(,)$ which preserve $\mathcal{E}$ and induces the identity map on $\mathcal{E}^\perp$. By [Bo, V,4.4,Cor.3] the powers of $w_0^I w_0^{I'}$ form a discrete subgroup of the group of isometries of $(,)$ hence a discrete subgroup of $\mathcal{G}$. This discrete subgroup is also infinite, as we have seen, and $\mathcal{G}$ is compact. This is a contradiction.

We see that $(,)$ is not positive definite on $\mathcal{E}$; however, as we have seen above, we have $(e_I, e_I) = 1$. It follows that the set of isotropic vectors in $\mathcal{E}$ is either a union of two lines L, L' (if $(,)$ is nondegenerate on $\mathcal{E}$) or a line L (if $(,)$ is degenerate on $\mathcal{E}$). In the first case both L, L' must be stable under the isometry $\sigma(w_0^I)\sigma(w_0^{I'})$ (which has determinant 1); it follows that $\sigma(w_0^I)\sigma(w_0^{I'})$ is diagonalizable over $\mathbf{R}$; hence it has real eigenvalues (with product 1). In the second case L must be stable under the isometry $\sigma(w_0^I)\sigma(w_0^{I'})$ hence $\sigma(w_0^I)\sigma(w_0^{I'})$ has again real eigenvalues (with product 1). Thus in both cases the trace of $\sigma(w_0^I w_0^{I'})$ on $\mathcal{E}$ must be ≥ 2. In other words we have $xy - 2 \ge 2$ that is $xy \ge 4$. Since $\sigma(w_0^I)$ is an isometry for $(,)$ we have $(e_{I'} + xe_I, e_{I'} + xe_I) = (e_{I'}, e_{I'})$ hence $2x(e_I, e_{I'}) + x^2(e_I, e_I) = 0$. Since $x > 0$ and $(e_I, e_I) = 1$ it follows that $x = -2(e_I, e_{I'})$. Similarly, we have $y = -2(e_I, e_{I'}$. Thus we have $x = y$ and $x^2 \ge 4$ hence $x \ge 2$. This proves (c).

Let $\bar{\mathbf{E}}$ be the subspace of $\mathbf{E}$ spanned by the vectors $e_I; I \in S'$. From (b) we see that the action of W' on $\mathbf{E}$ (by $w \mapsto \sigma(\lambda(w))$) leaves $\bar{\mathbf{E}}$ stable and the action of the generators g_I is given on $\bar{\mathbf{E}}$ by the formulas for the reflection representation of W' with the following modification: when $I, I' \in S'$ are such that $M_{I,I'} = \infty$ then

$$g_I(e_{I'}) = e_{I'} + x_{I,I'}e_I, g_{I'}(e_I) = e_I + x_{I,I'}e_{I'}$$

where $x_{I,I'} \in \mathbf{R}_{\geq 2}$, while in the actual reflection representation we would have $x_{I,I'} = 2$. Hence if $w \in W'$ satisfies $\lambda(w) = 1$ then w acts as identity on the (modified) reflection representation of W'. But the proof of faithfulness of the reflection representation in [Bo,Ch.V,4.4] extends to a proof of faithfulness of the modified reflection representation. (The only place where the proof must be changed is in [Bo,ChV, 4.5, case (a)] which is an easily verified statement about an infinite dihedral group.) This proves that λ is injective. The theorem is proved.

Theorem A.9. *Let $L : W \to \mathbf{N}$ be the restriction to W of the length function $\tilde{l} : \tilde{W} \to \mathbf{N}$. Then L is a weight function for W, S.*

Let $l : W \to \mathbf{N}$ be the length function of the Coxeter group W, S. It is enough to prove the following statement.

(a) If $w \in W$ and $I \in S$ are such that $l(w_0^I w) = 1 + l(w)$ then $\tilde{l}(w_0^I w) = \tilde{l}(w_0^I) + \tilde{l}(w)$.

Our assumption implies that $\sigma(w^{-1})e_I$ is an $\mathbf{R}_{\geq 0}$-linear combination of elements $e_{I'} \in \bar{\mathbf{E}}$ with $I' \in S$; hence for some $s \in I$, $\sigma(w^{-1})e_s$ is an $\mathbf{R}_{\geq 0}$-linear combination of elements $e_{s'} \in \mathbf{E}$ with $s' \in \tilde{S}$ (notation as in the proof of A.8). It follows that $\tilde{l}(sw) = \tilde{l}(w) + 1$. Hence for any i we have $\tilde{l}(\tau^i(sw)) = \tilde{l}(\tau^i(w)) + 1$ that is $\tilde{l}(\tau^i(s)w) = \tilde{l}(w) + 1$. Hence we have $s'w > w$ for any $s' \in I$. Using 9.7 we deduce that $\tilde{l}(w_0^I w) = \tilde{l}(w_0^I) + \tilde{l}(w)$, as required.

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Department of Mathematics, M.I.T., Cambridge, MA 02139