Hedging and Liquidity

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This article develops a model for evaluating alternative hedging strategies for financially constrained firms. A key advantage of the model is the ability to capture the intertemporal effects of hedging on the firm's financial situation. We characterize the optimal hedge. A wide range of alternative hedging strategies can be specified and the model allows us to determine in each case if the hedging strategy raises or lowers firm value and by how much. We show that hedging firm value, hedging cash flow from operations and hedging sales revenue are not optimal. The article highlights the fact that every hedging strategy comes packaged with a borrowing strategy which requires careful consideration.

Futures markets often provide the most liquid and convenient instruments for managing risk. However, because futures contracts are marked to market, it is often impossible to simultaneously hedge cash flows and values. For example, a futures contract that locked in the value of gold that a corporation planned to extract in one year would generate an uncertain cash flow pattern over the year.

This article examines how liquidity and cash flow timing problems associated with different hedging strategies can affect a firm's value. In our model, the objective for hedging is to increase the firm's financial flexibility. An optimal hedge maximizes the firm's liquidity—slack in the form of excess cash or unused debt capacity—when liquidity is most valuable. This lowers the danger of costly financial distress, reduces the effective cost of external financial constraints, and makes value maximizing investments affordable. A firm with no financial constraints does not gain from hedging, and the higher the firm's financial constraints the greater the potential value of hedging.

The value of hedging depends critically on the design of the hedging strategy. We show that the optimal hedge minimizes the variability in the marginal value of the firm's cash balances. Such a hedge efficiently redistributes cash balances across different states and periods, taking cash from those states for which the marginal cost of the financial constraint is low and giving cash to those states for which the marginal

cost is high, until the shadow value of cash across different price and cost paths is equalized.

Our model allows us to determine whether a particular hedging strategy creates value by increasing the return earned on the liquidity available to the firm. We show that a hedge that minimizes the variance of the firm's value is generally too large.

That a value hedge may be too large arises because the enormous funding requirements to implement the hedge can impose a dissipative cost on the firm. While an increase in price raises the value of the firm, only a small portion of this increase in value is reaped as an immediate cash flow. On the other hand, the full matching loss incurred on the hedge must be paid in cash immediately. One might think that a value hedge would create its own liquidity. If a hedge successfully locks in the firm's value, then by definition short-term losses on the hedge are exactly matched by an increase in anticipated future cash flows and consequently the short-term losses should be easily financed. We show that, in general, this is not the case. A hedge creates its own liquidity only if the firm is able to perfectly hedge all its different sources of risk. Any departure from this case requires a rigorous evaluation of how changes in the firm's debt capacity, in the firm's costs of external financing, and in the firm's value are all related to the hedge. We show that the financial risk created by the hedge itself is an important factor in determining the optimality of the hedge and how it contributes to value. A poorly conceived hedge can increase the expected costs of financing, tightening the financial constraints and lowering firm value. This problem is of great importance and calls attention to the fact that every hedging strategy comes packaged with a borrowing strategy. The advantages and disadvantages of the associated borrowing strategies are critical in determining both the value and success of alternative hedging strategies.[1]

The disaster at the German firm Metallgesellschaft is apposite. In our analysis of the case we identified the funding requirements of MG's hedging strategy as one of the central causes of the problem—see Mello and Parsons (1995a,b). Another problem was that basis risk in oil futures meant that the hedge did not successfully lock in value. Al-

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[1] Froot, Scharfstein, and Stein (1993) evaluate hedges in terms of the expected cost of external financing, but where this cost is specified exogenously. The exogenous specification of the costs is problematic because it is not possible to evaluate the effects of the hedge in equilibrium. The cost of external financing should depend on the effectiveness of the hedge. In our article the long-run effectiveness of the hedging determines the current marginal cost of external financing. The effect of the hedge on the value of the firm can then be weighed against the effect of the short-term funding costs. Several other articles have also addressed the effects of leverage constraints on the optimal hedge—see, for example, Grossman and Vila (1992), Naik and Uppal (1994) and Deep (1996). In all these models, hedging strategies are evaluated in the presence of exogenously imposed leverage costs or fixed constraints on the amount of hedging.

though several later studies have concurred that basis risk was a significant problem and that MG's hedge did not lock in value, these other studies overlook the issue of liquidity—see, for example, Brennan and Crew (1995), Hilliard (1995), Neuberger (1995), and Ross (1997). Others writing on the case explicitly denied that liquidity was a problem, arguing that a perfect hedge must create its own liquidity—see Culp and Miller (1995). Our result that a perfect hedge does not create its own liquidity sheds light on this debate and establishes more generally that the inability to fund a hedging strategy to its end can be a serious defect in the design of many popular hedging strategies.[2]

We also show that minimizing the variance of cash flow is not an optimal hedge. A cash flow hedge is generally too large. Hedging the next period cash flow minimizes the variability in the cash balance next period, but not the variability in the marginal value of the cash balance. A hedge designed to minimize the variance of the next period's cash flow assumes that the firm should hold the same cash balance next period, independent of the realized operating cash flow. It ignores the relationship between the next period cash flow and the marginal value of cash balances, and assumes instead that the marginal value of next period cash balances is independent of next period cash flows. However, variations in next period cash flows are informative of expected cash flows over a long horizon and therefore of the marginal value of cash balances. In our model, low cash flows next period reflect low prices or high costs so that the firm is more likely to exit or abandon operations regardless of its cash balances. Consequently, the marginal value of cash balances is less, and a hedge designed to secure the cash balance in this event would waste the firm's limited cash resources. Therefore the optimal hedge should not guarantee a constant cash balance next period, and the optimal hedge is not a cash flow hedge.

The article focuses on hedging with short-maturity futures contracts. Other hedging instruments such as forward contracts, swaps, options, and commodity linked bonds involve different packages of contingent payoffs and financing. In perfect, frictionless capital markets, each of these packages can be replicated by a strategy of short-term futures and riskless borrowing. But when capital markets are imperfect, the financing strategy embedded in these alternative instruments may prove to be strictly preferred by certain firms. By drawing out the significance of the financing strategy associated with any hedging strategy we hope to contribute to a better understanding of alternative hedging contracts, and to emphasize that the optimal hedging strategy is the one with a borrowing strategy that imposes the lowest financing costs on the firm.

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[2] Froot, Scharfstein, and Stein (1993) conjecture that in a multiperiod model interim funding requirements of an otherwise perfect hedge might lower the size of the optimal hedge. Our model proves this result under specific assumptions.

The structure of the article is as follows. Section 1 develops a dynamic model of a firm with financial constraints. Section 2 extends previous models that analyze the firm's optimal hedge ratio at a single point in time and in anticipation of a single random shock to the next period's cash flow or value—see, for example, Smith and Stulz (1985) and Froot, Scharfstein, and Stein (1993). Here we examine how hedging relaxes the effect of the financial constraints by increasing the firm's liquidity when it is most needed. A characterization of the optimal hedge ratio is provided and it is shown how this ratio varies with the firm's current margins and current cash balance. Section 3 evaluates the performance of different hedging strategies and shows how the value of a hedging strategy can be derived endogenously, by weighing the costs and the benefits of hedging as determined by the specific situation of the firm's investment and financial structure. Section 4 offers a discrete time example that highlights the liquidity problems associated with a hedge to minimize variance in firm value. Section 5 discusses extensions of these ideas, and Section 6 examines several implications for corporate risk management. Section 7 concludes the article with a brief summary and final remarks.

1. A Model of a Financially Constrained Firm

This section presents a dynamic model of a firm choosing an operating policy contingent on a pair of stochastic variables, the output price and the input cost. We begin with a benchmark case in which there is no financial constraint so that the first best operating policy is attainable. We then turn to the case in which the firm is constrained to finance cash flow deficits from retained cash balances and protected debt.

Consider a firm that produces a commodity at a constant annual rate of q units. The input cost per unit produced is c_t and the output price per unit is p_t. Both the cost and the price are stochastic. At any point in time the firm may either operate and realize an instantaneous cash flow equal to $q(p_t - c_t)$ or abandon production entirely. Abandonment is costless but irreversible. The input cost and the output price follow the exogenous processes

$$dc_t = \nu c_t dt + \sigma_c c_t dz_1(t), \quad \text{and} \tag{1}$$

$$dp_t = \mu p_t dt + \sigma_p p_t dz_2(t), \tag{2}$$

where $dz_1(t)$ and $dz_2(t)$ are each increments to a standard Gauss−Wiener process, with correlation ρ; σ_c and σ_p being the instantaneous standard deviations of the cost and price, respectively, assumed to be known and constant; and ν and μ are the instantaneous drifts in the cost and price. We assume a constant riskless interest rate r and

that κ_c and κ_p are the net convenience yields from owning an additional unit of the input and output, respectively. For simplicity, we assume that the convenience yields are constant proportions of c and p, $\kappa_c(c_t) = \kappa_c c_t$ and $\kappa_p(p_t) = \kappa_p p_t$. These assumptions are sufficient to define the complete set of Arrow–Debreu state prices with which to value any state contingent stream of cash flows generated by the firm.

Case 1. The first best: the unconstrained firm

The firm chooses an operating policy, φ, defining when it would abandon operation. The first best policy—the one chosen absent any financing constraints—is dependent only on the current price and the current cost, and can be represented as a function defining, for every cost, a critical price below which the firm would abandon, $\varphi^{fb}(c)$. The firm's value is a function of the current input cost, the current output price, and its operating policy: $V(c, p|\varphi^{fb})$. Applying Ito's lemma, the instantaneous change in the value of the firm is given by

$$dV = V_p\,dp + V_c\,dc + \frac{1}{2}\left[V_{pp}(dp)^2 + 2V_{pc}(dp)(dc) + V_{cc}(dc)^2\right] \quad (3)$$

and satisfies the partial differential equation

$$\frac{1}{2}\left[\sigma_p^2 p^2 V_{pp} + 2\rho\sigma_p\sigma_c\,pcV_{pc} + \sigma_c^2 c^2 V_{cc}\right] + p(r - \kappa_p)V_p$$
$$+ c(r - \kappa_c)V_c + q(p - c) - rV = 0. \quad (4)$$

One boundary condition is provided by the fact that the firm's value is zero when it is abandoned, that is, whenever p falls to $\varphi^{fb}(c)$:

$$V(c, p|\phi^{fb})|_{p = \varphi^{fb}(c)} = 0. \quad (5)$$

An additional boundary condition is given by requiring that as the ratio of price to cost grows, the ratio of value to price remains finite:

$$\lim_{(p/c)\to\infty} \frac{V(c, p|\phi^{fb})}{p} < \infty. \quad (6)$$

Optimality of the operating policy requires continuity of the slopes at the endogenously derived free boundary characterizing abandonment, that is, along $p = \varphi^{fb}(c)$:

$$V_c(c, p|\phi^{fb})|_{p = \phi^{fb}(c)} = 0, \quad (7)$$
$$V_p(c, p|\phi^{fb})|_{p = \phi^{fb}(c)} = 0. \quad (8)$$

An explicit solution for V can be derived by noting that $V(c, p|\varphi^{fb})$ is linearly homogeneous in c and p. The optimal policy is to abandon whenever the ratio of price and cost reaches a critical point, $\varphi^{fb}(c)/c = y^{fb}$, so that the two-variable problem can be represented as a

one-variable problem in the ratio of output price to input cost, $y = p/c$: $V(c, p|\varphi^{fb}) = cV(p/c, 1|\varphi^{fb}) = cU(y|y^{fb})$. The explicit solution to Equations (3)–(8) is

$$V^{fb}(c, p) = ck_1(p/c)^\gamma + q[p/\kappa_p - c/\kappa_c], \qquad (9)$$

with $k_1 = [(1/\kappa_c) - (y^{fb}/\kappa_p)](y^{fb})^{-\gamma}$, $y^{fb} = [\gamma/(1 - \gamma)](\kappa_p/\kappa_c)$, and $\gamma = \theta - (\theta^2 + 2\kappa_c/\sigma^2)^{1/2}$, where $\theta = [1/2 - (\kappa_c - \kappa_p)/\sigma^2]$ and $\sigma^2 = \sigma_p^2 - 2\rho\sigma_p\sigma_c + \sigma_c^2$.

Generally $y^{fb} < 1$, so that there are states in which the firm continues to operate when $p_t < c_t$ and the firm is making operating losses. To cover these losses, the firm requires a net cash inflow of $-q(p_t - c_t)$. Keeping the firm open and covering the current operating losses preserves the opportunity for future profit should the output price subsequently rise above the input cost. But to maintain the value of this option the firm must be able to cover the losses. The derivation of V^{fb} assumes that the firm is able to cover operating losses as long as it needs. This is possible if the owners of the firm themselves have unconstrained wealth with which to finance any cash requirements—we call this the deep pockets assumption—or, alternatively, if capital markets are frictionless so that outsiders may costlessly finance any cash requirements. Under either of these two assumptions it is difficult to understand the need for corporate hedging: the Modigliani–Miller proposition that the firm's capital structure is irrelevant applies to hedging as well.[3] In order to understand the role for corporate hedging, we next explore the consequences of abandoning these two assumptions.

Case 2. The financially constrained firm
In this case the firm is constrained to finance temporary cash shortfalls with retained cash or protected debt.[4] The firm begins with an initial

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[3] Of course, there are a number of other motivations for hedging, including taxes and managerial incentives, that may coexist with the assumption of frictionless capital markets. For a review of the determinants of hedging by corporations see Smith and Stulz (1985).

[4] Expanding the firm's options to include other forms of external finance, in particular risky debt, would relax the effective constraint facing the firm but otherwise leave the results unchanged. So long as financing is not frictionless the positive role for hedging can be analyzed exactly in the same way as we are describing here. And incorporating other forms of external financing is complicated. Allowing the firm to sell risky debt multiplies the complexity of the problem since the risk of the debt must be determined endogenously together with the value of the firm. Furthermore, allowing risky debt opens a pandora's box: the firm may wish to sell debt instruments of varying maturity and seniority since there may be some optimal debt structure contingent upon the values of the state variables. By restricting ourselves to riskless debt, we obtain a model in which the firm's financial structure can be intuitively described by a single state variable, W, and this allows us to evaluate the role of hedging, as well as the relationship between alternative hedging strategies and the firm's financial constraint.

cash balance of $W_0 > 0$. The firm's accumulated cash balance at time t is

$$W_t = \int_0^t e^{r(t-\tau)} q(p_\tau - c_\tau) d\tau + W_0 e^{rt} \qquad (10)$$

and its instantaneous cash flow or increment to the cash balance is $q(p_t - c_t) + rW_t$. If $q(p_t - c_t) > -rW_t$ then the cash balance is incremented, while if $q(p_t - c_t) < -rW_t$ the cash balance is decreased. When the accumulated cash balance falls below zero the firm is borrowing risklessly. The firm can continue to finance temporarily negative cash flows only so long as its accumulated debt is less than its liquidation value. We assume that liquidation occurs with some dissipative cost so that the firm is worth less than its first-best value: $\alpha V^{fb}(c, p), 0 \leq \alpha < 1$. Therefore, the firm can operate so long as $W_t \geq -\alpha V^{fb}(c, p)$.

The firm's operating policy is now a function of its cash balances, $\varphi^{fc}(c, W)$, and the value of the firm's operating assets therefore also become a function of the firm's cash balance, $V(c, p, W|\varphi^{fc})$. Since the firm's cash balance depends on the entire path of price and cost, the value of the firm is in turn path dependent. The valuation equation is

$$dV = V_p dp + V_c dc + V_W dW$$
$$+ \frac{1}{2}\left[V_{pp}(dp)^2 + 2V_{pc}(dp)(dc) + V_{cc}(dc)^2\right], \qquad (11)$$

where the term $V_W dW$ reflects the effect of the growth in cash balances on the value of the financially constrained firm. The value of the firm satisfies the partial differential equation

$$\frac{1}{2}\left[\sigma_p^2 p^2 V_{pp} + 2\rho\sigma_p\sigma_c pc V_{pc} + \sigma_c^2 c^2 V_{cc}\right]$$
$$+ p(r - \kappa_p)V_p + c(r - \kappa_c)V_c$$
$$+ \left[q\left(p[1 + r - \kappa_p] - c[1 + r - \kappa_c]\right) + rW\right]V_W - rV = 0. \quad (12)$$

As before, the boundary condition at the abandonment point is

$$V(c, p, W|\phi^{fc})|_{p = \phi^{fc}(c, W)} = 0. \qquad (13)$$

The boundary condition at the liquidation point is

$$V(c, p, W|\phi^*)|_{W = -\alpha V^{fb}(c, p)} = \alpha V^{fb}(c, p). \qquad (14)$$

Two additional boundary conditions are given by the requirement that

$$\lim_{(p/c) \to \infty} \frac{V(c, p, W|\phi^{fc})}{p} \quad \text{and} \qquad (15)$$

$$\lim_{W \to \infty} V(c, p, W|\varphi^{fc}) = V^{fb}(c, p). \qquad (16)$$

The last condition simply says that as the cash balances increase and so the significance of the financial constraint declines, the value of the financially constrained firm, net of its cash balances, approximates the value of the unconstrained firm. Optimality of the operating policy implies the additional boundary conditions:

$$V_c(c, p, W | \varphi^{fc})|_{p = \varphi^{fc}(c, W)} = 0, \tag{17}$$

$$V_p(c, p, W | \varphi^{fc})|_{p = \varphi^{fc}(c, W)} = 0, \tag{18}$$

$$V_W(c, p, W | \varphi^{fc})|_{p = \varphi^{fc}(c, W)} = 0. \tag{19}$$

For shorthand we denote the financially constrained firm's value under the optimal operating policy by V^{fc}. Because the valuation is path dependent, it is not generally possible to derive an explicit solution for V^{fc}, and numerical methods must be employed instead. Nevertheless, it is clear that the value of the constrained firm is less than the unconstrained firm:

$$V^{fc} < V^{fb}. \tag{20}$$

The difference comes from the advantage of having financial flexibility. While there will be cases in which the unconstrained firm would continue operating, $p > \varphi^{fb}(c)$, the constrained firm has exhausted its debt capacity and so must liquidate. For this reason it will generally be the case that $V_W > 0$ so that it is advantageous for the owners to retain all earnings and pay no dividends.

Figure 1 shows the relationship between the value of the financially constrained firm and the first best value of the firm at different levels of cash balances. The smaller the cash balances, the greater the shortfall in firm value relative to the first best value. As cash balances grow, the financially constrained firm's value approaches the first best value. As cash balances fall and the accumulated debt approaches the firm's liquidation value, the financially constrained firm's value declines and approaches the liquidation value. Note that the marginal value of a dollar added to the cash balance is large when the cash balance is small and that this marginal value quickly declines, approaching zero relatively swiftly. This effect is more pronounced for the firm with the greater bankruptcy cost. Lowering the firm's bankruptcy cost, that is, increasing the liquidation cost parameter α, increases the amount of riskless debt capacity and so relaxes the financial constraint. This raises the firm's value at every level of cash balance, bringing it everywhere closer to the first best.

The fact that the firm may be forced to liquidate feeds back to influence the firm's abandonment policy, even when it has available cash. Because the future value of the operating firm is lower, the firm's

Firm Value
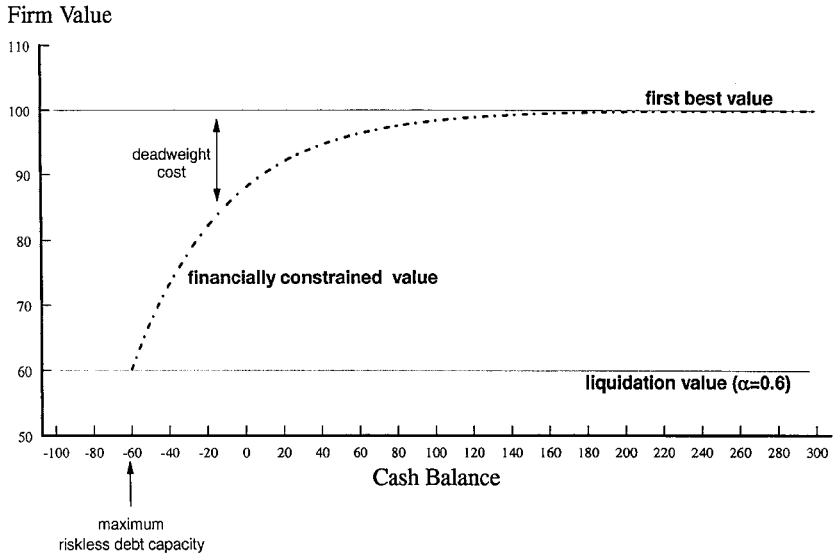
Figure 1
The value of the financially constrained firm
The value of the financially constrained firm is plotted for different levels of cash balances and given a liquidation cost parameter, α, of 0.6. The smaller the cash balances, the greater the deadweight cost of the firm's financial constraint, that is, the shortfall in firm value relative to the first best. As cash balances grow, the financially constrained firm's value approaches the first best value. As cash balances fall and the accumulated debt approaches the firm's liquidation value, the financially constrained firm's value declines and approaches the liquidation value.

abandonment option is less valuable than in the unconstrained case so that the firm abandons sooner, that is, at higher output prices and at lower input costs, than in the unconstrained case:

$$\varphi^{fc}(c, W) > \varphi^{fb}(c). \tag{21}$$

The deadweight cost of the firm's financial constraint is given as the firm's shortfall in value relative to the first best unconstrained benchmark:

$$V^{fb}(c, p) - V^{fc}(c, p, W). \tag{22}$$

Reducing this deadweight cost is the motivation for hedging.

2. The Value of a Hedging Strategy

For the financially unconstrained firm there is no advantage to hedging. Since all hedges are fairly priced, a hedge can only change the stochastic pattern of the firm's future cash flows, not the firm's value. However, this is not the case for the financially constrained firm. Because the

135

value of a dollar inside the firm can be greater than the value of that dollar outside the firm, it becomes possible that a hedge which is priced fairly on the market nevertheless adds value to the firm. A hedge is valuable if it moves cash from states in which the firm's own shadow value of liquidity is low to states in which the firm's own shadow value of liquidity is high. By reducing the expected costs of financing, hedging relaxes the financial constraints on the firm and increases the firm's debt capacity. On the other hand, a poorly conceived hedge may increase the expected costs of financing, tightening the financial constraints on the firm and lowering its value. In this section we extend our model to allow a careful examination of the effects of a hedge on firm value.

We allow the firm to maintain a dynamically rebalanced position in an instantaneously maturing futures contract written on the output price. In order to capture situations in which the firm cannot perfectly hedge, we preclude the use of futures contracts written on the input cost. Given the proportional convenience yield assumption made earlier, the futures prices for a contract maturing in τ periods $f(p_t, \tau) = p_t e^{(r - \kappa_p)\tau}$. A hedge or a position in h_τ futures contracts generates for the firm the instantaneous cash flow $h_\tau df(p_\tau)$, where $df(p_\tau) = f_p dp + 1/2 f_{pp} \sigma_p^2 p^2 dt - f_\tau dt$. The firm's dynamic hedging strategy is given as a quantity of instantaneous futures contracts held at any point in time, contingent on the current output and input prices, p_t and c_t, and possibly contingent on the firm's current cash balance, W_t: $h_t = h(c_t, p_t, W_t)$. At any time t, the hedged firm's accumulated cash balance is now given by

$$W_t = \int_0^t e^{r(t-\tau)} \left[q(p_\tau - c_\tau) + h_\tau df(p_\tau) \right] d\tau + W_0 e^{rt}. \quad (23)$$

The value of the hedged firm is now calculated using a program like that given in Section 1, case 2, but with this new cash balance equation:

$$dV = V_p dp + V_c dc + V_W dW$$

$$+ \frac{1}{2} \left[V_{pp}(dp)^2 + 2V_{pc}(dp)(dc) + V_{cc}(dc)^2 \right]. \quad (24)$$

Noting that $df(p_\tau) = f_p p[(\mu - r) + \kappa_p] dt + f_p p \sigma_p dz_2$ and that $\mu = r - \kappa_p$, the value of the firm satisfies

$$\frac{1}{2} \left[\sigma_p^2 p^2 V_{pp} + 2\rho\sigma_p \sigma_c pc V_{pc} + \sigma_c^2 c^2 V_{cc} \right]$$

$$+ p(r - \kappa_p)V_p + c(r - \kappa_c)V_c$$

$$+ \left[q(p[1 + r - \kappa_p] - c[1 + r - \kappa_c]) + rW \right] V_W - rV = 0. \quad (25)$$

The operating and financial policies of the hedged firm provide the following boundary conditions:

$$V(c, p, W | \phi^h)|_{p = \phi^h(c, W)} = 0, \tag{26}$$

$$V(c, p, W | \phi^h)|_{W = -\alpha V^{fb}(c, p)} = \alpha V^{fb}(c, p), \tag{27}$$

$$\lim_{(p/c) \to \infty} \frac{V(c, p, W | \phi^h)}{p} < \infty, \tag{28}$$

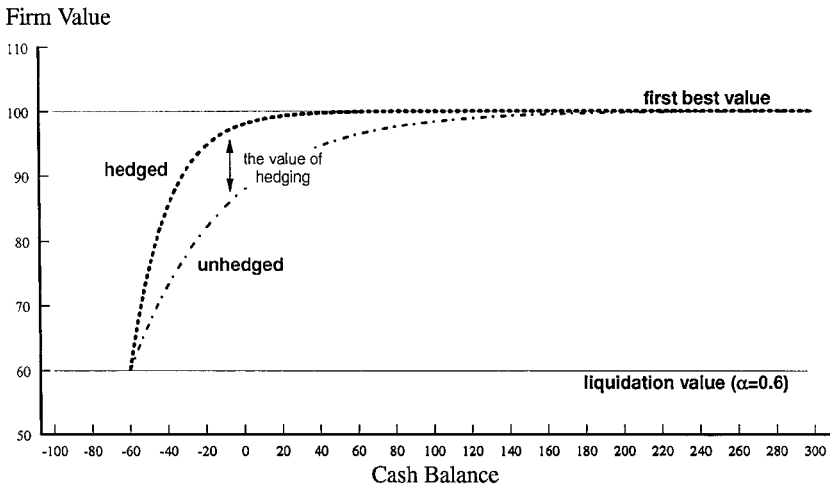$$\lim_{W \to \infty} V(c, p, W | \varphi^h) = V^{fb}(c, p), \tag{29}$$

$$V_c(c, p, W | \phi^h)|_{p = \phi^h(c, W)} = 0, \tag{30}$$

$$V_p(c, p, W | \phi^h)|_{p = \phi^h(c, W)} = 0, \tag{31}$$

$$V_W(c, p, W | \phi^h)|_{p = \phi^h(c, W)} = 0, \tag{32}$$

where φ^h denotes the optimal operating policy, given the hedging strategy implemented.

Figure 2 shows two graphs of the firm's value as a function of its current cash balance. The lowest graph is for the unhedged firm. The
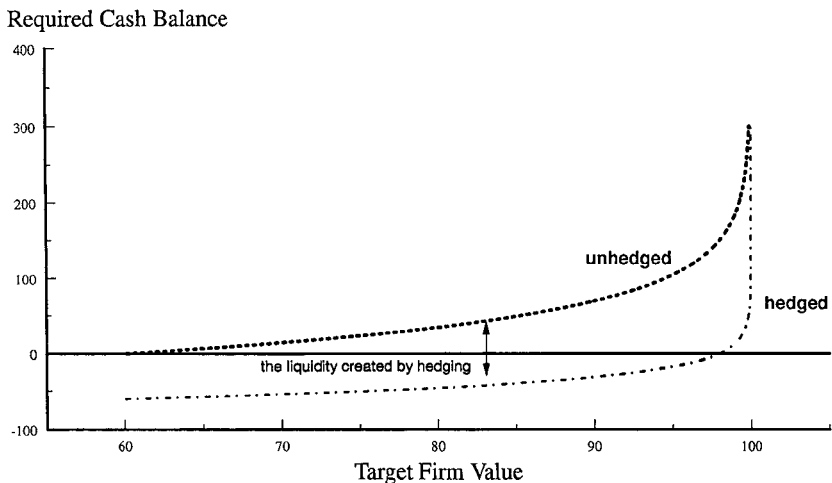

Firm Value

Figure 2
The effect of hedging on the value of the financially constrained firm
The value of a financially constrained firm, hedged and unhedged, is plotted for different levels of cash balances. At any given current level of cash balances, the hedged firm is less likely than the unhedged firm to have its future cash balances decline to the liquidation point. The value of hedging is the increment to firm value created by reducing the probability that the financial constraint is binding. At extremely high levels of cash balances, when the probability of the financial constraint binding is small regardless of the hedge, the values of the hedged and unhedged firms both approach the first best value.

highest graph is for the firm with a valuable hedge strategy. The boundary condition in Equation (27) requires that in both cases, if the firm has exhausted its debt capacity, $W = -\alpha V^{fb}$, its value is equal to its liquidation value $V = \alpha V^{fb}$. At extremely high levels of cash balances, when the probability of the financial constraint binding is small, the boundary condition in Equation (29) requires that the value functions for both the hedged and for the unhedged firm approach the first best value. At any given level of cash balances, a properly designed hedging strategy reduces the probability that the firm's cash balance will decline to the liquidation point. This raises the hedged firm's value above the value of the unhedged firm. The difference is the value of the hedge:

$$V^{h}(c, p, W) - V^{fc}(c, p, W). \tag{33}$$

A hedge can also be understood as a source of liquidity to the firm. The graphs displayed in Figure 3 are a simple transformation of the graphs shown in Figure 2 and show how a properly designed hedging strategy effectively adds to the firm's available liquidity. On the horizontal axis is the target firm value. On the vertical axis is the level of required cash balances. The graphs display the level of cash balances or liquidity required for the unhedged and for the hedged firm, respec-

Required Cash Balance

Figure 3
The liquidity created by hedging
The cash balance required to achieve a target firm value is displayed for both the hedged and the unhedged firms. Target firm value is on the horizontal axis, while the level of required cash balances is on the vertical axis. The hedged firm requires a lower cash balance in order to reach any given target value. The reduction in required cash balance is the liquidity created by hedging. The graphs in this figure are a simple transformation of those shown in Figure 2.

138

tively, to attain a given target value. The hedged firm requires a lower cash balance in order to reach any given target value. Hedging substitutes for cash balances, and in that sense hedging can be said to create liquidity. The difference between the two graphs displayed in Figure 3 is the liquidity created by the hedge.

For a particular operating strategy, hedging changes the likelihood of the firm exhausting its cash balance, and thereby changes the firm's value. This feeds back to affect the optimal operating policy given the specified hedging strategy, φ^h, which in turn yields the firm's hedged value. As a result, hedging and the value of the firm become interlinked variables. We denote by V^h the firm's value under this optimal operating policy, incorporating the hedge strategy and the external financial policy. The framework developed here enables us to evaluate hedging strategies by comparing the value created under each alternative.

An optimal hedge is one which maximizes Equation (33). This is equivalent to minimizing the deadweight loss in Equation (22) and also to maximizing the expected value of the hedged firm, V^h. As before, there is no explicit solution for the value of the firm in the problem outlined, and it is necessary to employ numerical solutions. This means that there is no direct way to solve for the optimal hedging strategy. Nevertheless, it is possible to develop insights into the determinants of an optimal hedging strategy and to characterize the optimal hedge ratio.

Suppose that $h^*(c, p, W)$ is a strategy for which V^* is the resulting maximized firm value function. For this hypothesized hedge strategy to be optimal, it must be the case that the hedge ratio employed at every time t separately maximizes the expected value of the firm given that the specified strategy is used at all future times:

$$h_t^* \text{ solves } \max_{h_t} E(V^*) \tag{34}$$

The solution to this problem is:

Proposition 1. *The hedging strategy that maximizes the value of the financially constrained firm satisfies*

$$h^*(c, p, W) = \frac{1}{f_p}\left[q\left(1 - \frac{c}{p}\frac{\sigma_c}{\sigma_p}\rho\right) + \left(\frac{V_{Wp}^*}{V_{WW}^*} + \frac{V_{Wc}^*}{V_{WW}^*}\frac{c}{p}\frac{\sigma_c}{\sigma_p}\rho\right)\right]. \tag{35}$$

Proof. See appendix.

The firm faces uncertainty regarding the realization of the two stochastic variables, p and c, which determine the firm's cash balances the next instant. The relative value of the cash balances under different realizations of the random variables is determined by V_W^*, and hedging

allows cash to be shifted from states in which the marginal value of cash balances is low to states in which the marginal value of cash balances is high. Then the optimal hedge ratio is set so as to minimize the variability in the marginal value of cash balances:

Proposition 2. *The optimal hedging strategy minimizes the variability in the marginal value of the firm's cash balances*:

$$h^* \, solves \, \min_{h_t} \, \mathrm{var}\,(V_W^*) \tag{36}$$

Proof. See appendix.

Equation (35) provides a characterization, but not a solution for the optimal hedge ratio, h. The functions V_{Wp}^*, V_{WW}^*, and V_{Wc}^* are themselves given by the complete hedge strategy and cannot be determined independently so as to yield an explicit expression for the optimal hedge ratio. Still, the characterization provides useful insights.

Comparative statics, summarized in Table 1, yield the following results: (i) For a positive correlation between costs and prices, as the volatility of input costs σ_c increases the optimal hedge ratio declines. With $\rho > 0$, the volatility of the firm's cash balances increases with the volatility of prices and decreases with the volatility of costs. A short hedge position produces an effect on the volatility of the firm's cash balances that resembles the effect of input costs. On the other hand, when prices and costs are independent or negatively correlated, a higher volatility of input costs σ_c increases the amount of optimal hedging. (ii) Whenever the volatility of output prices σ_p increases, the optimal hedge also increases. Higher volatility means more upside potential for firms with low asset value. Hence a greater amount of hedging is required to protect the option component. (iii) For the same

Table 1
Comparative statics for the optimal hedge, *h**

	Volatility of cost, σ_c	Volatility of price, σ_p	Correlation of cost and price, ρ	Cash balance, W	Liquidation cost, $(1 - \alpha)$	Interest rate, r
Change in optimal hedge, *h**	−	+	−	−	+	+

Using Equation (35) in Proposition 1 we derive comparative statics on the optimal hedge, *h**, for changes in the model parameters. An increase in the volatility of costs decreases the optimal hedge of price—when the correlation between price and cost is positive—while an increase in the volatility of price increases the optimal hedge. Increasing the correlation between cost and price decreases the optimal hedge. Increasing the firm's cash balance decreases the optimal hedge. Increasing the costs of liquidation increases the optimal hedge, as does increasing the interest rate.

reason as in (i) a higher correlation between costs and prices, ρ, decreases the optimal hedge ratio. (iv) The higher the cash balances W available to the firm, the less financially constrained the firm is and the lower the benefits of hedging. (v) When the loss from liquidation declines, that is, high α, the optimal hedge amount also declines. (vi) Higher interest rates reduce firm value and increase the costs of financing existing liabilities. The result is higher financial constraints and an increase in the amount of hedging.

3. Hedging and Liquidity: An Endogenous Evaluation

The framework developed here enables us to evaluate alternative hedging strategies. We consider three popular hedging strategies among practitioners and commonly referred to in the finance literature. The first is a hedge designed to minimize the variability in cash flows, the second is a hedge designed to minimize the variance in the firm's value, and the third is a hedge designed to minimize the variance of the firm's revenue.

A hedge designed to minimize the variance of the cash flow to the firm is given by

$$h^{cf}(c, p) = \frac{1}{f_p} q \left(1 - \frac{c}{p} \frac{\sigma_c}{\sigma_p} \rho \right). \tag{37}$$

A hedge designed to minimize the variance in the firm's revenue is given by

$$h^r(c, p) = \frac{1}{f_p} q, \tag{38}$$

and, of course, if $\rho = 0$, then the cash flow hedge is identical to the revenue hedge. Finally, a hedge designed to minimize the variance in the value of the financially constrained firm is given by

$$h^v(c, p, W) = \frac{1}{f_p} \left[\left(V_p + V_c \frac{c}{p} \frac{\sigma_c}{\sigma_p} \rho \right) + q V_W \left(1 - \frac{c}{p} \frac{\sigma_c}{\sigma_p} \rho \right) \right]. \tag{39}$$

Corollary 1. For a financially constrained firm, hedging strategies designed to minimize the variance of cash flow or revenue are both suboptimal.

Proof. The optimal hedge ratio from Equation (35) can be written as the sum of the cash flow hedge ratio plus a component related to the value of cash balances:

$$h^*(c, p, W) = h^{cf}(c, p) + \frac{1}{f_p} \left(\frac{V_{Wp}}{V_{WW}} + \frac{V_{Wc}}{V_{WW}} \frac{c}{p} \frac{\sigma_c}{\sigma_p} \rho \right). \tag{40}$$

So long as the second component is not identically zero, the cash flow hedge ratio is not generally optimal. Noting that $V_{wp}/V_{ww} < 0$ and $V_{wc}/V_{ww} > 0$, the second component is negative and $h^*(c, p, W) < h^{cf}(c, p)$ whenever $\rho \leq 0$. When $\rho > 0$ the relationship depends on the parameters, but only coincidentally would we have $h^* = h^{cf}$. That the revenue hedge is not generally optimal can be seen from the case in which $\rho = 0$ and the revenue hedge is identical to the cash flow hedge so that $h^* < h^r$. ∎

Corollary 2. *For a financially constrained firm, a hedging strategy designed to minimize the variance of firm value is suboptimal.*

Proof. Suppose that bankruptcy costs are significant, $\alpha = 0$, so that the firm's liquidation value is zero. Suppose also that the current cash balance is minuscule, so that the firm is virtually bankrupt, $W = \varepsilon$. By the boundary condition of Equation (14) we have $\lim_{W \to 0} V = 0$, $\lim_{W \to 0} V_p = 0$, and $\lim_{W \to 0} V_c = 0$, but $\lim_{W \to 0} V_W > 0$. From Equation (40) we can derive that the optimal hedge ratio approaches the cash flow hedge, $\lim_{W \to 0} h^* = h^{cf}$. From Equations (37) and (38) we can derive that the value hedge ratio is proportional to the cash flow hedge ratio, $\lim_{W \to 0} h^v = V_W h^{cf}$. Consequently, the value hedge will be greater than the optimal hedge, $h^v > h^*$, whenever the marginal value of liquidity is high. This occurs when the first best or unconstrained firm value is very large, that is, $p/c \to \infty$, so that the deadweight cost associated with the low cash balances is arbitrarily large, and correspondingly the marginal value of a dollar of cash balances is arbitrarily large, $V_W \to \infty$.

In certain other cases the value hedge is smaller than the optimal hedge. For example, whenever the price is so low that the firm is close to abandoning operations, for example, $p \to_+ \varphi^h(c, W)$, the variability in firm value at this point is very small. Hedging cannot reduce this variability to a significant degree so that the hedge ratio that minimizes variability in firm value approaches zero: $\lim_{p \to \phi^h(c, W)} V_p = \lim_{p \to \phi^h(c, W)} V_c = \lim_{p \to \phi^h(c, W)} V_W = 0$, and so $\lim_{p \to \phi^h(c, W)} h^v = 0$. However, the variability in the marginal value of cash balances remains substantial so that the optimal hedge ratio does not fall to zero: for simplicity let $\rho = 0$ so that we have $\lim_{p \to \phi^h(c, W)} h^{cf} = (1/f_p)q > 0$, and then note that $h^* \geq h^{cf}$ everywhere so that $\lim_{p \to \phi^h(c, W)} h^* \geq \lim_{p \to \phi^h(c, W)} h^{cf} = (1/f_p)q > 0$.[5] ∎

[5] To make this argument we have implicitly assumed that the price at which the firm abandons operations is greater than the price at which it is bankrupt and liquidation is forced. As the price falls to the abandonment point, both the firm's value and its debt capacity fall to zero. So the assumption that it not be forced into liquidation first entails the assumption that the firm is free of debt, $W \geq 0$. Therefore the case in which the value hedge is smaller than the optimal hedge ratio is a case in which we expect the value of hedging to be modest in the first place.

In the first case examined in the proof, the dangerously low cash balances mean that stability in the firm's immediate cash flow is essential to preserving whatever value the firm might have. Hedging value is the wrong thing to do because it wastes the firm's limited liquidity, increasing the variability of the marginal value of the firm's dangerously low cash balances.

We now turn to using the program given in Section 2 to calculate firm value under different hedging strategies and for different parameterizations of the model. In order that the hedge ratio for the minimization of the variance in firm value be well specified, we employ the ratio that minimizes the variability in the first best value of the firm:

$$h^{vfb}(c, p) = \frac{1}{f_p}\left(V_p^{fb} + V_c^{fb}\frac{c}{p}\frac{\sigma_c}{\sigma_p}\rho\right). \tag{41}$$

Figure 4 shows numerical estimates of the firm value for different initial cash balances and under alternative hedging strategies. Also shown is an estimate of the value of the unhedged firm. The parameters
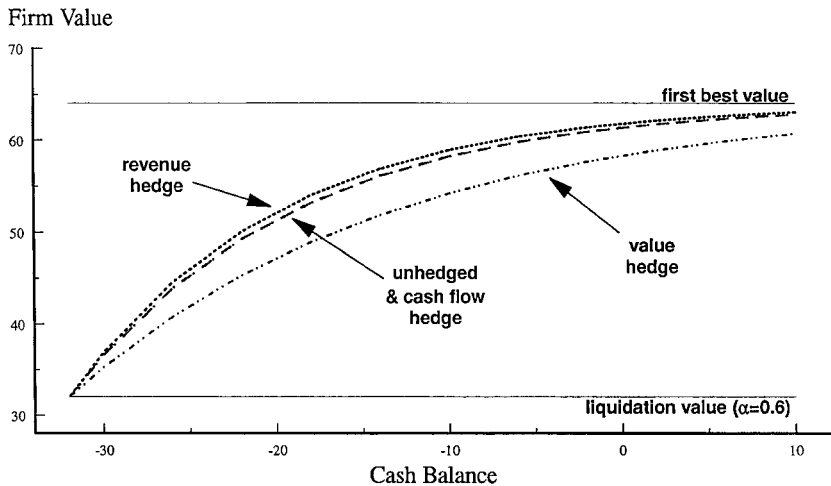
Figure 4
A comparison of alternative hedging strategies
The value of the firm unhedged and under three alternative hedging strategies are shown for various levels of cash balances. The example uses an output price per unit, $p = \$10.0$, a cost per unit produced, $c = \$10.0$, a riskless rate of interest, $r = 5\%$, an instantaneous standard deviation of price, $\sigma_p = 20\%$, an instantaneous standard deviation of cost, $\sigma_c = 20\%$, a correlation coefficient between shocks to price and cost, $\rho = 0.98$, and convenience yields of c and p, $\kappa_c = 4\%$ and $\kappa_p = 4\%$, respectively. The value of the firm using a revenue hedge is the highest. The value of the unhedged firm and the value of the firm using a cash flow hedge are virtually identical. A value hedge yields the lowest firm value.

143

used in example 1 are $p = \$10.0$, $c = \$10.0$, $r = 5\%$, $\sigma_p = 20\%$, $\sigma_c = 20\%$, $\kappa_p = 4\%$, $\kappa_c = 4\%$, and $\rho = 0.98$. The strategy of cash flow hedging appears to yield a value virtually identical to that of the unhedged firm. The strategy of hedging to minimize the variance in the first best value of the firm actually lowers firm value below that of the unhedged firm. While the revenue hedge has a positive value, the value hedge has a negative value.

4. A Discrete Time Example

A discrete time example is helpful in drawing out the liquidity costs of hedging. The example is a familiar lattice representation of the two

Table 2
State space and first best valuation

Node			Risk-neutral transition probability	Price	Cost	Futures price	Period operating cash flow	First best firm value
t	j	k						
0	0	0		10.0	10.0	10.6	0.0	158.0
1	0	0	0.246	7.3	5.3	7.7	2.0	141.2
1	0	1	0.238	7.3	18.8	7.7	− 11.5	42.9
1	0	2	0.262	13.7	5.3	14.6	8.4	310.2
1	0	3	0.254	13.7	18.8	14.6	− 5.1	156.3
2	0	0	0.246	5.3	2.8		120.8	120.8
2	0	1	0.238	5.3	10.0		55.2	55.2
2	0	2	0.262	10.0	2.8		254.5	254.5
2	0	3	0.254	10.0	10.0		144.3	144.3
2	1	0	0.246	5.3	10.0		55.0	55.0
2	1	1	0.238	5.3	35.4		0.0	0.0
2	1	2	0.262	10.0	10.0		144.3	144.3
2	1	3	0.254	10.0	35.4		22.8	22.8
2	2	0	0.246	10.0	2.8		263.2	263.2
2	2	1	0.238	10.0	10.0		165.8	165.8
2	2	2	0.262	18.8	2.8		476.1	476.1
2	2	3	0.254	18.8	10.0		346.5	346.5
2	3	0	0.246	10.0	10.0		164.4	164.4
2	3	1	0.238	10.0	35.4		27.9	27.9
2	3	2	0.262	18.8	10.0		346.7	346.7
2	3	3	0.254	18.8	35.4		124.3	124.3

The table shows the structure of the state space and the values of the parameters used in the discrete time example of Section 4. The table also shows the first best value of the firm. Columns 1–3 label the nodes on the tree, with column 1 denoting the period. In the column the dashed lines divide nodes by period, the dotted lines group nodes within a period which have a common antecedent node in the previous period. For example, nodes (2,0,1) through (2,0,4) follow from node (1,0,0). Generally, nodes (t,j,k), $k = 1$–4, emanate from node $(t − 1, 0, j)$. Column 4 gives the risk-neutral probability of arriving at that node from the previous one. Columns 5 and 6 show the price and cost variables at each node. Column 7 shows the futures price that prevails at a node for a contract settled in the next period. Column 8 shows the operating cash flow for that period and that node given that the firm is open. Column 9 shows the first best value of the firm. It is calculated by backward programming using the next periods' first best values and the risk-neutral probability of arriving at each node in the next period, and then adding the current period's operating cash flow.

variable, continuous time, path-dependent problem analyzed above. The firm begins at time 0 with both the output price and the input cost at $10.0/unit. The price has annual volatility of 10% so that at time period 1 the price may increase to $13.7/unit or decrease to $7.3/unit. The cost has annual volatility of 40%, so that at period 1 it increases to $18.8/unit or decreases to $5.3/unit. This yields four possible states as shown in Table 2 and denoted nodes $(1, 0, k), k = 0, 1, 2, 3$. The risk-neutral probabilities associated with each outcome shown in Table 2 are consistent with zero correlation between price and cost. At period 2, price and cost may each change by the same percentage amounts, yielding 16 possible states as shown in Table 2 and denoted nodes $(2, j, k), j = 0, 1, 2, 3$ and $k = 0, 1, 2, 3$, where the nodes $(2, j, k)$ emanate from $(1, 0, j)$. The fifth column of Table 2 shows the futures price on the output price prevailing at each node: this futures price is consistent with a 5% annualized rate of interest and a 4% annualized convenience yield.

The operating cash flows shown are simply $p_t - c_t$, since in the example $q = 1$. In period 2, it is assumed that the firm is sold at the full value of expected future profits. The firm could choose to shut down in either period 0 or period 1, however, for the parameter values given this is never optimal. The firm operates despite a negative cash flow in the hope of profiting from an increased price and decreased cost in period 2. Note that at node $(2, 1, 1)$ the capitalized value of the firm is zero: this is because it is optimal to shut the firm down at the low price and high cost even in an infinite horizon extension of the lattice. The final column of Table 2 presents the first best value of the firm at each node: this is a present value calculation by backward programming through the tree, period by period, recognizing the intermediate cash flows and using the risk-neutral probabilities and the risk-free interest rate of 5%.

Table 3 presents the valuation of an unhedged firm with an initial cash balance of $10.0. As shown in Table 2, the period 0 net operating cash flow is $0.0, and so the firm ends period 0 with a cash balance of $10.0, as shown in Table 3, and has no difficulty continuing into period 1. However, at node $(1, 0, 1)$ the negative cash flow of $11.5 ($p = 7.3, $c = 18.8) would leave the firm with a cash balance of $-$1.0. The firm can only continue operating by borrowing, but the firm's riskless debt capacity at node $(1, 0, 1)$ is $0.0. The riskless debt capacity is derived by looking forward to nodes $(2, 1, k), k = 0, 1, 2, 3$ and determining the maximum amount that could be repaid with certainty. Since the firm's value at node $(2,1,1)$ is zero, the maximum amount that could be repaid with certainty is $0.0, and this is the firm's riskless debt capacity at node $(1, 0, 1)$ as shown in column 4. The firm is assumed to liquidate at 60% of the first best value at that node, or $25.7. This yields a deadweight loss associated with the firm's financing constraint of $17.2 at node

Table 3
Unhedged firm valuation

Node			Starting cash balance	Ending cash balance	Debt capacity	Liquidation value	Constrained value	Deadweight loss
i	*j*	*k*						
0	0	0	10.0	10.0	34.5		154.1	3.9
1	0	0	10.5	12.5	52.6		141.2	
1	0	1	10.5	− 1.0	0.0	25.7	25.7	17.2
1	0	2	10.5	18.9	157.9		310.2	
1	0	3	10.5	5.4	26.6		156.3	

Columns 1–3 again label the nodes of the tree for the example. Column 4 shows the firm's cash balances at the start of the period. Column 5 shows the firm's ending cash balance which is equal to the starting cash balance plus the operating cash flow shown in Table 2. For nodes $(0, 0, 0)$, $(1, 0, 0)$, $(1, 0, 2)$, and $(1, 0, 3)$ the ending cash balance is positive and no financing is required. For node $(1, 0, 1)$ which is shaded, the ending cash balance would be negative. In order to operate the firm must be able to finance this deficit. Column 6 shows the firm's debt capacity at the node. For node $(1, 0, 1)$ the debt capacity is zero and the firm cannot finance the deficit, so it must be liquidated. Column 7 shows the firm's liquidation value at that node, which is 60% of the firm's first best value at that node as shown in Table 2. Column 8 calculates the firm's constrained value at each node by backward programming, using either the first best values in period 1 when the firm can continue to operate, or the liquidation value when the firm cannot continue. The deadweight loss at each node is equal to the firm's first best value less the firm's constrained value.

$(1, 0, 1)$. The deadweight loss is simply the difference between the first best value at that node as shown in Table 2 and the liquidation value shown in Table 3. Anticipation of this deadweight loss lowers the firm's value in period 0 to $154.1, down from a first best value of $158.0 as shown in Table 2, yielding a capitalized deadweight loss of $3.9.

A relatively modest hedge in period 0 can relax the firm's budget constraint at node $(1, 0, 1)$ as shown in Table 4. In period 0, the firm sells 0.31 futures contracts on the output price. At nodes $(1, 0, 0)$ and $(1, 0, 1)$ when the price falls, the hedge generates a positive cash flow of $1.0 and so increases the cash balance that period. At node $(1, 0, 1)$, this extra $1.0 in cash means that the firm can incur the negative net operating cash flow of $11.5 and end the period without any debt. By avoiding liquidation, the firm's value at $(1, 0, 1)$ is again the first best value of $42.9, an increase of $17.2 over the unhedged firm's value. This $17.2 is the source of value from hedging, and represents the gain from shifting $1 of cash to that state. The capitalized increase in value from hedging at period 0 is $3.9.

The small hedge shown in Table 4 completely restores the firm to its first best value and so is certainly optimal. There is no benefit to be gained from additional hedging. Nevertheless, it is interesting to note that other hedges are also optimal: so long as $h < −0.31$, the hedge generates enough cash at node $(1, 0, 1)$ to assure continued operation. Of course, as the hedge ratio increases in absolute value, the losses incurred at nodes $(1, 0, 2)$ and $(1, 0, 3)$ may themselves induce a new liquidity problem. While Table 4 shows the minimum—in absolute

Table 4
Hedge 1: minimum one-period optimal hedge

Node			Hedge ratio	Hedge cash flow	Starting cash balance	Ending cash balance	Debt capacity	Hedged value	Value of hedge
i	j	k							
0	0	0	−0.31		10.0	9.1	35.5	158.0	3.9
1	0	0		1.0	11.5	13.5	52.6	141.2	
1	0	1		1.0	11.5	0.0	0.0	42.9	17.2
1	0	2		−1.0	9.5	17.9	157.9	310.2	
1	0	3		−1.0	9.5	4.4	26.6	156.3	

Columns 1–3 again label the nodes of the tree for the example. Column 4 shows the hedge ratio opened in period 0 for contracts maturing in period 1. Column 5 shows the cash flows on the hedge. Column 6 shows the firm's cash balances at the start of the period. Column 7 shows the firm's ending cash balance which is equal to the starting cash balance plus the operating cash flow shown in Table 2 plus the hedge cash flow. The ending cash balance is nonnegative for all nodes. The firm does not need to finance a deficit and is not forced into liquidation. The hedge shown is the minimum hedge because it is just large enough to ensure that the firm is never forced into liquidation. Column 9 shows the value of the hedged firm which is equal to the first best value shown in Table 2. Column 10 shows the value of the hedge, which is equal to the difference between the value of the hedged firm shown in column 9 and the value of the constrained firm shown in Table 3.

Table 5
Hedge 2: maximum one-period optimal hedge

Node			Hedge ratio	Hedge cash flow	Starting cash balance	Ending cash balance	Debt capacity	Hedged value	Value of hedge
i	j	k							
0	0	0	−10.27		10.0	−20.4	66.9	158.0	3.9
1	0	0		34.1	44.6	46.5	52.6	141.2	
1	0	1		34.1	44.6	33.0	0.0	42.9	17.2
1	0	2		−32.0	−21.5	−13.1	157.9	310.2	
1	0	3		−32.0	−21.5	−26.6	26.6	156.3	

Column 4 shows the larger hedge ratio. This is the maximum optimal hedge, because in period 1, at node $(1, 0, 3)$, the firm's negative ending cash balance is exactly equal to its debt capacity. A larger hedge ratio would mean greater losses on the hedge at $(1, 0, 3)$ and a negative ending cash balance that is too large to be financed by debt, forcing liquidation and incurring a deadweight loss.

value—optimal hedge ratio, Table 5 shows the maximum optimal hedge ratio. By selling 10.27 futures at period 0, the firm incurs losses at $(1, 0, 2)$ and $(1, 0, 3)$ of $32.0. The firm's ending cash balance at node $(1, 0, 3)$ is −$26.6, which exactly matches its debt capacity. The debt capacity is determined by the minimum firm value over the nodes $(2, 3, k)$, $k = 0, 1, 2, 3$, discounted back one period. A higher hedge ratio at period 0 would mean a more severely negative cash balance at $(1, 0, 3)$, and therefore liquidation.[6]

Note that the firm's debt capacity at period 1 nodes is determined by its value at period 2 nodes, and that the firm's hedge ratio at period 1 affects the firm's value at period 2. Consequently, the firm can raise its

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[6] The set of optimal hedge ratios reduces with the length of the horizon for the problem and with the interval size.

Table 6
Hedge 3: minimum-variance two-period optimal hedge

Node			Hedge ratio	Hedge cash flow	Starting cash balance	Ending cash balance	Debt capacity	Hedged value	Value of hedge
i	*j*	*k*							
0	0	0	− 19.03		10.0	− 46.4	94.6	158.0	3.9
1	0	0	− 23.83	63.1	73.6	24.1	137.4	141.2	
1	0	1	− 12.06	63.1	73.6	36.0	21.7	42.9	17.2
1	0	2	− 22.33	− 59.2	− 48.7	− 131.2	330.0	310.2	
1	0	3	− 15.86	− 59.2	− 48.7	− 118.4	118.4	156.3	

Column 4 shows the hedge ratio used in period 0 and the hedge ratios used in period 1 contingent on the node. The hedge used at node $(1, 0, 3)$ raised the debt capacity at $(1, 0, 3)$. Consequently it is possible to put on a larger hedge at node $(0, 0, 0)$ without forcing liquidation at $(1, 0, 3)$.

debt capacity with the right hedge strategy. Table 6 shows the firm executing a hedge strategy over two periods. The period 1 hedge is state contingent and minimizes the variance in firm value at period 2. Note that the period 1 debt capacity is generally greater than when the firm is unhedged at period 1, as in the previous tables. This allows the firm to also increase its hedge ratio in period 0 without danger of forcing liquidation, for example, at node $(1, 0, 3)$. The hedge ratio of 19.03 contracts sold in period 0 is, however, the maximum feasible hedge. At node $(1, 0, 3)$ the firm accumulates debt of \$118.4 and reaches its expanded debt capacity.

   The hedge ratio in period 0 that minimizes the variance in firm value is 22.02 contracts sold, as shown in Table 7. If the firm were to implement this hedge, it would exceed its debt capacity at node $(1, 0, 3)$ and therefore reduce its value. The anticipated reduction in value would in turn lower the firm's debt capacity in period 0 to \$34.0, as shown in Table 7. In this case, the firm would already have exceeded its debt capacity in period 0 and would be liquidated for \$94.8, yielding a loss

Table 7
Hedge 4: minimum-variance hedge

Node			Hedge ratio	Hedge cash flow	Starting cash balance	Ending cash balance	Debt capacity	Hedged value	Value of hedge
i	*j*	*k*							
0	0	0	− 22.02		10.0	− 55.3	34.0	94.8	− 59.3
1	0	0	− 23.83	73.0	83.5	34.0	137.4		
1	0	1	− 12.06	73.0	83.5	45.9	21.7		
1	0	2	− 22.33	− 68.6	− 58.1	− 140.5	330.0		
1	0	3	− 15.86	− 68.6	− 58.1	− 127.7	26.6		

It is possible to reduce the variance in the firm's value beyond that achieved with hedge 4 shown in Table 6, but only by incurring the deadweight cost of liquidation in some events. The period 1 hedge ratios shown above are identical to those shown in Table 6. The period 0 hedge ratio is larger here. The losses on the hedge are so large at node $(1, 0, 3)$ that the firm does not have sufficient debt capacity to finance its negative cash balance. Instead it is forced to liquidate. This lowers the firm's value at node $(0, 0, 0)$ and so creates a negative value for the hedge.

from hedging of $59.3. Table 7 illustrates that the minimum variance hedge may absolutely lower firm value.

5. Hedging with Other Financial Contracts

Firms can hedge with many alternative contracts. Futures, forwards, and swaps offer substitution possibilities that appear to be redundant. The ability to replicate one instrument with another is a routine procedure used in pricing securities. Despite this formal equivalence, it has been recognized that alternative financial instruments differ at least in their institutional designs that motivate the choice of one over the other in different circumstances. For example, one key institutional difference between forward and futures contracts is the need for interim cash settlement. Futures contracts require daily cash settlement of the gains and losses, while forward contracts are often settled only at the maturity of the contract. Of course, when there is no interim cash settlement and the hedge is losing money, a liability accumulates and is carried over time. When granting a forward contract, the counterparty to the contract is aware that the firm gets an automatic line of credit in the form of a loan with the maturity of the hedging contract. Within the framework of our model, hedging contracts are default protected, and therefore, if the firm had hedged with a forward contract, both the amount granted and the maturity of the contract would be determined by the lowest firm value that would guarantee full payment of the contract. The line of credit accepted in the forward contract is exactly identical to the contingent debt accumulated from implementing a futures trading strategy. Consequently, trading in forwards does not give the firm any greater financial advantage or flexibility than that afforded by a futures hedge. Both the forward and the futures strategy involve an associated debt strategy. If the limits on debt apply equally to each contract, then the firm is indifferent between hedging with the one or the other.

In a somewhat different setting, which instrument is to be preferred? A hedge requiring explicit external funding or a prearranged credit line to cover the shortfalls over the intermediate horizon of the term to maturity of the hedging contract? To answer this, it is important to understand that the financing implicit in a forward contract is risky, while the daily settlement makes the risk of financing a futures position minimal. The forward contract is certainly less liquidity sensitive, but worries of default risk in forward contracts can force the posting of collateral, which reduces further the financial slack available to the firm. Then the choice among different hedging contracts depends on which type of debt package associated with a particular hedge imposes the lowest financing costs on the firm. Some firms may prefer to pay to get an up-front line of credit granted for multiple periods, while other firms

may prefer to get financing contingent on how things go. Even if forwards and swaps avoid the ex post funding needs associated with futures hedges, it is important to understand that the accumulated losses with such contracts affect the value of the firm by directly triggering debt covenants specific to these contracts, and by reshaping the incentives to manage the firm.

6. Implications for Corporate Risk Management

Our model postulates that the optimal hedge ratio is contingent upon the firm's financial constraints, so that the firm should hedge more as its leverage increases or its margins decline.

It is difficult to confirm whether such a prediction is consistent with the observed pattern of hedging by corporations. Nonfinancial firms do not report risk management instruments on their balance sheets and devising tests that capture hedging as value maximizing behavior based on the intertemporal costs of financial constraints is not a straightforward task. However, anecdotal evidence as well as recent surveys seem to confirm that firms are neither constrained nor unconstrained in an absolute sense, and act instead as if they are financially constrained at certain periods. That may explain why, for example, firms that are profitable and reputable, such as Merck & Co. and IBM, decide to hedge.[7]

Besides the findings reported in Wall and Pringle (1989), Nance, Smith, and Smithson (1993), and Dolde (1995) that weakly relate hedging to leverage across firms at a point in time, there is little evidence of how this relationship holds for a particular firm at different points of time. According to our model, financial constraints vary with time, and so should a corporation's need for hedging. This creates a time-varying pattern of hedging intensity that seems consistent with the findings that corporations do not hedge systematically and that hedging is often done on a short-term basis.

In addition, by endogenizing the costs of hedging, our model implies that hedging may not be possible if the firm is unable to provide sufficient evidence that it will honor the funding requirements implied by the hedge itself. Our prediction that severely constrained firms may not be able to hedge offers a new explanation to the reported weakness of the relationship between leverage and hedging, as well as why empirical research has frequently found an apparently stronger hedging activity among larger firms. It is possible that these results are influenced by characteristics of small and financially weak firms.

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[7] Dolde (1995) finds that among users of financial contracts, small firms hedge more frequently than larger firms.

Finally, the role of cash balances in the hedging decision was examined by Tufano (1996) and by Mian (1996). In both cases the authors seem to conclude that hedging is negatively correlated with the available liquidity, a result that fits well with our model.

7. Conclusion

This article examines how hedging can lower the effective cost of the firm's financial constraints. In a dynamic model of the firm's operation and financial policies, it shows how hedging adds financial flexibility, reduces the costs of financial distress, and allows the firm to take advantage of future investment opportunities. By shifting cash balances across states for which the shadow cost of financing differs, hedging maximizes its return on its cash balances. However, every hedging strategy implies a borrowing strategy for the firm, and designing the optimal hedge requires attention to the intertemporal financial requirements of the hedge. A hedge does not necessarily create its own liquidity, so that the financial risk created by the hedge itself is an important factor in determining the value of the hedge. This fact can be seen in two lights. On the negative side, many popular hedging strategies imply a borrowing strategy that actually undermines the firm's value. On the positive side, the costs of borrowing become a guide for designing the optimal hedging strategy.

Appendix

Proof of Proposition 1. The first-order condition of Equation (34) is

$$\frac{d}{dh}E(dV^*) = 0. \tag{A1}$$

Since $E(dV^*) = rV^*$, the first-order condition of Equation (A1) can be rewritten as $dV^*/dh^* = 0$. Substituting $dV = V_p dp + V_c dc + V_W dW + 1/2[V_{pp}(dp)^2 + 2V_{pc}pc(dp)(dc) + V_{cc}(dc)^2]$ and using the decomposition $\frac{d}{dh}V_i = \frac{d}{dw}V_i\frac{dw}{dh}$ we can rewrite the first-order condition as

$$E\left(\left[V_{pW}^* dp + V_{cW}^* dc + V_{WW}^* dW\right]\frac{dW}{dh} + V_W^*\frac{d}{dh}dW\right.$$
$$\left. + \frac{1}{2}\left[V_{ppW}^*(dp)^2 + 2V_{pcW}^* dp\,dc + V_{ccW}^*(dc)^2\right]\frac{dW}{dh}\right) = 0. \tag{A2}$$

Note that $dW = [q(p - c) + Wr]dt + q(dp - dc) + h(df)$ and therefore $\frac{d}{dh}dW = df$ and $E\left(\frac{d}{dh}dW\right) = E(df) = 0$. Note also that $\frac{dW}{dh} = f_p\sigma_p p dz_2$. Substituting we have

$$E\left(\left[V_{pW}^*\left(\mu p dt + \sigma_p p dz_2\right) + V_{cW}^*\left(\nu c dt + \sigma_c c dz_1\right)\right.\right.$$
$$+V_{WW}^*\left(\left[q(1 + \mu)p - q(1 + \nu)c + Wr\right]dt + q\sigma_p p dz_2 - q\sigma_c c dz_1 - h^*f_p\sigma_p p dz_2\right)$$
$$\left.\left. + \frac{1}{2}\left[V_{ppW}^*\sigma_p^2 p^2 + 2V_{pcW}^*\rho\sigma_p\sigma_c pc + V_{ccW}^*\sigma_c^2 c^2\right]dt\right]f_p\sigma_p p dz_2\right) = 0. \tag{A3}$$

Multiplying through, noting that $E(dz_1\, dz_2) = \rho dt$ and $E(dz_1^2) = E(dz_2^2) = dt$, and retaining only terms of order dt and lower, we obtain

$$V_{pW}^* \sigma_p p + V_{cW}^* \sigma_c c \rho + V_{WW}^* \big(q\sigma_p p - q\sigma_c c \rho - h^* f_p \sigma_p p \big) = 0. \tag{A4}$$

Solving for h^* yields Equation (35). ∎

Proof of Proposition 2. Since $V_{WW}^* < 0$, Equation (36) also yields the characterization of the optimal hedge ratio given in Equation (35). To see this, note that

$$
\begin{aligned}
dV_W^* = {}& V_{Wp}^* dp + V_{Wc}^* dc + V_{WW}^* dW \\
& + \frac{1}{2}\Big[V_{Wpp}^*(dp)^2 + 2V_{Wpc}^* pc (dp)(dc) + V_{Wcc}^*(dc)^2 \Big] \\
= {}& V_{Wp}^* \big(\mu p dt + \sigma_p p dz_2 \big) + V_{Wc}^* \big(\nu c dt + \sigma_c c dz_1 \big) \\
& + V_{WW}^* \big(\big[q(1+\mu)p - q(1+\nu)c + Wr \big] dt \\
& \qquad\quad + q\sigma_p p dz_2 - q\sigma_c c dz_1 - h f_p \sigma_p p dz_2 \big) \\
& + \frac{1}{2}\Big[V_{Wpp}^* \sigma_p^2 p^2 + 2V_{Wpc}^* \rho\sigma_p \sigma_c pc + V_{Wcc}^* \sigma_c^2 c^2 \Big] dt = 0. \tag{A5}
\end{aligned}
$$

Minimizing the variance of V_W^* is equivalent to minimizing the variance of

$$V_{Wp}^* \sigma_p p dz_2 + V_{Wc}^* \sigma_c c dz_1 + V_{WW}^* \big(q\sigma_p p dz_2 - q\sigma_c c dz_1 - h f_p \sigma_p p dz_2 \big). \tag{A6}$$

Applying the variance operator, noting that $\mathrm{var}(dz_1) = \mathrm{var}(dz_2) = dt$ and that $\mathrm{cov}(dz_1, dz_2) = \rho dt$, differentiating with respect to h and equating the result to zero gives Equation (35). ∎

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