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Heegaard and Regular Genus of 3-Manifolds with Boundary

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ABSTRACT. By means of branched coverings techniques, we prove that the Heegaard genus and the regular genus of an orientable 3-manifold with boundary coincide.

1. PRELIMINARIES

Throughout this paper the term "manifold" will denote a *compact, connected, orientable* PL-manifold with (possible empty) boundary.

An $(n+1)$-*coloured graph (with boundary)* is a pair (Γ, γ), where $\Gamma = (V(\Gamma), E(\Gamma))$ is a multigraph and $\gamma: E(\Gamma) \to \Delta_n = \{0, 1, \ldots, n\}$ is a map such that $\gamma(e) \neq \gamma(f)$, for each pair e, f of adjacent edges of

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Γ; γ is called *edge-coloration* on Γ. For each $B \subseteq \Delta_n$, the *B-residues* of (Γ, γ) are the connected components of the graph $\Gamma_B = (V(\Gamma), \gamma^{-1}(B))$. For each $c \in \Delta_n$, we set $\hat{c} = \Delta_n - \{c\}$.

The vertices of Γ whose degree is strictly less than $n+1$ are called the *boundary vertices* of Γ; if Γ has no boundary vertices, i.e. if Γ is regular of degree $n+1$, then (Γ, γ) is said to be an *$(n+1)$-coloured graph without boundary.*

The graph (Γ, γ) is called *regular with respect to the colour $c \in \Delta_n$* iff $\Gamma_{\hat{c}}$ is regular of degree n. From now on, we only consider $(n+1)$-coloured graphs which are *regular with respect to the "last" colour n.*

If (Γ, γ) is an $(n+1)$-coloured graph, the *boundary graph* $(\partial\Gamma, {}^{\partial}\gamma)$ is defined in the following way:

- the vertices of $\partial\Gamma$ are the boundary vertices of Γ;
- two vertices of $\partial\Gamma$ are joined by a c-coloured edge iff they belong to the same $\{n, c\}$-residue of (Γ, γ).

Given an $(n+1)$-coloured graph (Γ, γ), let us denote by $g(\Gamma)$ the number of components of Γ; by convention we set $g(\emptyset) = 1$. A connected $(n+1)$-coloured graph (Γ, γ) is called *contracted* iff $g(\Gamma_{\hat{n}}) = 1$ and, for every $c \in \Delta_{n-1}, g(\Gamma_{\hat{c}}) = g(\partial\Gamma)$.

Let K be an n-dimensional pseudocomplex [HW]. The *disjoint star* $\mathrm{std}(s, K)$ of a simplex s in K is the disjoint union of the n-simplexes containing s, with re-identification of the $(n-1)$-simplexes containing s and of all their faces; the *disjoint link* of s in K is the subcomplex $\mathrm{lkd}(s, K) = \{\tau \in \mathrm{std}(s, K) / s \cap \tau = \emptyset\}$.

A *vertex-coloration* on K is a map $\xi : V(K) \to \Delta_n$ which is injective on every simplex of K. If K is homogeneous, the pair (K, ξ) is called a *coloured n-complex.*

From now on, for sake of conciseness, we often drop edge - and vertex - colorations, writing Γ and K instead of (Γ, γ) and (K, ξ).

There exists a correspondence between $(n+1)$-coloured graphs and coloured n-complexes. In fact, given an $(n+1)$-coloured graph Γ, we can construct a coloured n-complex $K(\Gamma)$ in the following way:

- take an n-simplex $\sigma(v)$ for each $v \in V(\Gamma)$ and label its vertices by Δ_n;

- for each $c \in \Delta_n$ and each pair v, w of c-adjacent vertices in Γ, identify the $(n-1)$-faces of $\sigma(v)$ and $\sigma(w)$ opposite to the vertices labelled c, so that equally labelled vertices coincide.

The above construction can be easily reversed in order to associate an $(n+1)$-coloured graph $\Gamma(K)$ to each coloured n complex K.

Note that, by construction, each $(\Delta_n - \{c_0, \ldots, c_h\})$-residue Ξ of Γ corresponds to a unique h-simplex s of $K(\Gamma)$, whose vertices are labelled by $\{c_0, \ldots, c_h\}$ and viceversa; moreover $K(\Xi)$ =lkd$(s, K(\Gamma))$. It is easy to see that $\Gamma(K(\Gamma)) = \Gamma$; conversely $K(\Gamma(K)) = K$ iff the disjoint star of every simplex in K is strongly connected [G_1]. In this case K is said to be *representable.*

If $|K(\Gamma)|$ (the polyhedron determined by $K(\Gamma)$) is a manifold, then $\partial(|K(\Gamma)|) = |K(\partial\Gamma)|$ and we say that $|K(\Gamma)|$ is *represented* by Γ. Moreover $|K(\Gamma)|$ is orientable iff Γ is bipartite.

A contracted $(n+1)$-coloured graph representing a manifold M is called a *crystallization* of M.

For a general survey on manifold representation theory by means of edge-coloured graphs, see [FGG].

In [G_2], [G_4] a particular concept of imbedding of a coloured graph into a surface is defined. This naturally leads to the definition of an invariant for manifolds, the *regular genus*, which extends to dimension n the classical notion of genus of a surface.

In [G_3] the regular genus of a closed 3-manifold is proved to coincide with its classical Heegaard genus ([H] and [He]). In this paper we extend the result, for the orientable case, to manifolds with boundary. In particular, by means of branched coverings techniques, we prove that the Heegaard genus (see [M]) and the regular genus of an orientable 3-manifold with boundary coincide. About non-orientable 3-manifolds with boundary, we only know that the Heegaard genus is less than or equal to the regular genus, since part of the proof of the main result of this paper (Lemma 1 and 3) still holds. Unfortunately we lack, for the non-orientable case, results similar to those of [M], which are required in Lemma 2 to prove the opposite inequality.

2. REGULAR GENUS

Let Γ be an $(n+1)$-coloured graph (with boundary), regular with respect to the colour n.

We call *extended graph associated to* Γ the $(n+1)$-coloured graph Γ^* such that:

- $V(\Gamma^*) = V(\Gamma) \cup V^*$, where V^* is in one-to-one correspondence with $V(\partial\Gamma)$;
- $E(\Gamma^*) = E(\Gamma) \cup E^*$, where E^* is the set of all possible n-coloured edges whose endpoints are a boundary-vertex of Γ and its correspondent vertex in V^*.

A *regular imbedding* of Γ into a surface (with boundary) F, is an imbedding ι^* of Γ^* into F, satisfying the following conditions:

(a) $\iota^*(V^*) = \partial F \cap \iota^*(|\Gamma^*|)$

(b) the connected components of $(\text{int } F) - \iota^*(|\Gamma^*|)$ (the *regions* of the imbedding) are open balls;

(c) the boundary of any region $\mathcal{R}$ of ι^* is either the image of a cycle of Γ^* (*internal* region) or the union of the image $\mathcal{R}'$ of a path in Γ^* and an arc $\mathcal{R}''$ of ∂F, the intersection consisting of two (possibly coincident) vertices of $\iota^*(V^*)$ (*boundary* region);

(d) there exists a cyclic permutation $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n)$ of Δ_n such that for each internal region $\mathcal{R}$ (resp. boundary region $\mathcal{R}' \cup \mathcal{R}''$), the edges of $\partial\mathcal{R}$ (resp. of $\mathcal{R}'$) are alternatively coloured ε_i and ε_{i+1} $(i \in Z_{n+1})$.

According to [G$_2$] and [G$_4$], for each cyclic permutation $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, n)$ of Δ_n, there exists a surface (with boundary) F_ε and a regular imbedding ι_ε of Γ into F_ε; moreover F_ε is orientable iff Γ is bipartite. F_ε is called the *regular surface* associated to Γ and ε.

If F_ε is orientable, we have the following formula (see [G$_4$ Proposition 4]):

$$\text{genus}(F_\varepsilon) = 1 - \frac{1}{2}\left[\sum_{i \in Z_{n+1}} g_{\varepsilon_i \varepsilon_{i+1}}(\Gamma) + \right. \tag{*}$$

$$\left. + \frac{1}{2}(1-n)(p(\Gamma) - \bar{p}(\Gamma)) + (2-n)\frac{\bar{p}(\Gamma)}{2} \right] - \frac{1}{2} {}^{\partial}g_{\varepsilon_0 \varepsilon_{n-1}}(\Gamma)$$

where $g_{ij}(\Gamma)$ (resp. ${}^{\partial}g_{ij}(\Gamma)$) is the number of cycles of $\Gamma_{\{i,j\}}$ (resp. of $\partial\Gamma_{\{i,j\}}$) and $p(\Gamma)$ (resp. $\bar{p}(\Gamma)$) is the order of Γ (resp. of $\partial\Gamma$). Of course an analogous formula holds for non-orientable regular surfaces.

Set $\rho_\varepsilon(\Gamma)$= genus(F_ε). The *regular genus* $\rho(\Gamma)$ of Γ is defined as the minimum $\rho_\varepsilon(\Gamma)$ among all cyclic permutations ε of Δ_n.

Given an n-manifold M the *regular genus* of M is the non-negative integer:

$$\mathcal{G}(M) = \min\{\rho(\Gamma)/\Gamma \text{ represents } M\}$$

If Γ is a 4-coloured graph representing a 3-manifold M, let us describe an effective construction of F_ε.

Let K'_ε (resp. K''_ε) be the 1-dimensional subcomplex of K generated by the vertices coloured $(\varepsilon_1, 3)$ (resp. $(\varepsilon_0, \varepsilon_2)$). Denote by H_ε the largest 2-dimensional subcomplex of SdK (where Sd means "first barycentric subdivision") disjoint from $SdK'_\varepsilon \cup SdK''_\varepsilon$. Then $F_\varepsilon = |H_\varepsilon|$ is the regular surface associated to Γ and ε [G$_4$]. Moreover F_ε is an orientable surface with at least one boundary component on each component of ∂M; if Γ is a crystallization of M, then each boundary component of F_ε lies on a different component of ∂M (see Proposition 12 of [G$_4$]).

Note that H_ε splits $K(\Gamma)$ into two subcomplexes N'_ε and N''_ε which admit SdK'_ε and SdK''_ε as spines and such that $N'_\varepsilon \cap N''_\varepsilon = \partial N'_\varepsilon \cap \partial N''_\varepsilon = H_\varepsilon$. Let us set $\mathcal{A}'_\varepsilon = |N'_\varepsilon|$, $\mathcal{A}''_\varepsilon = |N''_\varepsilon|$.

The pair $(\mathcal{A}'_\varepsilon, \mathcal{A}''_\varepsilon)$ is called the *regular splitting* associated to Γ and ε.

Given a 3-manifold M by [B Theorem 1] the minimum genus can always be obtained by a crystallization of M. In this case we can set:

$$\mathcal{G}(M) = \min \{\rho(\Gamma)/\Gamma \text{ is a crystallization of } M\}.$$

Moreover, if M is closed, then $\mathcal{G}(M) = \mathcal{H}(M)$, where $\mathcal{H}$ denotes the Heegaard genus (see [G_3]).

3. HEEGAARD SPLITTINGS AND DIAGRAMS

A *singular 3-manifold* is a 3-dimensional polyhedron $N = |H|$, H being a simplicial complex, such that for each vertex v of H, the link $lk(v, H)$ is a combinatorial closed connected surface.

Note that if K is a pseudocomplex and $|K| = N$, then N is a singular 3-manifold iff for each vertex v of K, the disjoint link $lkd(v, K)$ is a combinatorial closed connected surface. In fact, the first barycentric subdivision K' of K is a simplicial complex and $lk(v, K')$ is isomorphic to a subdivision of $lkd(v, K)$.

Note also that if N is a singular 3-manifold and $|H| = N$, H being a simplicial complex (resp. $|K| = N$, K being a pseudocomplex), then for each h-simplex σ^h of H (resp. of K), with $h \geq 1$, the link $lk(\sigma^h, H)$ (resp. the disjoint link $lkd(\sigma^h, K)$) is a combinatorial $(3-h-1)$-sphere.

Hence, a polyhedron N is a singular 3-manifold iff each point x of H has a neighbourhood (PL) homeomorphic to a cone over a closed connected surface. If such a surface is not a sphere, then x is called a *singular point* of N.

Given a 3-manifold M, we obtain a singular 3-manifold $N = \hat{M}$ by capping off each component of ∂M by a cone over it. Conversely, given a singular 3-manifold N, we denote by $\check{N}$ the 3-manifold obtained from N by removing small open neighbourhoods of its singular points. If K is any (pseudo or simplicial) complex triangulating N, then $\check{N}$ can be simply obtained by deleting the open stars of the singular vertices in a suitable subdivision of K.

Note that, if we avoid 3-manifolds with spherical boundary components, the correspondence $N \to \check{N}$ becomes bijective.

Let now G be the 1-dimensional subset of S^3 pictured in Figure 1. By [M], given a 3-manifold M, there exist an integer $h \geq 1$ and a transitive pair of permutations $\sigma, \tau \in \Sigma_h$ such that $M \cong N(\sigma, \tau)$, where $N(\sigma, \tau)$ is the h-fold covering of S^3 branched over G, whose monodromy sends the two generators of $\pi_1(\mathrm{S}^3 - G)$ determined by the oriented meridians of Figure 1, to σ and τ.

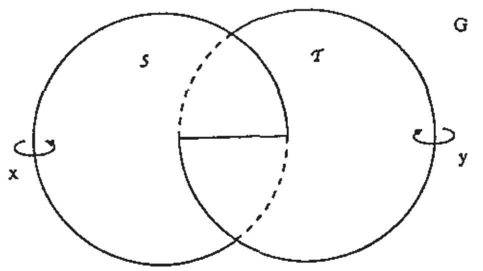

Figure 1

An effective construction of a triangulation $K(\sigma, \tau)$ of $N(\sigma, \tau)$ proceeds as follows:

- take h copies $t_1, \ldots, t_h$ of the standard tetrahedron t whose bidimensional faces are denoted by S, $\bar{S}$ and T, $\bar{T}$ as in Figure 2;
- for each $i = 1, \ldots, h$ call $S_{i\sigma(i)}$, $\bar{S}_{i\sigma^{-1}(i)}$, $T_{i\tau(i)}$, $\bar{T}_{i\tau^{-1}(i)}$ the faces S, $\bar{S}$, T, $\bar{T}$ (respectively) in the copy t_i;
- identify S_{ij} with $\bar{S}_{ji}$ and T_{ij} with $\bar{T}_{ji}$ by a linear homeomorphism respecting the edges $S \cap \bar{S}$ and $T \cap \bar{T}$.

Let T_g be the closed orientable surface of genus g, where, T_0 stays for the 2-sphere S^2. By a *proper handlebody of genus* $g \geq 0$ we mean an (orientable) 3-manifold X_g, obtained by attaching g 1-handles on the boundary of the 3-dimensional disk D^3. Note that two such handlebodies are homeomorphic iff they have the same genus. The boundary ∂X_g of X_g is the closed orientable surface T_g.

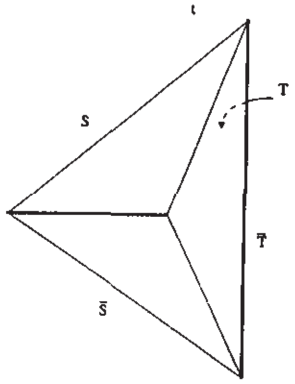

Figure 2.

By a *hollow handlebody of genus* $g \geq 0$ we mean an (orientable) 3-manifold X_g obtained from $T_g \times [0,1]$ by adding 2- and 3-handles along $T_g \times \{1\}$; $T_g \times \{0\}$ is called the *free boundary* of X_g.

Remark 1. Note that *a hollow handlebody* X_g *of genus* g *is proper iff its boundary* ∂X_g *coincides with its free boundary* $T_g \times \{0\}$.

In fact, if X_g is proper, then $X_g = D^3 \cup H_1^{(1)} \cup \ldots \cup H_g^{(1)}$, where for $1 \leq i \leq g$, $H_i^{(1)}$ is a 1-handle on the boundary of D^3. By adding a collar on $\partial X_g = T_g$ and dualizing the handle presentation, we have $X_g = (T_g \times [0,1]) \cup H_g^* \cup \ldots \cup H_1^* \cup D^3$, where each H_i^* is now a 2-handle on $T_g \times \{1\}$ and D^3 is a 3-handle.

Conversely, if X_g is a hollow handlebody and $\partial X_g = T_g \times \{0\}$, then

$$X_g = (T_g \times [0,1]) \cup H_1^{(2)} \cup \ldots \cup H_r^{(2)} \cup \tilde{H}_1^{(3)} \cup \ldots \cup \tilde{H}_s^{(3)}$$

where, for $1 \leq i \leq r$, $H_i^{(2)}$ is a 2-handle on $T_g \times \{1\}$ and for $1 \leq j \leq s$, $\bar{H}_j^{(3)}$ is a 3-handle. Hence, by dualizing the presentation and deleting a collar of the boundary, we obtain: $X_g = \bar{H}_1^* \cup \ldots \cup \bar{H}_s^* \cup H_r^* \cup \ldots \cup H_1^*$ where $\bar{H}_1^*, \ldots, \bar{H}_s^*$ are 0-handles and $H_r^*, \ldots, H_1^*$ are 1-handles. This proves that X_g is proper and we can simplify the presentation in the following way: $X_g = D^3 \cup \bar{H}_1^{(1)} \cup \ldots \cup \bar{H}_{r-s+1}^{(1)}$. Since $\partial X_g = T_g$, it follows that $r - s + 1 = g$.

A *generalized Heegaard splitting* of a 3-manifold M is a pair (X_g, Y_g) of hollow handlebodies of genus g such that:

- $M = X_g \cup Y_g$

- $T_g = X_g \cap Y_g$ is the free boundary of both X_g and Y_g.

The non-negative integer g is called the *genus* of the splitting.

If at least one of the two hollow handlebodies is proper, then (X_g, Y_g) is said to be a *proper Heegaard splitting* of M; in this case *we always suppose that X_g is proper.*

The *Heegaard genus* of M is defined to be the non-negative integer:

$$\mathcal{H}(M) = \min \{g/M \text{ admits a } \textit{proper} \text{ Heegaard splitting of genus } g\}.$$

By Remark 1, the above definitions coincide with the homonymous ones given by Montesinos in [M]; moreover, they generalize the classical ones given for closed manifolds.

A *generalized Heegaard diagram* is a triple $(T_g; v; w)$, where v and w are systems of simple closed curves on T_g.

Each generalized Heegaard splitting (X_g, Y_g) of a 3-manifold M produces a generalized Heegaard diagram, whose systems of curves are the attaching spheres of the 2-handles on X_g and Y_g.

Conversely, from a generalized Heegaard diagram $(T_g; v, w)$ we can obtain a hollow handlebody X (resp. Y) by considering $T_g \times [0, 1]$ (resp. $T_g \times [-1, 0]$) with 2-handles attached on $T_g \times \{1\}$ (resp. $T_g \times \{-1\}$) according to v (resp. to w) and possibly by capping off some of the spherical boundary components by 3-handles. If M is the 3-manifold obtained from $X \cup Y$ by identifying their free boundaries, then (X, Y)

is a generalized Heegaard splitting of the 3-manifold M. In this case we say that $(T_g; v, w)$ *represents* M.

A generalized Heegaard diagram $(T_g; v, w)$ is called a *proper Heegaard diagram* if its corresponding Heegaard splitting is proper.

Remark 2. If $(T_g; v = (v_1, \ldots, v_r),\ w = (w_1, \ldots, w_s))$ is a proper Heegaard diagram representing a 3-manifold M, then $r \geq g$; moreover, we can always find a subset v' of v, containing g curves, such that v' is a complete system of meridian curves for T_g, i.e. $T_g - v'$ is planar and connected. Since $(T_g; v', w)$ still represents M, from now on *we suppose* $r = g$ (or, equivalently, $T_g - v$ to be planar connected).

Proposition 1. *Every 3-manifold M admits a proper Heegaard splitting.*

Proof. The first part of this proof adapts an analogous one in [S]. Let K be a simplicial complex triangulating M and H_2 a tubular neighbourhood of the dual 1-skeleton of M. Set $H_1 = \overline{M - H_2}$, then H_1 and H_2 are proper handlebodies whose intersection is a proper subset of their boundaries. More precisely, ∂H_1 and ∂H_2 are not identified along ∂M, but $\partial M \cap \partial H_2 = \cup_i D_i$, where the D_i's are disks.

Let $N_i = D^2 \times [0, 1]$ be a collar of D_i in H_2 and H_1' the complex obtained by attaching the 2-handles N_i along ∂H_1, respecting the identification between ∂H_1 and ∂H_2. Hence $H_2' = H_2 - (\cup_i N_i)$ is a proper handlebody such that $H_1' \cap H_2' = \partial H_1' = \partial H_2'$ is a closed surface S. If C is a collar of S in H_2' then define $H_1'' = H_1' \cup C$, $H_2'' = \overline{H_2' - C}$.

H_2'' is a proper handlebody and H_1'' is a hollow handlebody obtained from $S \times [0, 1]$ by attaching the 2-handles N_i along $S \times \{0\}$. ∎

If M is a 3-manifold, then $M \cong \check{N}(\sigma, \tau)$ for a suitable transitive pair (σ, τ) of permutations. Let us describe a particular generalized Heegaard splitting of M arising from (σ, τ).

Let us call $\mathcal{S}$ and $\mathcal{T}$ the two disks embedded in S^3 as in Figure 1. Let F be the boundary of a tubular neighbourhood of $\partial\mathcal{S}$ in S^3 and let X and Y be the closures of the two components of $S^3 - F$. Then X and Y are regular neighbourhoods of $\partial\mathcal{S}$ and $\partial\mathcal{T}$ respectively and therefore (X, Y) is a proper genus one Heegaard splitting of S^3.

Let $\tilde{X}$ (resp. $\tilde{Y}$) be the hollow handlebody which is the preimage of X (resp. of Y) by the branched covering map $N(\sigma,\tau) \to S^3$ and let $\tilde{F}$ be the preimage of F; then $(\tilde{X},\tilde{Y})$ is a generalized Heegaard splitting (Theorem 10 of [M]), which is called *canonical Heegaard splitting of* $\check{N}(\sigma,\tau)$.

If the canonical splitting is proper, one of the hollow handlebodies, $\tilde{X}$ say, is proper; hence, all singular points of $N(\sigma,\tau)$ lie in $\tilde{Y}$. It is easy to see that, in this case, the singular vertices in $K(\sigma,\tau)$ are the endpoints of some of the edges $T \cap \bar{T}$.

Let H be a subgroup of $\Sigma_h(h \geq 1)$, generated by a certain set of permutations $\{\sigma_1,\ldots,\sigma_r\}$. We denote by $|\sigma_1,\ldots,\sigma_r|$ the number of orbits of the action of H on $\{1,2,\ldots,h\}$.

By [M], the canonical Heegaard splitting of $\check{N}(\sigma,\tau)$ is proper, i.e. $\tilde{X}$ (resp. $\tilde{Y}$) is a proper handlebody iff $|\sigma,\tau\sigma\tau^{-1}| = 1$ (resp. $|\tau,\sigma\tau\sigma^{-1}| = 1$).

A further result of [M] will be required later:

Proposition 2. *Let $(T_g;v,w)$ be a proper Heegaard diagram of a 3-manifold M, with $g > 0$ and $w \neq \emptyset$. There is an algorithm which determines an integer $h \geq 1$ and two permutations $\sigma,\tau \in \Sigma_h$ such that*

(i) $\check{N}(\sigma,\tau) \cong M$;

(ii) $|\sigma,\tau\sigma\tau^{-1}| = 1$ *(i.e. the canonical Heegaard splitting of $\check{N}(\sigma,\tau)$ is proper);*

(iv) $|\sigma| = g = 1 + \frac{1}{2}(h - |[\sigma,\tau]|)$.

Remark 3. If $g = 0$, then $M \cong S^3$ and we have directly $h = 1$ and $\sigma = \tau = \mathrm{id}_{\{1\}}$. If $w = \emptyset$, then M is a proper handlebody of genus g and it is very easy to construct a new proper Heegaard diagram representing M of genus $g+1$ and such that $w \neq \emptyset$.

4. THE MAIN RESULT

Proposition 3. *For every 3-manifold M, we have $\mathcal{G}(M) = \mathcal{H}(M)$.*

The proof requires three lemmas:

Lemma 1. *Let Γ be a crystallization of a 3-manifold M. For each cyclic permutation ε of Δ_3, there exists a proper Heegaard splitting of M whose genus is $\rho_\varepsilon(\Gamma)$.*

Proof. Let $(\mathcal{A}'_\varepsilon, \mathcal{A}''_\varepsilon)$ and F_ε be the regular splitting and the regular surface associated to Γ and ε. Note that $\mathcal{A}'_\varepsilon \cap \partial_i M$ (where $\partial_i M$ is the i-th boundary component of M) is a single disk B_i (since Γ is a crystallization) such that $\partial B_i = \partial_i F_\varepsilon$; moreover $\mathcal{A}''_\varepsilon \cap \partial_i M = \partial_i M - \mathrm{int} B_i$.

Let us consider the closed surface $S_\varepsilon = F_\varepsilon \cup (\cup_i B_i)$ and a collar C of S_ε in $\mathcal{A}'_\varepsilon$; define $Y = \mathcal{A}''_\varepsilon \cup C$ and $X = \overline{\mathcal{A}'_\varepsilon - C}$. X is a proper handlebody with $\partial X = C_1$ (where C_1 is the subset of ∂C corresponding to $S_\varepsilon \times \{1\}$) and $X \cap Y = C_1$.

Y is a hollow handlebody with free boundary C_1. In fact consider, for each edge e_i of $K(\Gamma)$ whose endpoints are coloured by ε_1 and 3, the 2-handle $H_i^{(2)}$ which is a regular neighbourhood of the dual 2-cell of e_i (see Figure 3); Y is obtained from $S_\varepsilon \times [0,1]$ by adding the $H_i^{(2)}$'s along $S_\varepsilon \times \{0\}$.

To complete the proof observe that the resulting proper Heegaard splitting (X, Y) of M has genus:
genus (C_1) = genus (S_ε) = genus $(F_\varepsilon) = \rho_\varepsilon(\Gamma)$.

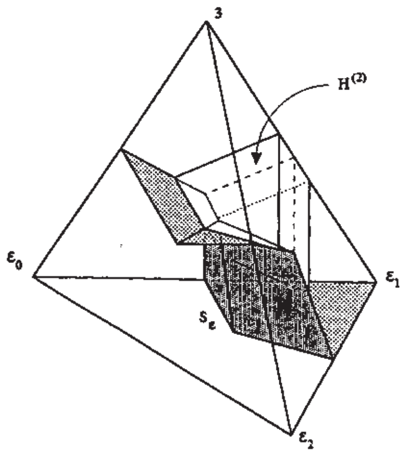

Figure 3.

Lemma 2. *Let M be a 3-manifold which is not a proper handlebody and let $N = \hat{M}$ be the singular 3-manifold associated to M. There exists a 4-coloured graph without boundary Γ representing N such that:*

- *if $\varepsilon = (0,1,2,3)$, then $\rho_\varepsilon(\Gamma) = \mathcal{H}(M)$;*
- *all singular vertices of $K(\Gamma)$ are 0-coloured.*

Proof. Suppose that $(T_g; v, w)$ is a proper Heegaard diagram representing M such that $g = \mathcal{H}(M)$.

By Proposition 2, we can algorithmically determine $h \geq 1$ and $\sigma, \tau \in \Sigma_h$ such that $\hat{N}(\sigma,\tau) \cong M$, $|\sigma, \tau\sigma\tau^{-1}| = 1$, $|\sigma| = g = 1 + \frac{1}{2}(h - |[\sigma,\tau]|)$.

Consider the triangulation $K(\sigma,\tau)$ of $N(\sigma,\tau)$ described in section 3 and subdivide it in the following way (see [Gr]):

- for each tetrahedron t, let V_S (resp. V_T) be the barycenter of $S \cap \bar{S}$ (resp. $T \cap \bar{T}$), join V_S and V_T by an edge lying in the interior of t and join V_S (resp. V_T) with the endpoints of $T \cap \bar{T}$ (resp. $S \cap \bar{S}$).

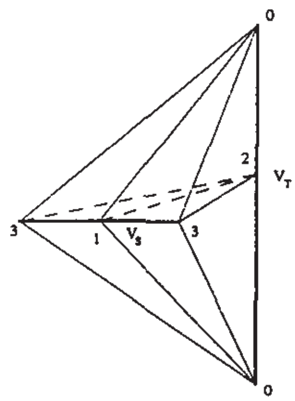

Figure 4.

Label now V_S (resp. V_T) by colour 1 (resp. by 2) and the endpoints of $S \cap \bar{S}$ (resp. of $T \cap \bar{T}$) by 3 (resp. by 0) (see Figure 4), thus obtaining a representable pseudocomplex K'. Let Γ be its associated 4-coloured graph (without boundary).

Note that, by Proposition 2, the canonical Heegaard splitting of $\check{N}(\sigma,\tau)$ is proper; therefore all the singular vertices of K' are 0-coloured.

Moreover $\sharp V(\Gamma) = 4h$, $g_{01}(\Gamma) = h$, $g_{12}(\Gamma) = \|[\sigma,\tau]\| = h + 2 - 2g$, $g_{23}(\Gamma) = h$, $g_{03}(\Gamma) = h$.

If $\varepsilon = (0,1,2,3)$, formula (*), for $n = 3$, gives:

$$\rho_\varepsilon(\Gamma) = 1 - \frac{1}{2}(g_{01}(\Gamma) + g_{12}(\Gamma) + g_{23}(\Gamma) - 4h) = g. \quad \blacksquare$$

Let us recall some definitions and results about subdivisions of coloured graphs (see [G_5]).

Given a 4-coloured graph without boundary Γ, two colours $\alpha,\beta \in \Delta_3$ and an α-coloured vertex w of $K(\Gamma)$, the *bisection of* Γ *of type* (α,β) *around* w is the 4-coloured graph $b\Gamma$ associated to the coloured complex $bK(\Gamma)$ obtained from $K(\Gamma)$ in the following way:

- consider the set $K_\beta(\Gamma)$ of all edges of $K(\Gamma)$ whose endpoints are w and a β-coloured vertex and perform a stellar subdivision on each edge of $K_\beta(\Gamma)$;
- colour w by β and the barycenters of the elements of $K_\beta(\Gamma)$ by α.

The coloration of $bK(\Gamma)$ agrees with that of $K(\Gamma)$ on the remaining vertices.

Let e be an edge of $K(\Gamma)$ whose endpoints, w_α and w_β, are α- and β-coloured respectively; the *trisection of* Γ *of type* (α,β) *on* e is the 4-coloured graph associated to the coloured complex obtained from $K(\Gamma)$ in the following way:

- perform two successive stellar subdivisions of $K(\Gamma)$: the first on e, introducing a new vertex w'_α, the second on the edge of endpoints w'_α and w_α, introducing another vertex w'_β;
- for $c \in \{\alpha,\beta\}$ colour w'_c by c, keeping the coloration of $K(\Gamma)$ for the remaining vertices.

We shall call *trisection of type* (α, β) *around the* α*-coloured vertex* w the graph $t\Gamma$ associated to the complex $tK(\Gamma)$, obtained by performing trisections of type (α, β) on all edges of $K(\Gamma)$, having w as endpoint.

If ε is a cyclic permutation of Δ_3, we have (see Proposition 7.1 and 7.2 of [G5]):

- if α and β are not consecutive in ε then $\rho_\varepsilon(t\Gamma) = \rho_\varepsilon(\Gamma) = \rho_\varepsilon(b\Gamma)$;
- if α and β are consecutive in ε then $\rho_\varepsilon(b\Gamma) = \rho_\varepsilon(\Gamma) + g(\Lambda_w) + g_{\beta\alpha'}(\Lambda_w) - 1$, where Λ_w is the $\hat{\alpha}$-residue of Γ representing $lkd(w, K(\Gamma))$ and α' is the colour non-consecutive to α in ε.

Lemma 3. *Let* Γ *be a 4-coloured graph without boundary representing a singular 3-manifold* N *such that all singular vertices in* $K(\Gamma)$ *are 0-coloured. If* $\varepsilon = (0,1,2,3)$*, there exists a 4-coloured graph with boundary* $\tilde{\Gamma}$*, regular with respect to 3, representing* $\check{N}$ *and such that* $\rho_\varepsilon(\tilde{\Gamma}) = \rho_\varepsilon(\Gamma)$.

Proof. If w is a (0-coloured) singular vertex of $K(\Gamma)$, let $\Gamma^{(1)}$ be the trisection of Γ of type (0,2) around w. Hence $\rho_\varepsilon(\Gamma^{(1)}) = \rho_\varepsilon(\Gamma)$ because 0 and 2 are *not consecutive in* ε.

Consider now the bisection $\Gamma^{(2)}$ of the previous graph of type (0,3) around w.

The genus of $\Gamma^{(2)}$ changes according to the following formula:

$$\rho_\varepsilon(\Gamma^{(2)}) = \rho_\varepsilon(\Gamma^{(1)}) + \rho(\Lambda_w) + g_{23}(\Lambda_w) - 1 = \rho_\varepsilon(\Gamma) + \rho(\Lambda_w) + g_{23}(\Lambda_w) - 1 \quad (')$$

where Λ_w is the $\hat{0}$-residue of $\Gamma^{(1)}$ representing $lkd(w, K(\Gamma^{(1)}))$. Note that w is now 3-coloured. Perform finally a trisection of type (3,1) around w, obtaining a 4-coloured graph $\Gamma^{(3)}$, with $\rho_\varepsilon(\Gamma^{(3)}) = \rho_\varepsilon(\Gamma^{(2)})$.

Delete now from $\Gamma^{(3)}$ the $\hat{3}$-residue Ξ representing $lkd(w, K(\Gamma^{(3)}))$ and the "hanging" 3-coloured edges and call Γ' the resulting 4-coloured graph with boundary. Clearly Γ' is regular with respect to the colour 3.

Note that:

(1) $g_{ij}(\Gamma') = g_{ij}(\Gamma^{(3)}) - g_{ij}(\Xi) \qquad \forall i, j \in \{0, 1, 2\}$

(2) $g_{3i}(\Gamma') = g_{3i}(\Gamma^{(3)}) - \frac{1}{2}\bar{p}(\Gamma') \qquad \forall i \in \{0,1,2\}$

(3) $p(\Gamma^{(3)}) = p(\Gamma') + \bar{p}(\Gamma')$

(4) $\bar{p}(\Gamma') = p(\Xi)$

(5) ${}^{\partial}g_{02}(\Gamma') = g_{02}(\Xi)$.

By formula (*) applied to Γ' we have:

$$\rho_\varepsilon(\Gamma') = 1 - \frac{1}{2}[g_{01}(\Gamma') + g_{12}(\Gamma') + g_{23}(\Gamma') + g_{03}(\Gamma') - (p(\Gamma') - \bar{p}(\Gamma')) - $$
$$-\frac{1}{2}\bar{p}(\Gamma')] - \frac{1}{2}{}^{\partial}g_{02}(\Gamma') \qquad (^{**})$$

By applying formula (*), for $n = 2$, to the 3-coloured graph Ξ and the permutation $\varepsilon' = (0,1,2)$, we obtain:

$$\rho_{\varepsilon'}(\Xi) = 1 - \frac{1}{2}[g_{01}(\Xi) + g_{12}(\Xi) + g_{02}(\Xi) - \frac{1}{2}p(\Xi)] \qquad (^{***})$$

By adding (**) and (***) and making use of (1), (2), (3) we have:

$$\rho_\varepsilon(\Gamma') + \rho_{\varepsilon'}(\Xi) = \rho_\varepsilon(\Gamma^{(3)}) + 1 - \frac{1}{2}\left[\frac{1}{2}\bar{p}(\Gamma') + g_{02}(\Xi) - \frac{1}{2}p(\Xi)\right] - \frac{1}{2}{}^{\partial}g_{02}(\Gamma').$$

By substituting equalities (4) and (5) we obtain:

$$\rho_\varepsilon(\Gamma') + \rho_{\varepsilon'}(\Xi) = \rho_\varepsilon(\Gamma^{(3)}) + 1 - \frac{1}{2}\left[\frac{1}{2}\bar{p}(\Gamma') + {}^{\partial}g_{02}(\Gamma') - \frac{1}{2}\bar{p}(\Gamma')\right] -$$
$$-\frac{1}{2}{}^{\partial}g_{02}(\Gamma') = \rho_\varepsilon(\Gamma^{(3)}) + 1 - {}^{\partial}g_{02}(\Gamma').$$

Finally formula (') gives:

$$\rho_\varepsilon(\Gamma') = \rho_\varepsilon(\Gamma) + \rho(\Lambda_w) + g_{23}(\Lambda_w) - \rho_{\varepsilon'}(\Xi) - {}^{\partial}g_{02}(\Gamma').$$

Note that both $g_{23}(\Lambda_w)$ and ${}^{\vartheta}g_{02}(\Gamma')$ equal the number of edges in $K(\Gamma)$ whose endpoints are w and a 1-coloured vertex. Moreover, since Ξ and Λ_w are 3-coloured graphs, they admit a unique regular imbedding, namely the one in the surface $|lkd(w, K(\Gamma))|$, which both represent (see [G$_4$] Corollary 5]), i.e. $\rho(\Lambda_w) = \rho(\Xi) = \rho_{\varepsilon'}(\Xi)$. Hence $\rho_\varepsilon(\Gamma') = \rho_\varepsilon(\Gamma)$.

By repeating the above procedure for all the singular vertices of $K(\Gamma)$, we obtain the required 4-coloured graph $\bar{\Gamma}$. ■

Proof of Proposition 3. If Γ is a crystallization of M, then, by Lemma 1, we have $\mathcal{H}(M) \leq \rho_\varepsilon(\Gamma)$, for every choice of ε and, by [B Theorem 1], it follows $\mathcal{H}(M) \leq \mathcal{G}(M)$.

If M is a proper handlebody of genus g, then $\mathcal{G}(M) = g$ (see [G$_4$ pg. 276]). Since rank$(M) \leq \mathcal{H}(M)$, we have $g \leq \mathcal{H}(M)$. Hence $g = \mathcal{H}(M)$.

Suppose now that M is not a proper handlebody and let $N = \hat{M}$ be its associated singular manifold. Then the 4-coloured graph Γ obtained by Lemma 2 satisfies the condition of Lemma 3. By applying Lemma 3 to Γ, we obtain a 4-coloured graph $\bar{\Gamma}$ representing M such that $\rho_\varepsilon(\bar{\Gamma}) = \rho_\varepsilon(\Gamma) = \mathcal{H}(M)$. Hence $\mathcal{G}(M) \leq \rho_\varepsilon = \mathcal{H}(M)$. ■

Remark 4. Let N be a singular 3-manifold and let $G^0(N)$ denote the set of all 4-coloured graphs Γ, representing N, such that the singular vertices of $K(\Gamma)$ are 0-coloured. Note that, by Lemma 2, $G^0(N)$ is non-empty. The *regular genus* of N is, by definition, the non-negative integer:

$$\mathcal{G}(N) = \min\{\rho(\Gamma)/\Gamma \in G^0(N)\}.$$

By Lemma 3 and Proposition 3, we have $\mathcal{G}(N) \geq \mathcal{H}(\check{N}) = \mathcal{G}(\check{N})$. If $\check{N}$ is not a proper handlebody, Lemma 2 gives $\mathcal{G}(N) \leq \mathcal{H}(\check{N})$. If $\check{N}$ is a proper handlebody of genus g, the same inequality can be obtained by directly constructing a 4-coloured graph, of genus g, representing N (see [FG]). Hence $\mathcal{G}(N) = \mathcal{H}(\check{N}) = \mathcal{G}(\check{N})$ for every singular 3-manifold N.

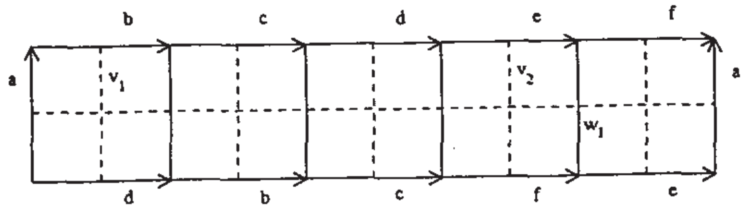

Figure 5.

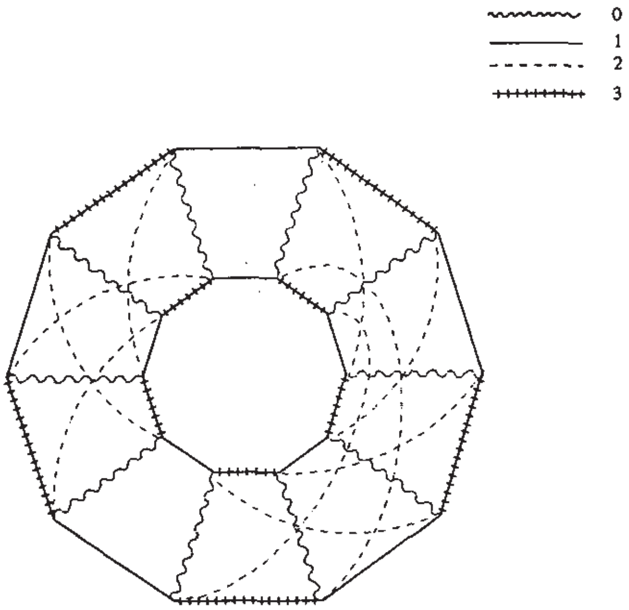

Figure 6.

An example. *The genus of the exterior of the trefoil knot.* Let M be the exterior of the trefoil knot. In Figure 5 a proper Heegaard diagram for M is shown (see example 2 of [M]). By applying the algorithm of Proposition 2, we have: $\sigma = (123)(45)$ and $\tau = (12345)$. Figure 6 shows the 4-coloured graph Γ obtained by using Lemma 2. Its genus is $\rho(\Gamma) = \rho_\varepsilon(\Gamma) = 2$, with $\varepsilon = (0,1,2,3)$. Therefore $\mathcal{G}(M) = \mathcal{H}(M) \leq 2$.

Actually $\mathcal{G}(M) = \mathcal{H}(M) = 2$, since any genus one 3-manifold whose boundary is a torus, is homeomorphic to a solid torus (see also the final remark of [C]). Moreover, the given Heegaard diagram describes the only genus two proper Heegaard splitting representing M (see [BRZ]).

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