Heffter Arrays and Biembedding Graphs on Surfaces*

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Abstract

A Heffter array is an $m \times n$ matrix with nonzero entries from $\mathbb{Z}_{2mn+1}$ such that
i) every row and column sum to 0, and *ii)* exactly one of each pair $\{x, -x\}$ of
nonzero elements appears in the array. We construct some Heffter arrays. These
arrays are used to build current graphs used in topological graph theory. In turn,
the current graphs are used to embed the complete graph K_{2mn+1} so that the faces
can be 2-colored, called a biembedding. Under certain conditions each color class
forms a cycle system. These generalize biembeddings of Steiner triple systems. We
discuss some variations including Heffter arrays with empty cells, embeddings on
nonorientable surfaces, complete multigraphs, and using integer arithmetic in place
of modular arithmetic.

1 Introduction

We study a relation between design theory, graph theory, and maps on surfaces. From
design theory a Heffter system is used to construct a cyclic k-cycle system. We introduce
orthogonal Heffter systems and represent them as a Heffter array. The array is related
to a current assignment on the complete bipartite graph $K_{m,n}$. The current graph with
certain conditions is then used to construct an orientable embedding of the complete
graph K_{2mn+1} that is face 2-colorable, the boundaries of each color class forming a cycle
system. A similar theorem is given for embeddings in nonorientable surfaces.

Heffter's First Difference Problem [16] asks if the numbers from 1 to $(m-1)/2$ can be
partitioned into $(m-1)/6$ triples (x, y, z) such that either $x+y = z$ or $x+y+z = m$. Heffter
used this partition to construct a *Steiner triple system*, $STS(m)$, a collection of triples
from an m-set that collectively contain every pair exactly once [8]. This corresponds to

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a set of 3-cycles whose edges partition $E(K_m)$. An *s-cycle system* partitions $E(K_m)$ into s-cycles. Buratti and Del Fra [6] proved the existence of k-cycle systems of K_m having a cyclic action on the parts whenever $m \equiv 1 \pmod{2k}$. Further information on Heffter's difference problem, Steiner triple systems and k-cycle systems can be found in [7].

A Heffter array is an $m \times n$ array with non-zero entries from $\mathbb{Z}_{2mn+1}$ such that the entries are all distinct up to sign and such that each row and column sum to 0. We explore Heffter arrays in Section 2. We also define two properties of orderings ω_r and ω_c on the cells of a Heffter array.

A current graph is an embedded graph where each directed edge has been assigned an element from a fixed current group. Under some special conditions current graphs can be used to construct embeddings of a complete graph. These embeddings were first used in the solution of the Map Color Theorem [20]. Details of this relation are given in [15]; we give a brief explanation in Section 3. The relationship between Heffter arrays and current graphs is described in Section 4. One main result is the following whose proof is given in Section 4.

Theorem 1.1 *Given a Heffter array $H(m, n; s, t)$ with compatible orderings ω_r on $D(m, s)$ and ω_c on $D(n, t)$, there exists an orientable embedding of K_{2ms+1} such that every edge is on a face of size s and a face of size t. Moreover, if ω_r and ω_c are both simple, then all faces are simple cycles.*

A *biembedding* of the complete graph K_m is one that is face 2-colorable. We are particularly interested when the face boundaries of the first color class form an s-cycle system and those of the other color class form a t-cycle system. These have most commonly been studied in the case $s = t = 3$, that is, both color classes are $STS(m)$'s. Triangular biembeddings on orientable surfaces were shown to exist in [20] for all $n \equiv 3 \pmod{12}$ and in [23] for all $n \equiv 7 \pmod{12}$, the two necessary cases. For nonorientable surfaces, the necessary conditions are *(i)* $n \equiv 1 \pmod 6$ and *(ii)* $n \equiv 3 \pmod 6$. The first of these cases was solved in [12] while the second was solved in [20]. Biembeddings of Steiner triple systems have been widely studied [12, 13] particularly for small values [11]. Grannell and Griggs [10] give a very nice survey. McCourt [18] has studied biembeddings where one color class gives an STS and the other half give a decomposition of $E(K_m)$ into Hamilton cycles. Brown [5] has a class of biembeddings where one color class contains triangles and the other contains quadrilaterals.

Section 5 discusses weak Heffter arrays and their use to construct biembeddings on nonorientable surfaces. Section 6 closes with some directions for future research, several of which will be the subject of subsequent papers.

2 Heffter systems and Heffter arrays

Let $\mathbb{Z}_m$ be the cyclic group of odd order m whose elements are denoted 0 and $\pm i$ where $i = 1, \ldots, (m-1)/2$. A *half-set L* is a subset of $(m-1)/2$ nonzero elements that contains exactly one of each pair $\{x, -x\}$. A *Heffter system*, $D(m, k)$, is a partition of L into

parts of size k such that the elements in each part sum to 0 modulo m. *Heffter's First Difference Problem* [16] asks if the numbers from 1 to $(m-1)/2$ can be partitioned into $(m-1)/6$ triples (x, y, z) such that either $x + y = z$ or $x + y + z = m$. This is equivalent to finding a $D(m, 3)$.

Two Heffter systems $D_m = D(2mn + 1, m)$ and $D_n = D(2mn + 1, n)$ on the same half-set L are *orthogonal* if each m-set of D_m intersects each n-set of D_n in a single element. A *Heffter array $H(m, n)$* is an $m \times n$ array whose rows form a $D(2mn + 1, n)$ and whose columns form a $D(2mn + 1, m)$; we call these the *row* and *column Heffter systems* respectively. A Heffter array $H(m, n)$ is equivalent to a pair of orthogonal Heffter systems: cell $a_{i,j}$ contains the common element in the ith part of the row system and the jth part of the column system. Figure 1 shows a Heffter array $H(3, 4)$.

1	-2	-10	11
-8	6	-3	5
7	-4	-12	9

Figure 1: A Heffter array $H(3, 4)$ over $\mathbb{Z}_{25}$

Let A be a subset of $\mathbb{Z}_m$ with $\sum_{a \in A} a \equiv 0 \pmod{m}$ such that no pair $\{x, -x\}$ is a subset of A. Consider a cyclic ordering $(a_1, \ldots, a_k)$ of the elements in A and let $s_i = \sum_{j=1}^{i} a_j$. The ordering is *simple* if $s_i \neq s_j$ for $i \neq j$. Equivalently, the cyclic ordering is simple if there is no consecutive subsequence of elements that sum to 0. A Heffter system $D(m, k)$ has a *simple ordering* if and only if each part has a simple ordering.

Proposition 2.1 *An ordered Heffter system $D(m, k)$ forms a decomposition of $E(K_m)$ into closed trails of length k. Moreover, if the ordering is simple, then its corresponding decomposition is a k-cycle system.*

Proof: Let $\{a_1, \ldots, a_k\}$ be a part of $D(m, k)$ under any ordering. Form a closed trail $(0, s_1, s_2, \ldots, s_k)$ in a complete graph K_m with vertex set $\mathbb{Z}_m$. Develop this trail modulo m and do the same for all other parts of $D(m, k)$. Since each pair $\{x, -x\}$ has exactly one element in $D(m, k)$, each difference appears once. Hence these closed trails partition $E(K_m)$. If the ordering on $D(m, k)$ is simple, then each of these trails are simple cycles. ∎

Let $D_1 = D(2mn + 1, s)$, $D_2 = D(2mn + 1, t)$ be two orthogonal Heffter systems with orderings ω_1, ω_2 respectively. The orderings are *compatible* if their composition $\omega_1 \circ \omega_2$ is a cyclic permutation on the half-set. The importance of compatible orderings will be examined in Section 4 when we relate Heffter arrays and current graphs.

A variation of Heffter arrays allows for some cells to be empty. Two Heffter systems $D(2ms + 1, s)$ and $D(2nt + 1, t)$ on the same half-set of order $ms = nt$ are *sub-orthogonal* if each s-set of $D(2ms + 1, s)$ intersects each t-set of $D(2nt + 1, t)$ in at most one element. As before, form an $m \times n$ array $H(m, n; s, t)$ where $a_{i,j}$ is the common element in the ith part

of $D(2ms+1, s)$ and the jth part of $D(2nt+1, t)$, if any, and the cell is empty otherwise. Necessary conditions for the existence of an $H(m, n; s, t)$ are $ms = nt$, $3 \leqslant s \leqslant m$, and $3 \leqslant t \leqslant n$. Figure 2 gives an example of a $H(6, 12; 8, 4)$.

		−1	2	5	−6			−25	26	29	−30
		3	−4	−7	8			27	−28	−31	32
9	−10			−13	14	33	−34			−37	38
−11	12			15	−16	−35	36			39	−40
−17	18	21	−22			−41	42	45	−46		
19	−20	−23	24			43	−44	−47	48		

Figure 2: A Heffter array $H(6, 12; 8, 4)$

If a Heffter array $H(m, n; s, t)$ is square, i.e., $m = n$, then necessarily $s = t$. In this case we denote the square by $H(n; k)$. The commonality in the notation is that parameters before the semicolon refer to sides of the squares, those after to the number of filled cells in a row or column. Figure 3 gives a $H(5; 4)$. Square arrays with empty cells are studied in [2].

	17	−8	−14	5
1		18	−9	−10
−6	2		19	−15
−11	−12	3		20
16	−7	−13	4	

Figure 3: A Heffter array $H(5; 4)$

Let $H(m_1 + m_2, n_1 + n_2; s, t)$ be a Heffter array. Suppose that the rows and columns of H can be permuted such that each nonempty cell $a_{i,j}$ has either $i \leqslant m_1$ and $j \leqslant n_1$, or it has $i > m_1$ and $j > n_1$. Then the array is called *block diagonal*. Constructing block diagonal arrays is convenient and powerful, but they are not suited for the application to graph embeddings and so are sometimes avoided.

3 Orientable embeddings and current graphs

In this section we describe the use of rotations to describe a cellular embedding of a graph on a fixed orientable surface. We also define current graphs and their usefulness in embedding complete graphs.

3.1 Orientable embeddings

Consider a graph G and for every edge e let e^+ and e^- denote its two possible directions. Let $D(G)$ be the set of all directed edges, so $|D(G)| = 2|E(G)|$, and define τ as the involution swapping e^+ and e^- for every e. Let $D_v = \{(v,u) \in D(G)\}$ denoted the edges directed out of v. A *local rotation* ρ_v is a cyclic permutation of D_v. If we select a local rotation for each vertex, then collectively they form a *rotation* ρ of $D(G)$. The orbits of ρ correspond bijectively to the set of local rotations on $V(G)$. The proof of the following is omitted; see [15, 19] for details.

Theorem 3.1 *A rotation on G is equivalent to a cellular embedding of G in an oriented surface. The face boundaries of the embedding corresponding to ρ are the orbits of $\rho \circ \tau$.*

Calculating $\rho \circ \tau$ is called the *face-tracing algorithm*. Knowing the number of faces lets you calculate the genus g of the surface using Euler's formula $|V| - |E| + |F| = 2 - 2g$.

We are especially interested in *monofacial* embeddings, those with a single face. By Euler's formula if a graph has a monofacial embedding, then $|V| \not\equiv |E| \pmod 2$. However, this necessary condition is not sufficient. The following is a special case of Xuong's Theorem [22]. The proof provides an algorithm for calculating a rotation yielding the monofacial embedding.

Theorem 3.2 *A graph G has a monofacial embedding on an orientable surface if and only if there is a spanning tree T such that every component of $G - T$ has an even number of edges.*

Kundu [17] showed that every 4-edge-connected graph has two disjoint spanning trees. This combined with Theorem 3.2 gives:

Corollary 3.3 *If G is 4-edge-connected and $|V| \not\equiv |E| \pmod 2$, then G has a monofacial embedding.*

3.2 Current graphs

Current graphs [14] were originally developed as quotients of surface embeddings. In particular, a monofacial embedding of a graph with currents added to the edges are used to construct a rotation on a derived graph, usually complete. We briefly describe this construction.

A *current assignment* on G with *currents* from $\mathbb{Z}_m$ is a function $\kappa : D(G) \to \mathbb{Z}_m$ such that $\kappa(e^-) = -\kappa(e^+)$. We frequently require the following conditions:

1. (Kirchoff's Current Law - *KCL*) For every vertex v, $\sum_{e \in D_v} \kappa(e) \equiv 0 \pmod m$,

2. (Unique Currents) κ is a bijection between $D(G)$ and $\mathbb{Z}_m \setminus \{0\}$,

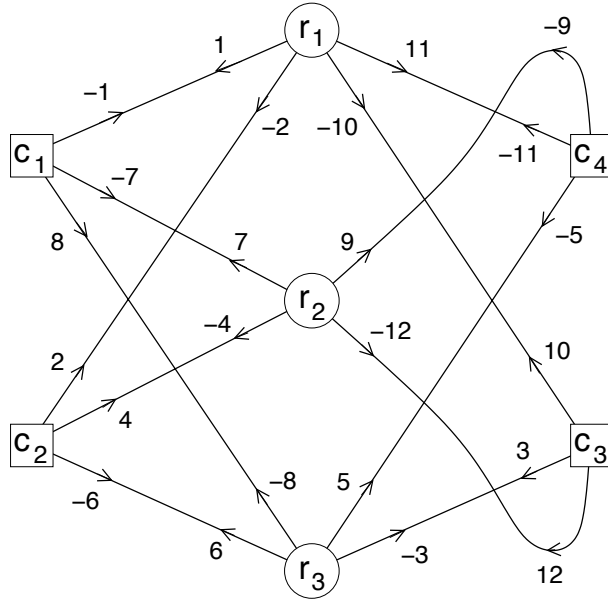3. (Monofacial) G is embedded on a surface with a single face (this property of G is independent of κ).

Figure 4: A current graph

Figure 4 shows a unique-current assignment that satisfies *KCL* in $\mathbb{Z}_{25}$. Here, as elsewhere, we denote a directed edge $\bar{e}$ by its unique current $\kappa(\bar{e})$. We read each local rotation anticlockwise as the edges emanate from the vertices; for example, around vertex C_4 the directed edges with currents $(-9, -11, -5)$ appear in that cyclic order. This gives the rotation:

$$\rho = (1, -2, -10, 11)(7, -4, -12, 9)(-8, 6, -3, 5) \\ (-1, 8, -7)(2, -6, 4)(10, 3, 12)(-11, -5, -9). \tag{1}$$

The face tracing algorithm gives a single face:

$$\rho \circ \tau = (1, 8, 6, 4, -12, 10, 11, -5, -8, -7, -4, 2, \\ -10, 3, 5, -9, 7, -1, -2, -6, -3, 12, 9, -11). \tag{2}$$

A rotation ρ on a current graph induces a local rotation on D_v for each vertex v, say $(e_1, \ldots, e_k)$. This local rotation is *simple with respect to* κ if the corresponding partial sums $s_i = \sum_{j=1}^{i} \kappa(e_j)$ are all distinct, similar to the definition of a simple ordering for a subset of a group.

An (s, t)-*biregular graph* with *biorder* (m, n) is a bipartite graph with one part having m vertices of degree s and the other part having n vertices of degree t.

Theorem 3.4 *Let G be an (s, t)-biregular graph of biorder (m, n). Suppose that G has a rotation ρ giving a monofacial embedding and a unique-current assignment κ from Z_{2ms+1} satisfying KCL. Then there is an embedding of K_{2ms+1} on an orientable surface such that each edge lies on a face of size s and a face of size t. Moreover, if each local rotation on G is simple with respect to κ, then the faces of K_{2ms+1} are simple cycles.*

Proof: This is the standard construction of a derived embedding from a current graph [15, 20]; a careful analysis is given in [14]. The vertex set of K_{2ms+1} will be the elements of $\mathbb{Z}_{2ms+1}$. Let $e_1, \ldots, e_{2ms}$ denote the directed edges traversed in the single face of the embedding of G. Define the local rotation at vertex $0 \in \mathbb{Z}_{2ms+1}$ as $(\kappa(e_1), \ldots, \kappa(e_{2ms}))$. Develop this rotation in $\mathbb{Z}_{2ms+1}$ by defining the rotation at vertex $i \in \mathbb{Z}_{2ms+1}$ as $(\kappa(e_1) + i, \ldots, \kappa(e_{2ms}) + i)$.

We use the face-tracing algorithm to show that a vertex of degree d in G satisfying *KCL* corresponds to $2ms + 1$ faces of size d in the embedding of K_{2ms+1}. Since the graph is (s, t)-biregular, each edge of K_{2ms+1} lies on faces of size s and t. Likewise, if each local rotation is simple, then the corresponding faces of the embedding of K_{2ms+1} are simple cycles. ∎

The rotation ρ on the current graph G plays two independent roles in this construction: *i)* ρ generates a monofacial embedding, and *ii)* each local rotation ρ_v is simple with respect to the current assignment κ. Any rotation on a vertex of degree $d \leqslant 5$ is simple.

There is a quick way to find the faces of the derived embedding arising from an embedded current graph. Let v be a vertex of the current graph G with degree k. The rotation ρ giving the monofacial embedding determines a local rotation $\rho_v = (e_1, \ldots, e_k)$ on D_v. Consider $s_i = \sum_{j=1}^{i} \kappa(e_j)$. By *KCL* $s_k \equiv 0 \pmod{2ms + 1}$. The faces of the embedding of K_{2ms+1} are precisely the cyclic shifts of $(0, s_1, \ldots, s_{k-1})$. Describing the faces is enough to determine the embedding. A full proof of Theorem 3.4 shows that the monofacial condition guarantees that these faces meet in a cyclic manner at each vertex, i.e., we have a surface without pinch points.

4 Relating Heffter arrays and current graphs

We have described two seemingly different objects: Heffter arrays and current graphs on biregular graphs. We show they are closely related.

Proposition 4.1 *A Heffter array $H = H(m, n; s, t)$ is equivalent to a unique-current assignment κ on a (s, t)-biregular graph G of biorder (m, n). This graph is connected if and only if H is not block diagonal.*

Two compatible orderings ω_r and ω_c on the row and column Heffter systems of H are equivalent to a monofacial rotation ρ on G. Moreover, if ω_r and ω_c are both simple, then ρ is simple with respect to κ.

Proof: Given a Heffter array $H(m, n; s, t)$, form a bipartite graph G whose vertex set is the rows of H together with its columns. For each non-empty $a_{i,j}$ in H add an edge in G labeled with current $a_{i,j}$ directed from the ith row of H to its jth column; as usual the reverse edge receives the negative current. Since H has s entries per row and t per column, the resulting graph is (s, t)-biregular graph of biorder (m, n). Each row and column of H sums to 0, so G satisfies *KCL*. The entries of H form a half-set L, so G has unique currents. If G is disconnected, then the components give a partition of rows and columns showing that H is block diagonal.

Relating the orderings ω_r and ω_c on H to the rotation ρ on G is more difficult. We use the unique currents to describe ρ not as a permutation of directed edges but rather as a permutation of their nonzero currents. We use $\tau(a)$ for $-a$, reflecting oppositely directed edges receive inverse group elements. Define $\rho : \mathbb{Z}_{2ms+1} \setminus \{0\} \to \mathbb{Z}_{2ms+1} \setminus \{0\}$ by:

$$\rho(a) = \left\{ \begin{array}{cc} \omega_r(a) & a \in L \\ \tau \circ \omega_c \circ \tau(a) & a \notin L \end{array} \right. .$$

Note that if $a \in L$, then $(\rho \circ \tau)^2(a) = \omega_r \circ \omega_c(a)$. Since ω_r is compatible with ω_c, $(\rho \circ \tau)^2$ acts cyclically on L. Also $\rho \circ \tau : L \leftrightarrow -L$. Hence ρ acts cyclically on $\mathbb{Z}_{2ms+1} \setminus \{0\}$ and the embedding is monofacial as desired.

The reverse of the construction above shows that a current assignment corresponds to a Heffter array, hence the equivalence. ∎

We combine our results for the following:

Proof: (of Theorem 1.1) Apply Theorem 4.1 to a Heffter array to build a current graph. Theorem 3.4 then gives the desired embedding. ∎

We illustrate the relation with an example. Figure 1 gives a Heffter array $H(3, 4)$ with entries from $\mathbb{Z}_{25}$. We simply order the parts of the row system $\omega_r = (1, -2, -10, 11)$ $(-8, 6, -3, 5)(7, -4, -12, 9)$ and simply order the column system $\omega_c = (1, -8, 7)(-2, 6, -4)$ $(-10, -3, -12)(11, 5, 9)$. Together these give the rotation ρ given in Equation 1, agreeing with the projected rotation on the $K_{3,4}$ of Figure 4. In turn this gives the monofacial face $\rho \circ \tau$ of Equation 2.

This current graph gives an embedding of K_{25} on an orientable surface with every edge on the boundary of a triangle and a quadrilateral, i.e., a biembedding of a 3-cycle system with a 4-cycle system. Euler's formula implies this surface is of genus 51.

5 Weak Heffter arrays and nonorientable embeddings

A variation of Heffter arrays corresponds to signed current graphs embedded on nonorientable surfaces. In turn, this can be used to construct nonorientable $\{s, t\}$-biembeddings of complete graphs. In this section we describe this relationship beginning with embeddings of signed graphs.

5.1 Signed current graphs

A *signed graph* $G^{\pm}$ is a graph G together with a signature $\sigma : E(G) \to \{+, -\}$. The *signature of a cycle* in G is the product of the signatures on its edges. A *local switch* at v toggles the sign of each edge incident with v. Two signatures are *equivalent* if and only if they are related by a sequence of local switches. A signature is equivalent to the all-positive signature if and only if the negative edges form a co-cycle, i.e., if there is no odd-length negative cycle. In this case the signature is called *balanced*, otherwise it is *unbalanced*.

We describe *signed-embeddings* of signed graphs. As before, for each vertex we give a cyclic permutation ρ_v of D_v. We keep track of the *local sense* of the orientation, one of two states *anticlockwise* or *clockwise*. When the local sense is anticlockwise, a face boundary entering a vertex v on a directed edge e^+ leaves along $\rho_v(e^-)$. When it is clockwise we exit v along the directed edge $\rho_v^{-1}(e^-)$. When traversing a negative edge we reverse our local sense of orientation. The face boundary closes when we reach a directed edge previously traversed in the same local sense. This process of tracing the orbits of $\rho \circ \tau$ while keeping sense of the local sense of orientation is called the *modified face-tracing algorithm*. The resulting surface is orientable if and only if the signature is balanced.

A *signed current assignment* is a function $\kappa : E(G^\pm) \to \mathbb{Z}_m$ such that $\kappa(e^-) = -\kappa(e^+)$ when $\sigma(e)$ is positive, and $\kappa(e^-) = \kappa(e^+)$ when $\sigma(e)$ is negative. In analogy with unsigned current assignments we frequently require the following:

1. (Kirchoff's Current Law - *KCL*) For every vertex v, $\sum_{e \in D_v} \kappa(e) \equiv 0 \pmod{m}$,

2. (Signed-Unique Currents) A current κ occurs on a unique directed edge unless that edge is signed negatively, where one of κ or $-\kappa$ appears twice and the other not at all, and

3. (Monofacial) $G^\pm$ is signed-embedded on a surface with a single face (this property of $G^\pm$ is independent of κ).
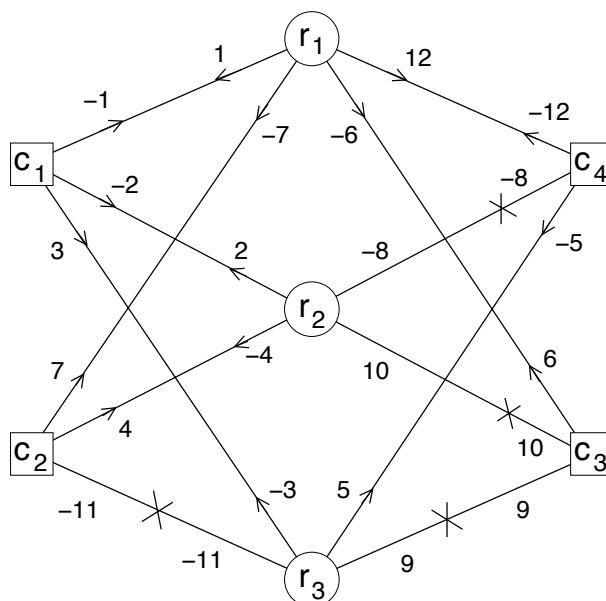

Figure 5: A signed current graph

For example, a signed current graph on $K_{3,4}$ is shown in Figure 5. Using the notation of Youngs [23] the negatively signed edges are indicated with an $\times$ in the middle; they

receive currents -8,9,10,-11. The current assignment in the figure satisfies KCL and has signed-unique currents. A local rotation at each vertex is read off anticlockwise in this projection diagram. Using this rotation and signature the modified face-tracing algorithm gives the single face:

$$(1, 3, -11, 7, 1, -2, -8, -5, -3, -2, -4, 7, -6, 10, -4, -11, 9, 10, -8, -12, -6, 9, 5, -12).$$

The restriction of ρ to D_v gives a cyclic permutation of the currents $\kappa(e^+)$, $e^+ \in D(v)$. As before, define ρ to be *simple at v* if the corresponding partial sums on D_v are all distinct, and to be *simple* if ρ is simple at each vertex v.

The following is analogous to Theorem 3.4.

Theorem 5.1 *Let $G^\pm$ be an (s, t)-biregular unbalanced signed graph of biorder (m, n). Suppose that $G^\pm$ has a monofacial nonorientable embedding and a signed unique-current assignment from Z_{2ms+1} satisfying* KCL. *Then there is an embedding of K_{2ms+1} on a nonorientable surface such that each edge lies on a face of size s and a face of size t. Moreover, if each local rotation on G is simple, then the faces of K_{2ms+1} are simple cycles.*

Proof: This is again a standard construction of a derived embedding from a current graph, see [15, 20] for the full proof. The vertex set of K_{2ms+1} is the elements of $\mathbb{Z}_{2ms+1}$. The monofacial signed embedding of the current graph is used to determine the rotation at a vertex of the derived graph. A lift of the signature on the current graph determines the signature on the derived complete graph.

Since the embedding underlying the current graph is unbalanced, it is in a nonorientable surface. Hence there is a negatively signed cycle. This in turn implies a negatively signed cycle in the derived embedding, i.e., it also is nonorientable. Finally, simple local rotations in the current graph correspond to simple cycles for faces in the derived graph as before. ∎

Using the modified face-tracing algorithm there is a quick way to find the faces of the derived embedding arising from a embedded signed-current graph. Let v be a vertex of the current graph $G^\pm$ with degree k. The rotation ρ gives determines a local rotation $\rho_v = (e_1, \ldots, e_k)$ on D_v. Consider $s_i = \sum_{j=1}^{i} \kappa(e_j)$. By KCL $s_k \equiv 0 \pmod{2ms + 1}$. The faces of the embedding of K_{2ms+1} are precisely the cyclic shifts of $(0, s_1, \ldots, s_{k-1})$. Describing the faces is enough to determine the embedding. A full proof of Theorem 5.1 shows that the monofacial condition guarantees that these faces again meet in a cyclic manner at each vertex.

A key feature in the above construction is a monofacial embedding of a signed graph G. The following theorem addresses when such embeddings exist. Let T be a positively signed spanning tree of G. A cotree component is *odd* if it has an odd number of edges and every negative edge is a bridge separating that component into two parts each having an odd number of edges.

Theorem 5.2 [21] *A signed graph has a monofacial signed embedding if and only if there is a spanning tree with no odd cotree components.*

To apply Theorem 5.1 this monofacial embedding need also induce a simple ordering at each vertex. We have no general theory for this and it remains a project for future research.

5.2 Weak Heffter arrays

Two Heffter systems $D_1(2ms + 1, s)$ and $D_2(2nt + 1, t)$ with $ms = nt$ are *weakly suborthogonal* if the ith part of D_1 has at most one element $a_{i,j}$ such that either $a_{i,j}$ or $-a_{i,j}$ is in the jth part of D_2. Form a *weak Heffter array* $H(m, n; s, t)$ by placing $a_{i,j}$ in row i column j. The upper sign on $\pm$ or $\mp$ is the *row sign* corresponding to its sign on $a_{i,j}$ in D_1, the lower sign is the *column sign* used in D_2. Using the row signs we get row sums 0 and the column signs give column sums 0. Figure 6 shows a weak Heffter array.

1	-7	-6	12
2	-4	± 10	∓ 8
-3	∓ 11	± 9	5

Figure 6: A weak Heffter array $H(3, 4)$ over $\mathbb{Z}_{25}$

We relate a weak Heffter array $H(m, n; s, t)$ to a signed current assignment. Form a bipartite G of order (m, n) whose vertices are the rows and columns of H. For each nonempty cell add an edge e labeled with the row-signed current $a_{i,j}$ directed from the ith row of H to its jth column. The same edge in the opposite direction is assigned $-a_{i,j}$ and signed positively unless the entry is signed $\pm$ or $\mp$, in which case it's assigned $a_{i,j}$ and is signed negatively. Figure 5 shows the signed current graph corresponding to the weak Heffter array of Figure 6; the negatively signed edges are marked with an $\times$.

6 Conclusion

We have introduced Heffter arrays and their relation with current graphs and with biembeddings. The following table summarizes these relations.

Heffter array	current graph	biembedding
group	current group	vertex set
rows and columns	bipartition of vertices	face-2-colorable
# entries row/col	biregular graph	biregular face
zero row/col sums	satisfying KCL	sizes s, t
simple order	rotation at a vertex	faces are simple cycles
compatible orders	monofacial embedding	no pinch points at vertices

In our definition of a Heffter array $H(m, n; s, t)$ we required the row and column sums to be 0 modulo $2ms + 1$. The following tighter requirement is useful in constructions. Let

L be a halfset of $\{\pm k \mid k = 1, \ldots, ms\}$. An *integer Heffter array* $H(m, n; s, t)$ is an $m \times n$ array with entries from L such that each row and each column sum to 0 over the integers.

Lemma 6.1 *If an integer Heffter array $H(m, n; s, t)$ exists, then $ms \equiv 0, 3 \pmod 4$.*

Proof: For a row to sum to 0 there must be an even number of odds. Hence the set of ms entries must have an even number of odds, implying the congruence. ∎

A third condition is also helpful in constructions. An integer Heffter array $H(m, n; s, t)$ is *shiftable* if each row and column contain the same number of positive as negative entries. The array of Figure 2 is integer and shiftable. Given a shiftable $H(m, n; s, t)$ with entries $a_{i,j}$, define $b_{i,j} = a_{i,j} + k$ if $a_{i,j} > 0$, and $b_{i,j} = a_{i,j} - k$ otherwise. The matrix $H \pm k$ whose entries are $b_{i,j}$ still has row and column sums 0 over the integers. Define the *support* of a matrix $A = \{a_{i,j}\}$ as $support(A) = \{|a_{i,j}|\}$. If the support of the original matrix is $\{1, \ldots, ms\}$, then the support of the new matrix is $\{1 + k, \ldots, ms + k\}$.

Lemma 6.2 *If a shiftable Heffter array $H(m, n; s, t)$ exists, then s and t are even both even, at least 4, and $ms \equiv 0 \pmod 4$.*

Proof: Since the number of positive and negative entries are the same in each row and column, s and t are both even. By Lemma 6.1 $ms \equiv 0, 3 \pmod 4$, so $ms \equiv 0 \pmod 4$. ∎

A natural question is the following, which we believe to be true.

Conjecture 6.3 *There exists a Heffter array $H(m, n; s, t)$ for all m, n, s, t with $s, t \geqslant 3$ and $ms = nt$. If the conditions of Lemma 6.1 are satisfied, then there is an integer Heffter array. If the conditions of Lemma 6.2 are satisfied, then there is a shiftable array.*

This conjecture is extensive since there are four parameters related by a single equation $ms = nt$. It is natural to examine special cases, such as *i)* when H has no empty cells ($n = t$ and $m = s$), *ii)* squares ($n = m$), or *iii)* fixing s and t.

In the case the array has no empty cells Conjecture 6.3 simplifies to the following.

Conjecture 6.4 *There exist Heffter arrays $H(m, n)$ for all $m, n \geqslant 3$. Moreover, they are integer Heffter arrays when $mn \equiv 0, 3 \pmod 4$, and are shiftable when m and n are both even and at least 4.*

The author with Tom Boothby and Jeff Dinitz believe we have a proof of Conjecture 6.4. We are writing up the details for publication in [1] and have a computer program for their construction. Another subsequent paper [2] examines square ($n \times n$) integer Heffter arrays including a complete characterization of shiftable arrays.

In our definition of a Heffter array we require at most one appearance of elements in $\{x, -x\}$. What if we allowed two appearances either the same or different, or a multiset of λ such elements? Using the difference-set construction this gives rise to λ-fold cycle systems. Let K_n^λ be the complete multigraph on n vertices where every pair of edges is

jointed by λ edges in parallel. The analogue of a Heffter square with higher λ gives an embedding of K_n^λ into a surface that is face 2-colorable, each color class being a λ-fold cycle system. For example, [9] uses current graphs to construct 2-fold embeddings with all faces triangles.

Recall that we need a simple ordering on the rows and one on the columns to ensure the resulting face boundaries are simple cycles. Moreover these two orderings should be compatible. These seem easy to find in practice, but hard to prove their existence in general. Alspach conjectures:

Conjecture 6.5 *Every $A \subset \mathbb{Z}_n \setminus \{0\}$ has a simple ordering, i.e., A can be ordered so that the partial sums are all distinct.*

The author together with Jeff Dinitz, and Doug Stinson have made some progress [3], including verifying Conjecture 6.5 for $n \leqslant 25$. Bode and Harborth showed it was true for $|A| = n - 1$. The general conjecture remains open.

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