

Invariant surfaces in Sol_3 with constant mean curvature and their computer graphics

Rafael López*

Departamento de Geometría y Topología
Universidad de Granada
18071 Granada. Spain
email: rcamino@ugr.es

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Abstract

In Sol_3 space there are three 1-parameter groups of isometries. In this work we study constant mean curvature surfaces invariant by one of these groups. We analyze the geometric properties of these surfaces by means of their computer graphics. We construct explicit examples of minimal surfaces and we shall relate them with recent examples of spheres with constant mean curvature.

1 Introduction

In recent years, the study of surfaces with constant mean curvature in homogeneous three-manifolds is a topic of great activity, specially after the extension by Abresch and Rosenberg of the Hopf theorem for this kind of spaces ([1]): see also [3] and references therein. Among the eight models of the geometry of Thurston ([14]), the

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space Sol_3 is the space with the smallest isometry group. As a Riemannian manifold, the space Sol_3 can be represented by \mathbb{R}^3 equipped with the metric

$$\langle, \rangle = e^{2z} dx^2 + e^{-2z} dy^2 + dz^2$$

where (x, y, z) are the canonical coordinates of \mathbb{R}^3 . The space Sol_3 is a Lie group with the operation

$$(x, y, z) * (x', y', z') = (x + e^{-z} x', y + e^z y', z + z')$$

and the metric \langle, \rangle is left-invariant with respect to the group operation.

It is natural to consider the study of curves and surfaces in Sol_3 under some additional geometric conditions. For example, the geodesics of Sol_3 were studied in [2], the totally geodesic surfaces had been classified in [15], the totally umbilical invariant surfaces in [13] and constant angle surfaces in [10]. The surfaces in Sol_3 with constant mean curvature have received special attention. In the search of explicit examples, some properties on the geometry of the surface have been assumed, for instance, to be invariant under some group of isometries ([9]) or to be a translation surface ([8]). Compact surfaces with constant mean curvature and non-empty boundary were studied in [7] and an integral representation formula for minimal surfaces in Sol_3 were obtained in [5] and generalized to constant mean curvature surfaces in [4].

Motivated by the extension of the Hopf theorem, an interesting problem posed in Sol_3 was whether exist closed surfaces with constant mean curvature. By the Alexandrov reflection method, an embedded compact surface with constant mean curvature must be of genus 0. In a first step, Daniel and Mira showed the existence of such surfaces for each value of H with $H > 1/\sqrt{3}$ ([4]). Finally Meeks extended the result for any real number H ([12]).

The main difficulty in order to obtain examples of surfaces with constant mean curvature is that Sol_3 has a small isometry group as for example, there are no rotations. The isometry group $\text{Iso}(\text{Sol}_3)$ has dimension 3 and the identity component is generated by the following two families of isometries (see [13]):

$$(x, y, z) \mapsto (\pm e^{-c} x + a, \pm e^c y + b, z + c)$$

$$(x, y, z) \mapsto (\pm e^{-c} y + a, \pm e^c x + b, -z + c),$$

where $a, b, c \in \mathbb{R}$.

We consider surfaces invariant under a 1-parameter group of isometries. In this ambient space, there are three types of such groups, namely, if $\{e_1 = (1, 0, 0), e_2 =$

$(0, 1, 0), e_3 = (0, 0, 1)\}$ is the canonical basis of \mathbb{R}^3 , each one of the above group is determined by the left translations by $te_i, 1 \leq i \leq 3$:

$$\mathcal{G}_i = \{p \in \text{Sol}_3 \mapsto te_i * p; t \in \mathbb{R}\}.$$

Constant mean curvature surfaces invariant by \mathcal{G}_1 and \mathcal{G}_2 were classified in [9]. In this work we focus in the third group \mathcal{G}_3 . Consider $L_t : \text{Sol}_3 \rightarrow \text{Sol}_3$ the left translation by $te_3 = (0, 0, t)$:

$$L_t(x, y, z) = (0, 0, t) * (x, y, z) = (e^{-t}x, e^ty, t + z).$$

A set $A \subset \text{Sol}_3$ will be called *3-invariant* if $L_t(A) \subset A$ for any $t \in \mathbb{R}$. Since we shall work with immersed surfaces in Sol_3 , we need to explicit the next definition:

Definition 1.1. *Let M be a surface and $\psi : M \rightarrow \text{Sol}_3$ an immersion. We say that ψ is a 3-invariant surface if it is the set $\psi(M)$.*

When the immersion is known in the context, we identify M with $\psi(M)$ and we abbreviate by saying that M is a 3-invariant surface.

In this article we study 3-invariant surfaces with constant mean curvature H . Since a 3-invariant surface is generated by a curve $\alpha = \alpha(s)$, the condition $H = ct$ expresses as an ordinary differential equation and for any initial condition we have a solution. However it is difficult to solve the equation $H = ct$ in all its generality, even in the minimal case ($H = 0$). Part of our study is supported in the use of a computer to make numerical pictures of the surfaces that in our case, it has been possible by using a symbolic program such as Mathematica. This contrast to with the minimal surfaces invariant by the groups \mathcal{G}_1 and \mathcal{G}_2 , where all solutions were done in [9].

The article is organized as follows. Section 2 takes up the most part of the article where we study 3-invariant surfaces with zero mean curvature. We shall obtain explicit examples and some geometric properties. At the end of Section 2, we will relate some of these minimal surfaces with spheres of Sol_3 with constant mean curvature. In Section 3, we consider surfaces with non-zero constant mean curvature. Due to the difficulty of the mean curvature equation, we give some graphics of such surfaces, showing by numerical computations the existence of generating curves that are simply closed curves. Last Section 4 is devoted to give explicit examples of constant Gauss curvature surfaces in Sol_3 .

2 Minimal surfaces

Let $\psi : M \rightarrow \text{Sol}_3$ be a 3-invariant surface. The orbit of a point (x, y, z) of Sol_3 is the curve $\{L_t(x, y, z); t \in \mathbb{R}\}$, which crosses once the plane $z = 0$ (at the time

$t = -z$). Therefore M parametrizes by the set of orbits of a curve α contained in the plane $z = 0$. This curve is called the *generating curve* of the surface. Let $\alpha(s) = (x(s), y(s), 0)$, $s \in I \subset \mathbb{R}$, be a parametrization of such curve. By the expression of L_t , the surface M has a single surface patch ψ given by

$$\psi(s, t) = (e^{-t}x(s), e^ty(s), t), \quad s \in I, t \in \mathbb{R}. \quad (1)$$

Remark 2.1. *The isometries $\phi(x, y, z) = (\pm e^{-c}y, \pm e^cx, -z + c)$, where c is a real parameter, carry 3-invariant surfaces into new 3-invariant surfaces. In fact, if α is the generating curve of a 3-invariant surface M , then $\phi \circ \alpha$ generates the 3-invariant surface $\phi(M)$. This is because $L_t \circ \phi = \phi \circ L_{-t}$.*

We compute the mean curvature of a given 3-invariant surface ψ . First, we recall that in Sol_3 there exists a left-invariant orthonormal frame $\{E_1, E_2, E_3\}$ given by

$$E_1 = e^{-z} \frac{\partial}{\partial x}, \quad E_2 = e^z \frac{\partial}{\partial y}, \quad E_3 = \frac{\partial}{\partial z}.$$

The Riemannian connection $\tilde{\nabla}$ of Sol_3 with respect to this frame is

$$\begin{aligned} \tilde{\nabla}_{E_1} E_1 &= -E_3 & \tilde{\nabla}_{E_1} E_2 &= 0 & \tilde{\nabla}_{E_1} E_3 &= E_1 \\ \tilde{\nabla}_{E_2} E_1 &= 0 & \tilde{\nabla}_{E_2} E_2 &= E_3 & \tilde{\nabla}_{E_2} E_3 &= -E_2 \\ \tilde{\nabla}_{E_3} E_1 &= 0 & \tilde{\nabla}_{E_3} E_2 &= 0 & \tilde{\nabla}_{E_3} E_3 &= 0. \end{aligned}$$

Consider N a Gauss map of M . The mean curvature H of ψ with respect to a local parametrization $\psi = \psi(s, t)$ is

$$H = \frac{1}{2} \frac{eG - fF + gE}{EG - F^2},$$

where, as usually, $\{E, F, G\}$ and $\{e, f, g\}$ stand for the coefficients of the first and second fundamental form, respectively, that is:

$$E = \langle \psi_s, \psi_s \rangle, \quad F = \langle \psi_s, \psi_t \rangle, \quad G = \langle \psi_t, \psi_t \rangle.$$

$$e = \langle N, \tilde{\nabla}_{\psi_s} \psi_s \rangle, \quad f = \langle N, \tilde{\nabla}_{\psi_s} \psi_t + \tilde{\nabla}_{\psi_t} \psi_s \rangle, \quad g = \langle N, \tilde{\nabla}_{\psi_t} \psi_t \rangle.$$

We point out here the difference of the coefficient f with respect to the Euclidean setting because the Lie bracket $[\psi_t, \psi_s]$ does not vanish necessarily in Sol_3 .

Without loss of generality, we assume that α is parametrized by the arc length. This means that if $\alpha' = (x', y', 0) = x'E_1 + y'E_2$, then $x'^2 + y'^2 = 1$. Let $\theta = \theta(s)$ be a differentiable function such that

$$x'(s) = \cos \theta(s), \quad y'(s) = \sin \theta(s). \quad (2)$$

The first derivatives of the parametrization ψ are:

$$\begin{aligned} \psi_s &= (-e^{-t}x', e^ty', 0) = \cos \theta E_1 + \sin \theta E_2 \\ \psi_t &= (-xe^{-t}, ye^t, 1) = -xE_1 + yE_2 + E_3. \end{aligned}$$

The first fundamental form is

$$E = 1, \quad F = -x \cos \theta + y \sin \theta, \quad G = 1 + x^2 + y^2.$$

Denote $W = EG - F^2$ the determinant of the first fundamental form. Then

$$W = 1 + A^2, \quad A = x \sin \theta + y \cos \theta.$$

Using the covariant derivatives $\tilde{\nabla}_{E_i} E_j$, we have

$$\begin{aligned} \tilde{\nabla}_{\psi_s} \psi_s &= -\theta' \sin \theta E_1 + \theta' \cos \theta E_2 - \cos(2\theta) E_3 \\ \tilde{\nabla}_{\psi_s} \psi_t &= (x \cos \theta + y \sin \theta) E_3 \\ \tilde{\nabla}_{\psi_t} \psi_t &= -xE_1 - yE_2 + (y^2 - x^2) E_3 \end{aligned}$$

A unit orthogonal vector field to M is

$$N = \frac{1}{\sqrt{W}} (\sin \theta E_1 - \cos \theta E_2 + AE_3).$$

Then the mean curvature of ψ computed with this choice of N is

$$H = \frac{\sin(2\theta)(-x \cos \theta + y \sin \theta) - (1 + x^2 + y^2)\theta'}{2W^{3/2}}. \quad (3)$$

Proposition 2.2. *The generating curve α of a 3-invariant minimal surface is obtained by a solution of the differential equations (2) and*

$$\theta' = \sin(2\theta) \frac{-x \cos \theta + y \sin \theta}{1 + x^2 + y^2}. \quad (4)$$

The solutions are obtained provided we give initial conditions:

$$x(0) = x_0, \quad y(0) = y_0, \quad \theta(0) = \theta_0. \quad (5)$$

We will denote the solution curve as $\alpha(s; x_0, y_0, \theta_0)$. A first geometric properties of the generating curve are:

Proposition 2.3. *Let $\alpha(s; x_0, y_0, \theta_0)$ be a generating curve of a 3-invariant minimal surface. Then the domain of α is \mathbb{R} . Moreover, a 3-invariant minimal surface can extend to be complete.*

Proof. The first assertion is a consequence that the derivatives x' , y' and θ' are bounded. For the second statement, we point out that $W = 1 + A^2 \geq 1$. The metric on the surface is the pullback $\psi^*(\langle, \rangle)$ of the metric \langle, \rangle of Sol_3 by the parametrization ψ . As ψ can extend to \mathbb{R}^2 , we compare $\psi^*(\langle, \rangle)$ with the Euclidean metric ds_0^2 of \mathbb{R}^2 by

$$ds_0^2 = (ds)^2 + (dt)^2 \leq \psi^*(\langle, \rangle).$$

This proves that the length of any divergent curve is unbounded, in particular, the surface is complete. \square

Once obtained the expression of the minimality condition of a 3-invariant surface, we begin obtaining solutions of (4). The first examples of such surfaces are the hyperbolic planes P of equation $y = 0$ and Q given by $x = 0$. These surfaces are invariant by the group \mathcal{G}_3 because

$$P = \mathcal{G}_3(\alpha(s)), \quad \alpha(s) = (s, 0, 0).$$

$$Q = \mathcal{G}_3(\alpha(s)), \quad \alpha(s) = (0, s, 0).$$

Other examples of 3-invariant minimal surfaces appear by choosing θ a constant function, such it occurs for the planes P and Q .

Proposition 2.4. *Given $x_0, y_0 \in \mathbb{R}$, we have the next solutions of (2)-(4):*

1. $\alpha(s; x_0, y_0, 0) = (s, 0, 0) + (x_0, y_0, 0)$. Here $\theta(s) = 0$ (type I).
2. $\alpha(s; x_0, y_0, \pi/2) = s(0, 1, 0) + (x_0, y_0, 0)$. Here $\theta(s) = \pi/2$ (type II).

Moreover (see Fig. 1):

1. The surfaces generated by the curves of type I (resp. type II) are invariant by the group \mathcal{G}_1 (resp. \mathcal{G}_2).

2. Both family of surfaces are ruled surfaces in the sense that they are generated by a 1-parameter family of geodesics.
3. There are two foliations of the Sol_3 space by 3-invariant minimal surfaces.

Proof. We only do the proof for the surfaces of type I. It is immediate that if $\theta_0 = 0$, then $x(s) = s + x_0$, $y(s) = y_0$ and $\theta(s) = 0$ is a solution of (2)-(4) with initial conditions (5). The parametrization of a such surface is $\psi(s, t) = (e^{-t}(s+x_0), y_0 e^t, t)$, which can be reparametrized as

$$\psi(u, v) = (u, y_0 e^v, v) = (0, y_0 e^v, v) + u(1, 0, 0), \quad u, v \in \mathbb{R}.$$

As a consequence, the surface is invariant by the 1-parameter group \mathcal{G}_1 because the isometries of \mathcal{G}_1 are the maps $(x, y, z) \mapsto (x + t, y, z)$, $t \in \mathbb{R}$. It is also trivial that the straight-lines $u \mapsto \psi(u, v)$ are geodesics of Sol_3 .

Finally, the foliations by 3-invariant minimal surfaces are generated by the family of curves $\{\alpha(s; 0, \lambda, 0); \lambda \in \mathbb{R}\}$ and $\{\alpha(s; \lambda, 0, \pi/2); \lambda \in \mathbb{R}\}$. \square

The generating curves of type I (resp. type II) are straight-lines parallel to the x -axis (resp. y -axis). In particular, the curve $\alpha(s; 0, 0, 0)$ (resp. $\alpha(s; 0, 0, \pi/2)$) generates the plane P (resp. the plane Q). On the other hand, the surfaces of type II can be obtained from the ones of type I after isometries of Sol_3 , exactly, by $\phi(x, y, z) = (y, x, -z)$.

We recall that all minimal surfaces in Sol_3 invariant under 1-parameter groups \mathcal{G}_1 and \mathcal{G}_2 were obtained in [9]. Besides the surfaces of type I and II, the classification completes with the horizontal planes $z = ct$.

We continue obtaining more solutions of (2)-(4). Again, we assume that θ is a constant function.

Proposition 2.5. *Given $x_0 \in \mathbb{R}$, the next curves generate 3-invariant minimal surfaces:*

1. $\alpha(s; x_0, x_0, \pi/4) = s(1, 1, 0) + (x_0, x_0, 0)$, where $\theta(s) = \pi/4$ (type III).
2. $\alpha(s; x_0, -x_0, -\pi/4) = s(1, -1, 0) + (x_0, -x_0, 0)$, where $\theta(s) = -\pi/4$ (type IV).

Moreover both surfaces are ruled, namely, the curves $s \mapsto \psi(s, t)$ are geodesic for any t .

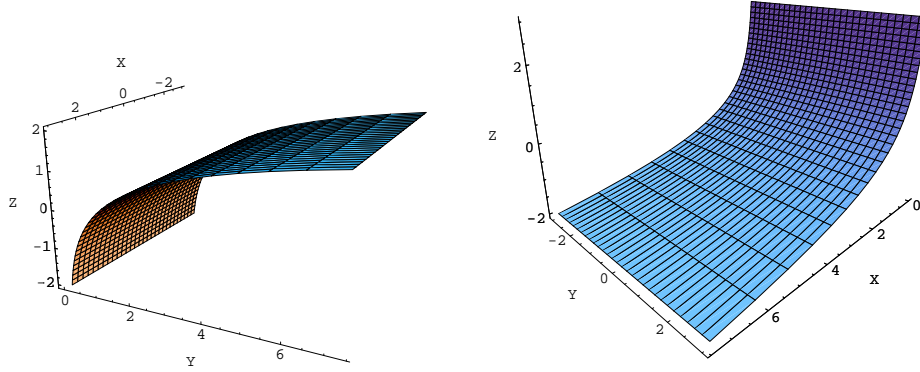


Figure 1: (left) A surface of type I parametrized $\psi(u, v) = (u, e^v, v)$; (right) a surface of type II given by $\psi(u, v) = (e^{-u}, v, u)$

In fact, the curve of type III (resp. type IV) is the straight-line $y = x$ (resp. $y = -x$) in the plane $z = 0$. The corresponding parametrizations of the surfaces are:

$$\psi(s, t) = (se^{-t}, se^t, t), \quad s, t \in \mathbb{R} \quad (\text{type III}).$$

$$\psi(s, t) = (se^{-t}, -se^t, t), \quad s, t \in \mathbb{R} \quad (\text{type IV}).$$

Proof. We only consider the surfaces of type III. Again, $x(s) = s + x_0$, $y(s) = s + x_0$ and $\theta(s) = \pi/4$ solve (2)-(4) with $\theta_0 = \pi/4$. It is also direct that the coordinate curve $t = ct$ are geodesics in the ambient space Sol_3 hence the surface is ruled. \square

The surfaces of type IV are obtained by the ones of type III after the isometry $\phi(x, y, z) = (y, x, -z)$. Except for those points of the z -axis, the surface of type III (resp. IV) is the Euclidean graph of the function $z = \frac{1}{2} \log \left| \frac{y}{x} \right|$ (resp. $z = \frac{1}{2} \log \left| \frac{x}{y} \right|$): see Fig. 2.

Remark 2.6. *The surfaces of type I, II, III and IV are the only ones that are obtained by choosing θ a constant function in the solutions of (4).*

From now, we consider now that $\theta' \neq 0$ at some point.

Proposition 2.7. *Let $\{x, y, \theta\}$ be a solution of (2)-(4). Then, up 2π multiple, the function θ never attains the values $0, \pi$ or $\pm\pi/2$ unless that α is of type I or type II. In particular, the generating curve is both a graph in the x -axis as in y -axis.*

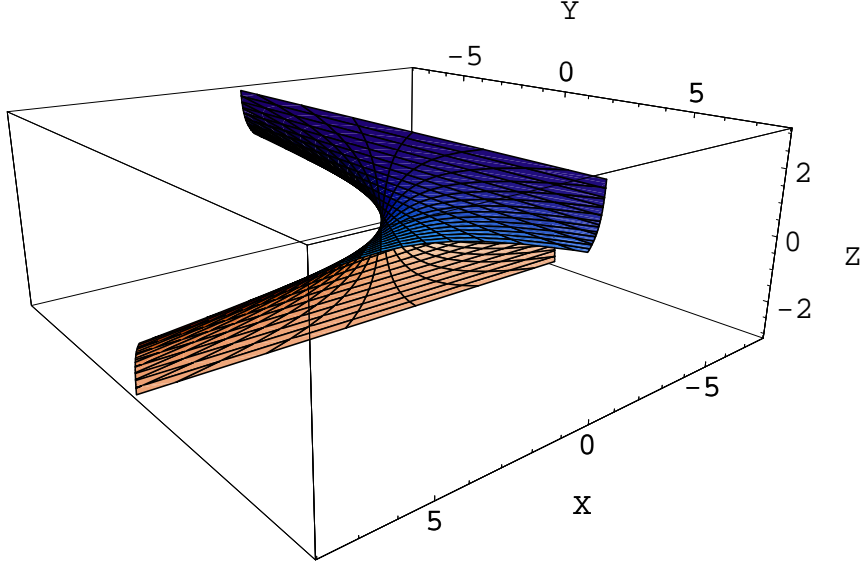


Figure 2: The surface of type III parametrized as $\psi(s, t) = (se^{-t}, se^t, t)$.

Proof. We only do the proof in the case that θ takes the value 0 at some point $s_0 \in \mathbb{R}$. Define the functions $\{x(s_0) + (s - s_0), y(s_0), 0\}$. This is a solution of (2)-(4) such that at $s = s_0$ the initial conditions are $\{x(s_0), y(s_0), 0\}$. Because the same occurs for the functions $\{x, y, \theta\}$, the uniqueness of ODE implies $x(s) = x(s_0) + (s - s_0)$, $y(s) = y(s_0)$ and $\theta(s) = 0$ around $s = s_0$. Since this occurs in the maximal interval of definition, we conclude that the curve α is parallel to the x -axis, that is, a curve of type I.

Except for the curves of type II and II, the functions $x'(s)$ and $y'(s)$ never vanish in the domain of α , which means that α is a graph on the x axis as well as a graph on the y -axis. \square

If α is a graph on the axis x , $\alpha(x) = (x, y(x), 0)$, a direct computation gives that the minimality condition (4) of the corresponding 3-invariant surface generated by α writes as:

$$y'' = 2y' \frac{yy' - x}{1 + x^2 + y^2}. \quad (6)$$

From this result and with appropriate isometries of Sol_3 , we can assume, without loss of generality, that the initial velocity vector $\alpha'(0)$ of the generating curve lies in the first quadrant, that is, $\theta(0) = \theta_0 \in (0, \pi/2)$. This condition on θ_0 will assumed

throughout the rest of the section.

The study of the solutions of (4) (or equivalently (6)) depends on the initial conditions. We have obtained good numerical pictures of solutions for many initial values. We carried out our graphics using Mathematica and the function `NDSolve` which numerically solves ordinary differential equations. On the basis of our computer pictures, the following result is illustrated in Figs. 3 and 4.

Experimental result 1. *Depending on the initial solutions and besides the surfaces I–IV, the graphics of solutions of (4) are of two types.*

1. *(Type A) The curve α is asymptotic to a quadrant of the plane $z = 0$ determined by two orthogonal Euclidean straight-lines parallel to the coordinate axis.*
2. *(Type B) The curve α is asymptotic to two straight-lines which are both parallel to the x axis or both parallel to the y -axis.*

In fact, if we fix the initial data $x(0)$ and $y(0)$ and we change the value of θ_0 , we expect that there exists a critical value θ_0 where the curve changes from one type to the other one. Assume that the function $y(x)$ describes the generating curve, that is, $y(x)$ is a solution of (6). For the curves of type A, the function $y(x)$ is convex or concave in all its domain. This would mean that the second derivative $y''(x)$ never vanish. On the other hand, the curves of type B would have only one inflection point, that is, the equation $y''(x) = 0$ would have only one solution x_0 and in each one of the intervals $(-\infty, x_0)$ and (x_0, ∞) the curve would change from convex (or concave) shape to concave (or convex).

In what follows, we assume that the starting point of the generating curve α is the origin $(0, 0)$ of \mathbb{R}^2 with initial velocity $\theta(0) = \theta_0 \in \mathbb{R}$, that, is,

$$x(0) = 0, \quad y(0) = 0, \quad \theta'(0) = \theta_0. \quad (7)$$

In particular, the z -axis is contained in the surface because it is the curve $\mathcal{G}_3(0, 0, 0)$.

Theorem 2.8. *Let α be a generating curve of a 3-invariant minimal surface satisfying the initial values (7).*

1. *The curve α is symmetric with respect to the origin $(0, 0)$.*
2. *The symmetry of α with respect to the line $y = x$ is a generating curve of other 3-invariant minimal surface.*

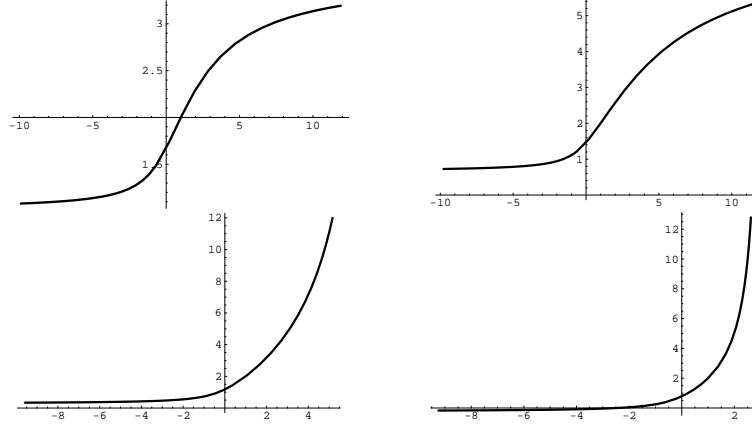


Figure 3: We fix the initial values x_0 and y_0 in (5) and we vary the value of θ_0 . Here $x(0) = 1$ and $y(0) = 2$. For θ_0 , we choose the values $\pi/10, \pi/6, \pi/4$ and $\pi/3$. At the beginning, the graphic presents one inflection point and it is asymptotic to two parallel lines. After a critical value for θ_0 , the shape of the curve changes to be asymptotic to a quadrant of the xy -plane.

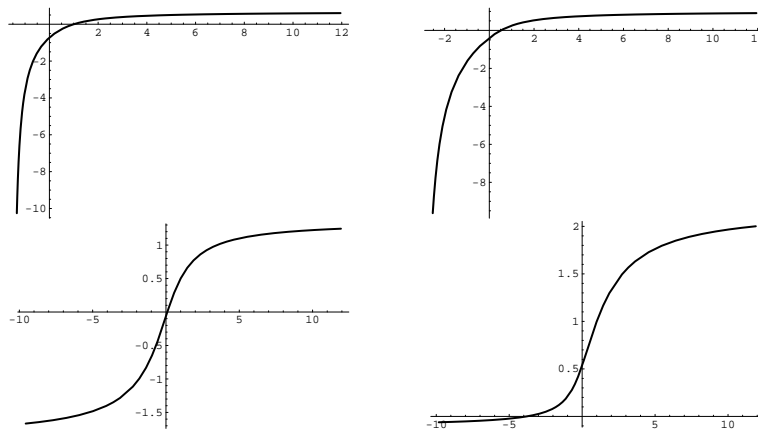


Figure 4: In this figure we have fixed x_0 and θ_0 in (5). Here $x_0 = 1$ and $\theta_0 = \pi/8$. The value $y(0)$ takes different values: $0, 1/4, 1/2$ and 1 . In the first two cases, the graphic is asymptotic to a quadrant of the xy -plane and for the next two values of y_0 , the graphic presents one inflection point and it is asymptotic to two horizontal straight-lines.

Proof. Denote $\alpha(s) = (x(s), y(s), 0)$ a solution of (2)-(4) with initial conditions (7).

1. The functions $\bar{x}(s) = -x(-s)$, $\bar{y}(s) = -y(-s)$ and $\bar{\theta}(s) = \theta(-s)$ are a solution of (2)-(4) with the same initial conditions (7). Let us apply uniqueness of ODE.
2. If $\theta_0 = \pi/4$, we know that α is the curve of type III, which agrees with its symmetry with respect to the line $y = x$. We assume $\theta_0 \in (0, \pi/4)$. The symmetry of α is the curve $\phi \circ \alpha$, where $\phi(x, y, z) = (y, x, -z)$. Now we use Remark 2.1.

□

Because the curve is symmetric with respect to the origin, then the curve is of type B depicted above, that is, the generating curve is asymptotic to two parallel straight-lines: see Fig. 5.

By the second statement of this result, we can assume that $\theta_0 \in (0, \pi/4)$.

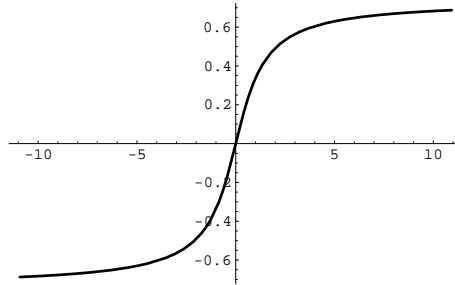


Figure 5: A generating curve for initial conditions $x(0) = y(0) = 0$ and $\theta(0) = \pi/8$. This curve is asymptotic to the lines $y \simeq \pm 0.736872$.

Theorem 2.9. *Assume $\theta_0 \in (0, \pi/4)$ and let $y = y(x)$ be the function which gives a generating curve α with initial conditions (7).*

1. *The function $y(x)$ is increasing in all its domain.*
2. *There exists only one inflection point, namely, $x = 0$. If $x > 0$ (resp. $x < 0$), then y is concave (resp. convex).*
3. *The curve α does not cross the line $y = x$.*

Proof. By the symmetry of α with respect to the origin, it suffices to do the study of the function $y(x)$ for $x \geq 0$. As $\theta_0 \in (0, \pi/4)$, then $0 < y'(0) < 1$. In particular, $y' > 0$ in some interval $(0, \epsilon)$, $\epsilon > 0$. If at some point, the derivative y' vanishes, Prop. 2.7 asserts that y is a constant function, which is impossible since $y'(0) \neq 0$. As a conclusion, $y' \neq 0$ and so, $y' > 0$ in $(0, \infty)$.

On the other hand, we know that $y''(0) = 0$. Define $h(x) = y(x)y'(x) - x$, which, by (6), gives us the sign of y'' . Then $h(0) = 0$ and $h'(0) = y'(0)^2 - 1 < 0$. This means that h is decreasing in some interval $(0, \delta)$, $\delta > 0$. From (6), $y'' < 0$ in $(0, \delta)$.

Claim. The function h is negative in $(0, \infty)$. On the contrary, let $x_1 > 0$ be the first point such that $h(x_1) = 0$. Then there exists $x_2 \in (0, x_1)$ such that $h'(x_2) = 0$, that is,

$$y''(x_2) = \frac{1 - y'(x_2)^2}{y(x_2)}.$$

As y' is decreasing on $(0, x_1)$, $y'(x_2) < 1$ and so, $y''(x_2) > 0$. This is a contradiction because y'' is negative in $(0, x_1)$. This proves the claim.

Then $y'' < 0$ in $(0, \infty)$ and y is concave. As a consequence, the function y' is decreasing in x : $y'(x) < y'(0) < 1$ on $(0, \infty)$. Near to $x = 0$ and for positive values of x , the curve α lies in the domain determined by the lines $y = x$ and $y = 0$. If the curve α attains the line $y = x$, by the intermediate value theorem, $y' = 1$ at some point of the interval $(0, \infty)$, which is impossible. \square

Recall that the preliminary numerical graphics of the solutions with initial conditions (7) suggest the following result:

Experimental result 2. *Assume that α is a solution of (2)-(4) with initial conditions (7) and $\theta(0) \in (0, \pi/4)$. Then the curve α lies between two parallel straight-lines to the x -axis, being α asymptotic to both ones.*

If the initial angle θ_0 moves in the range $(\pi/4, \pi/2)$, then α is asymptotic to two straight-lines parallel to the y -line.

Once assumed this result, let us consider α the generating curve of a 3-invariant minimal surface and let L_1 and L_2 be the two straight-lines in the plane $z = 0$ which α is asymptotic to. If we apply the group \mathcal{G}_3 to $L_1 \cup L_2$, we obtain two 3-invariant surfaces $P_i = \mathcal{G}_3(L_i)$, $i = 1, 2$, which lie separated by a constant distance. Then $P_1 \cup P_2$ determine a *slab* in Sol_3 . Because α lies between the strip of the plane $z = 0$ bounded by L_1 and L_2 , the surface that generates lies in the slab defined by P_1 and P_2 . Thus we conclude

Corollary 2.10. *In Sol_3 there exist many minimal surfaces contained in a slab.*

This result is the version in Sol_3 of the classical result of Jorge and Xavier about the existence of complete minimal surfaces in a slab in Euclidean space ([6])

We finish this section relating the minimal surfaces of type III and IV with the spheres of constant mean curvature and whose existence was proved in [4, 12]: see also [11] for any metric in Sol_3 . For each $H > 0$ there exists (up left translations) a unique sphere with constant mean curvature H which is embedded. We abbreviate by saying a H -sphere. Each one of these H -spheres admits certain symmetries of order 2. Exactly, at each point $p \in \text{Sol}_3$ there exists an orthonormal basis $\{v_1(p), v_2(p), v_3(p)\}$ such that the Ricci tensor diagonalizes. It is known that the Gauss map N of a H -sphere is a diffeomorphism on the unit sphere. Therefore in a given H -sphere there exists points where the Gauss map is a vector that diagonalizes the Ricci tensor, that is, $N(p) = \pm v_i(p)$, for some $1 \leq i \leq 3$. In fact, in the H -sphere there exist exactly 6 points. It is proved that the sphere is symmetric with respect to the rotation by angle π around the normal at p (see also [11]). At the origin $(0, 0, 0)$, the basis that diagonalizes the Ricci tensor is $\{(1/\sqrt{2})(E_1 + E_2), (1/\sqrt{2})(E_1 - E_2), E_3\}$.

Take H_n a sequence of positive numbers converging to 0 and the corresponding H_n -spheres M_n . For each M_n , there exists a point p_n where the unit normal vector is the translated of $(1/\sqrt{2})(E_1 - E_2)$. Let us do left translations of each M_n such that p_n agrees with the origin $p = (0, 0, 0)$ of \mathbb{R}^3 and thus $N_n(p) = (1/\sqrt{2})(E_1 - E_2)$. Then all spheres M_n are invariant with respect to the rotation by angle π around the normal at p that fixes p . In our case, the tangent plane $T_p M_n$ at p is spanned by $\{(1/\sqrt{2})(E_1 + E_2), E_3\}$ and the isometry that fixes $N_n(p)$ and which is a rotation by angle π in $T_p M_n$ is given by $\phi(x, y, z) = (-y, -x, -z)$. As $H_n \rightarrow 0$, the sequence M_n converges to a complete stable minimal surface M . Therefore, the limit surface, which is a minimal surface, inherits the same symmetry, that is, it is invariant by the action of ϕ . Of course, the normal vector of this surface at p is $(1/\sqrt{2})(E_1 - E_2)$.

On the other hand, the surface of type III whose parametrization is $\psi(s, t) = (se^{-t}, se^t, t)$, $s, t \in \mathbb{R}$ has the same tangent plane and normal vector as M_n at the point p . Moreover, this surface is invariant by the isometry ϕ because

$$\phi(\psi(s, t)) = (-se^t, -se^{-t}, -t) = \psi(-s, -t).$$

A similar process can do with the surfaces of type IV. As conclusion of the above reasoning,

Corollary 2.11. *The surfaces of type III and IV are candidate minimal surfaces to be the limit of a sequence of closed H_n -spheres, with $H_n \rightarrow 0$.*

Therefore it is an open question if there exists a sequence of H_n -spheres M_n , $H_n \rightarrow 0$, such that M_n converges to a surface of type III or IV.

3 Computer graphics of surfaces with constant mean curvature

We now consider 3-invariant surfaces with constant mean curvature H and $H \neq 0$. By using the value of H in (3) and that $W = 1 + A^2$, the equation that satisfies the angle function θ of the generating curve α is

$$\theta' = \frac{1}{1 + x^2 + y^2} \left(\sin(2\theta)(-x \cos \theta + y \sin \theta) - 2H(1 + A^2)^{3/2} \right).$$

In this case, the term $2H(1 + A^2)^{3/2}$ adds extra difficulties in the study of the geometric properties of the solution curves. By means of the computer, one can produce approximate solutions. In Fig. 6, we present two solutions for different initial data. In both cases, the initial angle θ_0 is the same, namely, $\theta_0 = 0$.

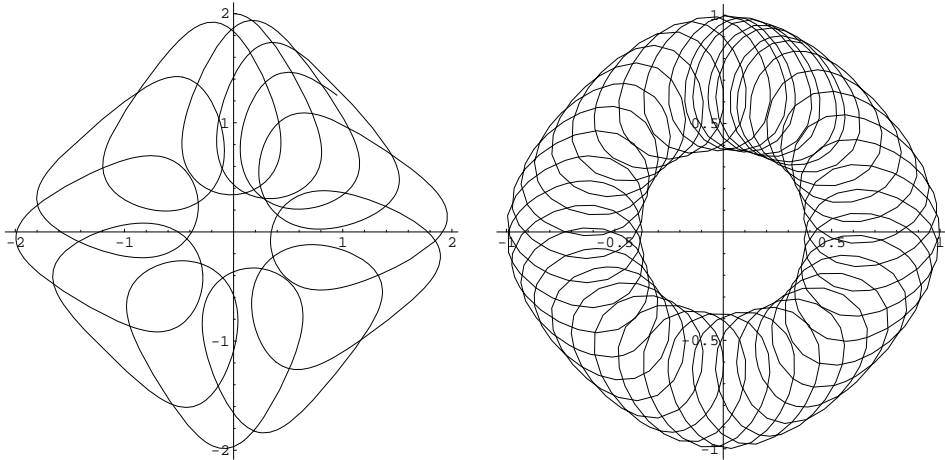


Figure 6: Two generating curves of 3-invariant surfaces with constant mean curvature: (left) $H = 1$ and $\alpha(0) = (0, 2, 0)$; (right) $H = 2$ and $\alpha = (0, 1, 0)$.

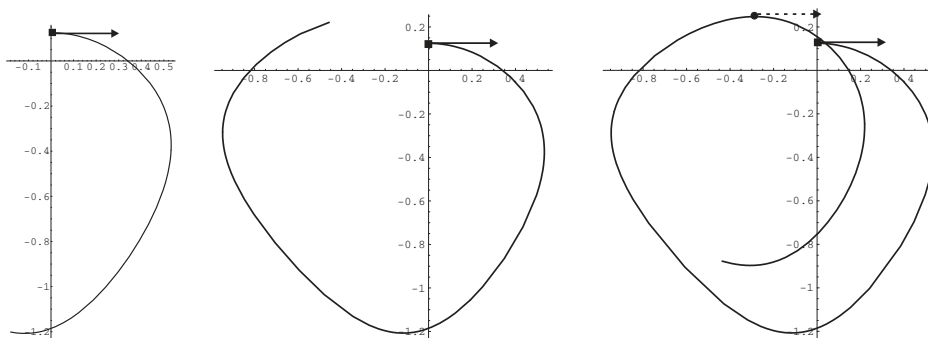
Although the variety of cases, our numerical computations give evidences, as can be seen in the the accompanying computer graphics. What we observe is the following. We fix the value of the (constant) mean curvature, namely, $H = 1$ and the initial velocity of the curve at $s = 0$, $\theta_0 = 0$, that is, $\alpha'(0)$ is the horizontal vector $(1, 0)$

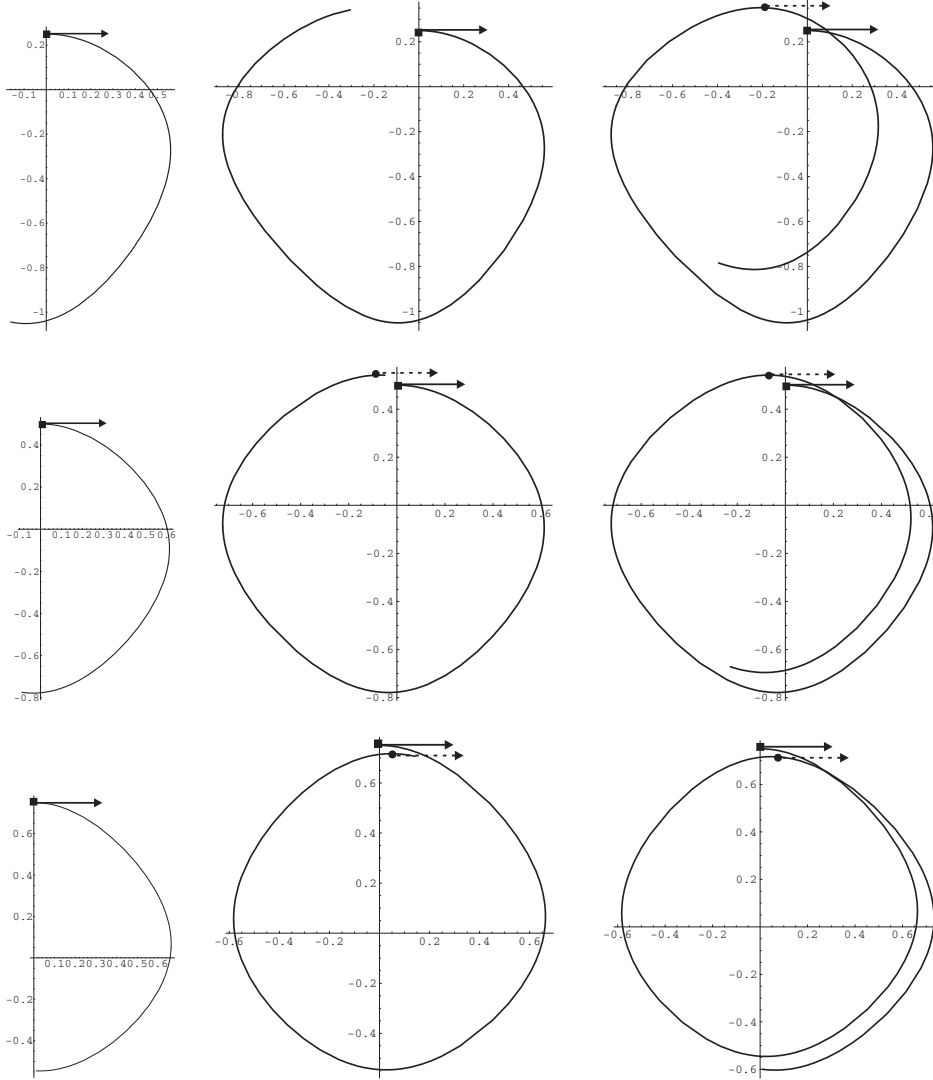
viewing the curve α in the xy -plane. With respect to the initial conditions (5), we take $x(0) = 0$ and $y(0) = y_0$, where y_0 is a one real parameter. This means that $\alpha(0)$ belongs to the y -axis.

The twelve curves pictured below were produced with different values of y_0 . The rows correspond to the values $1/8, 1/4, 1/2$ and $3/4$ of the parameter y_0 and the columns correspond to the values $2, 4$ and 6 of s_0 , where $[0, s_0]$ is the domain of the generating curve $\alpha = \alpha(s)$. We observe that as s_0 increases, the trace of α begins doing a turn in such way that α crosses the positive y -axis again, and many times as s_0 goes to ∞ . We have pointed in the graphic with a black small square the initial point of α , that is, $\alpha(0)$, and with a black small disc the following point where α has tangent horizontal vector $(1, 0)$. We denote this new point as $\alpha(s_1; y_0) = (x(s_1; y_0), y(s_1; y_0))$ where s_1 indicates the time where it reaches this point and y_0 the dependence on y_0 . The corresponding velocity vectors at $s = 0$ and $s = s_1$ appear as a non-dashed and dashed arrow, respectively. For values s_0 close to 0 , $x(s_1; y_0) < 0$ and $y(s_1; y_0) > y_0$. However, as y_0 increases, the function $x(s_1; y_0)$ is increasing on y_0 and $y(s_1; y_0)$ is decreasing on y_0 . The pictures indicate that we attain a value y_0^* such that

$$\begin{cases} \alpha(s_1; y_0^*) = \alpha(0; y_0^*) \\ \alpha'(s_1; y_0^*) = \alpha'(0; y_0^*). \end{cases}$$

This would say that there exists a *closed simply generating curve*.





Experimental result 3. *Let $H \neq 0$. Then there exists a simply closed curve in the xy -plane which is the generating curve of a 3-invariant surface with constant mean curvature.*

Although we do not know a proof of this statement, we have carried out different numerical graphics that seem to indicate that this result is true. Moreover, the pictures obtained of the closed solution suggest that the curve is also symmetric with respect to the straight-lines $y = \pm x$. For example, in Fig. 7, the intersection points with the coordinate points are equidistant from the origin. Recall that at the origin of Sol_3 , the directions of both straight-lines together the vertical direction $(0, 0, 1)$ correspond with the directions where the Ricci tensor diagonalizes.

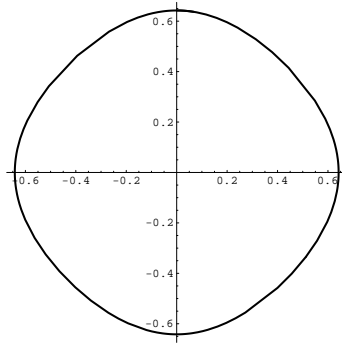


Figure 7: The generating curve of a 3-invariant surface with $H = 1$. The curve is closed. Here the initial conditions are $x(0) = 0$ and $y(0) \simeq 0.6425$. We remark the symmetry of α with respect to straight-line $y = \pm x$.

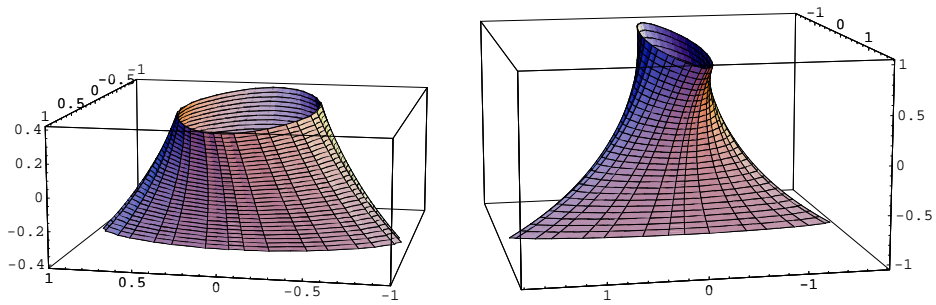


Figure 8: Different views of the 3-invariant surface obtained by the generating curve of Fig. 7.

4 Surfaces with constant Gauss curvature

In this section we present an example of a 3-invariant surface in Sol_3 with zero Gauss curvature. Surfaces in the homogenous space Sol_3 with constant Gauss curvature have not received yet the interest of geometers. In this sense, the surfaces with constant Gauss curvature K which are invariant by the groups \mathcal{G}_1 and \mathcal{G}_2 were studied in [9] and the corresponding surfaces with $K = 0$ were all obtained. The invariance of the surface makes that the Gauss curvature equation converts into an ordinary differential equation which can be partially studied.

In our context of 3-invariant surfaces, we compute the Gauss curvature K . The expression of K is

$$K = K_{ext} + K(\psi_s \wedge \psi_t), \quad (8)$$

where K_{ext} is the extrinsic curvature of the surface and $K(\psi_s \wedge \psi_t)$ is the sectional curvature of the tangent plane at $\psi(s, t)$. The value K_{ext} is

$$K_{ext} = \frac{eg - f^2}{EG - F^2}$$

and the sectional curvature $K(\psi_s \wedge \psi_t)$ is given by

$$K(\psi_s \wedge \psi_t) = \frac{\langle \tilde{\nabla}_{\psi_s} \tilde{\nabla}_{\psi_t} \psi_t - \tilde{\nabla}_{\psi_t} \tilde{\nabla}_{\psi_s} \psi_t - \tilde{\nabla}_{[\psi_s, \psi_t]} \psi_t, \psi_s \rangle}{W}.$$

From the computations of the first and second fundamental form realized in Sect. 2, we have

$$K_{ext} = -\frac{A^2(x \cos \theta + y \sin \theta)^2 + (\theta' + A \cos(2\theta))(-x \sin \theta + y \cos \theta + A(-x^2 + y^2))}{W^2} \quad (9)$$

On the other hand,

$$\begin{aligned} \langle \tilde{\nabla}_{\psi_s} \tilde{\nabla}_{\psi_t} \psi_t, \psi_s \rangle &= -1 + \cos(2\theta)(y^2 - x^2) \\ \langle \tilde{\nabla}_{\psi_t} \tilde{\nabla}_{\psi_s} \psi_t, \psi_s \rangle &= -(x \cos \theta + y \sin \theta)^2 \\ [\psi_s, \psi_t] &= 0. \end{aligned}$$

Hence we have

$$K(\psi_s \wedge \psi_t) = \frac{-1 + A^2}{W}. \quad (10)$$

Thus, by combining (9) and (10), we obtain from (8)

$$K = \frac{-(\theta' + A \cos(2\theta))(-x \sin \theta + y \cos \theta + A(-x^2 + y^2)) - 1 - \cos(2\theta)(x^2 - y^2)A^2}{W^2}. \quad (11)$$

We search 3-invariant surfaces in Sol_3 with constant Gauss curvature. Assume that θ is a constant function, $\theta(s) = \theta_0$. When the value of θ_0 is 0 or $\pi/2$, we obtain from the expression of the Gauss curvature:

1. If $\theta_0 = 0$, we get $K = -\frac{1}{1+y_0^2}$. Thus the surface has constant negative Gauss curvature. Because $\theta(s) = 0$, the generating curve is a straight-line parallel to the x -axis. In the particular case that $y_0 = 0$, the surface is the hyperbolic plane P , which it has Gauss curvature $K = -1$, as it is well known.
2. If $\theta_0 = \pi/2$, the result is similar, but now $K = -\frac{1}{1+x_0^2}$. Now the generating curves are straight-lines parallel to the y -axis and when $x_0 = 0$, the surface is the plane Q .

These surfaces are the surfaces of type I and II that appeared in Prop. 2.4, which we know that they are also invariant by the 1-parameter groups \mathcal{G}_1 and \mathcal{G}_2 . Finally, these surfaces appeared in the classification given in [9] of constant Gauss curvature surfaces invariant by both groups.

More surprisingly is that we can obtain a surface with zero Gauss curvature. From (11), if $K = 0$, the generating curve satisfies (2) and

$$(\theta' + A \cos(2\theta))(-x \sin \theta + y \cos \theta + A(-x^2 + y^2)) + 1 + \cos(2\theta)(x^2 - y^2)A^2 = 0. \quad (12)$$

Although Eq. (12) is a bit cumbersome, a solution of this equations is given by an Euclidean circle in the xy -plane centered at the origin. Exactly, if $r > 0$, take

$$x(s) = r \sin\left(\frac{s}{r}\right), \quad y(s) = -r \cos\left(\frac{s}{r}\right), \quad \theta(s) = \frac{s}{r}.$$

Then it is direct that $\{x(s), y(s), \theta(s)\}$ is a solution of (2)-(12) with initial conditions $x(0) = 0$, $y(0) = -r$ and $\theta(0) = 0$. A picture of the corresponding surface appears in Fig. 9.

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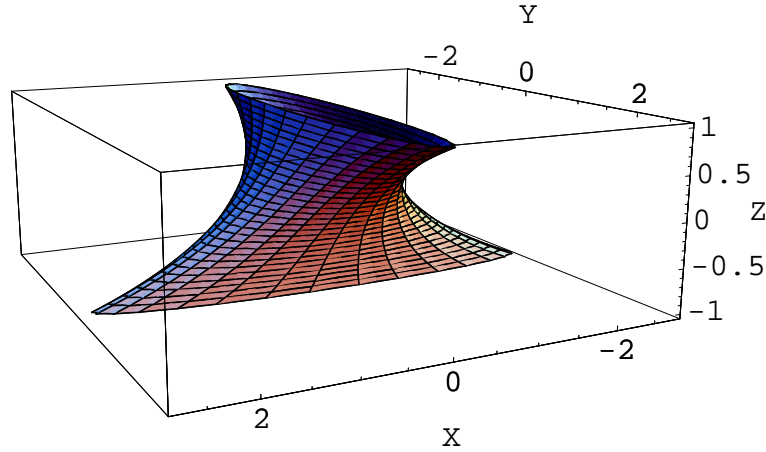


Figure 9: A 3-invariant surface with zero Gauss curvature, which it is generated by the curve $\alpha(s) = (\sin(s), -\cos(s), 0)$.

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