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HELICOIDAL SURFACES WITH POINTWISE 1-TYPE GAUSS MAP

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ABSTRACT. The helicoidal surfaces with pointwise 1-type or harmonic gauss map in Euclidean 3-space are studied. The notion of pointwise 1-type Gauss map is a generalization of usual sense of 1-type Gauss map. In particular, we prove that an ordinary helicoid is the only genuine helicoidal surface of polynomial kind with pointwise 1-type Gauss map of the first kind and a right cone is the only rational helicoidal surface with pointwise 1-type Gauss map of the second kind. Also, we give a characterization of rational helicoidal surface with harmonic or pointwise 1-type Gauss map.

1. Introduction

Since the late 1970's, the study of submanifolds of Euclidean space or pseudo-Euclidean space with the notion of finite type immersion has been extensively carried out. An isometric immersion $\mathbf{x} : M \to \mathbb{E}^m$ of a submanifold M in Euclidean m-space $\mathbb{E}^m$ is said to be of finite type if $\mathbf{x}$ identified with the position vector field of M in $\mathbb{E}^m$ can be expressed as a finite sum of $\mathbb{E}^m$-valued eigenfunctions of the Laplacian Δ of M, acting on $\mathbb{E}^m$-valued functions (cf. [4, 5]). Granted, this notion of finite type immersion is naturally extended to any differential maps defined on the submanifold M, in particular, to the Gauss map G on M in Euclidean space ([8]). Thus, if a submanifold M of Euclidean space has 1-type Gauss map G, then G satisfies $\Delta G = \lambda(G + C)$ for some $\lambda \in \mathbb{R}$ and some constant vector C (cf. [1, 2, 3, 11]). However, the Laplacian of the Gauss map of some typical well-known surfaces such as a helicoid, a catenoid and a right cone in Euclidean 3-space $\mathbb{E}^3$ take a somewhat different form; namely, $\Delta G = f(G + C)$ for some non-constant function f and some constant vector C. Therefore, it is worth studying the class of solution surfaces satisfying such an equation.

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A submanifold M of a Euclidean space $\mathbb{E}^m$ is said to have *pointwise 1-type Gauss map* if its Gauss map G satisfies

$$\Delta G = f(G + C) \tag{1.1}$$

for some non-zero smooth function f on M and a constant vector C. A pointwise 1-type Gauss map is called *proper* if the function f defined by (1.1) is non-constant. A submanifold with pointwise 1-type Gauss map is said to be *of the first kind* if the vector C in (1.1) is zero vector. Otherwise, the pointwise 1-type Gauss map is said to be *of the second kind* ([6, 9, 12, 13]).

In [9], two of the present authors characterized the minimal helicoid in terms of pointwise 1-type Gauss map of the first kind. Also, together with B. Y. Chen, they proved that surfaces of revolution with pointwise 1-type Gauss map of the first kind coincides with surfaces of revolution with constant mean curvature. Moreover, they characterized the rational surfaces of revolution with pointwise 1-type Gauss map ([6]).

On the other hand, the class of helicoidal surfaces includes surfaces of revolution and ordinary helicoid. Thus, we need to consider the helicoidal surfaces in $\mathbb{E}^3$ with pointwise 1-type Gauss map.

In this paper, we study the helicoidal surface of polynomial kind with pointwise 1-type Gauss map. In particular, we prove that an ordinary helicoid is the only genuine helicoidal surface of polynomial kind with pointwise 1-type Gauss map of the first kind. Also, we characterize helicoidal surfaces with pointwise 1-type Gauss map of rational kind. As a result, we show that a right cone is the only rational helicoidal surface with pointwise 1-type Gauss map of the second kind.

Here, we give examples of helicoidal surfaces with proper pointwise 1-type Gauss map of the first kind and of the second kind, respectively.

Example 1.1. An ordinary helicoid is, up to a rigid motion, parameterized by

$$x(t,\theta) = (t\cos\theta, t\sin\theta, h\theta), \quad h \neq 0$$

with respect to a surface patch (t,θ). Then the Gauss map is given by

$$G = \frac{1}{\sqrt{h^2+t^2}}(h\sin\theta, -h\cos\theta, t)$$

and the Laplacian ΔG of the Gauss map G is obtained as

$$\Delta G = \frac{2h^2}{(h^2+t^2)^2}G.$$

Therefore, an ordinary helicoid has pointwise 1-type Gauss map of the first kind.

Example 1.2. Consider the right cone C_a which is parameterized by

$$x(u,v) = (v\cos u, v\sin u, av), \quad a \geq 0.$$

Then the Gauss map G and its Laplacian ΔG are respectively given by

$$G = \frac{1}{\sqrt{1+a^2}}(a\cos u, a\sin u, -1)$$

and

$$\Delta G = \frac{1}{v^2}\Big(G + \Big(0, 0, \frac{1}{\sqrt{1+a^2}}\Big)\Big).$$

It implies that the right cone has pointwise 1-type Gauss map of the second kind.

2. Preliminaries

Let M be a surface of the Euclidean 3-space $\mathbb{E}^3$ (surfaces are assumed to be smooth and connected unless otherwise mentioned). The map $G : M \to S^2 \subset \mathbb{E}^3$ which sends each point of M to the unit normal vector to M at the point is called the *Gauss map* of the surface M, where S^2 is the unit sphere in $\mathbb{E}^3$ centered at the origin. For the matrix $\tilde{g} = (\tilde{g}_{ij})$ consisting of the components of the metric on M, we denote by $\tilde{g}^{-1} = (\tilde{g}^{ij})$ (resp. $\mathcal{G}$) the inverse matrix (resp. the determinant) of the matrix $(\tilde{g}_{ij})$. The Laplacian Δ on M is, in turn, given by

$$\Delta = -\frac{1}{\sqrt{\mathcal{G}}}\sum_{i,j}\frac{\partial}{\partial x^i}\Big(\sqrt{\mathcal{G}}\,\tilde{g}^{ij}\frac{\partial}{\partial x^j}\Big). \tag{2.1}$$

Let (x, y, z) be the standard coordinates of $\mathbb{E}^3$. Consider the one-parameter subgroup $g_t : \mathbb{E}^3 \to \mathbb{E}^3$ is given by

$$g_t(x, y, z) = (x\cos t + y\sin t, -x\sin t + y\cos t, z + ht), \quad t \in \mathbb{R},$$

where h is a constant.

The rigid motion g_t is called the *helicoidal motion* with axis Oz and pitch h. A *helicoidal surface* with axis Oz and pitch h is a surface which is invariant by one-parameter subgroup g_t.

The helicoidal surface M in $\mathbb{E}^3$ is then parameterized by

$$x(\rho, \varphi) = (\rho\cos\varphi, \rho\sin\varphi, \alpha(\rho) + h\varphi), \tag{2.2}$$

where (ρ, φ) is the polar coordinates in the xy-plane with the origin of φ as x-axis and the curve $z = \alpha(\rho)$ determines the profile of the surface M.

In such a case that ρ in the above parametrization is not constant, the helicoidal surface M can be parameterized by

$$x(t, \theta) = (t\cos\theta, t\sin\theta, \lambda(t) + h\theta)$$

in terms of the adapted coordinates (t, θ) for some function λ of t. If $h = 0$, then M is the surface of revolution. For this reason the helicoidal surface except the surface of revolution, that is, $h \neq 0$, is called the *genuine helicoidal surface.*

Furthermore, a helicoidal surface M is said to be *of polynomial kind* if $\lambda(t)$ is a polynomial and *of rational kind* if $\lambda(t)$ is a rational function. A helicoidal surface of rational kind is simply called a *rational helicoidal surface* (cf. [7]).

From now on, we handle non-trivial helicoidal surfaces, i.e., ρ is not constant in (2.2) unless otherwise stated.

3. Helicoidal surfaces with pointwise 1-type Gauss map

We now consider a helicoidal surface M in $\mathbb{E}^3$ parameterized by

$$x(t,\theta) = (t\cos\theta, t\sin\theta, \lambda(t) + h\theta) \tag{3.1}$$

for the adapted coordinate system (t,θ). A direct computation shows that the Gauss map G of M is given by

$$G = \frac{1}{\sqrt{h^2 + (1+\lambda'^2(t))t^2}}(h\sin\theta - t\lambda'(t)\cos\theta, -h\cos\theta - t\lambda'(t)\sin\theta, t) \tag{3.2}$$

and the Laplacian ΔG of the Gauss map G satisfies

(3.3)
$$\Delta G = -\frac{1}{(h^2 + (1+\lambda'^2(t))t^2)^{\frac{7}{2}}}(A(t)\cos\theta + B(t)\sin\theta, A(t)\sin\theta - B(t)\cos\theta, D(t)),$$

where we have put

(3.4)
$$\begin{aligned}
A(t) = &- 3h^6\lambda'' + (6h^4\lambda' + 8h^4\lambda'^3 - h^6\lambda''')t + (7h^4\lambda'^2\lambda'' - 7h^4\lambda'')t^2 \\
&+ (7h^2\lambda' + 12h^2\lambda'^3 + 5h^2\lambda'^5 - 3h^4\lambda''' + 4h^4\lambda'\lambda''^2 - h^4\lambda'^2\lambda''')t^3 \\
&+ (-5h^2\lambda'' + 6h^2\lambda'^2\lambda'' + 2h^2\lambda'^4\lambda'')t^4 \\
&+ (8h^2\lambda'\lambda''^2 + \lambda'(1+\lambda'^2)^3 - 3h^2\lambda''' - 2h^2\lambda'^2\lambda''')t^5 - (\lambda'' + \lambda'^2\lambda'')t^6 \\
&+ (4\lambda'\lambda''^2 - \lambda''' - \lambda'^2\lambda''')t^7,
\end{aligned}$$

(3.5)
$$\begin{aligned}
B(t) = &- 2h^5 - 4h^5\lambda'^2 - 7h^5\lambda'\lambda''t - (2h^3 + 2h^3\lambda'^2 + h^5\lambda''^2 + h^5\lambda'\lambda''')t^2 \\
&+ (h^3\lambda'^3\lambda'' - 8h^3\lambda'\lambda'')t^3 \\
&+ (3h^3\lambda'^2\lambda''^2 - 2h^3\lambda'\lambda''' - 2h^3\lambda''^2 - h^3\lambda'^3\lambda''')t^4 \\
&- (h\lambda'\lambda'' + h\lambda'^3\lambda'')t^5 + (3h\lambda'^2\lambda''^2 - h\lambda''^2 - h\lambda'\lambda''' - h\lambda'^3\lambda''')t^6,
\end{aligned}$$

(3.6)
$$\begin{aligned}
D(t) = &(-2h^4 - 4h^4\lambda'^2)t - 7h^4\lambda'\lambda''t^2 - (2h^2 + 2h^2\lambda'^2 + h^4\lambda''^2 + h^4\lambda'\lambda''')t^3 \\
&+ (h^2\lambda'^3\lambda'' - 8h^2\lambda'\lambda'')t^4 \\
&+ (3h^2\lambda'^2\lambda''^2 - 2h^2\lambda'\lambda''' - 2h^2\lambda''^2 - h^2\lambda'^3\lambda''')t^5 \\
&- (\lambda'\lambda'' + \lambda'^3\lambda'')t^6 + (3\lambda'^2\lambda''^2 - \lambda''^2 - \lambda'\lambda''' - \lambda'^3\lambda''')t^7.
\end{aligned}$$

We now prove

Lemma 3.1. *Let M be a helicoidal surface in $\mathbb{E}^3$. If the Gauss map G of M satisfies the equation $\Delta G = f(G + C)$ for some smooth function f and a constant vector C, then either the Gauss map is harmonic, that is, $\Delta G = 0$ or the function f defined by (1.1) depends only on t and the vector C in (1.1) is parallel to the axis of the helicoidal surface.*

Proof. If M has pointwise 1-type Gauss map, then (1.1) holds for some function f and some vector C. When the Gauss map is not harmonic, (1.1), (3.2) and (3.3) imply that f depends only on t, that is, f is independent of the parameter θ. Moreover we obtain

(3.7)
$$A(t) = ft\lambda'(t)(h^2 + (1 + \lambda'^2(t))t^2)^3 \quad \text{and} \quad B(t) = -fh(h^2 + (1 + \lambda'^2(t))t^2)^3.$$

It implies that the first two components of C are zero, that is, $C = (0, 0, c)$ for some constant c. □

Now we suppose that M is a genuine helicoidal surface in $\mathbb{E}^3$ with pointwise 1-type Gauss map, i.e., $h \neq 0$. Then, (3.3) and Lemma 3.1 give

$$D(t) = -f(h^2 + (1 + \lambda'^2(t))t^2)^3\Big(t + c\sqrt{h^2 + (1 + \lambda'^2(t))t^2}\Big). \tag{3.8}$$

By direct computation, (3.5), (3.6), (3.7) and (3.8) imply

$$fc\Big(h^2 + (1 + \lambda'^2(t))t^2\Big)^{\frac{7}{2}} = 0$$

on an open set $U = \{p \in M \mid f(p) \neq 0\}$. Since $h^2 + (1 + \lambda'^2(t))t^2 \neq 0$, we conclude that the third component of the constant vector C is zero, i.e., $c = 0$ and so C is zero vector. Thus we have

Theorem 3.2. *Let M be a genuine helicoidal surface in Euclidean 3-space $\mathbb{E}^3$. If M has pointwise 1-type Gauss map, then it is of the first kind, that is, it satisfies the condition $\Delta G = fG$ for some smooth function f.*

By using Lemma 5.1 in [13] and calculating the Laplacian of the Gauss map G, we get

Proposition 3.3. *Let M be a surface in Euclidean 3-space $\mathbb{E}^3$. Then, the Gauss map G is of pointwise 1-type of the first kind or harmonic if and only if M has constant mean curvature.*

Thus, we have

Corollary 3.4. *Let M be a genuine helicoidal surface in Euclidean 3-space. Then, M has pointwise 1-type or harmonic Gauss map if and only if M has constant mean curvature.*

Remark. A helicoidal surface with constant mean curvature was studied by M. P. do Carmo, M. Dajczer, and W. Seaman (cf. [10, 14]).

We now consider the case of a genuine helicoidal surface with pointwise 1-type Gauss map, that is, $h \neq 0$. Applying (3.7) and Theorem 3.2, we have

$$A(t) + \frac{B(t)}{h} t\lambda'(t) = 0.$$

A straightforward computation with the help of (3.4) and (3.5) gives the following equation :

(3.9)

$$\begin{aligned}
&-3h^6\lambda'' + (4h^4\lambda' + 4h^4\lambda'^3 - h^6\lambda''')t - 7h^4\lambda''t^2 \\
&+ (5h^2\lambda' + 10h^2\lambda'^3 + 5h^2\lambda'^5 - 3h^4\lambda''' + 3h^4\lambda'\lambda''^2 - 2h^4\lambda'^2\lambda''')t^3 \\
&- (5h^2\lambda'' + 2h^2\lambda'^2\lambda'' - 3h^2\lambda'^4\lambda'')t^4 + (6h^2\lambda'\lambda''^2 + \lambda' + 3\lambda'^3 + 3\lambda'^5 + \lambda'^7 \\
&- 3h^2\lambda''' - 4h^2\lambda'^2\lambda''' + 3h^2\lambda'^3\lambda''^2 - h^2\lambda'^4\lambda''')t^5 - (\lambda'' + 2\lambda'^2\lambda'' + \lambda'^4\lambda'')t^6 \\
&+ (3\lambda'\lambda''^2 - \lambda''' - 2\lambda'^2\lambda''' + 3\lambda'^3\lambda''^2 - \lambda'^4\lambda''')t^7 = 0.
\end{aligned}$$

Suppose that M is of polynomial kind, that is, $\lambda(t)$ is a polynomial in t. Denote by $\deg \lambda(t)$ the degree of $\lambda(t)$.

If $\deg \lambda(t) \geq 2$, then the term $\lambda'^7(t)t^5$ in (3.9) has the highest degree in t and so the leading coefficient of $\lambda'^7(t)t^5$ must be zero, which is a contradiction.

Now, we assume that $\deg \lambda(t) = 1$. We may put $\lambda(t) = at + b$ for some nonzero constant a and $b \in \mathbb{R}$. If we make use of (3.9) again, we must have $a = 0$, which is also a contradiction. Therefore λ is a constant. Putting together (3.5) and (3.7), we obtain $f(t) = \frac{2h^2}{(h^2+t^2)^2}$ (See Example 1.1.).

Thus, the parametrization of M is reduced to

$$x(t,\theta) = (t\cos\theta, t\sin\theta, a + h\theta), \quad h \neq 0$$

for some constant a. It is nothing but part of an ordinary helicoid. Conversely, by straightforward computation, an ordinary helicoid has pointwise 1-type Gauss map of the first kind. Consequently, we have

Theorem 3.5. *A genuine helicoidal surface of polynomial kind has pointwise 1-type Gauss map if and only if it is part of an ordinary helicoid.*

Next, we suppose that M is of rational kind. Then the function $\lambda(t)$ in (3.1) and $\lambda'(t)$ are both rational functions in t. If $\lambda'(t)$ is not a polynomial, we may put

$$\lambda'(t) = p(t) + \frac{r(t)}{q(t)}, \tag{3.10}$$

where $p(t)$ is a polynomial in t and the polynomials $r(t)$ and $q(t)$ are relatively prime. Let $\deg p(t) = l$, $\deg r(t) = n$ and $\deg q(t) = m$ with $\deg r(t) = n < \deg q(t) = m$, where l and n are some nonnegative integers. Putting (3.10) in (3.9) and multiplying $q^7(t)$ with thus obtained equation, we get a polynomial in t in the left hand side of (3.9). A long algebraic computation shows that the degree of the polynomial is $7l + 7m + 5$. So the leading coefficient must be zero. But it contradicts the character of functions p and q. Consequently, we have

Theorem 3.6. *There is no genuine rational helicoidal surface with pointwise 1-type Gauss map except that of polynomial kind.*

Combining Theorem 3.5 and Theorem 3.6, we get

Theorem 3.7 (Characterization). *A genuine rational helicoidal surface M with pointwise 1-type Gauss map if and only if M is part of an ordinary helicoid.*

On the other hand, in [6], it was proved that rational surfaces of revolution with pointwise 1-type Gauss map is part of a circular cylinder or a right cone. Part of a circular cylinder has 1-type Gauss map in usual sense that means the function f in (1.1) is constant. In particular, a circular cylinder is a trivial helicoidal surface, i.e., ρ is constant in (2.2). Consequently, we give the following

Theorem 3.8 (Characterization). *A rational helicoidal surface M with pointwise 1-type Gauss map if and only if M is part of a circular cylinder, a right cone or an ordinary helicoid.*

From this, we immediately get the following

Corollary 3.9. *A right cone is the only rational helicoidal surface with pointwise 1-type Gauss map of the second kind.*

4. Helicoidal surfaces with harmonic Gauss map

In this section, we consider the helicoidal surfaces with harmonic Gauss map. We now suppose that M is a helicoidal surface of polynomial kind in $\mathbb{E}^3$, which has harmonic Gauss map, that is, its Gauss map G satisfies $\Delta G = 0$. Then the polynomials $A(t), B(t)$ and $D(t)$ in (3.3) are vanishing.

If $\deg \lambda'(t) \geq 1$, then the term ${\lambda'}^7(t)t^5$ of $A(t)$ has the highest degree in t and its leading coefficient must be zero, which is a contradiction. Therefore $\deg \lambda'(t) = 0$, i.e., λ' is constant. A direct computation gives that $A(t) = 0$ implies that $\lambda'(t) = 0$. Hence, λ is constant. Moreover with the help of (3.5), $B(t) = 0$ implies that $h = 0$. Thus, M is a non-genuine helicoidal surface, i.e., a surface of revolution. Therefore, the parametrization of M is reduced to

$$x(t, \theta) = (t \cos\theta, t \sin\theta, \lambda)$$

for some constant λ, which means M is part of a plane. Consequently, we have

Theorem 4.1. *Let M be a helicoidal surface of polynomial kind in $\mathbb{E}^3$. Then M has harmonic Gauss map if and only of M is part of a plane.*

Next we suppose that M is a helicoidal surface of rational kind in $\mathbb{E}^3$ with harmonic Gauss map. Then $\lambda(t)$ and $\lambda'(t)$ are rational functions in t. If $\lambda'(t)$ is not a polynomial, we may put $\lambda'(t) = p(t) + \frac{r(t)}{q(t)}$, where $p(t)$ is a polynomial in t, $r(t)$ and $q(t)$ are relatively prime polynomials with $\deg q \geq 1$. From a straightforward computation, $q^7(t)A(t)$ is a polynomial in t and the leading coefficient must be zero. This is a contradiction and thus M is of polynomial kind. Consequently, we have

Theorem 4.2. *A helicoidal surface of polynomial kind is the only rational helicoidal surface with harmonic Gauss map.*

Combining the above theorems and Theorem 3.8 we have

Theorem 4.3 (Characterization). *Let M be a rational helicoidal surface in Euclidean 3-space $\mathbb{E}^3$. Then, the Gauss map G is either harmonic or of pointwise 1-type if and only if M is part of a plane, a circular cylinder, a helicoid and a right cone.*

We finally propose a problem to classify helicoidal surfaces with pointwise 1-type Gauss map.

Problem: Classify all helicoidal surfaces with pointwise 1-type Gauss map.

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