Helicoids and axially symmetric minimal surfaces in 3-dimensional homogeneous spaces

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Abstract. The Bianchi-Cartan-Vranceanu spaces are Riemannian 3-manifolds whose isometry groups have at least 4-dimension and not of constant negative curvature. In this paper we study helicoids and axially symmetric minimal surfaces in the Bianchi-Cartan-Vranceanu spaces. In particular, axially symmetric minimal surfaces are explicitly classified in terms of elliptic functions. Moreover the non-existence of totally umbilical surfaces in the irreducible Bianchi-Cartan-Vranceanu spaces is proved.

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Introduction.

It is a classical result that the maximum dimension of the isometry groups of Riemannian 3-manifolds is 6. The maximum dimension 6 is attained by 3-dimensional Riemannian space forms. There are no Riemannian 3-manifolds with 5-dimensional isometry group. Moreover, if a Riemannian 3-manifold has 4-dimensional isometry group, then it is a homogeneous Riemannian manifold (See Kobayashi [28]).

Homogeneous Riemannian manifolds with 4-dimensional isometry group has been appeared in many contexts of differential geometry.

L. Bianchi gave a local classification of such homogeneous metrics [9]. The following 2-parameter family of homogeneous Riemannian metrics on the Cartesian 3-space $\mathbb{R}^3(x, y, z)$ are found by Bianchi [10], E. Cartan [12] and G. Vranceanu [50]:

$$\mathrm{d}s^2_{\lambda,\mu} = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{(1+\mu(x^2+y^2))^2} + \left(\mathrm{d}z + \frac{\lambda}{2}\frac{y\mathrm{d}x - x\mathrm{d}y}{1+\mu(x^2+y^2)}\right)^2, \quad \lambda, \mu \in \mathbb{R}.$$

The metric $\mathrm{d}s^2_{\lambda,\mu}$ is defined on the region $\mathcal{D} = \{(x, y, z) \mid 1 + \mu(x^2 + y^2) > 0\}$. Note that for $\mu \geqq 0$, $\mathrm{d}s^2_{\lambda,\mu}$ is defined on the whole 3-space $\mathbb{R}^3$.

This 2-parameter family contains all the Riemannian metrics with 4-dimensional isometry groups and metrics of constant non-negative curvature.

We denote by $\mathcal{M}^3_{\lambda,\mu}$ the homogeneous Riemannian 3-manifold $(\mathcal{D}, \mathrm{d}s^2_{\lambda,\mu})$ and call it the *Bianchi-Cartan-Vranceanu space* (BCV-space, in short).

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The BCV-spaces has been appeared in some contexts of 3-dimensional Riemannian geometry. For instance, 3-dimensional naturally reductive homogeneous spaces are either constant curvature or a BCV-space. Three dimensional D'Atri spaces (Riemannian manifolds whose geodesic symmetries are volume preserving up to sign) are of constant curvature or BCV-spaces (see [48]). Every BCV-space is a pseudo-symmetric space of constant type (see [14]). Moreover, BCV-spaces provide model spaces of *Thurston's 3-dimensional model geometries* except *solvegeometry* (see [47]).

Although, differential geometry of curves and surfaces in 3-dimensional Riemannian space forms has been studied extensively, such geometry in 3-dimensional homogeneous spaces of *non-constant curvature* has started recently. In these fifteen years differential geometry of BCV-spaces has been paid much attention of differential geometers (See [8] and references therein). In particular, many differential geometers investigate curves and surfaces in BCV-spaces. Here we emphasise that the BCV-family contains Euclidean 3-space $\mathbb{E}^3$, spherical space forms, the Heisenberg group $\mathbb{H}_3$ (the model space of nilgeometry) and product symmetric spaces $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$.

For example, some fundamental examples of minimal surfaces in Heisenberg group $\mathbb{H}_3$ are constructed in our previous works [3] [5] and [24, Part II]. Surfaces with parallel fundamental form in the BCV-space are classified by M. Belkhelfa, F. Dillen and the fourth named author [8]. More generally surfaces with higher order parallel second fundamental form in the BCV-space are recently classified by J. Van der Veken [49].

R. Caddeo, P. Piu and A. Ratto [13] and P. Tomter [46] investigated surfaces of revolution with constant mean curvature in $\mathbb{H}_3$. Grassmann geometry of surfaces in $\mathbb{H}_3$ is investigated by Naitoh, Kuwabara and the fourth named author [25]. Note that Kuwabara studied Grassmann geometries of the Euclidean motion group $E(2)$ and the Minkowski motion group $E(1,1)$. Surfaces with parallel second fundamental form in $E(1,1)$, $E(2)$ are classified by Van der Veken and the fourth named author [27]. The harmonicity of Gauss maps for surfaces in the BCV-space with $\mu \neq 0$ was studied by M. Tamura [45].

Minimal or constant mean curvature surfaces in the product space $\mathbb{H}^2 \times \mathbb{R}$ are discussed in [32]–[40], [42].

S. Montaldo and I. Onnis [33] investigated invariant constant curvature surfaces in $\mathbb{H}^2 \times \mathbb{R}$. Integral representation formulas for minimal surfaces in $\mathbb{H}_3$ and $\mathbb{H}^2 \times \mathbb{R}$ are obtained by Merucuri-Montaldo-Piu [31] and the fourth named author [21], [22].

For integral representation formulas of minimal surfaces in the model space Sol of solvegeometry, we refer to [20]–[21], [26].

In [6], D. A. Berdinskiĭ and I. A. Taĭmanov obtained a Weierstrass type representation for minimal surfaces in BCV-spaces and Sol in terms of spinors and Dirac operators.

Biharmonic curves in BCV-space are classified by Caddeo, Montaldo Oniciuc and Piu [11] and Cho, Lee and the fourth named author [15].

Moreover, since the discovery of holomorphic quadratic differential (called *generalized Hopf differential* or *Abresch-Rosenberg differential*) for constant mean curvature surfaces in the BCV-space, global geometry of constant mean curvature surfaces in the BCV-space has been extensively studied [1]–[2].

In this paper we shall study minimal surface equation in $\mathcal{M}^3_{\lambda,\mu}$ for a function

$z = f(x, y)$ in the BCV-space. The minimal surface equation for f is

$$\left(\frac{1}{\delta^2} + Q^2\right) f_{xx} - 2PQ f_{xy} + \left(\frac{1}{\delta^2} + P^2\right) f_{yy} - \frac{2\mu}{\delta}(Px + Qy)\left[P^2 + Q^2 + \tfrac{\lambda}{2\delta}(Qx - Py)\right] = 0,$$

where

$$\delta = 1 + \mu(x^2 + y^2),\ P = f_x + \frac{\lambda y}{2\delta},\ Q = f_y - \frac{\lambda x}{2\delta}.$$

We shall exhibit explicit examples of axially symmetric minimal surfaces in the BCV-space.

As an application of our result, we shall give a characterization of Euclidean and Heisenberg metrics among the BCV-family. More precisely, in the BCV-family, the only metrics which admit all the affine planes $z = f(x, y) = ax + by + c$ as minimal surfaces are Euclidean and Heisenberg metrics.

It should be remarked that the Euclidean helicoid $z = f(x, y) = h(\frac{y}{x}) = a \tan^{-1} g(\frac{y}{x}) + b;\ a, b \in \mathbb{R}$ are minimal in general BCV-spaces.

§ 1. Preliminaries

subsection Take two real numbers λ and μ and define the region $\mathcal{D}$ of the Cartesian 3-space $\mathbb{R}^3(x, y, z)$ by

$$\mathcal{D} = \begin{cases} \text{the whole space } \mathbb{R}^3,\ \mu \geq 0, \\ x^2 + y^2 = -1/\mu^2,\ \mu < 0. \end{cases}$$

The *Bianchi-Cartan-Vranceanu space* (BCV-space, in short) is a 3-dimensional Riemannian manifold $\mathcal{M}^3_{\lambda,\mu} := (\mathcal{D}, \mathrm{d}s^2_{\lambda,\mu})$ with homogeneous metric:

$$\mathrm{d}s^2_{\lambda,\mu} = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{\{1 + \mu(x^2 + y^2)\}^2} + \left(\mathrm{d}z + \frac{\lambda}{2}\frac{y\mathrm{d}x - x\mathrm{d}y}{1 + \mu(x^2 + y^2)}\right)^2.$$

This 2-parameter family of metrics includes all 3-dimensional homogeneous Riemannian metrics with isometry group of dimension 4 or 6 except hyperbolic space forms. More precisely, $\mathcal{M}^3_{\lambda,\mu}$ is (locally) isometric to the following homogeneous Riemannian 3-manifolds (See [8], [41]):

- If $\lambda = \mu = 0$, then $\mathcal{M}^3_{0,0}$ is the Euclidean space $\mathbb{E}^3$.
- If $\lambda \neq 0$, $\mu = 0$, then $\mathcal{M}^3_{\lambda,0}$ is the Heisenberg group $\mathbb{H}_3$.
- If $\lambda \neq 0$, $\mu > 0$, then $\mathcal{M}^3_{\lambda,\mu}$ is locally isometric to SU(2) with left invariant metric.
- If $\lambda \neq 0$, $\mu < 0$, then $\mathcal{M}^3_{\lambda,\mu}$ is locally isometric to $\mathrm{SL}_2\mathbb{R}$ with left invariant metric.
- If $\lambda = 0$, $\mu > 0$, then $\mathcal{M}^3_{\lambda,\mu}$ is locally isometric to $\mathbb{S}^2(1/2\sqrt{\mu}) \times \mathbb{R}$.

- If $\lambda = 0$, $\mu < 0$, then $\mathcal{M}^3_{\lambda,\mu}$ is $\mathbb{H}^2(1/2\sqrt{-\mu}) \times \mathbb{R}$.
- If $4\mu - \lambda^2 = 0$ then $\mathcal{M}^3$ is locally isometric to $\mathbb{S}^3(2/\lambda)$.

Here $\mathbb{S}^n(r)$ and $\mathbb{H}^n(r)$ denote the n-sphere of radius $r = 1/2\sqrt{\mu}$ and the hyperbolic n-space of radius $r = 1/2\sqrt{-\mu}$ $(n = 2, 3)$, respectively. This explicit classification implies the following fact:

Proposition 1.1. *A BCV-space $\mathcal{M}^3_{\lambda,\mu}$ is locally symmetric if and only if $\lambda^2(4\mu - \lambda^2) = 0$. More precisely, $\mathcal{M}^3_{\lambda,\mu}$ is locally symmetric if and only if it is locally isometric to one of the following model spaces; $\mathbb{E}^3$, $\mathbb{S}^3(2/\lambda)$ or the product spaces $\mathbb{S}^2(1/2\sqrt{\mu})\times\mathbb{R}$, $\mathbb{H}^2(1/2\sqrt{-\mu}) \times \mathbb{R}$.*

1.1

(1) Take an orthonormal frame field $\mathcal{E} = (e_1, e_2, e_3)$, defined by

$$e_1 = \delta\frac{\partial}{\partial x} - \frac{\lambda y}{2}\frac{\partial}{\partial z}, \quad e_2 = \delta\frac{\partial}{\partial y} + \frac{\lambda x}{2}\frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial z}$$

with $\delta := 1+\mu(x^2+y^2)$. One can easily check that $\mathcal{E}$ satisfies $\mathrm{d}s^2_{\lambda,\mu}(e_i, e_j) = \delta_{ij}$. Here δ_{ij} denotes the *Kronecker's delta*;

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

(2) The *dual coframe field* $\vartheta = (\theta^1, \theta^2, \theta^3)$ associated to $e = (e_1, e_2, e_3)$ is a triplet of 1-forms which satisfies the condition $\theta^i(e_j) = \delta_{ij}$. This coframe field is computed as given by

$$\theta^1 = \frac{\mathrm{d}x}{\delta}, \ \theta^2 = \frac{\mathrm{d}y}{\delta}, \ \theta^3 = \mathrm{d}z + \frac{\lambda}{2}\frac{y\mathrm{d}x - x\mathrm{d}y}{\delta}.$$

Note that the 1-form θ^3 is a *contact form* on $\mathcal{M}^3_{\lambda,\mu}$, *i.e.*, $\mathrm{d}\theta^3 \wedge \theta^3 \neq 0$, if and only if $\lambda \neq 0$.

(3) Denote by $\mathfrak{X}(\mathcal{M}_{\lambda,\mu})$ the space of all smooth vector fields on the BCV-space. The *Levi-Civita connection* D of $\mathcal{M}_{\lambda,\mu}$ is the differential operator

$$D : \mathfrak{X}(\mathcal{M}_{\lambda,\mu}) \times \mathfrak{X}(\mathcal{M}_{\lambda,\mu}) \to \mathfrak{X}(\mathcal{M}_{\lambda,\mu}); \quad (V, W) \longmapsto D_V W$$

defined by the following *Koszul formula* (See [38, p. 61]):

$$\begin{aligned} 2\mathrm{d}s^2_{\lambda,\mu}(D_V W, Z) \ &= \ V\,\mathrm{d}s^2_{\lambda,\mu}(W, Z) + W\,\mathrm{d}s^2_{\lambda,\mu}(Z, V) - Z\,\mathrm{d}s^2_{\lambda,\mu}(V, W) \\ &\quad -\mathrm{d}s^2_{\lambda,\mu}(V, [W, Z]) + \mathrm{d}s^2_{\lambda,\mu}(W, [Z, V]) + \mathrm{d}s^2_{\lambda,\mu}(Z, [V, W]) \end{aligned}$$

for all V, W, $Z \in \mathfrak{X}(\mathcal{M}^3_{\lambda,\mu})$. By using the Koszul formula, the Levi-Civita connection D of $\mathcal{M}^3_{\lambda,\mu}$ is explicitly given as follows;

$$D_{e_1}e_1 = 2\mu y e_2, \quad D_{e_1}e_2 = -2\mu y e_1 + \frac{\lambda}{2}e_3, \quad D_{e_1}e_3 = -\frac{\lambda}{2}e_2,$$

(1.1.1) $$D_{e_2}e_1 = -2\mu x e_2 - \frac{\lambda}{2}e_3, \quad D_{e_2}e_2 = 2\mu x e_1, \quad D_{e_2}e_3 = \frac{\lambda}{2}e_1,$$

$$D_{e_3}e_1 = -\frac{\lambda}{2}e_2, \quad D_{e_3}e_2 = \frac{\lambda}{2}e_1, \quad D_{e_3}e_3 = 0.$$

(1.1.2) $$[e_1, e_2] = -2\mu y e_1 + 2\mu x e_2 + \lambda e_3, \quad [e_2, e_3] = [e_3, e_1] = 0.$$

(4) The *Riemannian curvature tensor* R is a tensor field on $\mathcal{M}^3_{\lambda,\mu}$ defined by

$$R(V, W)Z = D_V D_W Z - D_W D_V Z - D_{[V,W]}Z.$$

The components $\{R_{lkij}\}$ of R relative to $\mathcal{E}$ are defined by

$$\mathrm{d}s^2_{\lambda,\mu}(R(e_i, e_j)e_k, e_l) = R_{lkij}.$$

The *Ricci tensor* Ric is defined by

$$\mathrm{Ric}(V, W) = \mathrm{trace}\,\{Z \longmapsto R(Z, V)W\}.$$

The components $\{R_{ij}\}$ of Ric relative to $\mathcal{E}$ are defined by

$$R_{ij} = \mathrm{Ric}(e_i, e_j) = \sum_{k=1}^{3} R_{kjki}.$$

The components R_{ij} are computed as

(1.1.3) $$R_{1212} = 4\mu - \frac{3}{4}\lambda^2, \quad R_{1313} = R_{2323} = \frac{\lambda^2}{4},$$

$$R_{11} = R_{22} = 4\mu - \lambda^2, \quad R_{33} = \frac{\lambda^2}{2}.$$

The *scalar curvature* $\mathrm{s} = \sum_{i=1}^{3} R_{ii}$ is $\mathrm{s} = 8\mu - \lambda^2/2$.

For more informations on the BCV-space, we refer to [8]. Our general reference of Riemannian geometry is O'Neill's book [38].

1.2

Let us denote by $\overline{\mathcal{M}}^2_\mu$ the Riemannian 2-manifold defined by

$$\overline{\mathcal{M}}^2_\mu = (\{(x, y) \in \mathbb{R}^2 \mid 1 + \mu(x^2 + y^2) > 0\}, \mathrm{d}\bar{s}^2)$$

with metric

$$\mathrm{d}\bar{s}^2_\mu = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{\{1 + \mu(x^2 + y^2)\}^2}.$$

Here μ is a real constant. Then the map

$$\pi : \mathcal{M}^3_{\lambda,\mu} \to \overline{\mathcal{M}}^2_\mu; \quad \pi(x, y, z) = (x, y)$$

is a *Riemannian submersion* with totally geodesic fibers ([37]). One can easily check that the base space $\overline{\mathcal{M}}^2_\mu$ is of constant curvature 4μ.

Now let us take a curve γ in $\overline{\mathcal{M}}^2_\mu$ with signed curvature $\bar{\kappa}$. Then its inverse image $S_\gamma = \pi^{-1}\{\gamma\}$ is a flat surface in $\mathcal{M}^3_{\lambda,\mu}$ with mean curvature $H = \frac{1}{2}\bar{\kappa}\circ\pi$. This surface S_γ is called the *vertical cylinder* over γ ([8]). The definition of mean curvature and flat surface will be given in the next section.

§ 2. Minimal surface equation

2.1

Let S be an immersed surface in $\mathcal{M}^3_{\lambda,\mu}$ which is given as a graph of a function $z = f(x,y)$. The position vector field $\boldsymbol{X}(x,y)$ of S is expressed as a vector valued function $\boldsymbol{X}(x,y) = (x, y, f(x,y))$.

The tangent vector fields $\boldsymbol{X}_x = \partial\boldsymbol{X}/\partial x$ and $\boldsymbol{X}_y = \partial\boldsymbol{X}/\partial y$ are described by

$$\boldsymbol{X}_x(x,y) = \tfrac{1}{\delta}e_1 + Pe_3 \quad \boldsymbol{X}_y(x,y) = \tfrac{1}{\delta}e_2 + Qe_3,$$

in terms of the orthonormal frame field $\mathcal{E}$. Here the functions P and Q are defined by

$$P = f_x + \frac{\lambda}{2\delta}y, \quad Q = f_y - \frac{\lambda}{2\delta}.$$

The *first fundamental form* I of S is a Riemannian metric on S defined by

$$\mathrm{I} = E\mathrm{d}x^2 + 2F\mathrm{d}x\mathrm{d}y + G\mathrm{d}y^2,$$

where

$$E = \mathrm{d}s^2_{\lambda,\mu}(\boldsymbol{X}_x, \boldsymbol{X}_x), \quad F = \mathrm{d}s^2_{\lambda,\mu}(\boldsymbol{X}_x, \boldsymbol{X}_y), \quad G = \mathrm{d}s^2_{\lambda,\mu}(\boldsymbol{X}_y, \boldsymbol{X}_y).$$

The coefficient functions E, F and G are computed as

$$E = \frac{1}{\delta^2} + P^2, \quad F = PQ, \quad G = \frac{1}{\delta^2} + Q^2.$$

2.2

Take a unit vector field $\boldsymbol{N}$ normal to S. Namely $\boldsymbol{N}$ is a vector field along S which satisfies

$$\mathrm{d}s^2_{\lambda,\mu}(\boldsymbol{X}_x, \boldsymbol{N}) = \mathrm{d}s^2_{\lambda,\mu}(\boldsymbol{X}_y, \boldsymbol{N}) = 0, \quad \mathrm{d}s^2_{\lambda,\mu}(\boldsymbol{N}, \boldsymbol{N}) = 1.$$

The *second fundamental form* II derived from $\boldsymbol{N}$ is defined by

$$\mathrm{II} = L\mathrm{d}x^2 + 2M\mathrm{d}x\mathrm{d}y + N\mathrm{d}y^2,$$

where

$$L = \mathrm{d}s^2_{\lambda,\mu}\left(D_{\frac{\partial}{\partial x}}\boldsymbol{X_x}, \boldsymbol{N}\right), \quad M = \mathrm{d}s^2_{\lambda,\mu}\left(D_{\frac{\partial}{\partial y}}\boldsymbol{X_x}, \boldsymbol{N}\right), \quad L = \mathrm{d}s^2_{\lambda,\mu}\left(D_{\frac{\partial}{\partial y}}\boldsymbol{X_y}, \boldsymbol{N}\right).$$

Since S is a graph of a function f, we can choose a unit normal vector field $\boldsymbol{N}$ as

$$\boldsymbol{N} = -\frac{1}{W}(\delta P e_1 + \delta Q e_2 - e_3), \quad W = \sqrt{1 + \delta^2(P^2 + Q^2)}.$$

The second fundamental form derived from this unit normal vector field is given by

$$\begin{aligned} WL &= f_{xx} + \lambda PQ + \frac{2\mu}{\delta}(Px - Qy) - \frac{\lambda\mu}{\delta^2}xy, \\ WM &= f_{xy} + \frac{\lambda}{2}(Q^2 - P^2) + \frac{2\mu}{\delta}(Py + Qx) + \frac{\lambda\mu}{2\delta^2}(x^2 - y^2), \\ WN &= f_{yy} - \lambda PQ - \frac{2\mu}{\delta}(Px - Qy) + \frac{\lambda\mu}{\delta^2}xy. \end{aligned}$$

2.3

Let us denote the following matrix valued functions associated to I and II by the same letters I and II, respectively;

$$\mathrm{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \quad \mathrm{II} = \begin{pmatrix} L & M \\ M & N \end{pmatrix}.$$

The solutions λ_1 and λ_2 to the characteristic equation $\det(\mathrm{II} - t\mathrm{I}) = 0$ are called the *principal curvatures* of S. Recall that the average $H = (k_1 + k_2)/2$ of k_1 and k_2 is called the *mean curvature* of S. The mean curvature H is computed by the formula:

$$H = \frac{EN + GL - 2FM}{2(EG - F)^2}.$$

A surface $S : z = f(x, y)$ is said to be *minimal* if $H = 0$.

On the other hand, the *Gaussian curvature* K of S is given by [38, p. 101]

$$K = K_{\mathcal{M}} + k_1 k_2,$$

where

$$K_{\mathcal{M}} = \frac{\mathrm{d}s^2_{\lambda,\mu}(R(\boldsymbol{X}_x, \boldsymbol{X}_y)\boldsymbol{X}_y, \boldsymbol{X}_x)}{EG - F^2}.$$

A surface $S : z = f(x, y)$ is said to be *flat* if $K = 0$.

2.4

The differential equation $H = 0$ for a surface S defined as a graph $(x, y, f(x, y))$ is called the *minimal surface equation* in $\mathcal{M}^3_{\lambda,\mu}$. The minimal surface equation is given explicitly by

$$(\mathcal{A}_{\lambda,\mu}) \qquad \begin{aligned} &(\tfrac{1}{\delta^2} + Q^2)f_{xx} - 2PQf_{xy} + (\tfrac{1}{\delta^2} + P^2)f_{yy} \\ &\qquad - \frac{2\mu}{\delta}(Px + Qy)\left[P^2 + Q^2 + \frac{\lambda}{2\delta}(Qx - Py)\right] = 0. \end{aligned}$$

Clearly, this equation reduces to the one for the Heisenberg group $\mathbb{H}_3$ if $\mu = 0, \ \lambda = 1$ ([3]), and for the minimal surface equation of $\mathbb{E}^3$ if $\lambda = \mu = 0$, respectively.

2.5

The minimal surface equation $(\mathcal{A}_{\lambda,\mu})$ can be rewritten as the following *divergence form*:

$$\frac{\partial}{\partial x}\left(\frac{P}{W}\right)+\frac{\partial}{\partial y}\left(\frac{Q}{W}\right)=0.$$

§ 3. Minimal surfaces in BCV-spaces

Minimal surface theory in Euclidean 3-space $\mathbb{E}^3$ was started with constructing/classifying fundamental examples of minimal surfaces, *eg.*, minimal surfaces of revolution, ruled minimal surfaces, or translation minimal surfaces etc. (For more informations on minimal surface theory in $\mathbb{E}^3$, we refer to Nitsche's book [35] and Osserman's book [39]).

In this section we study elementary and fundamental examples of minimal surfaces in BCV-spaces.

3.1

The following 2-parameter family of Riemannian metrics on $\mathbb{R}^3$ are investigated by the first named author [4] and T. Hangan [19].

$$g_{\eta,\xi}=\frac{1}{1-(\eta+\xi^2)r^2}\left[\mathrm{d}x^2+\mathrm{d}y^2-\eta(y\mathrm{d}x-x\mathrm{d}y)^2+2\xi(y\mathrm{d}x-x\mathrm{d}y)\mathrm{d}z+\mathrm{d}z^2\right],\quad \eta,\xi\in\mathbb{R},$$

where $r^2=x^2+y^2$. These metrics have a characteristic property; with respect to which all the planes $z=ax+by+c$ are minimal surfaces. Note that Euclidean metrics and the canonical left invariant metrics on the Heisenberg group belong to this family. In fact, the Euclidean metric is $g_{0,0}$. The left invariant metrics on the Heisenberg group are characterized as a parabola $\eta=-\xi^2$.

It is easy to observe that in the BCV-space, linear function $f(x,y)=ax+by+c$ are not solutions to the minimal surface equation, for general λ and μ.

This implies that general BCV-metrics are not the member of the family $\{g_{\eta,\xi}\}$.

The Heisenberg metrics $g_{\xi,-\xi^2}$ $(\xi\neq 0)$ and Euclidean metric belong to $\{g_{\eta,\xi}\}$. This property characterizes these two metrics as follows:

Proposition 3.2. *Let $\mathcal{M}^3_{\lambda,\mu}$ be a BCV-space. Then the surface $z=f(x,y)=ax+by+c$ is minimal for arbitrary a,b and c if and only if $\mu=0$, i.e, the space $\mathcal{M}^3_{\lambda,\mu}$ is isomorphic to Euclidean 3-space or the Heisenberg group.*

Proof. $(\Leftarrow)$ If $\mu=0$, then the minimal surface equation $(\mathcal{A}_{\lambda,\mu})$ admits all linear functions as its solutions (*cf.* [3]).
$(\Rightarrow)$ Conversely, let us assume that all the linear functions are solutions to the minimal surface equation $(\mathcal{A}_{\lambda,\mu})$. Then the minimal surface equation $(\mathcal{A}_{\lambda,\mu})$ with respect to $f(x,y)=ax+by+c$ is

$$\tfrac{2\mu}{\delta}\,(ax+by)\left[a^2+b^2+\tfrac{\lambda}{2\delta}(bx-ay)\right]=0.$$

Since this equation holds for all a and b, we have $\mu=0$. ■

3.2 Helicoids in BCV-spaces

Euclidean helicoids can be characterized as a minimal surface in $\mathbb{E}^3$ which is a graph of a function of the form $f(x,y) = h(\frac{y}{x})$. In this subsection we look for minimal surfaces determined by the solution $f(x,y) = h(\frac{y}{x})$ to the minimal surface equation in BCV-space.

Let S be a surface which is a graph of a function of the form $f(x,y) = h(\frac{y}{x})$. Put $u = \frac{y}{x}$ for $x \neq 0$. Then we have

$$f_x = -\frac{y}{x^2}h', f_y = \frac{1}{x}h',$$
$$f_{xx} = \frac{2y}{x^3}h' + \frac{y^2}{x^4}h'', f_{xy} = -\frac{1}{x^2}h' - \frac{y}{x^3}h'', f_{yy} = \frac{1}{x^2}h'',$$

where h' and h'' are the derivatives with respect to u.

Now we insert these data to the minimal surface equation $(\mathcal{A}_{\lambda,\mu})$ and multiply x^2 to it, then we have the differential equation

$$(1+u^2)h'' + 2uh' = 0.$$

One can see that general solutions to this ODE is given explicitly by $f(x,y) = h(\frac{y}{x}) = \alpha \tan^{-1}(\frac{y}{x}) + \beta$, where $\alpha, \beta \in \mathbb{R}$.

Proposition 3.3. *The only minimal surfaces in BCV-space which has the form $z = f(x,y) = h(\frac{y}{x})$ are the surfaces $f(x,y) = \alpha \tan^{-1}(y/x) + \beta$, $\alpha, \beta \in \mathbb{R}$.*

Remark 3.1. *If $\lambda = \mu = 0$, the minimal surface $z = \alpha \tan^{-1}(y/x) + \beta$ is a helicoid in $\mathbb{E}^3$. This surface is minimal in the Heisenberg space as well as in $\mathbb{R}^3$ with metric $g_{\eta,\xi}$.*

3.3 Axially symmetric minimal surfaces

It is easy to see that the metrics $ds^2_{\lambda,\mu}$ are invariant under rotations about z-axis and translations along the same axis. Based on this fundamental property, in this subsection, we classify *axially symmetric* minimal graphs in $\mathcal{M}^3_{\lambda,\mu}$.

A surface $S : z = f(x,y)$ is said to be *axially symmetric* if f depends only on $r = \sqrt{x^2+y^2}$.

Now let S be an axially symmetric graph of a function $f(x,y) = T(r)$. Then we have

$$f_x = \frac{x}{r}T', \quad f_y = \frac{y}{r}T',$$
$$f_{xx} = \frac{y^2}{r^3}T' + \frac{x^2}{r^2}T'', \quad f_{yy} = \frac{x^2}{r^3}T' + \frac{y^2}{r^2}T'', \quad f_{xy} = \frac{-xy}{r^3}T' + \frac{xy}{r^2}T'',$$

where T' and T'' are the derivatives with respect to r. From these, we get the following minimal surface equation:

$$r(4+\lambda^2r^2)T'' + 4T'[1 + (1-\mu^2r^4)T'^2] = 0. \tag{3.3.4}$$

To solve this equation we put $T' = u$. Since $r(4+\lambda^2 r^2) \neq 0$, then (3.3.4) is rewritten as:

$$u' + \frac{4}{r(4+\lambda^2 r^2)} u = -\frac{4(1-\mu^2 r^4)}{r(4+\lambda^2 r^2)} u^3.$$

This is a *Bernouilli's differential equation.* Now we put $v = 1/u^2$ then the preceding equation is rewritten as:

$$v' - \frac{8}{r(4+\lambda^2 r^2)} v = \frac{8(1-\mu^2 r^4)}{r(4+\lambda^2 r^2)}.$$

General solutions to this equation are given by

$$v = \frac{\alpha r^2 - 4(1+\mu^2 r^4)}{(4+\lambda^2 r^2)}, \quad \alpha > 0.$$

Hence

$$(T')^2 = \frac{4+\lambda^2 r^2}{\alpha r^2 - 4(1+\mu^2 r^4)} = -\frac{1+(\lambda/2)^2 r^2}{1+r^2(\mu^2 r^2 - \beta)}, \quad \beta = \alpha/4 > 0.$$

To solve this equation, we need separate discussions according to the values of λ and μ. Our general reference on the elliptic integrals is [17].

(1) $\lambda = \mu = 0$ ($\mathcal{M}^3_{0,0} = \mathbb{E}^3$): In this case we have $(T')^2 = 1/(\beta r^2 - 1)$, $r^2 > 1/\beta$, $\beta \in \mathbb{R}^+$. The solution is the axially symmetric surface:

$$T(r) = \int_{\frac{1}{\sqrt{\beta}}}^{r} \frac{\mathrm{d}t}{\sqrt{\beta t^2 - 1}} + c_1 = \frac{1}{\sqrt{\beta}} \cosh^{-1} \sqrt{\beta} r + c_1.$$

Hence the surface is a *catenoid* in $\mathbb{E}^3$.

(2) $\lambda \neq 0$, $\mu = 0$ ($\mathcal{M}^3_{\lambda,\mu} = \mathbb{H}_3$): In the Heisenberg group $\mathbb{H}_3$, we have

$$(T')^2 = \frac{1+(\lambda/2)^2 r^2}{\beta r^2 - 1} = \left(\frac{k\lambda}{2}\right)^2 \frac{r^2 + (2/\lambda)^2}{r^2 - k^2}, \quad r^2 > \frac{1}{\beta} = k^2, \ k > 0.$$

The solution is the axially symmetric surface

$$T(r) = \frac{k\,|\lambda|}{2} \int_k^r \sqrt{\frac{(t^2 + 4/\lambda^2)}{(t^2 - k^2)}} \mathrm{d}t + c_2.$$

This elliptic integral can be expressed by the elliptic integrals in Legendre form of first and second kinds. In fact, the integral

$$\mathcal{I} = \int_b^u \sqrt{\frac{(t^2 + a^2)}{(t^2 - b^2)}} \mathrm{d}t$$

is represented as

$$\mathcal{I} = \sqrt{a^2 + b^2}\ (F(\varepsilon, s) - E(\varepsilon, s)) + \frac{1}{u}\sqrt{(u^2 + a^2)(u^2 - b^2)}$$

with $u > b > 0$. Here $F(\varepsilon, s)$ and $E(\varepsilon, s)$ are the elliptic integrals in Legendre form of first and second kind, respectively;

$$F(\varepsilon, s) = \int_0^\varepsilon (1 - s^2 \sin^2 \alpha)^{-\frac{1}{2}} d\alpha, \quad E(\varepsilon, s) = \int_0^\varepsilon (1 - s^2 \sin^2 \alpha)^{\frac{1}{2}} d\alpha.$$

The modulus ε and the variable s are given by $\varepsilon = \arccos \dfrac{b}{u}$, $s = a/\sqrt{a^2 + b^2}$.

Note that this minimal surface was already given by the first named author's paper [3].

(3) $\lambda = 0$, $\mu \neq 0$: In this case $\mathcal{M}^3_{\lambda,\mu}$ is (locally) isometric to the product spaces. The minimal surface is determined by the solution T to

$$(T')^2 = -\frac{1}{1 + r^2(\mu^2 r^2 - \beta)}, \quad \beta > 0,\ 1 + r^2(\mu^2 r^2 - \beta) < 0.$$

Now we put $R = r^2$. Since $1 + r^2(\mu^2 r^2 - \beta) < 0$, we have $\Delta_{\beta,\mu} := \beta^2 - 4\mu^2 > 0$ and $R \in (R_1, R_2)$, where $R_1 = (\beta - \sqrt{\Delta_{\beta,\mu}})/2\mu^2$, $0 < R_1 < R_2 = (\beta + \sqrt{\Delta_{\beta,\mu}})/2\mu^2$. Hence the axially symmetric surface is given by

$$T(R) = \frac{1}{2\,|\mu|} \int_{R_1}^{R} \frac{\mathrm{d}t}{\sqrt{t(t - R_1)(R_2 - t)}} + c_3, \quad R < R_2.$$

This is an elliptic integral which can be expressed by the elliptic integral in Legendre form of the first kind as follows:

$$\mathcal{I} = \int_b^u \frac{\mathrm{d}t}{\sqrt{(a - t)(t - b)(t - c)}} = \frac{2}{\sqrt{a - c}} F(\varepsilon, s),$$

where $a \geq u > b > c$, $\varepsilon = \sin^{-1} \sqrt{\dfrac{(a - c)(u - b)}{(a - b)(u - c)}}$ and $s = \sqrt{\dfrac{a - b}{a - c}}$.

(4) $\lambda \neq 0$, $\mu \neq 0$. In this case $\mathcal{M}^3_{\lambda,\mu}$ is locally isometric to SU(2) or $\mathrm{SL}_2\mathbb{R}$.

$$(T')^2 = -\left(\frac{\lambda}{2}\right)^2 \frac{r^2 + 4/\lambda^2}{1 + r^2(\mu^2 r^2 - \beta)}, \quad \beta > 0,\ 1 + r^2(\mu^2 r^2 - \beta) < 0.$$

As in the preceding case, one can solve this ODE in the following way:

$$T(R) = \frac{|\lambda|}{4\,|\mu|} \int_{R_1}^{R} \sqrt{\frac{(t + 4/\lambda^2)}{t(t - R_1)(R_2 - t)}}\, \mathrm{d}t + c_4, \quad R \leq R_2.$$

This elliptic integral is expressed in Legendre form of first and third kinds. In fact, the elliptic integral of the form

$$\mathcal{I} = \int_b^u \sqrt{\frac{(t - d)}{(a - t)(t - b)(t - c)}}\, \mathrm{d}t$$

is represented as

$$\mathcal{I} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{ (b-c)\Pi(\varepsilon, \frac{a-b}{a-c}, s) + (c-d)F(\varepsilon, s) \right\},$$

where

$$a \geq u > b > c > d, \quad \varepsilon = \sin^{-1}\sqrt{\frac{(a-c)(u-b)}{(a-b)(u-c)}}\ , \ s = \sqrt{\frac{(a-b)(c-d)}{(a-c)(b-d)}}.$$

Here we recall that the elliptic integral in Legendre form of third kind $\Pi(\varepsilon, \frac{a-b}{a-c}, s)$ is given by

$$\Pi(\varepsilon, \frac{a-b}{a-c}, s) = \int_0^\varepsilon \frac{\mathrm{d}\alpha}{\left(1 + \frac{a-b}{a-c}\sin^2\alpha\right)\sqrt{1 - s^2\sin^2\alpha}}.$$

Theorem 3.1. *The only axially symmetric minimal surfaces in $\mathcal{M}^3_{\lambda,\mu}$ are the graphs of functions $f(x,y) = T(r) = T(\sqrt{x^2+y^2})$ with $r^2 = x^2 + y^2$, where*

(1) $T(r) = \frac{1}{\sqrt{\beta}}\cosh^{-1} > \sqrt{\beta}r + c_1$, $\beta > 0$ *for* $\lambda = \mu = 0$.

(2) $T(r) = \frac{k|\lambda|}{2}\int_k^r \sqrt{\frac{(t^2+4/\lambda^2)}{(t^2-k^2)}}\mathrm{d}t + c_2$, $k \geqslant 0$ *for* $\lambda \neq 0$, $\mu = 0$.

(3) $T(R) = \frac{1}{2|\mu|}\int_{R_1}^R \frac{\mathrm{d}t}{\sqrt{t(t-R_1)(R_2-t)}} + c_3$; $R = r^2 \leq R_2$ *for* $\lambda = 0$, $\mu \neq 0$.

(4) $T(R) = \frac{|\lambda|}{4|\mu|}\int_{R_1}^R \sqrt{\frac{(t+4/\lambda^2)}{t(t-R_1)(R_2-t)}}\mathrm{d}t + c_4, R = r^2 \leq R_2$, *for* $\lambda \neq 0$, $\mu \neq 0$ *where* $R_1 = (\beta - \sqrt{\beta^2 - 4\mu^2})/2\mu^2$, $R_2 = (\beta + \sqrt{\beta^2 - 4\mu^2})/2\mu^2, \beta > 0$.

Remark 3.2. *Rotationally symmetric surfaces,* i.e., *surfaces which are invariant under rotations in z-axis in* $\mathrm{SL}_2\mathbb{R}$ *are investigated in [29], [7], [23].*

§ 4. Umbilical surfaces

4.1

A point of a surface S in $\mathcal{M}^3_{\lambda,\mu}$ is said to be an *umbilical point* (or *umbilic*) if the second fundamental form is proportional to the first fundamental form. A surface S is said to be (totally) *umbilical* if all the points are umbilical.

In particular, S is said to be *totally geodesic* if its second fundamental form vanishes. Totally geodesic property is equivalent to the condition: every geodesic in S is a geodesic in the ambient space, too.

Recall that in the Euclidean 3-space $\mathbb{E}^3$, the only umbilical surfaces are the planes and spheres. Note that planes are totally geodesic in $\mathbb{E}^3$.

In the 3-sphere $\mathbb{S}^3(r)$ of radius r, the only umbilical surfaces are small spheres and great spheres. In particular, great spheres are the only totally geodesic surfaces.

In the Heisenberg space, Hangan [18] proved the non-existence of totally geodesic surfaces. A. Sanini [43] gave a geometric proof of the non-existence of umbilical surfaces in $\mathbb{H}_3$. In [25], a Grassmannian geometric proof for the non-existence of totally geodesic surfaces is obtained.

More generally, the following result is known (see *eg.*, [8]):

Proposition 4.4. *The BCV-space* $\mathcal{M}^3_{\lambda,\mu}$ *has totally geodesic surfaces if and only if* $\lambda = 0$ *or* $\lambda^2 = 4\mu$.

In other words, $\mathcal{M}^3_{\lambda,\mu}$ admits totally geodesic surfaces if and only if it is locally symmetric. Totally geodesic surfaces in $\mathcal{M}^3_{\lambda,0}$ are classified in [8] as follows:

Proposition 4.5. *The only totally geodesic surfaces in the product space* $\mathcal{M}^3_{\lambda,0}$ *with* $\lambda \neq 0$ *are leaves* $z =$ *constant and vertical cylinders over geodesics.*

4.2

Now we study totally umbilical surfaces in the *non locally symmetric* BCV-space $\mathcal{M}^3_{\lambda,\mu}$.

Let S be an umbilical surface in $\mathcal{M}^3_{\lambda,\mu}$ with $\lambda^2 - 4\mu \neq 0$. Then, by definition, there exists a function ρ such that $W\,\mathrm{I\!I} = \rho\,\mathrm{I}$. Namely,

$$\frac{L}{E} = \frac{M}{F} = \frac{N}{G} = \frac{\rho}{W}.$$

By the implicit function theorem, S is locally expressed as a graph of some function $z = f(x,y)$ defined on a region in $\mathbb{R}^2(x,y)$.

The umbilical condition is given by the following system of partial differential equations:

$$(4.4.5)\qquad \begin{cases} f_{xx} = -\lambda PQ - \frac{2\mu}{\delta}(xP - yQ) + \frac{\lambda\mu}{\delta^2}xy + \rho\left(\frac{1}{\delta^2} + P^2\right), \\ f_{xy} = \frac{\lambda}{2}(P^2 - Q^2) - \frac{2\mu}{\delta}(yP + xQ) - \frac{\lambda\mu}{2\delta^2}(x^2 - y^2) + \rho PQ, \\ f_{yy} = \lambda PQ + \frac{2\mu}{\delta}(xP - yQ) - \frac{\lambda\mu}{\delta^2}xy + \rho(\frac{1}{\delta^2} + Q^2). \end{cases}$$

The third derivatives of f are given explicitly by

$$\begin{aligned} f_{xxy} &= \rho_y(\tfrac{1}{\delta^2} + P^2) + \rho(2PP_y - \tfrac{4\mu}{\delta^3}y) + \tfrac{\lambda\mu}{\delta^3}x(1 + \mu(x^2 - 3y^2)) + \tfrac{4\mu^2}{\delta^2}y(Px - Qy), \\ &\quad - \tfrac{2\mu}{\delta}(P_y x - Q_y y - Q) - \lambda(P_y Q + PQ_y), \\ f_{xyx} &= \rho_x PQ + \rho(P_x Q + PQ_x) + \tfrac{\lambda\mu}{\delta^3}x(-1 + \mu(x^2 - 3y^2)) + \tfrac{4\mu^2}{\delta^2}x(Py + Qx), \\ &\quad - \tfrac{2\mu}{\delta}(P_x y + Q_x x + Q) + \lambda(PP_x - QQ_x), \\ f_{xyy} &= \rho_y PQ + \rho(P_y Q + PQ_y) + \tfrac{\lambda\mu}{\delta^3}y(1 + \mu(3x^2 - y^2)) + \tfrac{4\mu^2}{\delta^2}y(Py + Qx) \\ &\quad - \tfrac{2\mu}{\delta}(P_y y + Q_y x + P) + \lambda(PP_y - QQ_y), \\ f_{yyx} &= \rho_x(\tfrac{1}{\delta^2} + Q^2) + \rho(2QQ_x - \tfrac{4\mu}{\delta^3}x) - \tfrac{\lambda\mu}{\delta^3}y(1 - \mu(3x^2 - y^2)) - \tfrac{4\mu^2}{\delta^2}x(Px - Qy), \\ &\quad + \tfrac{2\mu}{\delta}(P_x x - Q_x y + P) + \lambda(P_x Q + PQ_x). \end{aligned}$$

From these the *integrability condition*:

$$(s_1)\qquad \begin{cases} f_{xxy} = f_{xyx} \\ f_{yyx} = f_{xyy} \end{cases}$$

of f are equivalent to the system:

$$(4.4.6)\qquad -PQ\rho_x + (\frac{1}{\delta^2} + P^2)\rho_y - \frac{1}{\delta^2}Q\rho^2 - \frac{\lambda}{2}P(\frac{1}{\delta^2} + P^2 + Q^2)\rho + \frac{4\mu - \lambda^2}{\delta^2}Q = 0,$$

$$(4.4.7)\qquad (\frac{1}{\delta^2} + Q^2)\rho_x - PQ\rho_y - \frac{1}{\delta^2}P\rho^2 + \frac{\lambda}{2}Q(\frac{1}{\delta^2} + P^2 + Q^2)\rho + \frac{4\mu - \lambda^2}{\delta^2}P = 0.$$

We multiply P to (4.4.6). Next, multiply Q to (4.4.7) and subtracting it from $P\times$ (4.4.6). Then we obtain $P\rho_y = Q\rho_x + \frac{\lambda}{2}(P^2 + Q^2)\rho$. Inserting this into (4.4.7), we get

$$(4.4.8)\qquad \rho_x = P(\rho^2 - 4\mu + \lambda^2) - \frac{\lambda}{2}Q\rho.$$

Here we used a fact $P^2 + Q^2 + \frac{1}{\delta^2} \neq 0$. Analogously we have

$$(4.4.9)\qquad \rho_y = Q(\rho^2 - 4\mu + \lambda^2) + \frac{\lambda}{2}P\rho.$$

The equations (4.4.8) and (4.4.9) together with the integrability condition for ρ ($\rho_{xy} = \rho_{yx}$) imply that

(4.4.10)

$$(P_y - Q_x)(\rho^2 - 4\mu + \lambda^2) + 2\rho(P\rho_y - Q\rho_x) - \frac{\lambda}{2}(Q_y + P_x)\rho - \frac{\lambda}{2}(Q\rho_y + P\rho_x) = 0.$$

Inserting the equations

$$P = f_x + \frac{\lambda y}{2\delta}, \quad Q = f_y - \frac{\lambda x}{2\delta},$$

into (4.4.10) we have

$$(\mathcal{O}_{\lambda,\mu})\qquad \lambda(4\mu - \lambda^2)(P^2 + Q^2 - \frac{2}{\delta^2}) = 0.$$

Since we assumed that $\mathcal{M}^3_{\lambda,\mu}$ is not of constant curvature and $\lambda \neq 0$, $(\mathcal{O}_{\lambda,\mu})$ reduces to the equation $\lambda(P^2 + Q^2 - \frac{2}{\delta^2}) = 0$.

Hence the umbilical surface S satisfies $P^2 + Q^2 = 2/\delta^2 \neq 0$. Differentiating this equation we have

$$(s_2)\qquad PP_x + QQ_x = -\frac{4\mu}{\delta^3}x, \quad PP_y + QQ_y = -\frac{4\mu}{\delta^3}y.$$

Substituting P_x, P_y, Q_x, and Q_y into this system we have

$$(s_3) \qquad P\rho - \frac{\lambda}{2}Q = 0, \quad Q\rho + \frac{\lambda}{2}P = 0.$$

We multiply P the first equation of (s_3). Next multiplying Q to the second equation of (s_3). Adding the resulting two equations, we have $\rho(P^2+Q^2) = 0$. Since $P^2+Q^2 = 2/\delta^2 \neq 0$ we have necessarily $\rho = 0$. Hence the surface is totally geodesic. However $\mathcal{M}^3_{\lambda,\mu}$ with $\lambda^2 - 4\mu \neq 0$ and $\lambda \neq 0$ has no totally geodesic surfaces.

Thus we obtain the following Theorem.

Theorem 4.2. *The non locally symmetric BCV-space $\mathcal{M}^3_{\lambda,\mu}$ has no totally umbilical surfaces.*

Remark 4.3. *If $\mathcal{M}^3_{\lambda,\mu}$ is a product space,* i.e, $\lambda = 0$ *and* $\mu \neq 0$, *the derivatives of* ρ *are given by*

$$\rho_x = (\rho^2 - 4\mu)f_x, \quad \rho_y = (\rho^2 - 4\mu)f_y.$$

Hence we obtain

$$(4.4.11) \qquad f(x,y) = \frac{1}{2\sqrt{-\mu}}\tan^{-1}\frac{\rho(x,y)}{2\sqrt{-\mu}} + c, \ \textit{for}\ \mu < 0,$$

$$(4.4.12) \qquad f(x,y) = \frac{1}{2\sqrt{\mu}}\tanh^{-1}\frac{\rho(x,y)}{2\sqrt{\mu}} + c, \ \textit{for}\ \mu > 0.$$

Here c is a real constant.

Inserting (4.4.11) or (4.4.12) into the umbilical condition (4.4.5), we get the following system of PDE's with respect to ρ:

$$(4.4.13) \qquad \rho_{xx} = -\frac{2\mu}{\delta}(x\rho_x - y\rho_y) + \frac{\rho}{\delta^2}(\rho^2 - 4\mu) + \frac{3\rho\rho_x^2}{\rho^2 - 4\mu},$$

$$(4.4.14) \qquad \rho_{xy} = -\frac{2\mu}{\delta}(y\rho_x + x\rho_y) + \frac{\rho\rho_x\rho_y}{\rho^2 - 4\mu}$$

$$(4.4.15) \qquad \rho_{yy} = \frac{2\mu}{\delta}(x\rho_x - y\rho_y) + \frac{\rho}{\delta^2}(\rho^2 - 4\mu) + \frac{3\rho\rho_y^2}{\rho^2 - 4\mu}.$$

The system (4.4.13)–(4.4.15) for $\mu < 0$ is solved by Montaldo and Onnis [34]. Namely, totally umbilical surfaces in $\mathbb{H}^2(1/2\sqrt{-\mu}) \times \mathbb{R}$ are explicitly given in [34]. J. Van der Veken classfied all totally umbilical surfaces in the BCV-space, independently [49].

Remark 4.4. *Sanini obtained the following result.*

Lemma 4.1. *Let N be a Riemannian* 3*-manifold and S a surface in N. Denote by ψ the tangential Gauss map of S:*

$$\psi : S \to \mathrm{Gr}_2(TN), \ \psi(p) := T_pS \subset T_pN.$$

Here $\mathrm{Gr}_2(TN)$ denotes the Grassmannian bundle of all 2*-planes in the tangent bundle of N. Then the tangential Gauss map is conformal if and only if S is totally umbilical or minimal.*

The following result is essentially due to Tamura [45]:

Theorem 4.3. *The only minimal surfaces with vertically harmonic tangential Gauss map in the BCV-space of non-constant curvature are totally geodesic leaves or vertical cylinders over geodesics. Moreover their tangential Gauss maps are conformal and harmonic.*

These two results together with our Theorem 4.2 imply the following characterization.

Corollary 4.1. *The only surfaces in non-locally symmetric BCV-space $\mathcal{M}^3_{\lambda,\mu}$ with conformal tangential Gauss map are the vertical cylinders over geodesics.*

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