Helmholtz's decomposition theorem, *etc*.

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The fact is used that electromagnetic fields are covariant (antisymmetric) tensors or contravariant (antisymmetric) tensor densities, which are mutual conjugated. The conjugation allows many-fold specific differentiation of the fields and leads to field chains. An integral operation, named the generation, is considered, which is reverse to the specific differentiation. The double generation yields zero as well as the double differentiation. The Helmholtz decomposition is compared with the Poincare decomposition, and many ways of the Helmholtz decomposition are presented. Laplace operator and the inverse Laplace operator are expressed in terms of the differential and integral operations. All results are illustrated by simple examples.

1. Introduction. Helmholtz's decomposition and Poincare's decomposition

The Helmholtz's theorem is familiar to physicists [1] and mathematics [2]. The essence of the theorem is as follows. A field, e.g. an electric vector field $\mathbf{E}$, can be written as the sum of two terms, the transverse or solenoidal field $\underset{\circ}{\mathbf{E}}$ and the longitudinal or irrotational field $\underset{\times}{\mathbf{E}}$:

$$\mathbf{E} = \underset{\circ}{\mathbf{E}} + \underset{\times}{\mathbf{E}}, \quad \mathrm{div}\,\underset{\circ}{\mathbf{E}} \equiv \nabla\cdot\underset{\circ}{\mathbf{E}} = 0, \quad \mathrm{curl}\,\underset{\times}{\mathbf{E}} \equiv \nabla\times\underset{\times}{\mathbf{E}} = 0. \qquad (1.1)$$

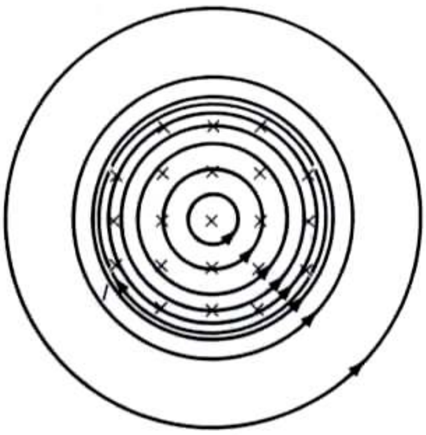

Fig. 32.22 Electric field lines (black) within solenoid and outside of solenoid.

Fig. 1. It is from [3]
The electric field is generated by a time-dependent magnetic field. The field lines form closed loop, the field lines do not start on electric charges.

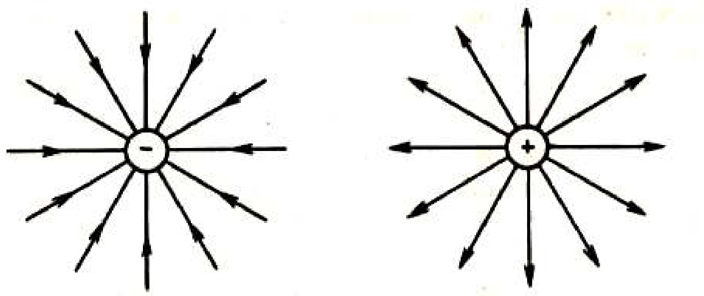

Рис. 1

Fig. 2. It is from [4]. Irrotational vector fields.
The field lines start on electric charges and diverge.

Transverse or solenoidal fields are usually denoted by $\mathbf{E}_t$ or $\mathbf{E}_\perp$, and longitudinal or irrotational fields are denoted by $\mathbf{E}_l$ or $\mathbf{E}_\parallel$, however, we use the circle $\circ$ and the cross $\times$ for marking solenoidal and irrotational vector fields respectively because this notations remind pictures of field tubes (or lines) of these fields (see Fig. 1 from [3] for a solenoidal field and Fig. 2 from [4] for irrotational fields). We are sure these visual notations are appropriate for a pedagogical paper. Moreover, in accordance with Fig. 1, we name divergence-free fields *closed* fields. Thus, the circle $\circ$ marks a closed field.

Note that

$$\mathrm{curl}\,\underset{\circ}{\mathbf{E}} \equiv \nabla\times\underset{\circ}{\mathbf{E}} = -\dot{\mathbf{B}}, \quad \mathrm{div}\,\underset{\times}{\mathbf{E}} \equiv \nabla\cdot\underset{\times}{\mathbf{E}} = \rho/\varepsilon_0. \qquad (1.2)$$

Helmholtz's decomposition is well known. The irrotational part of a vector field $\mathbf{E}$ is

$$\underset{\times}{\mathbf{E}}(x) = -\nabla\int\frac{\nabla'\cdot\mathbf{E}(x')dV'}{4\pi r(x,x')}. \qquad (1.3)$$

Meanwhile, two different expressions are known for the solenoidal part of $\mathbf{E}$:

$$\underset{\circ}{\mathbf{E}}(x)=\nabla\times\left(\nabla\times\int\frac{\mathbf{E}(x')dV'}{4\pi r(x,x')}\right)\ [1, 5], \tag{1.4}$$

$$\underset{\circ}{\mathbf{E}}(x)=\nabla\times\int\frac{\nabla'\times\mathbf{E}(x')dV'}{4\pi r(x,x')}\ [2, 6 - 9]. \tag{1.5}$$

Here x means the coordinates x, y, z; the prime marks the variables of integrating or differentiating under the integral sign; $r(x.x') = |x - x'|$ is the distance between x' and x. Thus, a field is integrated and differentiated by turns when the field is under the Helmgoltz's decomposition.

We pay attention that a similar consecution of integrating and differentiating takes place when an exterior differential form ω (briefly, form) is decomposed into closed and not closed parts [10, 11]:

$$\omega = dK\omega + Kd\omega = \underset{\circ}{\omega} + \underset{+}{\omega} \tag{1.6}$$

(we mark closed parts of forms by the circle and not closed parts by the plus sign). This formula is very important in the theory of exterior differential forms. In Eq. (1.6), d means the exterior derivative, and K is an operation which is an inverse operation to the exterior derivative in the following sense: if $\omega = \underset{\circ}{\omega}$ is a closed form, i.e. $d\omega = 0$, then

$$\underset{\circ}{\omega} = d\underset{+}{\alpha} \quad \& \quad K\underset{\circ}{\omega} = \underset{+}{\alpha} \quad \text{entails} \quad \underset{+}{\alpha} = Kd\underset{+}{\alpha} \quad \& \quad \underset{\circ}{\omega} = dK\underset{\circ}{\omega}. \tag{1.7}$$

K-operator exists in domains which are not too complicated topologically, according to the Poincare theorem.

We present here an example of K-operator using tensor indices. If ω is a 3-form, $\omega_{ijk}(x)$, then

$$K^i\omega_{ijk} = \int_0^1 t^2 x^i \omega_{ijk}(tx)dt. \tag{1.8}$$

We name K-operator the Poincare generative operator, we name $K^i\omega_{ijk}$ the Poincare generation from ω_{ijk}, and we name ω_{ijk} a source of $K^i\omega_{ijk}$. Note that the Poincare generative operator does not contain a metric tensor.

It is easy to show that a double application of the K-operator yields zero, i.e. $KK = 0$. For example,

$$K^j K^i \omega_{ijk} = \int_0^1 \tau x^j \left[\int_0^1 t^2 x^i \omega_{ijk}(tx)dt\right]_{x=\tau x} d\tau = 0 \tag{1.9}$$

because $x^j x^i \omega_{ijk} = 0$. We say that the generation from the generation is zero, or that the generation is *sterile*. So, K eliminates the sterile part of a form ω while d eliminates the closed part of a form:

$$d(\underset{\circ}{\omega} + \underset{+}{\omega}) = d\underset{+}{\omega}, \quad K(\underset{\circ}{\omega} + \underset{+}{\omega}) = K\underset{\circ}{\omega}. \tag{1.10}$$

Thus, Eq. (1.6) is the decomposition of a form ω into the closed part $\underset{\circ}{\omega}$ and the Poincare sterile part $\underset{+}{\omega}$.

A purpose of this paper is to show that the Helmholtz's decomposition is a decomposition into closed and sterile fields as well.

2. Differential forms, tensor densities, the boundaries, the conjugation, *etc*

It is important to recognize that the electromagnetism involves geometrical quantities [12] of two different types [13]. These are covariant (antisymmetric) tensors φ, E_i, A_i, B_{ij}, which are named also exterior differential forms or simply forms, and contravariant (antisymmetric) tensor densities: $B^{ik}_{\wedge}$, $j^i_{\wedge}$, $E^i_{\wedge}$, $\rho_{\wedge}$.

The distinction between forms and tensor densities is known long since. For example, professor J. A. Schouten delivered lectures on this subject before the war at Delft, and after the war at Amsterdam (see the classical monograph [12], which was grown from the lectures, and Fig. 3). A similar interpretation for covectors is presented in [14] (Fig. 4). Note that the magnitude of a covector is proportional to the *density* of sheets. Therefore, a covector fields must be depicted by a family of *bisurfaces* with an outer orientation and bivector densities must be depicted by a family of bisurfaces with an inner orientation rather than by lines or tubes.

The geometric images of **F**, **D**, **H**, and **B** are

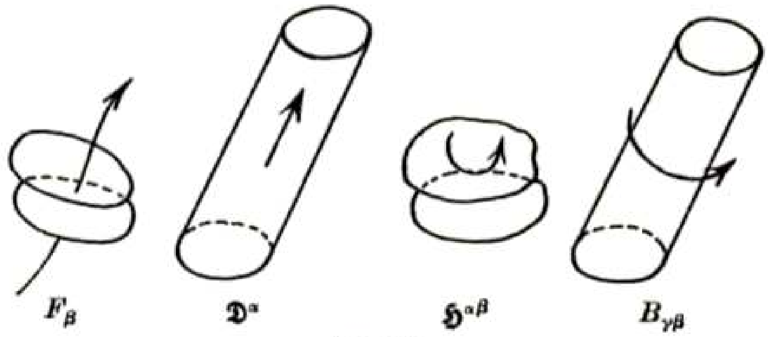

FIG. 23.

Fig. 3. It is from [12].
The distinction between differential forms and contravariant tensor densities is presented. We denote $\mathbf{F} \longrightarrow E_i$, $\mathbf{D} \longrightarrow E^i_\wedge$, $\mathbf{H} \longrightarrow B^{ij}_\wedge$, $\mathbf{B} \longrightarrow B_{ij}$

Unfortunately, this distinction is ignored by most of the physicists.

So, common boldfaced characters do not represent the quantities adequately, and we are forced to use tensor indices. Besides, instead of using of Gothic characters (as in Fig. 3), it is convenient to mark the density by the symbol 'wedge' $\wedge$ at the level of bottom indices for a density of weight $+1$ and at the level of top indices for a density of weight -1. For example, volume element $dV^\wedge$ is a density of weight -1. Also we mark pseudo forms by the asterisk * and pseudo densities by the tilde ~: E^*_{ij}, $\varepsilon_{ijk}^{\sim}$.

Now consider a covector. This should be familiar to most students in terms of a gradient. We can picture a gradient best in terms of the equipotential surfaces to which it refers, and this is the basis of the pictorial representation. That is, draw the surfaces themselves, along with some sense of direction, which might be indicated by a wavy line with an arrow at the end, or with a whorl on one of the sheets:

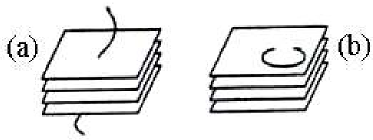

Fig. 4. It is from [14]. The authors refer to this pictorial representation of a covector as a "lasagna vector". Covector (a) has an outer orientation. (b) represents a pseudo covector; it has an inner orientation.

The exterior derivation of the forms is used in the electrodynamics. The exterior derivation of a scalar is the common partial derivation,

$$E_i = \partial_i \phi \Leftrightarrow \mathbf{E} = \operatorname{grad} \phi = \nabla \cdot \phi \tag{2.1}$$

(we do not write minus in this formula, so we write $\rho = \nabla^2 \phi$), but in a general case an antisymmetrization is implied. The notion "curl" can be used with a covector,

$$B_{ij} = 2\partial_{[i} A_{j]} \Leftrightarrow \mathbf{B} = \operatorname{curl} \mathbf{A} = \nabla \times \mathbf{A}\,, \quad \dot{B}_{ij} = -2\partial_{[i} E_{j]} \Leftrightarrow \dot{\mathbf{B}} = -\operatorname{curl} \mathbf{E} = -\nabla \times \mathbf{E}\,, \tag{2.2}$$

we denote $\partial_{[i} A_{j]} = \frac{1}{2}(\partial_i A_j - \partial_j A_i)$. The exterior derivation of a covariant tensor of valence 2 is a divergence,

$$3\partial_{[k} B_{ij]} = 0 \Leftrightarrow \operatorname{div} \mathbf{B} = \nabla \cdot \mathbf{B} = 0\,, \tag{2.3}$$

we denote $\partial_{[k} B_{ij]} = \frac{1}{3}(\partial_k B_{ij} + \partial_i B_{jk} + \partial_j B_{ki})$.

As to tensor densities, a transvection over a (last) contravariant index is used in the electrodynamics when the specific derivation is performed. The derivation of a vector density is named divergence:

$$\rho_\wedge = \partial_i E^i_\wedge \Leftrightarrow \rho = \operatorname{div} \mathbf{E}\,, \tag{2.4}$$

but the derivation of a bivector density is denote by curl,

$$j^i_\wedge = \partial_k B^{ik}_\wedge \Leftrightarrow \mathbf{j} = \operatorname{curl} \mathbf{B}\,. \tag{2.3}$$

The derivation of a scalar density is zero,

$$\partial_i \rho_\wedge \equiv 0, \tag{2.4}$$

because $\rho_\wedge$ has no contravariant indexes. We emphasize that all presented differential operations are covariant operations. Their writing is valid no matter what coordinates are in use, Euclidean or curvilinear. The Cristoffel symbols are not needed.

For short, we will designate the derivations of the both types by the symbol ∂, curly d, without indices. We name a derived field a *boundary*, and we name the field under derivation the *filling* of the boundary, i.e. (boundary) = ∂ (filling), for example,

$$\rho_\wedge = \partial E^i_\wedge, \quad E_i = \partial\phi, \quad \dot{B}_{ij} = -\partial E_j, \quad j^i_\wedge = \partial B^{ik}_\wedge, \quad 0 = \partial B_{ij}. \tag{2.5}$$

The term "boundary" is justified, for example, by the fact that lines (or tubes) of force of $E^i_\wedge$-field are bounded by a charge density $\rho_\wedge$, according to $\rho_\wedge = \partial E^i_\wedge$. This example is depicted in Fig. 2 where the electric charge bounds the electric vector field **E.** Thus, the symbol ∂ expresses the relation between a boundary and its filling, i.e. ∂ is a boundary operator.

Our symbol ∂, instead of d, means the exterior derivation when it is applied to a form. We are convinced it is anti-pedagogical to use the symbol d as a designation of the exterior derivation. The symbol d is used for infinitesimal quantities in physics and mathematics, not for a derivation. For example, $dq = \rho(x)dV$ denotes the charge of an infinitesimal volume dV. Another example: one can write $\mathbf{r} + \mathbf{v}dt = \mathbf{r} + d\mathbf{r}$ where $\mathbf{v}$ is a velocity, $d\mathbf{r} = \mathbf{i}dx + \mathbf{j}dy + \mathbf{k}dz$ is an infinitesimal increment of the vector $\mathbf{r}$, and dx, dy, dz are the infinitesimal increments of the coordinates. Also $df(x) = f'(x)dx$, and we can write $d(df) = d^2 f = f''(dx)^2$ or even $d^2 f = f''(dx)^2 + f' d^2 x$.

Contrary to this, d is used as an operator which takes each p-form ω to a $(p+1)$-form $d\omega$, and $d(d\omega) = 0$ forever [10, 11]. Accordingly, the expressions dx, dy, dz are known as a nonindex notation for the coordinate 1-forms, i.e., covectors, rather than as the components of the infinitesimal vector $d\mathbf{r}$, i.e. dx, dy, dz are known as $dx = \partial_i x = \delta^1_i,\ dy = \partial_i y = \delta^2_i,\ dz = \partial_i z = \delta^3_i$. This mishmash is inadmissible.

On the other side, the symbol ∂ means "boundary" in the theory of sets. And this is the very meaning that our symbol ∂ has.

If the boundary of a field is zero, we say that the field is *closed,* for example, B_{ij} is closed: $\partial B_{ij} = 3\partial_{[k} B_{ij]} = 0$. An example of a closed electric field is presented in Fig. 1, $\partial_i \underset{\circ}{E}{}^i_\wedge = 0$. Lines of force of the induced (solenoidal) vector field $\underset{\circ}{\mathbf{E}} = \underset{\circ}{E}{}^i_\wedge$ have no boundaries, and this field has no boundary. In accordance with Section 1, we mark a closed field by a circle.

The double derivation gives zero, $\partial\partial = 0$. For example, if $\underset{\circ}{E}{}_i = \partial_i\phi$, then $\partial_{[k} \underset{\circ}{E}{}_{i]} = \partial^2_{[ki]}\phi = 0$. We say that the boundary of a boundary is zero, or that a boundary is closed. A boundary has no boundary, but a boundary has a filling, according to the Poincare theorem. In the case, $\underset{\circ}{E}{}_i = \partial_i\phi$, ϕ is a filling of the boundary $\underset{\circ}{E}{}_i$. This case is depicted in Fig. 11. Another example: if $\underset{\circ}{E}{}^i_\wedge = \partial_j \Pi^{ij}_\wedge$, then $\partial_i \underset{\circ}{E}{}^i_\wedge = 0$, i.e. $\underset{\circ}{E}{}^i_\wedge$ is a boundary, and the vector electric potential $\Pi^{ij}_\wedge$ is the filling.

The raising and lowering of tensor indices is usually performed by a metric tensor g^{ik} or g_{ik}. But, in the electrodynamics, this process is accompanied by the transition between differential forms and contravariant densities, for example, between the covector E_i and the vector density $E^i_\wedge$. So this process uses the root of the metric tensor determinant $\sqrt{g}_\wedge$, which is a scalar density of weight $+1$. So, the tensor densities $g^{ik}_\wedge = g^{ik}\sqrt{g}_\wedge$ or $g^\wedge_{ik} = g_{ik}/\sqrt{g}_\wedge$ is used instead of g^{ik} or g_{ik},. If Cartesian coordinates are in use, the absolute value of the determinant equals one, but the root has a specific geometrical properties. The process of the raising or lowering of tensor indices changes the geometrical sense of a field; it is referred to as *the conjugation* here, and we designate the process by the *Courier* star * (in contrast to the Hodge star operation *), for example,

$$\star E_i = g^{ik}_{\wedge} E_i = E^k_{\wedge}, \quad \star E^k_{\wedge} = g^{\wedge}_{ik} E^k_{\wedge} = E_i, \quad \star B^{mn}_{\wedge} = g_{mi} g^{\wedge}_{nj} B^{mn}_{\wedge} = B_{ij}, \quad \star B_{ij} = g^{im} g^{jn}_{\wedge} B_{ij} = B^{mn}_{\wedge}. \tag{2.6}$$

The conjugation is obviously involutory: $\star\star = \mathbf{1}$. We say that a field and the conjugate field make up a tandem. For example (E_i & $E^i_{\wedge} = \star E_i$) and ($B^{ij}_{\wedge}$ & $B_{ij} = \star B^{ij}_{\wedge}$) are tandems (see Fig. 5 and Fig. 6).

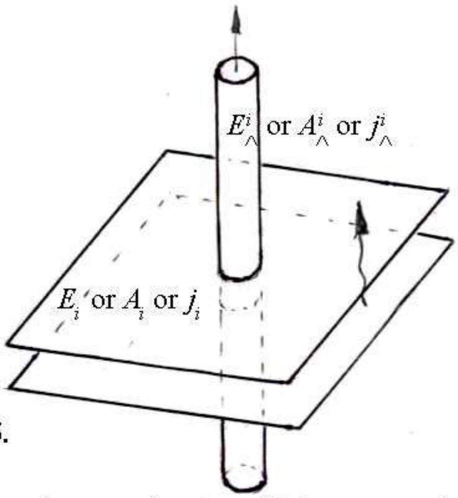

Fig. 5.

Covector and vector density, which are mutual conjugated. For example, $E^i_{\wedge} = \star E_i$
The element of a bisurface is orthogonal to the field tube. An outer orientation of the bisurface is coordinated with the inner orientation of the tube.

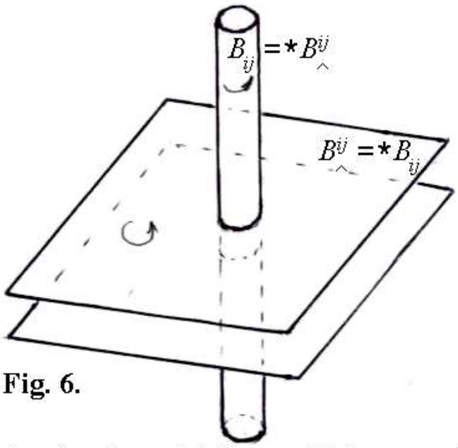

Fig. 6.

Bivector density and 2-form, which are mutual conjugated. The element of a bisurface is orthogonal to the field tube. An inner orientation of the bisurface is coordinated with the outer orientation of the tube.

The conjugation * differ from the Hodge star operation * [10, 11, 15]. Hodge operator performs our conjugation of a field and then renumbers components of the field by the antisymmetric tensor pseudo density $\varepsilon^{\sim}_{ijk}$ (Levi Civita density). For example, $* E_i = \varepsilon^{\sim}_{mnj} g^{ij}_{\wedge} E_i = E^*_{mn}$. Here Hodge operator transforms a 1-form E_i into the pseudo 2-form E^*_{mn}. However, the renumbering has no physical and geometrical meaning because E^*_{mn} has the same geometrical meaning as the vector density $E^j_{\wedge} = \star E_i$ (Fig. 5). Therefore, the addition of the renumbering to the conjugation has no sense. We do not use the renumbering and so we reduced Hodge operation to the conjugation. Note, the Hodge operator cannot be applied to a tensor density.

It is important that when the Hodge operator is applied two times in the structure $*\partial*$, the result differs from $\star\partial\star$ by a sign only [15, p. 315]:

$$* \partial * \overset{p}{\omega} = (-1)^{np+n+p+1} \star \partial \star \overset{p}{\omega}, \tag{2.7}$$

where n is the dimension of the space and p is the degree of the form $\overset{p}{\omega}$. Because a so-called *codifferential* is defined as

$$\delta \overset{p}{\omega} = (-1)^{np+n+1} * \partial * \overset{p}{\omega}, \tag{2.8}$$

we have for the codifferential

$$\delta \overset{p}{\omega} = (-1)^{p} \star \partial \star \overset{p}{\omega}. \tag{2.9}$$

It is remarkable that the conjugation often transforms a closed field into a not closed field. For example,

$$\partial B_{ij} = 0, \text{ but } \partial \star B_{ij} = \partial_n (g^{im} g^{jn}_{\wedge} B_{ij}) = \partial_n B^{mn}_{\wedge} = \mu_0 j^m_{\wedge}. \tag{2.10}$$

Such fields, closed before or after conjugation, is named *conjugate-closed* fields, or, simply, *coclosed* fields: B_{ij} is closed, but $B^{mn}_{\wedge}$ is coclosed because $\partial \star B^{mn}_{\wedge} = \partial B_{ij} = 0$.

Now recall Helmholtz's theorem (1.1). It must be written down in terms of vector densities,

$$E^i_{\wedge} = \underset{\circ}{E}{}^i_{\wedge} + \underset{\times}{E}{}^i_{\wedge}. \tag{2.11}$$

The solenoidal vector field $\underset{\circ}{\mathbf{E}} = \underset{\circ}{E}{}^{i}_{\wedge}$, which satisfies $\mathrm{div}\underset{\circ}{\mathbf{E}} \equiv \nabla \cdot \underset{\circ}{\mathbf{E}} = 0$, $\mathrm{curl}\underset{\circ}{\mathbf{E}} \equiv \nabla \times \underset{\circ}{\mathbf{E}} = -\dot{\mathbf{B}}$ is closed, $\partial_i \underset{\circ}{E}{}^{i}_{\wedge} = 0$. The irrotational Coulomb vector field $\underset{\times}{\mathbf{E}} = \underset{\times}{E}{}^{i}_{\wedge}$, which satisfies $\mathrm{curl}\underset{\times}{\mathbf{E}} \equiv \nabla \times \underset{\times}{\mathbf{E}} = 0$, $\mathrm{div}\underset{\times}{\mathbf{E}} \equiv \nabla \cdot \underset{\times}{\mathbf{E}} = \rho / \varepsilon_0$, is coclosed:

$$\nabla \times \underset{\times}{\mathbf{E}} = 2\partial_{[k}(g_{j]i}\, \underset{\times}{E}{}^{i}_{\wedge}) = \partial \star \underset{\times}{\mathbf{E}} = 0 . \tag{2.12}$$

Thus, the mark $\times$, which was used already in (1.1), marks the coclosed fields.

As a result, we see that Helmholtz's decomposition (1.1) is a decomposition into closed and coclosed components.

It is important that Helmholtz's decomposition (1.1) can be rewritten in the conjugate form in term of covectors

$$\star\mathbf{E} = \star\underset{\circ}{\mathbf{E}} + \star\underset{\times}{\mathbf{E}} \quad \Leftrightarrow \quad \star E^{i}_{\wedge} = \star\underset{\circ}{E}{}^{i}_{\wedge} + \star\underset{\times}{E}{}^{i}_{\wedge} \quad \Leftrightarrow \quad E_i = \underset{\times}{E}{}_i + \underset{\circ}{E}{}_i . \tag{2.13}$$

Here $\underset{\times}{E}{}_i$, the first component of decomposition (2.13), which corresponds to the induced solenoidal closed vector field, is not closed now; it has a boundary: $-\underset{\circ}{\dot{B}}{}_{ji} = 2\partial_{[j}\, \underset{\times}{E}{}_{i]}$ (1.2). This boundary is the time-dependent magnetic field, and the bisurfaces, which depict the covector field $\underset{\times}{E}{}_i$, are bounded by tubes of force of the magnetic field, the bisurfaces start from tubes of force of the magnetic field in Fig. 7. These bisurfaces is orthogonal to lines of force in Fig. 1.The field $\underset{\times}{E}{}_i$, is obviously coclosed: $\partial_k g^{ki}_{\wedge}\, \underset{\times}{E}{}_i = 0$.

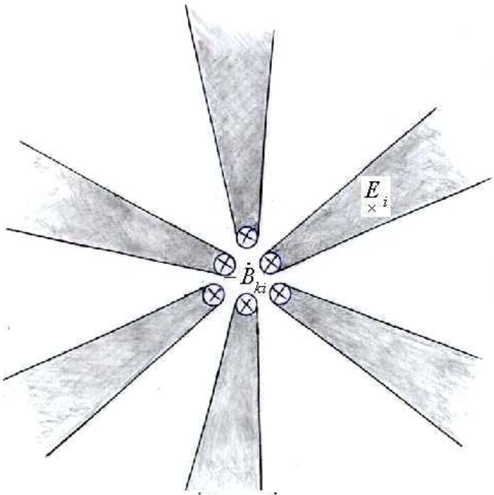

Fig. 7. The induced covector field has the boundary. However, as a vector field, this field is a solenoidal one, and it has no boundary in Fig. 1.

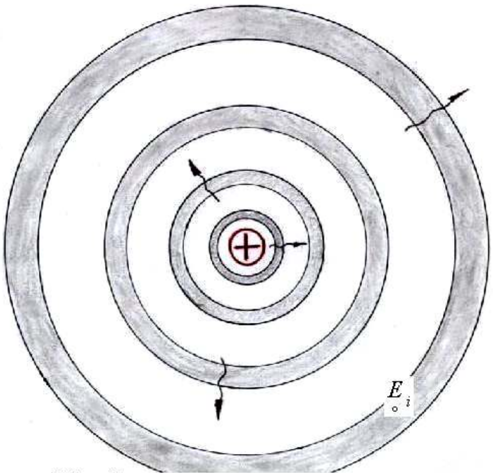

Fig. 8. This field was an irrotational vector field in Fig. 2. However, as a covector field, this field is closed here.

The second component in (2.13), $\underset{\circ}{E}{}_i$, which corresponds to the irrotational vector field $\underset{\times}{\mathbf{E}}$, is closed now because Eq. (1.1), $\mathrm{curl}\underset{\times}{\mathbf{E}} \equiv \nabla \times \underset{\times}{\mathbf{E}} = 0$, means $2\partial_{[i} g^{\wedge}_{j]k}\, \underset{\times}{E}{}^{k}_{\wedge} = 2\partial_{[i}\, \underset{\circ}{E}{}_{j]} = 0$. Accordingly, $\underset{\circ}{E}{}_i$-field is depicted by spherical bisurfaces in Fig. 8. These bisurfaces is orthogonal to lines of force in Fig. 2.

3. Chains of fields

The property of the conjugation to transform closed fields into not closed fields leads to an existence of infinite or finite chains of fields. We present here, as an example, the infinite chain of the electrostatic.

$$\cdots(\partial_i)\delta(x)(\star)\delta_{\wedge}(x)(\partial_i)\underset{\times}{G}{}^{i}_{\wedge}(\star)\underset{\circ}{G}{}_i(\partial_i)\rho(\star)\rho_{\wedge}(\partial_i)\underset{\times}{E}{}^{i}_{\wedge}(\star)\underset{\circ}{E}{}_i(\partial_i)\phi(\star)\phi_{\wedge}(\partial_i)\underset{\times}{F}{}^{i}_{\wedge}(\star)\underset{\circ}{F}{}_i(\partial_i)\cdots \tag{3.1}$$

The sections of the chain, i.e. $\delta_{\wedge}(x)$, $\underset{\times}{G}{}^{i}_{\wedge}$, ρ, *etc.*, are joined by the symbols (∂_i) and (*). It means, for example, $\rho_{\wedge} = \partial_i \underset{\times}{E}{}^{i}_{\wedge}$, $\underset{\times}{E}{}^{i}_{\wedge} = \star \underset{\circ}{E}{}_i$, $\underset{\circ}{E}{}_i = \partial_i \phi$. In (3.1), $\delta(x)$ is the Dirac delta function if the electric charge density $\rho_{\wedge}$ equals $\rho_{\wedge} = -1/4\pi r$. The explicit form of the chain in this case is

$$\cdots(\partial_i)\delta(x)(\star)\delta_\wedge(x)(\partial_i)\frac{r^i}{4\pi r^3}(\star)\frac{r_i}{4\pi r^3}(\partial_i)\frac{-1}{4\pi r}(\star)\frac{-1}{4\pi r}(\partial_i)\frac{-r^i}{8\pi r}(\star)\frac{-r_i}{8\pi r}(\partial_i)\frac{-r}{8\pi}(\star)\frac{-r}{8\pi}(\partial_i)\frac{-rr^i}{32\pi}(\star)\frac{-rr_i}{32\pi}(\partial_i)\cdots \tag{3.2}$$

Really, a coclosed electric intensity, corresponding to the density $\rho_\wedge = -1/4\pi r$, is $\underset{\times}{E}{}^{i}_{\wedge} = -x^i/8\pi r$ because
$\partial_i(-r^i/8\pi r) = -1/4\pi r$; the corresponding potential is $\phi = -r/8\pi$; the density $\phi_\wedge = \star\phi = -r/8\pi$ is the boundary of a hypothetical coclosed field $\underset{\times}{F}{}^{i}_{\wedge} = -rr^i/32\pi$; *etc*. On the other hand, the boundary of $\rho = \star\rho_\wedge = -1/4\pi r$ is $\underset{\circ}{G}{}_i = \partial_i(-1/4\pi r) = r_i/4\pi r^3$; the boundary of $\star\underset{\circ}{G}{}_i = \underset{\times}{G}{}^{i}_{\wedge} = r^i/4\pi r^3$ is the δ-function: $\delta_\wedge(x) = \partial_i(r^i/4\pi r^3)$; *etc*.:

We can present a complementary electrostatic chain:

$$\cdots(\partial_k)\underset{\times}{G}{}_i(\star)\underset{\circ}{G}{}^{i}_{\wedge}(\partial_k)\underset{\times}{J}{}^{ik}_{\wedge}(\star)\underset{\circ}{J}{}_{ik}(\partial_i)\underset{\times}{E}{}_k(\star)\underset{\circ}{E}{}^{k}_{\wedge}(\partial_i)\underset{\times}{\Pi}{}^{ki}_{\wedge}(\star)\underset{\circ}{\Pi}{}_{ki}(\partial_k)\underset{\times}{F}{}_i(\star)\underset{\circ}{F}{}^{i}_{\wedge}(\partial_k)\cdots. \tag{3.3}$$

Here $\underset{\circ}{E}{}^{k}_{\wedge}$ is a closed vector density, $\underset{\circ}{E}{}^{k}_{\wedge} = \partial_i\underset{\times}{\Pi}{}^{ki}_{\wedge}$, i.e. $\mathbf{E} = \mathrm{curl}\Pi$, and Π is so-called electric vector potential. Magnetic current density $\underset{\circ}{J}{}_{ik}$ is the boundary of the coclosed covector electric intensity $\underset{\times}{E}{}_k = \star\underset{\circ}{E}{}^{k}_{\wedge}$: $\underset{\circ}{J}{}_{ik} = 2\partial_{[i}\underset{\times}{E}{}_{k]}$, i.e. $\mathbf{J} = \mathrm{curl}\,\mathbf{E}$.

We can present also a chain that is complementary to chain (3.3):

$$\cdots(\partial_k)\underset{\times}{\varsigma}{}^{ijk}_{\wedge}(\star)\underset{\circ}{\varsigma}{}_{ijk}(\partial_i)\underset{\times}{J}{}_{jk}(\star)\underset{\circ}{J}{}^{jk}_{\wedge}(\partial_i)\underset{\times}{\eta}{}^{jki}_{\wedge}(\star)\underset{\circ}{\eta}{}_{jki}(\partial_j)\underset{\times}{\Pi}{}_{ki}(\star)\underset{\circ}{\Pi}{}^{ki}_{\wedge}(\partial_j)\underset{\times}{\theta}{}^{kij}_{\wedge}(\star)\underset{\circ}{\theta}{}_{kij}(\partial_k)\cdots. \tag{3.4}$$

Here the closed electric bivector potential $\underset{\circ}{\Pi}{}^{ki}_{\wedge}$ is the boundary of a hypothetic $\underset{\times}{\theta}{}^{kij}_{\wedge}$-field: $\underset{\circ}{\Pi}{}^{ki}_{\wedge} = \partial_j\underset{\times}{\theta}{}^{kij}_{\wedge}$. This can be expressed as $\Pi^* = \mathrm{grad}\,\theta^*$ where Π^* is a pseudo covector and θ^* is a pseudo scalar because $\Pi^*_j = \varepsilon^{\sim}_{jki}\underset{\circ}{\Pi}{}^{ki}_{\wedge}/2$, $\theta^* = \varepsilon^{\sim}_{kij}\underset{\times}{\theta}{}^{kij}_{\wedge}/6$.

Two complementary magnetostatic chains can be obtained by renaming of sections of chains (3.3) and (3.4),

$$E \Rightarrow j,\ \Pi \Rightarrow B,\ F \Rightarrow A,\ \eta \Rightarrow \underset{m}{\rho}:$$

$$\cdots(\partial_i)\underset{\times}{j}{}_k(\star)\underset{\circ}{j}{}^{k}_{\wedge}(\partial_i)\underset{\times}{B}{}^{ki}_{\wedge}(\star)\underset{\circ}{B}{}_{ki}(\partial_k)\underset{\times}{A}{}_i(\star)\underset{\circ}{A}{}^{i}_{\wedge}(\partial_k)\cdots, \tag{3.5}$$

$$\cdots(\partial_i)\underset{m}{\rho}{}^{jki}_{\wedge}(\star)\underset{m}{\rho}{}_{jki}(\partial_j)\underset{\times}{B}{}_{ki}(\star)\underset{\circ}{B}{}^{ki}_{\wedge}(\partial_j)\underset{\times}{\theta}{}^{kij}_{\wedge}(\star)\underset{\circ}{\theta}{}_{kij}(\partial_k)\cdots. \tag{3.6}$$

Here $\underset{m}{\rho}{}_{jki} = 3\partial_{[j}\underset{\times}{B}{}_{ki]}$ is the magnetic charge, $\underset{m}{\rho} = \mathrm{div}\mathbf{B}$, and $\underset{\times}{\theta}{}^{kij}_{\wedge}$ is the magnetic pseudo scalar potential: $\underset{\circ}{B}{}^{ki}_{\wedge} = \partial_j\underset{\times}{\theta}{}^{kij}_{\wedge}$, i.e. $\mathbf{B} = \mathrm{grad}\,\theta$.

4. The Laplace operator

The derivation of a sum of a closed and a coclosed field equals the derivation of the coclosed term only, because the derivation eliminates closed term, like (1.10),

$$\partial_i(\underset{\circ}{E}{}^{i}_{\wedge} + \underset{\times}{E}{}^{i}_{\wedge}) = \partial_i\underset{\times}{E}{}^{i}_{\wedge}. \tag{4.1}$$

However, Laplacian, the second order operator, $\nabla^2 = g^{ij}\partial_i\partial_j$, treats both terms of such a sum. As is known [15, p. 316],

$$\nabla^2 = -\delta\partial - \partial\delta \tag{4.2}$$

(the signature of $g^{ij} = +++$). Here δ is the codifferential (2.8), (2.9). Because of (2.9),

$$\delta\partial\overset{p}{\omega} = \delta\overset{p+1}{\alpha} = (-1)^{p+1}\star\partial\star\overset{p+1}{\alpha} = -(-1)^p\star\partial\star\partial\overset{p}{\omega},\quad \partial\delta\overset{p}{\omega} = (-1)^p\partial\star\partial\star\overset{p}{\omega}. \tag{4.3}$$

Therefore, for a p-form $\overset{p}{\omega}$,

$$\nabla^2 \overset{p}{\omega} = (-1)^p (\star\partial\star\partial - \partial\star\partial\star)\overset{p}{\omega}, \tag{4.4}$$

see also [16]. It is easy to show that for a contravariant density of valence p $\overset{p}{\beta}_\wedge$,

$$\nabla^2 \overset{p}{\beta}_\wedge = (-1)^{p+1} (\star\partial\star\partial - \partial\star\partial\star)\overset{p}{\beta}_\wedge . \tag{4.5}$$

Thus, Laplacian realizes a transition to four sections of a chain to the left and, maybe, changes the sign. For example, according to (3.2) and (4.5) for $p = 0$,

$$\delta_\wedge(x) = \partial\star\partial\star \frac{-1}{4\pi r} = \nabla^2 \frac{-1}{4\pi r}, \tag{4.6}$$

according to (3.5) and (4.4) for $p = 1$,

$$\underset{\times}{j}_i = \star\partial\star\partial \underset{\times}{A}_i = -\nabla^2 \underset{\times}{A}_i . \tag{4.7}$$

or, according to (3.5) and (4.5) for $p = 1$ [1, (5.31)]

$$\underset{\circ}{j}{}^i_\wedge = \partial\star\partial\star \underset{\circ}{A}{}^i_\wedge = -\nabla^2 \underset{\circ}{A}{}^i_\wedge . \tag{4.8}$$

If the vector potential is not satisfied (3-dimension) Lorentz gauge, $\partial_i A^i_\wedge \neq 0$, then, according to (4.5) and [1, (5.30)],

$$\nabla^2 A^i_\wedge = \star\partial\star\partial A^i_\wedge - \partial\star\partial\star A^i_\wedge = \operatorname{grad}\operatorname{div}\mathbf{A} - \mathbf{j}. \tag{4.9}$$

5. The generations

Given a closed differential form or contravariant density, we can find their fillings by an integral generative operator

$$\dagger^i = \int \frac{dV^{\wedge'} r^i(x,x')}{4\pi r^3(x,x')}, \tag{5.1}$$

instead of K^i from (1.7). For example, the filling of $\delta_\wedge(x)$ (see (3.2)) is

$$\int \delta_{\wedge'}(x') \frac{r^i_\wedge(x,x')\, dV^{\wedge'}}{4\pi r^3(x,x')} = \frac{r^i_\wedge(x)}{4\pi r^3(x)} . \tag{5.2}$$

We say that $\delta_\wedge(x)$ is a *source* which *generates* $\frac{r^i_\wedge(x)}{4\pi r^3}$ and write $\dagger^i \delta_\wedge(x) = \frac{r^i_\wedge(x)}{4\pi r^3}$, or $\delta_\wedge(x) \to \frac{r^i_\wedge(x)}{4\pi r^3}$.
Generally, we say that a source generates the generation, i.e. †(source) = (generation).

Next example: $\frac{r_i}{4\pi r^3}$ generates $\frac{-1}{4\pi r}$, i.e. $\dagger^i \frac{r_i}{4\pi r^3} = \frac{-1}{4\pi r}$, i.e. $\frac{r_i}{4\pi r^3} \to \frac{-1}{4\pi r}$, i.e.

$$\int \frac{r_{i'}(x')}{4\pi r^3(x')} \frac{r^i(x,x')dV'}{4\pi r^3(x,x')} = -\int \partial_{i'} \frac{1}{4\pi r(x')} \frac{r^i(x,x')dV'}{4\pi r^3(x,x')} = \int \frac{1}{4\pi r(x')} \partial_{i'} \frac{r^i(x,x')dV'}{4\pi r^3(x,x')}$$
$$= -\int \frac{1}{4\pi r(x')} \delta(x,x')dV' = \frac{-1}{4\pi r(x)}, \tag{5.3}$$

because

$$\partial_{i'} \frac{r^i(x,x')dV'}{4\pi r^3(x,x')} = -\partial_i \frac{r^i(x,x')dV'}{4\pi r^3(x,x')}$$

Thus, we can rewrite the chain (3.2), for example, in terms of the generation instead of the bourdary operator ∂.

$$\cdots \to \delta(x)(\star)\delta_\wedge(x) \to \frac{r^i}{4\pi r^3}(\star)\frac{r_i}{4\pi r^3} \to \frac{-1}{4\pi r}(\star)\frac{-1}{4\pi r} \to \frac{-r^i}{8\pi r}(\star)\frac{-r_i}{8\pi r} \to \frac{-r}{8\pi}(\star)\frac{-r}{8\pi} \to \frac{-rr^i}{32\pi}(\star)\frac{-rr_i}{32\pi} \to \cdots \tag{5.4}$$

We can say, in particular, that ρ is a source of the vector field $\mathbf{E}$, or ρ generates $\mathbf{E}$, or $\mathbf{E}$ is the generation from ρ:

$$\dagger^i \rho_{\wedge} = \int \rho_{\wedge'}(x') \frac{r^i_{\wedge}(x,x')\,dV^{\wedge'}}{4\pi r^3} = \underset{\times}{E}{}^i_{\wedge}\,. \tag{5.5}$$

This generating is depicted in Fig. 2. Electric charge emits lines of force of $\mathbf{E}$. It is a source of $\mathbf{E}$. This example shows visually that *generations are coclosed*, i.e.

$$\partial \star \dagger = 0\,. \tag{5.6}$$

Really, bisurfaces, which are orthogonal to the emitted lines, are closed (Fig. 8). This assertion is proved by a simple identity $g_{i[j}\partial_{k]}\dfrac{r^i}{r^3} = 0$. Accordingly, we mark generations by the cross $\times$.

The generating $\mathbf{B} \to \mathbf{A}$ (magnetic field generates the magnetic covector potential), according to (3.5),

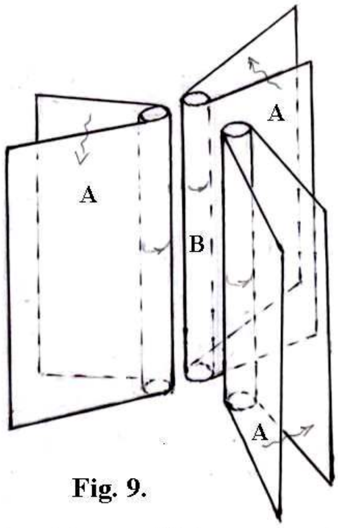

Fig. 9.

Tubes of force B, which is a source, emit bisurfaces A, which is the generation

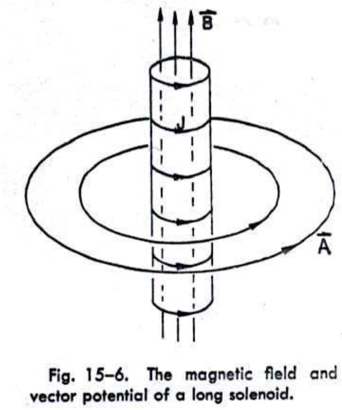

Fig. 10. This is from [17]
Lines of the vector field **A** depict the field $A^i_{\wedge} = \star A_i$, which is the conjugation of the covector field A_i from Fig. 9.

$$\dagger^i B_{ik} = \int B_{i'k} \frac{r^{i'}_{\wedge'}(x,x')\,dV^{\wedge'}}{4\pi r^3} = \underset{\times}{A}{}_k \quad (\mathbf{B} \text{ generates } \mathbf{A}), \tag{5.7}$$

is depicted in Fig. 9. B_{ik} determines the potential $\underset{\times}{A}{}_k$ uniquely. This potential stands out against a background of all gauge equivalent vector potentials [16].

However, the magnetic vector potential usually is depicted by lines of force of the vector $\underset{\circ}{A}{}^i$ as in Fig. 10 from [17], but Fig. 9 shows visually how **B-**tubes emit bisurfaces of $\underset{\times}{A}{}_i$.

Formula (5.7) is analogous to the Biot-Savarat law.

$$\dagger^k j^i_{\wedge} = 2\int j^{[i'}_{\wedge'} \frac{r^{k']}_{\wedge}(x,x')\,dV^{\wedge'}}{4\pi r^3} = \underset{\times}{B}{}^{ik}_{\wedge} \quad (\mathbf{j} \text{ generates } \mathbf{B}), \tag{5.8}$$

The only distinction between (5.8) and (5.7) is that B_{ik}-tubes and A_k-bisurfaces have an outer orientation, but $j^i_{\wedge}$-tubes and $B^{ik}_{\wedge}$-bisurfaces have an inner orientation.

A scalar field $\phi(x)$ is depicted by hatching or darkening of the space. Fig. 11 shows how the closed electric field $\underset{\circ}{E}{}_i = \partial_i\phi$ bounds $\phi(x)$, i.e. how ϕ-field fills the closed $\underset{\circ}{E}{}_i$-field, or how $\underset{\circ}{E}{}_i$ generates ϕ,

$$\dagger^i E_i = \int E_{i'} \frac{r^{i'}_{\wedge'}(x,x')\,dV^{\wedge'}}{4\pi r^3} = \phi(x) \quad (\mathbf{E} \text{ generates } \phi); \tag{5.9}$$

E_i is a source of ϕ and, at the same time, ϕ is a filling of $\underset{\circ}{E}{}_i$. Note, Eq. (5.9) determines the potential $\phi(x)$ uniquely as well as Eq. (5.7) determines $\underset{\times}{A}{}_k$.

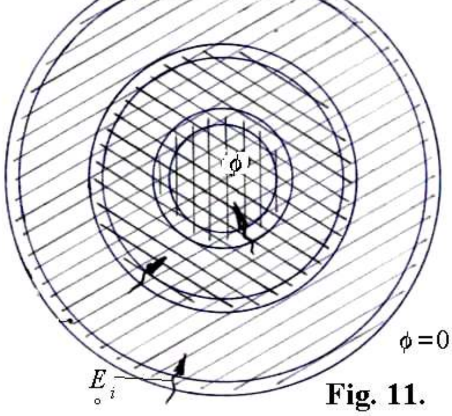

Fig. 11.
The closed covector field $\underset{\circ}{E}_i$ generates the scalar function ϕ

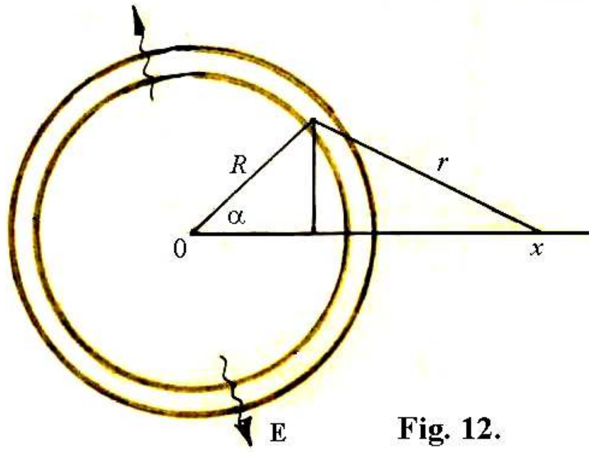

Fig. 12.
A two-dimensional "spherical" capacitor generates potential $\phi(x)$

As an example, we apply Eq. (5.9) for a solution of the problem: “What potential is generated by a thin two-dimensional ‘spherical’ capacitor?” So, in a two-dimensional (for simplicity) space there are two concentric circles between which a given radial electric field **E** exists. It is necessary to find the potential $\phi(x)$ in this space by the formula

$$\phi(x) = \int \frac{(\mathbf{E}\cdot\mathbf{r})da}{2\pi r^2}, \tag{5.10}$$

where da is an element of the space (plane). We have (see Fig 12):

$$\mathbf{E}\cdot\mathbf{r} = E_x r^x + E_y r^y = E\cos\alpha\cdot(x - R\cos\alpha) - E\sin\alpha\cdot R\sin\alpha = E(x\cos\alpha - R), \tag{5.11}$$

$$r^2 = R^2\sin^2\alpha + (x - R\cos\alpha)^2 = R^2 + x^2 - 2xR\cos\alpha.. \tag{5.12}$$

If we write δ for the small gap between the circles, then $da = R\delta d\alpha.$, and

$$\phi = \frac{ER\delta}{2\pi x}\int_0^{2\pi}\frac{\cos\alpha + v/2}{u + v\cos\alpha}d\alpha, \quad u = R^2/x^2 + 1, \quad v = -2R/x. \tag{5.13}$$

Integrating yields:

$$\phi(x) = \frac{ER\delta}{x}\left[-\frac{x}{2R} + \frac{(R^2+x^2)x}{2R(|R^2-x^2|)} - \frac{Rx}{|R^2-x^2|}\right]. \tag{5.14}$$

I.e. $\phi = 0$ on the outside of the circles, that is at $R < x$, and $\phi = -E\delta$ inside the circles, that is at $R > x$, just as expected.

The generations have an important property: the generation generates zero,

$$\dagger\dagger = 0. \tag{5.15}$$

In other words, the generations are *sterile* as well as the Poincare generation (1.7). We prove this assertion here for $\underset{\times}{E}{}^{i}_{\wedge}$ from (5.5), i.e. we prove that the integral

$$\dagger^{[i}\underset{\times}{E}{}^{k]}_{\wedge} \equiv \int \frac{\underset{\times}{E}{}^{[i}_{\wedge}(x')r^{k]}_{\wedge'}(x,x')dV^{\wedge'}}{4\pi r^3(x,x')} \tag{5.16}$$

equals zero. Indeed, inserting $\underset{\times}{E}{}^{i}_{\wedge}$ from (5.5) into (5.16) gives

$$\dagger^{[i}\underset{\times}{E}{}^{k]}_{\wedge} \equiv \iint \frac{\rho_{\wedge''}(x'')r^{[i}_{1\wedge}(x',x'')r^{k]}_{\wedge'}(x,x')dV^{\wedge''}dV^{\wedge'}}{4\pi r_1^3(x',x'')\cdot 4\pi r^3(x,x')} = 0. \tag{5.17}$$

To prove the last equality, fix the points x'' and x. Then, because of the symmetry of the space, for each x' exists such $\tilde{x}'$ that the vector product $r_1^{[i}r^{k]}$ in x' and $\tilde{x}'$ differ in the sign only. So, integrating over $dV^{\wedge'}$ gives zero.

It can be shown that coclosed fields are sterile, i.e.

$$\dagger\star\partial = 0\,. \tag{5.18}$$

As a simple example, consider the constant coclosed density $\underset{\times}{\psi}_{\wedge} = 1$. We have:

$$\dagger^{i}\underset{\times}{\psi}_{\wedge} = \int\frac{r^{i}(x,x')dV'}{r^{3}} = 0\,, \tag{5.19}$$

because of the symmetry of the space.

Thus, generating eliminates the sterile part of a source as well as the derivation eliminates the closed part of a filling (4.1). Only closed part of a source generates. For example,

$$\dagger^{i}\,(\underset{\times}{E}_{i} + \underset{\circ}{E}_{i}) = \dagger^{i}\,\underset{\circ}{E}_{i} = \underset{\times}{\phi}. \tag{5.20}$$

Therefore, we can calculate the potential of a non-potential field by the generative operator. The solenoidal part $\underset{\circ}{\mathbf{E}}$ of the vector field (1.1), which satisfies $\mathrm{div}\underset{\circ}{\mathbf{E}} = 0$, $\mathrm{curl}\underset{\circ}{\mathbf{E}} = -\dot{\mathbf{B}}$, and which correspond to sterile $\underset{\times}{E}_{i}$ in Eq. (2.13), may be present in the integrand of Eq. (5.9), but its contribution is zero. In contrast to Eq. (5.8), the standard formula $\varphi = \int_{x}^{0}\mathbf{E}d\mathbf{l}$ gives an ambiguous result for a non-potential field $\mathbf{E}$.

According to Eq. (5.20),

$$\underset{\circ}{E}_{i} = \partial_{i}\underset{\times}{\phi} \;\;\&\;\; \dagger^{i}\,\underset{\circ}{E}_{i} = \underset{\times}{\phi} \;\;\text{entails}\;\; \underset{\times}{\phi} = \dagger^{i}\,\partial_{i}\,\underset{\times}{\phi} \;\;\&\;\; \underset{\circ}{E}_{i} = \partial_{i}\dagger^{j}\,\underset{\circ}{E}_{j}\,. \tag{5.21}$$

In that sense, $\dagger$ and ∂ are the mutually inverse operators (compare with (1.7)). For example, we have $\dagger^{i}\,\partial_{j}\dfrac{r^{j}_{\wedge}}{4\pi r^{3}} = \dfrac{r^{i}_{\wedge}}{4\pi r^{3}}$ and $\dagger^{i}\,\partial_{i}\dfrac{-1}{4\pi r} = \dfrac{-1}{4\pi r}$.

If relations between E and ϕ are $E = \partial\phi$, as in (3.2) – (3.6), *and* $E \to \phi$, as in (5.4), we will write $E \mapsto \phi$ and so on. For example ,

$$\cdots \mapsto \underset{\times}{G}{}^{k}_{\wedge}\star\underset{\circ}{G}_{k} \mapsto \rho\star\rho_{\wedge} \mapsto \underset{\times}{E}{}^{i}_{\wedge}\star\underset{\circ}{E}_{i} \mapsto \phi\star\phi_{\wedge} \mapsto \underset{\times}{F}{}^{k}_{\wedge}\star\underset{\circ}{F}_{k} \mapsto \cdots \tag{5.22}$$

$$\cdots \mapsto \underset{\times}{G}_{k}\star\underset{\circ}{G}{}^{k}_{\wedge} \mapsto \underset{\times}{J}{}^{ki}_{\wedge}\star\underset{\circ}{J}_{ki} \mapsto \underset{\times}{E}_{i}\star\underset{\circ}{E}{}^{i}_{\wedge} \mapsto \underset{\times}{\Pi}{}^{ik}_{\wedge}\star\underset{\circ}{\Pi}_{ik} \mapsto \underset{\times}{F}_{k}\star\underset{\circ}{F}{}^{k}_{\wedge} \mapsto \cdots. \tag{5.23}$$

instead of (3.1), (3.3)

6. The generative operator squared

We define the generative operator squared, $\ddagger$, by the equations (compare with (4.4), (4.5)) $\dagger$

$$\ddagger\overset{p}{\omega} = (-1)^{p+1}(\star\dagger\star\dagger - \dagger\star\dagger\star)\overset{p}{\omega}\,, \tag{6.1}$$

$$\ddagger\overset{p}{\beta}_{\wedge} = (-1)^{p}(\star\dagger\star\dagger - \dagger\star\dagger\star)\overset{p}{\beta}_{\wedge}\,. \tag{6.2}$$

The result follows from the definition:

$$\ddagger\nabla^{2} = \nabla^{2}\ddagger = 1\,. \tag{6.3}$$

Really, for examlpe,

$$\ddagger\nabla^{2}\overset{p}{\omega} = (-1)^{p+1}(\star\dagger\star\dagger - \dagger\star\dagger\star)(-1)^{p}(\star\partial\star\partial - \partial\star\partial\star)(\underset{\circ}{\overset{p}{\omega}} + \underset{\times}{\overset{p}{\omega}}) = \star\dagger\star\dagger\,\partial\star\partial\star\underset{\circ}{\overset{p}{\omega}} + \dagger\star\dagger\star\star\partial\star\partial\underset{\times}{\overset{p}{\omega}} = \overset{p}{\omega}\,, \tag{6.4}$$

because of (4.1), (5.18). Eqs. (6.3) yields

$$\ddagger = -\int\frac{dV^{\wedge'}}{4\pi r(x,x')}\,. \tag{6.5}$$

Thus, the generative operator squared makes a transition to four sections of a chain to the right and, maybe, changes the sign. For example, according to (3.2) and (6.2) for $p = 0$,

$$\ddagger\underset{\circ}{\delta}_{\wedge}(x) = \star\dagger\star\dagger\underset{\circ}{\delta}_{\wedge}(x) = \frac{-1}{4\pi r}\,,\;\text{ i.e. }\;\int\delta_{\wedge'}(x')\frac{dV^{\wedge'}}{4\pi r(x,x')} = \frac{1}{4\pi r}\,. \tag{6.6}$$

According to (3.5) and (6.1) for $p = 1$,

$$\ddagger \underset{\times}{j}_{i} = -\dagger\star\dagger\star \underset{\times}{j}_{i} = -\underset{\times}{A}_{i}, \text{ i.e. } \int \frac{j_{i'}dV^{\wedge'}}{4\pi r(x,x')} = A_i \text{ [1, (5.32)].} \tag{6.7}$$

7. Various variants of Helmholtz's decomposition

There are many different ways of the Helmholtz's decomposition. Recall formula (1.1) $\mathbf{E} = \underset{\circ}{\mathbf{E}} + \underset{\times}{\mathbf{E}}$.

The simplest decomposition of the vector (density) field is

$$\mathbf{E} = \underset{\circ}{\mathbf{E}} + \underset{\times}{\mathbf{E}} = (\partial\dagger + \dagger\partial)\mathbf{E}, \quad \text{i.e.} \quad \underset{\circ}{\mathbf{E}} = \partial\dagger\mathbf{E}, \;\; \underset{\times}{\mathbf{E}} = \dagger\partial\mathbf{E}. \tag{7.1}$$

This decomposition does not use the conjugation. An explicit form of the decomposition is

$$\underset{\circ}{\mathbf{E}} = \underset{\circ}{E}{}^{i}_{\wedge} = \partial\dagger E^i_\wedge = 2\partial_k \int \frac{E^{[i'}_{\wedge'} r^{k]}_{\wedge} dV^{\wedge'}}{4\pi r^3} = \nabla\times\int\frac{\mathbf{E}'\times\mathbf{r}\,dV'}{4\pi r^3}, \tag{7.2}$$

$$\underset{\times}{\mathbf{E}} = \underset{\times}{E}{}^{i}_{\wedge} = \dagger\partial E^i_\wedge = \int \frac{\partial_{k'} E^{k'}_{\wedge'} r^i_\wedge dV^{\wedge'}}{4\pi r^3} = \int \frac{(\nabla'\cdot\mathbf{E}')\mathbf{r}dV'}{4\pi r^3}. \tag{7.3}$$

It is depicted in Figures 13 and 14.

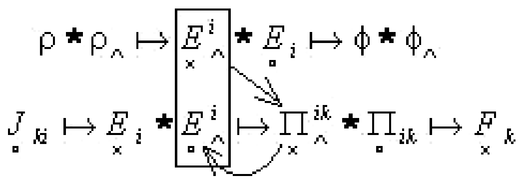

Fig. 13. $\underset{\square}{\mathbf{E}} = \partial\dagger\mathbf{E}$

$\mathbf{E}$ generates Π, and the boundary of Π is $\underset{\square}{\mathbf{E}}$

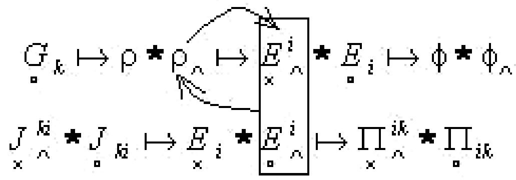

Fig. 14. $\underset{\times}{\mathbf{E}} = \dagger\partial\mathbf{E}$

The boundary of $\mathbf{E}$ is ρ, which generates $\underset{\times}{\mathbf{E}}$

There is another decompisition of a vector field, which uses operators $\partial, \dagger$ only:

$$\mathbf{E} = \underset{\circ}{\mathbf{E}} + \underset{\times}{\mathbf{E}} = (\star\dagger\partial\star + \star\partial\dagger\star)\mathbf{E}, \quad \text{i.e.} \quad \underset{\circ}{\mathbf{E}} = \star\dagger\partial\star\mathbf{E}, \;\; \underset{\times}{\mathbf{E}} = \star\partial\dagger\star\mathbf{E}. \tag{7.4}$$

An explicit form of the decomposition is

$$\underset{\circ}{E}{}^{i}_{\wedge} = \star\dagger\partial\star E^l_\wedge = 2g^{ij}\int\frac{\partial_{[k'} g_{j]l'} E^{l'}_{\wedge} r^{k'}_{\wedge'} dV^{\wedge'}}{4\pi r^3} = \int\frac{(\nabla'\times\mathbf{E}')\times\mathbf{r}dV'}{4\pi r^3}, \tag{7.5}$$

$$\underset{\times}{E}{}^{i}_{\wedge} = \star\partial\dagger\star E^i_\wedge = g^{il}\partial_l \int\frac{g_{k'j'} E^{j'}_{\wedge} r^{k'}_{\wedge'} dV^{\wedge'}}{4\pi r^3} = \nabla\int\frac{(\mathbf{E}'\cdot\mathbf{r})dV'}{4\pi r^3}. \tag{7.6}$$

Eq. (7.5) is depicted in Fig.15.

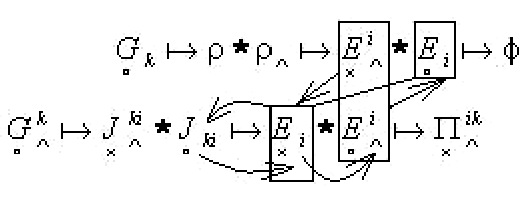

Fig. 15. $\underset{\square}{\mathbf{E}} = \star\dagger\partial\star\mathbf{E}$

$E^i_\wedge$ is conjugated to $E_i = \star E^i_\wedge$.

The boundary of E_i is $\underset{\square}{J}_{ki}$,

which generates $\underset{\times}{E}_i$, and $\underset{\square}{E}{}^{i}_{\wedge} = \star\underset{\times}{E}_i$

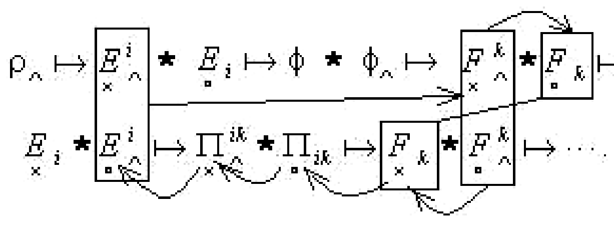

Fig. 16. $\underset{\square}{\mathbf{E}} = -\partial\star\partial\star\ddagger\mathbf{E}$.

$-\ddagger$ makes the transition from $\mathbf{E}$ to $\mathbf{F}$.

$\underset{\square}{\Pi}$ is the boundary of $\star\mathbf{F}$, and

$\underset{\square}{E}{}^{i}_{\wedge}$ is the boundary of $\underset{\times}{\Pi}$

Eqs. (1.3), (1.4) use the generative operator squared, ‡, and two the boundary operators, ∂, e.g., for (1.4), $\underset{\circ}{E}{}^{i}_{\wedge} = -\partial\star\partial\star\ddagger E^i_\wedge$. It is depicted in Fig. 16. (Note, according to (6.2), $\ddagger\underset{\circ}{E}{}^{i}_{\wedge} = -\star\dagger\star\dagger\underset{\circ}{E}{}^{i}_{\wedge}$).

Eq. (1.5) uses the operators ‡ and ∂ in another order then Eq. (1.4): $\underset{\circ}{E}{}^{i}_{\wedge} = -\partial\star\ddagger\partial\star E^i_\wedge$.

Also we can offer the operator ∇^2 and two operators † for obtaining $\underset{\circ}{E}{}^{i}_{\wedge}$, for example,

$$\underset{\circ}{E}{}^{i}_{\wedge} = \star\dagger\star\dagger\nabla^2 E^{i}_{\wedge} = \int\left(\int\frac{(\nabla^2\mathbf{E}''\times\mathbf{r}')dV''}{4\pi(r')^3}\right)\times\mathbf{r}\frac{dV'}{4\pi r^3}. \tag{7.7}$$

8. An example of the use of the simplest decomposition (7.1)

Consider a semi-infinite straight thin wire carrying an electric current I along the positive z-axis. Let the current density $\mathbf{j}$ is singular in the wire territory:

$$j_z = I\,\delta(R,0) \ \text{ if } \ z>0, \ \text{ and } \ j_z = 0 \ \text{ if } \ z<0, \tag{8.1}$$

where $R=\sqrt{x^2+y^2}$, and $\delta(R,0)$ satisfies $\int\delta(R,0)\,2\pi R dR = 1$. So,

$$\nabla\mathbf{j} = -\partial_t\rho = I\,\delta(x,0), \tag{8.2}$$

where $\delta(x,0)$ satisfies $\int\delta(x,0)dV = 1$.

Our aim is to decompose the density $\mathbf{j}$ into solenoidal and irrotational parts by applying Eq. (7.1) to $\mathbf{j}$:

$$\mathbf{j} = \underset{\circ}{\mathbf{j}} + \underset{\times}{\mathbf{j}} = \nabla\times\int\frac{\mathbf{j}'\times\mathbf{r}'(x,x')dV'}{4\pi r^3(x,x')} + \int\frac{(\nabla'\mathbf{j}')\mathbf{r}(x,x')dV'}{4\pi r^3(x,x')} = \nabla\times\mathbf{B} - \partial_t\mathbf{E}. \tag{8.3}$$

We have step by step. The Biot-Savarat law yields:

$$\mathbf{B} = \int\frac{\mathbf{j}'\times\mathbf{r}'(x,x')dV'}{4\pi r^3(x,x')} = \int\frac{I\,\mathbf{dl}'\times\mathbf{r}'(x,x')}{4\pi r^3(x,x')}, \tag{8.4}$$

$$B_x = \frac{I\,y}{4\pi}\int_{z'=0}^{\infty}\frac{dz'}{[x^2+y^2+(z-z')^2]^{3/2}} = -\frac{I\,y}{4\pi R^2}\left(1+\frac{z}{r}\right), \quad B_y = \frac{I\,x}{4\pi R^2}\left(1+\frac{z}{r}\right), \quad R^2 = (x^2+y^2). \tag{8.5}$$

$\underset{\circ}{\mathbf{j}} = \nabla\times\mathbf{B}$ for all points except the semi axis z ($z>0,\ x=y=0$), where the magnetic field is singular, is

$$\underset{\circ}{j}_x = -\partial_z B_y = -\frac{I\,x}{4\pi r^3}, \quad \underset{\circ}{j}_y = \partial_z B_x = -\frac{I\,y}{4\pi r^3}, \quad \underset{\circ}{j}_z = \partial_x B_y - \partial_y B_x = -\frac{I\,z}{4\pi r^3}, \quad \text{i.e.} \quad \underset{\circ}{\mathbf{j}} = -\frac{I\,\mathbf{r}}{4\pi r^3}. \tag{8.6}$$

$\nabla\times\mathbf{B}$ for $z>0,\ x=y=0$ can be determined from (8.5) by the Stokes theorem for a circle of radius R when $R\to 0$:

$$\nabla\times\mathbf{B} \qquad \int(\nabla\times\mathbf{B})\mathbf{da} = \oint\mathbf{B}\,\mathbf{dl} = \frac{I}{2\pi R^2}\int -y dx + x dy = \frac{I}{2\pi}\int_0^{2\pi}d\varphi = I. \tag{8.7}$$

Thus, according to (8.7), $(\nabla\times\mathbf{B})_z = I\delta(R,0)$. In other words, the solenoidal part $\underset{\circ}{\mathbf{j}}$ of the current $\mathbf{j}$,

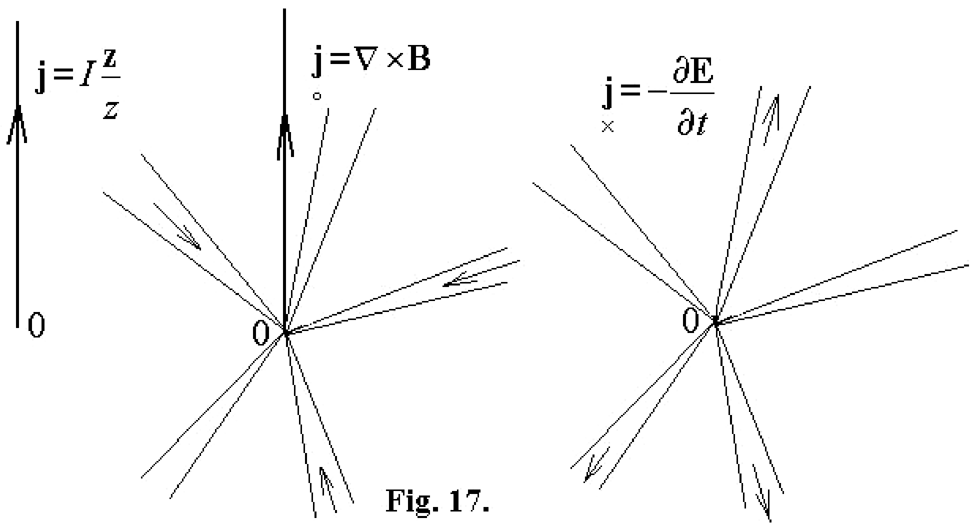

Fig. 17.

The decomposition of a straight current into solenoidal and irrotational parts

$$\underset{\circ}{\mathbf{j}} = -\frac{I\,\mathbf{r}}{4\pi r^3} + I\frac{\mathbf{z}}{z},\ \text{if } z>0,\quad \underset{\circ}{\mathbf{j}} = -\frac{I\,\mathbf{r}}{4\pi r^3},\ \text{if } z<0, \tag{8.8}$$

consists of radial converged field tubes (8.6) and the semi axis $z > 0$.

The irrotational part is

$$\underset{\times}{\mathbf{j}}(x) = \int \frac{I\,\delta(x',0)\mathbf{r}(x,x')dV'}{4\pi r^3(x,x')} = \frac{I\,\mathbf{r}(x)}{4\pi r^3(x)}. \tag{8.9}$$

The decomposition is depicted at Fig. 17. Field tubes are used instead of common field lines because the current density **j** is a vector density. One can see that the components of the decomposition extend over all space, despite **j** is localized. Many authors pointed out this fact [1].

9. The Minkowski space. Tandem-closed fields

Chains (3.1) – (3.4) use the Euclidean metric tensor for conjugating, $g_{ij} = \mathrm{diag}\{+1,+1,+1\}$, but chains of electromagnetic fields use $g_{\mu\nu} = \mathrm{diag}\{+1,-1,-1,-1\}$:

$$\dots\underset{\times}{j}_{\alpha}(\star)\underset{\circ}{j}{}^{\alpha}_{\wedge}(\partial)\underset{\times}{B}{}^{\alpha\beta}_{\wedge}(\star)\underset{\circ}{B}_{\mu\nu}(\partial)\underset{\times}{\mathrm{A}}_{\nu}(\star)\underset{\circ}{\mathrm{A}}{}^{\alpha}_{\wedge}(\partial)\underset{\times}{Q}{}^{\alpha\beta}_{\wedge}(\star)\underset{\circ}{Q}_{\mu\nu}(\partial)\underset{\times}{C}_{\nu}(\star)\underset{\circ}{C}{}^{\alpha}_{\wedge}(\partial)\dots \tag{9.1}$$

We denote the electromagnetic tensor $B_{\mu\nu} = -F_{\mu\nu}$ instead of $F_{\mu\nu}$ in order to $j^{\alpha}_{\wedge} = \partial_{\beta}B^{\alpha\beta}_{\wedge}$ instead of common $j^{\alpha}_{\wedge} = -\partial_{\beta}F^{\alpha\beta}_{\wedge}$, and our magnetic vector 4-potential A_{ν} satisfies $B_{\mu\nu} = 2\partial_{[\mu}\mathrm{A}_{\nu]}$. So, e.g., $B_{31} = B^2 = \partial_3\mathrm{A}_1 - \partial_1\mathrm{A}_3$, and $\mathrm{A}_i = A_i$, $\mathrm{A}^i = -A^i$; $B_{10} = E_1 = \partial_1\mathrm{A}_0 - \partial_0\mathrm{A}_1$, and $\mathrm{A}_0 = \mathrm{A}^0 = \phi$.

In (9.1), Q and C are hypothetic fields.

Besides closed and coclosed fields, there are *tandem-closed* fields, i.e.fields, which are closed and coclosed simultaneously. We mark these fields by the pair of signs $\times\circ$. For example, an electromagnetic plane wave makes up an tandem-closed field $\underset{\times\circ}{B}{}^{\alpha\beta}_{\wedge}$, $\underset{\times\circ}{B}_{\mu\nu}$. Indeed, if

$$\underset{\times\circ}{B}{}^{01}_{\wedge} = E^1 = e^{iz-it},\quad \underset{\times\circ}{B}{}^{31}_{\wedge} = B_2 = e^{iz-it},\quad \underset{\times\circ}{B}_{01} = -e^{iz-it},\quad \underset{\times\circ}{B}_{31} = -\underset{\times\circ}{B}_{13} = e^{iz-it}, \tag{9.2}$$

then

$$\partial_{\beta}\underset{\times\circ}{B}{}^{1\beta}_{\wedge} = \partial_0\underset{\times\circ}{B}{}^{10}_{\wedge} + \partial_3\underset{\times\circ}{B}{}^{13}_{\wedge} = 0,\quad \partial_{[\lambda}\underset{\times\circ}{B}_{\mu\nu]} = \partial_3\underset{\times\circ}{B}_{10} + \partial_0\underset{\times\circ}{B}_{31} = 0. \tag{9.3}$$

In this case, both the fields of a tandem are closed. We name such a tandem an *end-tandem* because a chain ends at the tandem. Obviously, an end-tandem is the end of two (complementary) chains. For example

$$0(\partial)\underset{\times\circ}{B}{}^{\alpha\beta}_{\wedge}(\star)\underset{\times\circ}{B}_{\mu\nu}(\partial)\underset{\times}{\mathrm{A}}_{\nu}(\star)\underset{\circ}{\mathrm{A}}{}^{\alpha}_{\wedge}(\partial)\underset{\times}{Q}{}^{\alpha\beta}_{\wedge}(\star)\underset{\circ}{Q}_{\mu\nu}(\partial)\underset{\times}{C}_{\nu}(\star)\underset{\circ}{C}{}^{\alpha}_{\wedge}(\partial)\dots \tag{9.4}$$

$$0(\partial)\underset{\times\circ}{B}_{\mu\nu}(\star)\underset{\times\circ}{B}{}^{\alpha\beta}_{\wedge}(\partial)\underset{\times}{\Pi}{}^{\alpha\beta\gamma}_{\wedge}(\star)\underset{\circ}{\Pi}_{\lambda\mu\nu}(\partial)\underset{\times}{Q}_{\mu\nu}(\star)\underset{\circ}{Q}{}^{\alpha\beta}_{\wedge}(\partial)\underset{\times}{R}{}^{\alpha\beta\gamma}_{\wedge}(\star)\underset{\circ}{R}_{\lambda\mu\nu}(\partial)\dots \tag{9.5}$$

Electromagnetic field $\underset{\times\circ}{B}_{\mu\nu}$ in chain (9.4) is the boundary of the magnetic vector potential $\underset{\times}{\mathrm{A}}_{\nu}$: $\underset{\times\circ}{B}_{\mu\nu} = 2\partial_{[\mu}\underset{\times}{\mathrm{A}}_{\nu]}$, $\underset{\times}{\mathrm{A}}_1 = -\underset{\circ}{\mathrm{A}}{}^{1}_{\wedge} = -ie^{iz-it}$, however the same electromagnetic field $\underset{\times\circ}{B}{}^{\alpha\beta}_{\wedge}$ in chain (9.5) is the boundary of the electric three-vector potential $\underset{\times}{\Pi}{}^{\alpha\beta\gamma}_{\wedge}$: $\underset{\times\circ}{B}{}^{\alpha\beta}_{\wedge} = \partial_{\gamma}\underset{\times}{\Pi}{}^{\alpha\beta\gamma}_{\wedge}$, $\underset{\times}{\Pi}{}^{103}_{\wedge} = \underset{\circ}{\Pi}_{103} = ie^{iz-it}$. Some next sections of the chains are

$$\underset{\times}{Q}{}^{10}_{\wedge} = -\underset{\circ}{Q}_{10} = \frac{-1+i(t+z)}{4}e^{iz-it},\quad \underset{\times}{Q}{}^{13}_{\wedge} = \underset{\circ}{Q}_{13} = \frac{1+i(t+z)}{4}e^{iz-it},\quad \underset{\times}{C}_{1} = -\underset{\circ}{C}{}^{1}_{\wedge} = -\frac{(t+z)}{4}e^{iz-it}, \tag{9.6}$$

$$\underset{\times}{Q}_{10} = -\underset{\circ}{Q}{}^{10}_{\wedge} = \frac{1+i(t+z)}{4}e^{iz-it},\quad \underset{\times}{Q}_{13} = \underset{\circ}{Q}{}^{13}_{\wedge} = \frac{1-i(t+z)}{4}e^{iz-it},\quad \underset{\times}{R}{}^{103}_{\wedge} = \underset{\circ}{R}_{103} = -\frac{(t+z)}{4}e^{iz-it}, \tag{9.7}$$

Because Laplacian realizes a transition to four sections of a chain to the left and, maybe, changes the sign, we have, according to (9.4),

$$\underset{\times\circ}{B}{}^{\alpha\beta}_{\wedge} = \star\partial\star\partial\underset{\times}{Q}{}^{\alpha\beta}_{\wedge} = -\nabla^2\underset{\times}{Q}{}^{\alpha\beta}_{\wedge}. \tag{9.8}$$

But, the same field $\underset{\times\circ}{B}{}^{\alpha\beta}_{\wedge}$,according to (9.5), equals

$$\underset{\times\circ}{B}{}^{\alpha\beta}_{\wedge}=\partial\star\partial\star\underset{\circ}{Q}{}^{\alpha\beta}_{\wedge}=\nabla^2\underset{\circ}{Q}{}^{\alpha\beta}_{\wedge}\,. \tag{9.9}$$

Now we have arrived at an interesting conclusion:

Sum of similar terms of complementary chains is a harmonic field if the transition to four positions to the left from the terms comes to an end in an end-tandem. (9.10)

Really,

$$\nabla^2 Q^{\alpha\beta}_{\wedge}=\nabla^2(\underset{\times}{Q}{}^{\alpha\beta}_{\wedge}+\underset{\circ}{Q}{}^{\alpha\beta}_{\wedge})=-\underset{\times\circ}{B}{}^{\alpha\beta}_{\wedge}+\underset{\times\circ}{B}{}^{\alpha\beta}_{\wedge}=0\,. \tag{9.11}$$

By the way, we can consider the harmonic field $Q^{\alpha\beta}_{\wedge}$ as a plane polarized electromagnetic plane wave because it satisfies the wave equation (9.11). By analogy with (9.2) we have from (9.6), (9.7)

$$Q^{01}_{\wedge}=\underset{\times}{Q}{}^{01}_{\wedge}+\underset{\circ}{Q}{}^{01}_{\wedge}=E^1=\frac{1}{2}e^{iz-it},\quad Q^{31}_{\wedge}=\underset{\times}{Q}{}^{31}_{\wedge}+\underset{\circ}{Q}{}^{31}_{\wedge}=B_2=-\frac{1}{2}e^{iz-it}\,. \tag{9.12}$$

But it is a very strange wave. The Poynting vector, $\mathbf{S}=(\mathbf{E}\times\overline{\mathbf{B}})/2$, has the z-component directed opposite to the direction of wave propagation: $S_z=-1/8$, and the wave is accompanied by electric $\mathrm{j}^{\alpha}_{\wedge}$ and magnetic $\mathbf{J}_{\lambda\mu\nu}$ currents:

$$\mathrm{j}^{\alpha}_{\wedge}=\partial_\beta\underset{\times}{Q}{}^{\alpha\beta}_{\wedge}=\underset{\circ}{\mathrm{A}}{}^{\alpha}_{\wedge},\quad \mathbf{J}_{\lambda\mu\nu}=3\partial_{[\lambda}\underset{\times}{Q}{}_{\mu\nu]}=\underset{\circ}{\Pi}{}_{\lambda\mu\nu}\,. \tag{9.13}$$

We present here another example of the end-tandem which is very simple. Let $g_{ij}=\mathrm{diag}\{+1,+1\},\ i,j=1,2$. Then $a^i_{\wedge}$ and a_i are the tandem-closed fields if $a^1_{\wedge}=y,\ a^2_{\wedge}=x$. Really: $a_1=\star a^1_{\wedge}=y,\ a_2=\star a^2_{\wedge}=x$, and $\partial_i a^i_{\wedge}=0,\ \partial_{[i}a_{j]}=0$. So, we have two complementary chains:

$$0(\partial)\underset{\times\circ}{a}{}^{i}_{\wedge}(\star)\underset{\times\circ}{a}{}_{i}(\partial)b(\star)b_{\wedge}(\partial)\underset{\times}{c}{}^{i}_{\wedge}(\star)\underset{\circ}{c}{}_{i}(\partial)\cdots, \tag{9.14}$$

$$0(\partial)\underset{\times\circ}{a}{}_{i}(\star)\underset{\times\circ}{a}{}^{i}_{\wedge}(\partial)d^{ik}_{\wedge}(\star)d_{ik}(\partial)\underset{\times}{c}{}_{i}(\star)\underset{\circ}{c}{}^{i}_{\wedge}(\partial)\cdots, \tag{9.15}$$

where

$$b=xy,\ \underset{\times}{c}{}^{1}_{\wedge}=\underset{\circ}{c}{}_{1}=\frac{x^2y}{4}+\frac{y^3}{12},\ \underset{\times}{c}{}^{2}_{\wedge}=\underset{\circ}{c}{}_{2}=\frac{y^2x}{4}+\frac{x^3}{12},\ d^{12}_{\wedge}=d_{12}=-\frac{x^2}{2}+\frac{y^2}{2},\ \underset{\times}{c}{}_{1}=\underset{\circ}{c}{}^{1}_{\wedge}=-\frac{y^3}{6},\ \underset{\times}{c}{}_{2}=\underset{\circ}{c}{}^{2}_{\wedge}=-\frac{x^3}{6}\,. \tag{9.16}$$

According to (9.10), $c_i=\underset{\circ}{c}{}_i+\underset{\times}{c}{}_i$ must be harmonic. Really,

$$c_1=\underset{\circ}{c}{}_1+\underset{\times}{c}{}_1=\frac{x^2y}{4}-\frac{y^3}{12},\quad c_2=\underset{\circ}{c}{}_2+\underset{\times}{c}{}_2=\frac{y^2x}{4}-\frac{x^3}{12}, \tag{9.17}$$

and $\partial_{xx}c_1+\partial_{yy}c_1=\partial_{xx}c_2+\partial_{yy}c_2=0$. This is OK.

10. Conclusion

Usefulness of concepts of differential forms and tensor densities in the electromagnetism is shown. Concepts of boundary and its filling, source and its generation are introduced. These concepts extend an understanding of electrodynamics because they explain mutual relations between the electromagnetic fields.

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