Herding and Contrarian Behaviour in Financial Markets[*]

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Abstract

Rational herd behaviour and informationally efficient security prices have long been considered to be mutually exclusive but for exceptional cases. In this paper we describe the conditions on the underlying information structure that are necessary and sufficient for informational herding and contrarianism. In a standard sequential security trading model, subject to sufficient noise trading, people herd if and only if, loosely, their information is sufficiently dispersed so that they consider extreme outcomes more likely than moderate ones. Likewise, people act as contrarians if and only if their information leads them to concentrate on middle values. Both herding and contrarianism generate more volatile prices, and they lower liquidity. They are also resilient phenomena, although by themselves herding trades are self enforcing whereas contrarian trades are self-defeating. We complete the characterization by providing conditions for the absence of herding and contrarianism.

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1 Introduction

In times of great economic uncertainty, financial markets often appear to behave frantically, displaying substantial price spikes as well as drops. Such extreme price fluctuations are possible only if there are dramatic changes in behaviour with investors switching from buying to selling or the reverse. This pattern of behaviour and the resulting price volatility is often claimed to be inconsistent with rational traders and informationally efficient asset prices and is attributed to investors' animal instincts. We argue in this paper, however, that such behaviour can be the result of fully rational social learning where agents change their beliefs and behaviour as a result of observing the actions of others.

One example of social learning is herd behaviour in which agents switch behaviour (from buying to selling or the reverse) *following the crowd*. So-called "rational herding" can occur in situations with information externalities, when agents' private information is swamped by the information derived from observing others' actions. Such "herders" rationally act against their private information and follow the crowd.[1]

It is not clear, however, that such herd behaviour can occur in informationally efficient markets, where prices reflect all public information. For example, consider an investor with unfavorable private information about a stock. Suppose that a crowd of people buys the stock frantically. Such an investor will update his information, and upon observing many buys, his expectation of the value of the stock will rise. At the same time, prices also adjust upward. Then it is not clear that the investor buys — to him the security may still be overvalued. So, for herding private expectations and prices must diverge.

In models with only two states of the world, such divergence is impossible as prices always adjust so that there is no herding.[2] Yet two state models are rather special and herding can emerge once there are at least three states. In this paper we characterize the possibility of herding in the context of a simple, informationally efficient financial market. Moreover, we show that (i) during herding prices can move substantially and (ii) herding can induce lower liquidity and higher price volatility than if there were no herding.

Herd behaviour in our set-up is defined as any history-switching behaviour in the direction of the crowd (a kind of momentum trading).[3] Social learning can also arise as a result of traders switching behaviour by acting *against the crowd*. Such contrarian behaviour is the natural counterpart of herding, and we also characterize conditions for such behaviour. Contrary to received wisdom that contrarian behaviour is stabilizing, we also show that

[1] See Banerjee (1992) or Bikhchandani, Hirshleifer, and Welch (1992) for early work on herding.

[2] With two states the price will adjust so that it is always below the expectation of traders with favourable information and above the expectation of those with unfavourable information *irrespective* of what is observed. As we show, with more than two states, this strict separation no longer applies.

[3] We are concerned with short-term behavior. In the literature, there are also other definitions of herding; see Section 4 for a discussion.

contrarian behaviour leads to higher volatility and lower liquidity, just as herd behaviour.

The key insight of our characterization result is that social learning in financial markets occurs if and only if investors receive information that satisfies a compelling and intuitive property. Loosely, herding arises if and only if private information satisfies a property that we call "U-shaped." An investor who receives such information believes that extreme states are more likely to have generated the information than more moderate ones. Therefore, when forming his posterior belief, the recipient of such a signal will shift weight away from the center to the extremes so that the posterior distribution of the trader is "fat-tailed." The recipient of a U-shaped signal thus discounts the possibility of the intermediate value and as a consequence will update the probabilities of extreme values faster than an agent who receives only the public information.

Contrarianism occurs if and only if the investor's signal indicates that moderate states are more likely to have generated the signal than extreme states. We describe such signals as being "Hill-shaped." The recipient of a Hill-shaped signal always puts more weight on middle outcomes relative to the market so that this trader's posterior distribution becomes "thin-tailed." He thus discounts the possibility of extreme states and, therefore, updates extreme outcomes slower than the market maker.

We follow the microstructure literature and establish our results in the context of a stylized trading model in the tradition of Glosten and Milgrom (1985). In such models, the bid and ask prices are set by a competitive market maker. Investors trade with the market maker either because they receive private information about the asset's fundamental value or because they are "noise traders" and trade for reasons outside of the model.

The simplest possible Glosten-Milgrom trading model that allows herding or contrarianism is one with at least three states. For this case, we show that (i) a U-shaped (Hill-shaped) signal is necessary for herding (contrarianism) and (ii) herding (contrarianism) occurs with positive probability if there exists at least one U-shaped (Hill-shaped) signal and there is a sufficient amount of noise trading. The latter assumption on the minimum level of noise trading is not required in all cases and is made as otherwise the bid and ask spread may be too large to induce appropriate trading. In Section 9 we show that the intuition for our three states characterization carries over to a setup with an arbitrary number of states.

We obtain our characterization results without restrictions on the signal structure. In the literature on asymmetric information, it is often assumed that information structures satisfy the monotone likelihood ratio property (MLRP). Such information structures are "well-behaved" because, for example, investors' expectations are ordered. It may appear that such a strong monotonicity requirement would prohibit herding or contrarianism. Yet MLRP does not only admit the possibility of U-shaped signals (and thus herding) or Hill-shaped signals (and thus contrarianism), but also the trading histories that generate herding and contrarianism are significantly simpler to describe with than without MLRP signals.

Our second set of results concerns the impact of social learning on prices. We first show that the range of price movements can be very large during both contrarianism and herding. We then compare price movements in our set-up where agents observe one another with those in a hypothetical economy, that is otherwise identical to our set-up except that the informed traders do not switch behaviour. We refer to the former as the *transparent* economy and to the latter as the *opaque* economy. In contrast to the transparent economy, in the opaque economy there is no social learning by assumption. We show, for the case of MLRP, that once herding or contrarianism begins, prices respond more to individual trades relative to the situation without social learning so that price rises *and* price drops are greater in the transparent set-up than in the opaque one.[4] As a corollary, liquidity, measured by the inverse of the bid-ask-spread, is lower with social learning than without.

The price volatility and liquidity results may have important implications for the discussion on the merits of "market transparency." The price path in the opaque economy can be interpreted as the outcome of a trading mechanism in which people submit orders without knowing the behaviour of others or the market price. Our results indicate that in the less transparent setup, price movements are less pronounced and liquidity is higher.

While the results on price ranges, volatility and liquidity indicate similarities between herding and contrarianism, there is also a stark difference. Contrarian trades are self-defeating because a large number of such trades will cause prices to move "against the crowd" thus ending contrarianism. During herding, on the other hand, investors continue to herd when trades are "in the direction of the crowd," so herding is self-enforcing.

Examples of situations that generate U- and Hill-shaped signals. First, a U-shaped signal may be interpreted as a "volatility signal."[5] Very informally, an example is a signal that generates a mean preserving spread of a symmetric prior distribution. Conversely, a mean preserving signal that decreases the variance is Hill-shaped.

Second, U-shaped and Hill-shaped signals may also be good descriptions of situations with a potential upcoming event that has an uncertain impact. For example, consider the case of a company or institution that contemplates appointing a new leader who is an uncompromising "reformer". If this person takes power, then either the necessary reforms take place or there will be strife with calamitous outcomes. Thus the institution will not be the same as the new leader will be either very good or disastrous. Any private information signifying that the person is likely to be appointed exemplifies a U-shaped signal and any information revealing that this person is unlikely to be appointed (and thus the institution

[4]The increase in price-volatility associated with herding is only relative to a hypothetical scenario. Even when herding is possible, in the long-run volatility settles down and prices react less to individual trades. It is well known that the variance of Martingale price-processes such as ours is bounded by model primitives.

[5]We thank both Markus Brunnermeier and an anonymous referee for this interpretation.

will carry on as before) represents a Hill-shaped signal.[6]

Third, consider a financial institution FI that is a competitor to a bank that has recently failed. Suppose there are three possible scenarios: (i) FI will also fail because it has deals with the failed bank that will not be honored and/or that the business model of FI is as bad as that of the failed bank; (ii) FI's situation is entirely unrelated to the bank and the latter's collapse will not affect FI; and (iii) FI may benefit greatly from the bank's collapse as it is able to attract the failed bank's customers and most capable employees. Cases (i) and (iii) resemble extreme outcomes and case (ii) a middle outcome.

In this environment, some investor's information might have implied that the most likely outcome is either that FI will also go down as well or that it will benefit greatly from the failed banks' demise. Such information is an example of a U-shaped signal. Alternatively, some investors' assessments might have implied that the most likely outcome is that FI is unaffected. Such information is an example of a Hill-shaped signal.

It is conceivable that in the Fall of 2008 (after the collapse of Lehman) and early 2009 many investors believed that for individual financial institutions the two extreme states (collapse or thrive) were the most likely outcomes. Then our theory predicts the potential for herd behaviour, with investors changing behaviour in the direction of the crowd, causing strong short-term price fluctuations. Hill-shaped private signals, signifying that the institutions were likely to be unaffected, may also have occurred, inducing contrarianism and changes of behaviour against the crowd.

The mechanism that induces herding and contrarianism. Consider the above banking example and assume that all scenarios are equally likely. Let the value of the stock of FI in each of the three scenarios (i), (ii) and (iii) be $V_1 < V_2 < V_3$, respectively. We are interested in the behaviour of an investor, who has a private signal S, after different public announcements. Specifically, consider a good public announcement G that rules out the worst state, $\mathsf{Pr}(V_1|G) = 0$, and a bad public announcement B that rules out the best state, $\mathsf{Pr}(V_3|B) = 0$. Assume that the price of the stock is equal to the expected value of the asset conditional on the public information and that the investor buys (sells) if his expectation exceeds (is less than) the price. Note that the price will be higher after G and lower after B, compared to the ex-ante situation when all outcomes are equally likely.

Both G and B eliminate one state, so that, after each such announcement there are only two states left. In two state models, an investor has a higher (lower) expectation than the market if and only if his private information is more (less) favourable towards the better state than towards the worse state. Thus, in the cases of G and B, $\mathsf{E}[V|G] \lesseqgtr \mathsf{E}[V|S,G]$ is equivalent to $\mathsf{Pr}(S|V_2) \lesseqgtr \mathsf{Pr}(S|V_3)$ and $\mathsf{E}[V|S,B] \lesseqgtr \mathsf{E}[V|B]$ is equivalent to $\mathsf{Pr}(S|V_2) \lesseqgtr$

[6] Other examples are an upcoming merger or takeover with uncertain merits, the possibility of a government stepping down, announcements of FDA drug approvals, outcomes of lawsuits etc; degenerate examples for such signals were first discussed in Easley and O'Hara (1992) and referred to as "event uncertainty".

$\mathsf{Pr}(S|V_1)$. Hence, for example, after good news G, an investors buys (sells) if he thinks, relative to the market, that it is more (less) likely that FI will thrive than being unaffected.

It follows from the above that the investor buys after G and sells after B if and only if $\mathsf{Pr}(S|V_3) > \mathsf{Pr}(S|V_2)$ and $\mathsf{Pr}(S|V_1) > \mathsf{Pr}(S|V_2)$. Such an investor, loosely, herds in the sense that he acts like a momentum trader, buying with rising and selling with falling prices. The private information (conditional probabilities) that is both necessary and sufficient for such a behaviour has thus a *U shape*. Conversely, the investor sells after G and buys after B if and only if $\mathsf{Pr}(S|V_3) < \mathsf{Pr}(S|V_2)$ and $\mathsf{Pr}(S|V_1) < \mathsf{Pr}(S|V_2)$. Such an investor, loosely, trades contrary to the general movement of prices. The private information that is both necessary and sufficient to generate such a behaviour has thus a *Hill shape*.[7]

There are several points to note about this example. First, the public announcements G and B are degenerate as they each exclude one of the extreme states. Yet the same kind of reasoning holds if we replace G by an announcement that attaches arbitrarily small probability to the worst outcome, V_1, and if we replace B by an announcement that attaches arbitrarily small probability to the best outcome, V_3. Second, in the above illustration, G and B are *exogenous* public signals. In the security model described in this paper, on the other hand, public announcements or, more generally, public information are created *endogenously* by the history of publicly observable transactions. Yet the intuition behind our characterization results is similar to the above illustration. The analysis in the paper involves describing public histories of trades that allow investors to almost rule out some extreme outcome, either V_1 or V_3.[8] Such histories are equivalent to public announcements G and B (or, to be more precise, to perturbations of G and B), and demonstrating their existence is crucial for demonstrating the existence of herding and contrarianism.

Overview. The next section discusses some of the related literature. Section 3 outlines the setup. Section 4 defines herding and contrarian behaviour. Section 5 discusses the necessary and sufficient conditions that ensure herding and contrarianism. Section 6 discusses the special case of MLRP signals. Section 7 considers the resiliency, and fragility of herding and contrarianism and describes the range of prices for which there may be herding and contrarianism. Section 8 discusses the impact of social learning on prices with respect to volatility and liquidity. Section 9 extends the result to a setting with an arbitrary number of states. Section 10 discusses the relation of our findings to an earlier important paper on financial market herding. Section 11 concludes. Proofs that are not in the text are either in the appendix or in the supplementary material.

[7] In our formal definition of herding and contrarianism, we benchmark behaviour against the decision that the trader would take at the *initial* history, but the switching mechanism is akin to what we describe here.

[8] In the asset market described in the paper, every state has a positive probability at all finite trading histories because of the existence of noise traders. Therefore, we describe public histories at which the probabilities of extreme states are arbitrarily small, but not zero.

2 Related Literature

Extensive literature surveys on herding in financial markets are in Brunnermeier (2001), Chamley (2004) and Vives (2008). Our work relates to the part of the literature that focusses on short-run herding. The work closest to ours is Avery and Zemsky (1998), henceforth AZ, who were the first to present an intuitively appealing example of informational herding in financial markets. They argue that herd behaviour with informationally efficient asset prices is not possible unless signals are "non-monotonic" and they attribute the herding result in their example to "multidimensional uncertainty" (investors have a finer information structure than the market). In their main example, however, prices hardly move under herding. To generate extreme price movements (bubbles) with herding AZ expand their example to a second level of information asymmetry that leads to an even finer information partition. Yet even with these further informational asymmetries, the likelihood of large price movements during herding is extremely small (see Chamley (2004)).

The profession, for instance Brunnermeier (2001), Bikhchandani and Sunil (2000), Chamley (2004), has derived three messages from AZ's paper. First, with "uni-dimensional" or "monotonic" signal structures, herding is impossible. Second, the information structure needed to induce herding is very special. Third, herding does not involve violent price movements except in the most unlikely environments.

AZ's examples are special cases of our framework. Our paper demonstrates that the conclusions derived from AZ's examples should be reconsidered. First, we show that it is U-shaped signals, and not multi-dimensionality or non-monotonicity of the information structure, that is both necessary and sufficient for herding. Second, while AZ's examples are intuitively appealing, due to their extreme nature (with several degenerate features) it may be argued that they are very special and therefore have limited economic relevance. Our results show instead that herding may apply in a much more general fashion and therefore, there may be a great deal more rational informational herding than is currently expected in the literature. Third, we show that extreme price movements with herding are possible under not so unlikely situations, even with MLRP signals and without "further dimensions of uncertainty." In Section 10 below, we discuss the above in detail by comparing and contrasting the work and conclusions of AZ with ours.

A related literature on informational learning explains how certain facets of market organization or incentives can lead to conformism and informational cascades. In Lee (1998), fixed transaction costs temporarily keep traders out of the market. When they enter suddenly and en masse, the market maker absorbs their trades at a fixed price, leading to large price jumps after this "avalanche." In Cipriani and Guarino (2008), traders have private benefits from trading in addition to the fundamental value payoff. As the private and public expectations converge, private benefits gain importance to the point when they overwhelm the informational rents. Then learning breaks down and an informational cascade arises. In Dasgupta and Prat (2008) an informational cascade is triggered by traders' reputation concerns, which eventually outweigh the possible benefit from trading on information. Chari and Kehoe (2004) also study a financial market with efficient prices; herding in their model arises with respect to a capital investment that is made outside of the financial market.

Our work also relates to the literature that shows how public signals can have a larger influence on stock price fluctuations than warranted by their information content. Beginning with He and Wang (1995) who describe the relation between public information and non-random trading volume patterns caused by the dynamic trading activities of long-lived traders, this literature has identified how traders who care for future prices (as opposed to fundamentals) rely excessively on public expectations (see also Allen, Morris, and Shin (2006), Bacchetta and Wincoop (2006), Bacchetta and Wincoop (2008), Ozdenoren and Yuan (2007), Goldstein, Ozdenoren, and Yuan (2010)).

All of the above contributions highlight important aspects, facets, and mechanisms that can trigger conformism in financial markets. Our findings complement the literature in that the effects that we identify may be combined with many of the above studies and they may amplify the effects described there.

3 The Model

We model financial market sequential trading in the tradition of Glosten and Milgrom (1985).

Security: There is a single risky asset with a value V from a set of three potential values $\mathbb{V} = \{V_1, V_2, V_3\}$ with $V_1 < V_2 < V_3$. Value V is the liquidation or true value when the game has ended and all uncertainty has been resolved. States V_1 and V_3 are the extreme states, state V_2 is the moderate state. The prior distribution over $\mathbb{V}$ is denoted by $\mathsf{Pr}(\cdot)$. To simplify the computations we assume that $\{V_1, V_2, V_3\} = \{0, \mathcal{V}, 2\mathcal{V}\}$, $\mathcal{V} > 0$ and that the prior distribution is symmetric around V_2; thus $\mathsf{Pr}(V_1) = \mathsf{Pr}(V_3)$.[9]

Traders: There is a pool of traders consisting of two kinds of agents: *Noise Traders* and *Informed Traders*. At each discrete date t one trader arrives at the market in an exogenous and random sequence. Each trader can only trade once at the point in time at which he arrives. We assume that at each date the entering trader is an informed agent with probability $\mu > 0$ and a noise trader with probability $1 - \mu > 0$.

The informed agents are risk neutral and rational. Each receives a private, conditionally i.i.d. signal about the true value of the asset V. The set of possible signals or *types* of informed agents is denoted by $\mathbb{S}$ and consists of three elements S_1, S_2 and S_3. The signal structure of the informed can therefore be described by a 3-by-3 matrix $\mathcal{I} = \{\mathsf{Pr}(S_i|V_j)\}_{i,j=1,2,3}$ where $\mathsf{Pr}(S_i|V_j)$ is the probability of signal S_i if the true value of the asset is V_j.

[9]The ideas of this paper remain valid without these symmetry assumptions.

Noise traders have no information and trade randomly. These traders are not necessarily irrational, but they trade for reasons not included in this model, such as liquidity.[10]

Market Maker: Trade in the market is organised by a market maker who has no private information. He is subject to competition and thus makes zero-expected profits. In every period t, prior to the arrival of a trader, he posts a bid-price bid^t at which he is willing to buy the security and an ask-price ask^t at which he is willing to sell the security. Consequently he sets prices in the interval $[V_1, V_3]$.

Traders' Actions: Each trader can buy or sell *one* unit of the security at prices posted by the market maker, or he can be inactive. So the set of possible actions for any trader is $\{\text{buy}, \text{hold}, \text{sell}\}$. We denote the action taken in period t by the trader that arrives at that date by a^t. We assume that noise traders trade with equal probability. Therefore, in any period, a noise-trader buy, hold or sale occurs with probability $\gamma = (1-\mu)/3$ each.

Public History: The structure of the model is common knowledge among all market participants. The identity of a trader and his signal are private information, but everyone can observe past trades and transaction prices. The history (public information) at any date $t > 1$, the sequence of the traders' past actions together with the realised past transaction prices, is denoted by $H^t = ((a^1, p^1), \ldots, (a^{t-1}, p^{t-1}))$ for $t > 1$, where a^τ and p^τ are traders' actions and realised transaction prices at any date $\tau < t$ respectively. Also, H^1 refers to the initial history before any trade takes place.

Public Belief and Public Expectation: For any date t and any history H^t, denote the public belief/probability that the true liquidation value of the asset is V_i by $q_i^t = \mathsf{Pr}(V_i|H^t)$, for each $i = 1, 2, 3$. The public expectation, which we sometimes also refer to as the market expectation, of the liquidation value at H^t is given by $\mathsf{E}[V|H^t] = \sum q_i^t V_i$. Also, we shall respectively denote the probability of a buy and the probability of a sale at any history H^t, when the true value of the asset is V_i, by $\beta_i^t = \mathsf{Pr}(\text{buy}|H^t, V_i)$ and $\sigma_i^t = \mathsf{Pr}(\text{sell}|H^t, V_i)$. For instance, suppose that at history H^t the only informed type that buys is S_j. Then when the true value is V_i, the probability of a buy is given by the probability that there is a noise trader who buys plus the probability that there is an informed trader with signal S_j: $\beta_i^t = (1-\mu)/3 + \mu\mathsf{Pr}(S_j|V_i)$.

The Informed Trader's Optimal Choice: The game played by the informed agents is one of incomplete information; therefore the optimal strategies correspond to a Perfect Bayesian equilibrium. Here, the equilibrium strategy for each trader simply involves comparing the quoted prices with his expected value taking into account both the public history and his own private information. For simplicity, we restrict ourselves to equilibria in which each agent trades *only if* he is strictly better off (in the case of indifference the agents do

[10] As is common in the microstructure literature with asymmetric information, we assume that noise traders have positive weight ($\mu < 1$) to prevent "no-trade" outcomes a la Milgrom and Stokey (1982).

not trade). Therefore, the equilibrium strategy of an informed trader that enters the market in period t, receives signal S^t and observes history H^t is (i) to *buy* if $\mathsf{E}[V|H^t, S^t] > \mathsf{ask}^t$, (ii) to *sell* if $\mathsf{bid}^t > \mathsf{E}[V|H^t, S^t]$, and (iii) to *hold* in all other cases.

The Market Maker's Price-Setting: To ensure that the market maker receives zero expected profits, the bid and ask prices must satisfy the following at any date t and any history H^t: $\mathsf{ask}^t = \mathsf{E}[V|a^t = \text{buy at } \mathsf{ask}^t, H^t]$ and $\mathsf{bid}^t = \mathsf{E}[V|a^t = \text{sell at } \mathsf{bid}^t, H^t]$. Thus if there is a trade at H^t, the public expectation $\mathsf{E}[V|H^{t+1}]$ coincides with the transaction price at time t (ask^t for a buy, bid^t for a sale).

If the market maker always sets prices equal to the public expectation, $\mathsf{E}[V|H^t]$, he makes an expected loss on trades with an informed agent. However, if the market maker sets an ask-price and a bid-price respectively above and below the public expectation, he gains on noise traders, as their trades have no information value. Thus, in equilibrium the market maker must make a profit on trades with noise traders to compensate for any losses against informed types. This implies that if at any history H^t, there is a possibility that the market maker trades with an informed trader, then there is a spread between the bid and ask prices at H^t and the public expectation $\mathsf{E}[V|H^t]$, satisfies $\mathsf{ask}^t > \mathsf{E}[V|H^t] > \mathsf{bid}^t$.

Trading by the Informed Types and No Cascade Condition: At any history H^t either informed types do not trade and every trade is by a noise trader or there is an informed type that would trade at the quoted prices. The game played by the informed types in the former case is trivial as there will be no trade by the informed from H^t onwards and an informational cascade occurs. The reason is that if there were no trades by the informed at H^t, no information will be revealed and the expectations and prices remain unchanged; hence, by induction, we would have no trading by the informed and no information revelation at any date after H^t. In this paper, we thus consider only the case in which at every history there is an informed type that would trade at the quoted prices.

Informative Private Signals: The private signals of the informed traders are informative at history H^t if

$$\text{there exists } S \in \mathbb{S} \text{ such that } \mathsf{E}[V|H^t, S] \neq \mathsf{E}[V|H^t]. \tag{1}$$

First note that (1) implies that at H^t there is an informed type that buys and an informed type that sells. To see this observe that by (1) there must exist two signals S' and S'' such that $\mathsf{E}[V|H^t, S'] < \mathsf{E}[V|H^t] < \mathsf{E}[V|H^t, S'']$. If no informed type buys at H^t then there is no informational content in a buy and $\mathsf{ask}^t = \mathsf{E}[V|H^t]$. Then, by $\mathsf{E}[V|H^t] < \mathsf{E}[V|H^t, S'']$, type S'' must be buying at H^t; a contradiction. Similarly, if no informed type sells at H^t then $\mathsf{bid}^t = \mathsf{E}[V|H^t]$. Then, by $\mathsf{E}[V|H^t] > \mathsf{E}[V|H^t, S']$, type S' must be selling at H^t; a contradiction. Second, it is also the case that if there is an informed type who trades at H^t, then (1) must hold. Otherwise, for every signal $S \in \mathbb{S}$, $\mathsf{E}[V|H^t, S] = \mathsf{E}[V|H^t] = \mathsf{ask}^t = \mathsf{bid}^t$ and the informed types would not trade at H^t.

It follows from the above that (1) is both necessary and sufficient for trading by an informed type at H^t. Since we are interested in the case when the informed types trade, we therefore assume throughout this paper that (1) holds at every history H^t.[11]

One important consequence of condition (1) is that past behaviour can be inferred from past transaction prices alone: since the bid and ask prices always differ by condition (1), one can infer behaviour iteratively, starting from date 1. Therefore, all the results of this paper are valid if traders observe only past transaction prices and no-trades.

Long-run behaviour of the model. Since price formation in our model is standard, (1) also ensures that standard asymptotic results on efficient prices hold. More specifically, by standard arguments as in Glosten and Milgrom (1985) we have that transaction prices form a martingale process. Since by (1) buys and sales have some information content (at every date there is an informed type that buys and one that sells), it also follows that beliefs and prices converge to the truth in the *long-run* (see, for instance, Proposition 4 in AZ). However, here we are solely interested in *short-run* behaviour and fluctuations.

Conditional signal distributions. As we outlined in the introduction, the possibility of herding or a contrarian behaviour for any informed agent with signal $S \in \mathbb{S}$ depends critically on the shape of the conditional signal distribution of S. Henceforth, we refer to the conditional signal distribution of the signal as the *csd.* Furthermore, we will also employ the following terminology to describe four different types of csds:

increasing: $\Pr(S|V_1) \leq \Pr(S|V_2) \leq \Pr(S|V_3)$; *decreasing:* $\Pr(S|V_1) \geq \Pr(S|V_2) \geq \Pr(S|V_3)$;
U-shaped: $\Pr(S|V_i) > \Pr(S|V_2)$ for $i = 1, 3$; *Hill-shaped:* $\Pr(S|V_i) < \Pr(S|V_2)$ for $i = 1, 3$.

An increasing csd is strictly increasing if all three conditional probabilities for the signal are distinct; a strictly decreasing csd is similarly defined.

For the results in our paper it is also important whether the likelihood of a signal is higher in one of the extreme states V_1 or V_3 relative to the other extreme state. We thus define the *bias* of a signal S as $\Pr(S|V_3) - \Pr(S|V_1)$. A U-shaped csd with a negative bias, $\Pr(S|V_3) - \Pr(S|V_1) < 0$, will be labeled as an nU-shaped csd and a U-shaped csd with a positive bias, $\Pr(S|V_3) - \Pr(S|V_1) > 0$, will be labeled as a pU-shaped csd. Similarly, we use nHill (pHill) to describe a Hill-shaped csd with a negative (positive) bias.

In describing the above properties of a type of csd for a signal we shall henceforth drop the reference to the csd and attribute the property to the signal itself, when the meaning is clear. Similarly, when describing the behaviour of a signal recipient we attribute the behaviour to the signal itself.

[11] A sufficient condition for (1) to hold at every H^t is that all minors of order two of matrix $\mathcal{I}$ are non-zero.

4 Definitions of Herding and Contrarian Behaviour

In the literature, there are several definitions of herding. Some require "action convergence" or even complete informational cascades where all types take the same action in each state, irrespective of their private information; see Brunnermeier (2001), Chamley (2004), Vives (2008). The key feature of this early literature was that herding can induce, after some date, the loss of all private information and wrong or inefficient decisions henceforth.

A situation like an informational cascade in which all informed types act alike may not, however, be very interesting in an informationally efficient financial market setting. In such a framework prices account for the information contained in the traders' actions. If all informed types act alike then their actions would be uninformative, and as result, prices would not move. Therefore, such uniformity of behaviour cannot explain prices movements, which is a key feature of financial markets. Moreover, if the uniform action involves trading, then a large imbalance of trades would accumulate without affecting prices — contrary to common empirical findings.[12]

Furthermore, as we have explained in the previous section, in our "standard" microstructure trading model, at any history uniform behaviour by the informed types is possible if and only if *all* private signals are uninformative, in the sense that the private expectations of all informed types are equal that of the market expectation (condition (1) is violated). The case of such uninformative private signals is trivial and uninteresting as it implies that no further information is revealed, that all informed types have the same expectation and that, because we assume that informed agents trade only if trading makes them strictly better off, a no-trade cascade results.

In this paper we thus focus on the social learning (learning from others) aspect of behaviour for individual traders that is implied by the notion of herding from the earlier literature. Specifically, we follow Brunnermeier (2001)'s (Ch. 5) description of herding as a situation in which "an agent imitates the decision of his predecessor even though his own signal might advise him to take a different action" and we consider the behaviour of a *particular* signal type by looking at how the history of past trading can induce a trader to change behaviour and trade against his private signal.[13]

Imitative behaviour is not the only type of behaviour that learning from others' past trading activities may generate: a trader may also switch behaviour and go against what most have done in the past. We call such social learning contrarianism and differentiate it from herding by describing the latter as a history-induced switch of opinion *in the direction of the crowd* and the former as a history-induced switch *against the direction of the crowd.*

[12] See, for instance, Chordia, Roll, and Subrahmanyam (2002).

[13] Vives (2008) (Ch. 6) adopts a similar view of herding, defining it as a situation in which agents put "too little" weight on their private signals with respect to a well-defined welfare benchmark.

The direction of the crowd here is defined by the "recent" price movement. Thus, there is a symmetry in our definitions, making herding the intuitive counterpart to contrarianism.

Definition *Herding.* A trader with signal S buy herds in period t at history H^t if and only if (i) $\mathsf{E}[V|S] < \mathsf{bid}^1$, (ii) $\mathsf{E}[V|S,H^t] > \mathsf{ask}^t$, $(iii\text{-}h)$ $\mathsf{E}[V|H^t] > \mathsf{E}[V]$. Sell herding at history H^t is defined analogously with the required conditions $\mathsf{E}[V|S] > \mathsf{ask}^1$, $\mathsf{E}[V|S,H^t] < \mathsf{bid}^t$, and $\mathsf{E}[V|H^t] < \mathsf{E}[V]$. Type S herds if he either buy herds or sell herds at some history.

Contrarianism. A trader with signal S engages in buy contrarianism in period t at history H^t if and only if (i) $\mathsf{E}[V|S] < \mathsf{bid}^1$, (ii) $\mathsf{E}[V|S,H^t] > \mathsf{ask}^t$, $(iii\text{-}c)$ $\mathsf{E}[V|H^t] < \mathsf{E}[V]$. Sell contrarianism at history H^t is defined analogously with the required conditions $\mathsf{E}[V|S] > \mathsf{ask}^1$, $\mathsf{E}[V|S,H^t] < \mathsf{bid}^t$, and $\mathsf{E}[V|H^t] > \mathsf{E}[V]$. Type S engages in contrarianism if he engages either in buy contrarianism or sell contrarianism at some history.

Both with buy herding and buy contrarianism, type S prefers to sell at the initial history, before observing other traders' actions (condition (i)), but prefers to buy after observing the history H^t (condition (ii)). The key differences between buy herding and buy contrarianism are conditions $(iii\text{-}h)$ and $(iii\text{-}c)$. The former requires the public expectation, which is the last transaction price and an *average* of the bid and ask prices, to rise at history H^t so that a change of action from selling to buying at H^t is *with* the general movement of the prices (crowd), whereas the latter condition requires the public expectation to have dropped so that a trader who buys at H^t acts *against* the movement of prices.

Henceforth, we refer to a "buy herding history" as one at which some type, were they to trade, would buy herd at that history; similarly for a "buy contrarianism history."

Our definition of herding is identical to that in Avery and Zemsky (1998),[14] and it has also been used in other work on social learning in financial markets (see, for instance, Cipriani and Guarino (2005) or Drehmann, Oechssler, and Roider (2005)). It describes histories at which a trader acts "against his signal" (judgement) and follows "the trend", where "against his signal" is defined by comparing the herding action to the benchmark without public information. "The trend" is identified by price movements based on the idea that prices rise (fall) when there are more (less) buys than sales. The contrarianism definition, on the other hand, captures the contra-trend action that is also against one's signal.[15]

Our definitions also capture well-documented financial market trading behaviour. In particular, our herding definition is a formalization of the idea of rational *momentum* trad-

[14]Avery and Zemsky (1998)'s definition of contrarianism is stronger than ours (they also impose an additional bound on price movements that reflects the expectations that would obtain if the traders receives an infinite number of draws of the same signal). We adopt the definition of contrarianism above because, as we explained before, it is a natural and simple counterpart to the definition of herd behaviour.

[15]Herding and contrarianism here refer to extreme switches of behaviour from selling to buying or the reverse. One could expand the definition to switches from holding to buying or to selling (or the reverse). For consistency with the earlier literature, we use the extreme cases where switches do not include holding.

ing. It also captures the situation in which traders behave as if their demand functions are increasing. Contrarianism has a *mean-reversion* flavour. Both momentum and contrarian trading have been analyzed extensively in the empirical literature and have been found to generate abnormal returns over some time horizon.[16] Our analysis thus provides a characterization for momentum and mean reversion behaviour and shows that it can be rational.

As we have argued above, in our set-up we assume that trades are informative to avoid the uninteresting case of a no-trade cascade. Thus, we cannot have a situation in which *all* informed types act alike.[17] Nevertheless, our set-up allows for the possibility that a very large portion of informed traders is involved in herding or contrarianism. The precise proportion of such informed traders is, in fact, determined exogenously by the information structure through the likelihood that a trader receives the relevant signal. We discuss this point further in Section 11 and show that this proportion can be arbitrarily large.

Although the informative trade assumption ensures that asymptotically the true value of the security is learned, our definition of herding and contrarianism admits the possibility of switches of trades "in the wrong direction" in the *short* run. For instance, traders may buy herd even though the true value of the asset is V_1 (the lowest).

Finally, social learning can have efficiency consequences. In any setting where agents learn from the actions of others, an informational externality is inevitable as future agents are affected when earlier agents take actions that reveal private information. For example, in the early herding models of Bikhchandani, Hirshleifer, and Welch (1992) and Banerjee (1992) once a cascade begins, no further information is revealed. However, in this paper we do not address the efficiency element, because it requires a welfare benchmark, which "is generally lacking in asymmetric information models of asset markets" (see Avery and Zemsky (1998), p. 728). Instead, our definition is concerned with observable behaviour and outcomes that are sensitive to the details of the trading history. Such sensitivities may dramatically affect prices in financial markets.

[16] In the empirical literature, contrarian behaviour is found to be profitable in the very short run (1 week and 1 month, Jegadeesh (1990) and Lehmann (1990)) and in the very long run (3-5 years, de Bondt and Thaler (1985))). Momentum trading is found to be profitable over the medium term (3-12 months, Jegadeesh and Titman (1993)) and exceptionally unprofitable beyond that (the 24 months following the first 12, Jegadeesh and Titman (2001)). Sadka (2006) studies systematic liquidity risk and links a component of it, which is caused by informed trading, to returns on momentum trading.

[17] Note that even though with no cascades (the market expects that) the different types do not take the same actions, in degenerate settings when a particular type is not present in certain states, it is possible that in these particular states all informed types that occur with positive probability take the same action. An example of such a degenerate situation is the information structure in Avery and Zemsky (1998) that we discuss in Section 10.

5 Characterization Results: The General Case

The main characterization result for herding and contrarianism is as follows:

Theorem 1 (a) Herding. (i) *Necessity: If type S herds, then S is U-shaped with a non-zero bias.* (ii) *Sufficiency: If there is a U-shaped type with a non-zero bias, there exists $\mu_h \in (0,1]$ such that some informed type herds when $\mu < \mu_h$.*
(b) Contrarianism. (i) *Necessity: If type S acts as a contrarian, then S is Hill-shaped with a non-zero bias.* (ii) *Sufficiency: If there is a Hill-shaped type with a non-zero bias, there exists $\mu_c \in (0,1]$ such that some informed type acts as a contrarian when $\mu < \mu_c$.*

Theorem 1 does not specify when we have buy or sell herding or when we have buy or sell contrarianism. In what follows, we first consider the necessary and then the sufficient conditions for each of these cases. The proof of Theorem 1 then follows from these characterization results at the end of this section.

We begin by stating three useful lemmas. The first lemma provides a useful characterization for the difference between private and public expectations.

Lemma 1 *For any S, time t and history H^t, $\mathsf{E}[V|S,H^t] - \mathsf{E}[V|H^t]$ has the same sign as*

$$q_1^t q_2^t\ [\mathsf{Pr}(S|V_2) - \mathsf{Pr}(S|V_1)] + q_2^t q_3^t\ [\mathsf{Pr}(S|V_3) - \mathsf{Pr}(S|V_2)] +\ 2q_1^t q_3^t\ [\mathsf{Pr}(S|V_3) - \mathsf{Pr}(S|V_1)]. \quad (2)$$

Second, as the prior on the liquidation values is symmetric, it follows that the expectation of the informed is less (greater) than the public expectation at the initial history if and only if his signal is negatively (positively) biased.

Lemma 2 *For any signal S, $\mathsf{E}[V|S]$ is less than $\mathsf{E}[V]$ if and only if S has a negative bias, and $\mathsf{E}[V|S]$ is greater than $\mathsf{E}[V]$ if and only if S has a positive bias.*

An immediate implication of this lemma is that someone sells at the initial history only if this type's signal is negatively biased and buys only if the signal is positively biased.

Third, note that herding and contrarianism involve switches in behaviour after changes in public expectations relative to the initial period. A useful way to characterize these changes is the following.

Lemma 3 *If $\mathsf{E}[V|H^t] > \mathsf{E}[V]$ then $q_3^t > q_1^t$ and if $\mathsf{E}[V] > \mathsf{E}[V|H^t]$ then $q_1^t > q_3^t$.*

Thus the public expectation rises (falls) if and only if the public belief attaches a lower (higher) probability to the lowest value, V_1, than to the highest value, V_3.

5.1 Necessary Conditions

Herding and contrarianism by a given signal type involve buying at some history and selling at another. Our first result establishes that this cannot happen if the signal is decreasing or increasing. Consider a decreasing type S. Since the ask price always exceeds the public expectation, it follows that, at any H^t, type S does not buy if the expectation of S, $\mathsf{E}[V|S,H^t]$, is no more than the public expectation, $\mathsf{E}[V|H^t]$. The latter must indeed hold because, for any two valuations V_ℓ, V_h such that $V_\ell < V_h$, the likelihood that a decreasing type S attaches to V_ℓ relative V_h at any history H^t cannot exceed that of the market: $\frac{\mathsf{Pr}(V_l|S,H^t)}{\mathsf{Pr}(V_h|S,H^t)} = \frac{\mathsf{Pr}(V_l|H^t)\mathsf{Pr}(S|V_l)}{\mathsf{Pr}(V_h|H^t)\mathsf{Pr}(S|V_h)} \leq \frac{\mathsf{Pr}(V_l|H^t)}{\mathsf{Pr}(V_h|H^t)}$. Formally, every term in (2) is non-positive when S is decreasing; therefore, it follows immediately from Lemma 1 that $\mathsf{E}[V|S,H^t] \leq \mathsf{E}[V|H^t]$.

An analogous set of arguments demonstrates that an increasing type does not sell at any history. Therefore, we can state the following.

Proposition 1 *If S is decreasing then type S does not buy at any history. If S is increasing then type S does not sell at any history. Thus recipients of such signals cannot herd or behave as contrarians.*

Proposition 1 demonstrates that any herding and contrarianism type must be either U or Hill-shaped. We next refine these necessary conditions further and state the main result of this subsection as follows.

Proposition 2 (a) *Type S buy herds only if S is nU-shaped and sell herds only if S is pU-shaped.* (b) *Type S acts as a buy contrarian only if S is nHill-shaped and acts as a sell contrarian only if S is pHill-shaped.*

A sketch of the proof of Proposition 2 for buy herding and buy contrarianism is as follows. Suppose that S buy herds or acts as a buy contrarian. Then it must be that at H^1 type S sells and thus his expectation is below the public expectation. By Lemma 2, this implies that S is negatively biased.[18] Thus, by Proposition 1, S is either nU or nHill-shaped.

The proof is completed by showing that buy herding is inconsistent with an nHill-shaped csd and that buy contrarianism is inconsistent with an nU-shaped csd. To see the intuition, for example, for the case of buy herding, note that in forming his belief a trader with an nHill-shaped csd puts less weight on the tails of his belief (and thus more on the center) relative to the public belief; furthermore, the shift from the tails towards the center is more for value V_3 than for V_1 because of the negative bias. When buy herding occurs, the public belief must have risen and thus, by Lemma 3, the public belief attaches more weight to V_3 relative to V_1 (i.e. $q_1^t < q_3^t$). Such a redistribution of probability mass ensures that S's expectation is less than that of the public. Hence an nHill-shaped S cannot be buying.

The arguments for sell herding and sell contrarianism are analogous except that here the bias has to be positive to ensure that the informed type buys at the initial history.

5.2 Sufficient Conditions

The above necessary conditions — U shape for herding and Hill shape for contrarianism — turn out to be almost sufficient as well. Before stating the sufficiency results, it will be

[18] The bias is only required because we assume that priors are symmetric.

useful to discuss two of the ideas that provide insight into the analysis.

(I) With a U-shaped signal S, the difference between the private and the public expectation, $\mathsf{E}[V|S,H^t]-\mathsf{E}[V|H^t]$*, is positive at histories at which the public probability of state V_1 is sufficiently small relative to the probability of the other states (both q_1^t/q_2^t and q_1^t/q_3^t are close to zero) and is negative at histories at which the public probability of state V_3 is sufficiently small relative to the probability of the other states (both q_3^t/q_1^t and q_3^t/q_1^t are close to zero). For a Hill-shaped signal S the sign of this difference is the reverse.*

The intuition for this statement relates to the example from the introduction. Consider first any history H^t at which both q_1^t/q_2^t and q_1^t/q_3^t are close to zero. Then there are effectively two states V_2 and V_3 at H^t. This means that at such a history the difference between the expectations of an informed type S and of the public (which has no private information), $\mathsf{E}[V|S,H^t]-\mathsf{E}[V|H^t]$, has the same sign as $\mathsf{Pr}(S|V_3)-\mathsf{Pr}(S|V_2)$.[19] The latter is positive for a U-shaped S and negative for a Hill-shaped S. Thus, at such history, $\mathsf{E}[V|S,H^t]-\mathsf{E}[V|H^t]$ is positive if S is U-shaped and is negative if S is Hill-shaped.

By a similar reasoning the opposite happens at any history H^t at which both q_3^t/q_1^t and q_3^t/q_2^t are close to zero. At such a history there are effectively two states, V_1 and V_2; therefore, $\mathsf{E}[V|S,H^t]-\mathsf{E}[V|H^t]$ has the same sign as $\mathsf{Pr}(S|V_2)-\mathsf{Pr}(S|V_1)$.[20] Since the latter is negative for a U-shaped S and positive for a Hill-shaped S, it follows that at such a history $\mathsf{E}[V|S,H^t]-\mathsf{E}[V|H^t]$ is negative if S is U-shaped and is positive if S is Hill-shaped.

(II) *The probability of noise trading may have to be sufficiently large to ensure that the bid-ask spread is not too wide both at the initial initial history and later at the history at which the herding or contrarian candidate changes behaviour.*

In (I) we have compared the private expectation of the informed trader with that of the public. To establish the existence of herding or contrarian behaviour, however, we must compare the private expectations with the bid- and ask-*prices*. The difference is that bid- and ask-prices form a spread around the public expectation. To ensure the possibility of herding or contrarianism, this spread must be sufficiently "tight". Tightness of the spread, in turn, depends on the extent of noise trading: the more noise there is (the smaller the likelihood of the informed types, μ), the tighter the spread.

More specifically, to ensure buy herding or buy contrarianism (the other cases are similar) by an informed type, a minimal amount of noise trading may be necessary, so that the informed type (i) sells initially and (ii) switches to buying after some history H^t. Below, we formalize these minimal noise trading restrictions by introducing two bounds,

[19] Formally, at any H^t at which both q_1^t/q_2^t and q_1^t/q_3^t are close to zero, the first and the third terms in (2) are arbitrarily small; moreover, the second term in (2) has the same sign as $\mathsf{Pr}(S|V_3)-\mathsf{Pr}(S|V_2)$; therefore, it follows from from Lemma 1 that $\mathsf{E}[V|S,H_t]-\mathsf{E}[V|H_t]$ has the same sign as $\mathsf{Pr}(S|V_3)-\mathsf{Pr}(S|V_2)$.

[20] This claim also follows from Lemma 1: at such history, the second and the third terms in (2) are arbitrarily small and the second term in (2) has the same sign as $\mathsf{Pr}(S|V_3)-\mathsf{Pr}(S|V_2)$.

one to ensure a tight spread for the *initial* sell and the second to ensure a tight spread for the subsequent *switch* to buying, and require the likelihood of informed trading μ to be less than both these bounds.

Appealing to the above ideas, we can now state a critical lemma. To save space we state the result only for the case of buy herding and buy contrarianism.

Lemma 4 (*i*) *Suppose that signal S is nU-shaped. Then there exist μ^i and $\mu^s_{bh} \in (0,1]$ such that S buy herds if $\mu < \mu_{bh} \equiv \min\{\mu^i, \mu^s_{bh}\}$ and if the following holds:*

$$\text{For any } \epsilon > 0 \text{ there exists a history } H^t \text{ such that } q^t_1/q^t_l < \epsilon \text{ for all } l = 2, 3. \quad (3)$$

(*ii*) *Suppose signal S is nHill-shaped. Then there exist μ^i and $\mu^s_{bc} \in (0,1]$ such that S acts as a buy contrarian if $\mu < \mu_{bc} \equiv \min\{\mu^i, \mu^s_{bc}\}$ and if the following holds:*

$$\text{For any } \epsilon > 0 \text{ there exists a history } H^t \text{ such that } q^t_3/q^t_l < \epsilon \text{ for all } l = 1, 2. \quad (4)$$

Conditions (3) and (4) above ensure that there are histories at which the probability of an extreme state, V_1 or V_3, can be made arbitrarily small relative to the other states. Value μ_{bh} is the minimum of the two bounds μ^i and μ^s_{bh}, mentioned in (*II*), that respectively ensure that spreads are small enough at the initial history and at the time of the switch of behaviour by a buy herding type. Similarly, μ_{bc} is the minimum of the two bounds μ^i and μ^s_{bc} that respectively ensure that spreads are small enough at the initial history and at the time of the switch of behaviour by a buy contrarian type S. Below we will discuss how restrictive these bounds are, and if they are necessary for the results.

A sketch of the proof for part (*i*) of Lemma 4 is as follows. First, by Lemma 2, S having a negative bias implies that $\mathsf{E}[V|S] < \mathsf{E}[V]$. Then it follows from the arguments outlined in (*II*) above that one can find an upper bound $\mu^i > 0$ on the size of the informed trading such that if $\mu < \mu^i$ then at H^1 the bid price bid^1 is close enough to the public expectation $\mathsf{E}[V]$ so that $\mathsf{E}[V|S]$ is also below bid^1, i.e. S sells at H^1.

Second, since S is U-shaped, it follows from the arguments outlined in (*I*) that at any history H^t at which q^t_1/q^t_2 and q^t_1/q^t_3 are sufficiently small $\mathsf{E}[V|S, H^t] > \mathsf{E}[V|H^t]$ (by condition (3) such history H^t exists). Then, by the arguments outlined in (*II*) above, one can find an upper bound $\mu^s_{bh} > 0$ such that if $\mu < \mu^s_{bh}$ then at H^t the ask price ask^t is sufficiently close to $\mathsf{E}[V|H^t]$ so that $\mathsf{E}[V|S, H^t]$ also exceeds ask^t, i.e. S switches to buying.

Finally, since q^t_1/q^t_3 is small at H^t, by Lemma 3, $\mathsf{E}[V|H^t] > \mathsf{E}[V]$. Thus, the switch to buying at H^t by S is in the direction of the crowd.

The argument for contrarianism in part (*ii*) is analogous except that to ensure $\mathsf{E}[V|S, H^t]$ exceeds $\mathsf{E}[V|H^t]$ at some history H^t for a Hill-shaped S, by (*I*), H^t must be such that q^t_3/q^t_1 and q^t_3/q^t_2 are sufficiently small (condition (4) ensures that such history H^t exists). Then by (*II*) there exists $\mu^s_{bc} > 0$ such that if $\mu < \mu^s_{bc}$, ask^t is sufficiently close to $\mathsf{E}[V|H^t]$ so that $\mathsf{E}[V|S, H^t] > \mathsf{ask}^t$. Furthermore, at such a history H^t, S will be buying *against* the crowd because when q^t_3/q^t_1 is small, which is the case at H^t, by Lemma 3, $\mathsf{E}[V|H^t] < \mathsf{E}[V]$.

A similar set of results to Lemma 4 can be obtained for the cases of sell herding and sell contrarianism except that to ensure sell herding the appropriate assumptions are that S is pU and (4) holds, and to ensure sell contrarian we need that S is pHill and (3) holds.[21]

The sufficiency results in Lemma 4 are non-vacuous provided that conditions (3) and (4) are satisfied. As these conditions depict properties of endogenous variables, to complete the analysis we need to show the existence of the histories assumed by conditions (3) and (4).

In some cases, this is a straightforward task. The easiest case arises when at any t there is a trade that reduces *both* q_1^t/q_2^t and q_1^t/q_3^t or *both* q_3^t/q_1^t and q_3^t/q_2^t uniformly (independent of time). For example, suppose that the probability of a buy is uniformly increasing in the liquidation value at any date and history:

$$\text{for some } \epsilon > 0,\ \beta_j^t > \beta_i^t + \epsilon \text{ for any } j > i \text{ and any } t, \tag{5}$$

Since $q_i^{t+1}/q_j^{t+1} = (q_i^t\beta_i^t)/(q_j^t\beta_j^t)$ when there is a buy at date t, it follows that in this case a buy reduces *both* q_1^t/q_2^t and q_1^t/q_3^t uniformly. Thus if (5) holds, a sufficiently large number of buys induces the histories described in (3). Similarly, with a sale $q_i^{t+1}/q_j^{t+1} = (q_i^t\sigma_i^t)/(q_j^t\sigma_j^t)$. Thus, if the probability of a sale is uniformly decreasing in the liquidation value at any H^t,

$$\text{for some } \epsilon > 0,\ \sigma_i^t > \sigma_j^t + \epsilon \text{ for any } j > i \text{ and any } t, \tag{6}$$

then a sale reduces *both* q_3^t/q_1^t and q_3^t/q_2^t uniformly. Thus, if (6) holds, a sufficiently large number of sales induces the histories described in (4).[22]

Demonstrating conditions (3) and (4) generally, however, requires a substantially more complex construction. In particular, in some cases there are no paths that result in both q_1^t/q_2^t and q_1^t/q_3^t decreasing at *every* t or in both q_3^t/q_1^t and q_3^t/q_2^t decreasing at *every* t.[23] For these cases, we construct outcome paths consisting of two different stages. For example, to ensure (3), the path is constructed so that in the first stage q_1^t/q_2^t becomes small while ensuring that q_1^t/q_3^t does not increase by too much. Then in the second stage, once q_1^t/q_2^t is sufficiently small, the continuation path makes q_1^t/q_3^t small while ensuring that q_1^t/q_2^t does not increase by too much. A similar construction is used for (4).

Such constructions work for most signal distributions. The exceptions are cases with two U-shaped signals with opposite biases or two Hill-shaped signals with opposite biases. In these cases we can show, depending on the bias of the third signal, that either (3) or (4)

[21]These differences arise because with sell herding or sell contrarianism the informed needs to buy initially and switch to selling later. To ensure the former, by Lemma 2, we need to assume a positive bias and, to ensure the latter, the appropriate "extreme" histories at which the switches happen are the opposite of the buy herding and buy contrarian case. Also, the values for the bounds on the size of informed trading might be different for sell herding and sell contrarianism from those for buy herding and buy contrarianism.

[22]These monotonicity properties of the probability of a buy and the probability of a sale as defined in (5) and (6) are satisfied, for instance, by MLRP information structures; see the next section.

[23]For instance, if for every action a^t (=buy, sell or hold) the probability of a^t in state V_1 is no less than the probability of a^t in the other two states, there will not be a situation that reduces both q_1^t/q_2^t and q_1^t/q_3^t.

holds, but we cannot show both. For example, if one of the three signals is nU and another is pU, then we can show that (3) holds if the third signal has a non-negative bias, and (4) holds if the third signal has a non-positive bias. This implies, by Lemma 4 (i), that in the former case the nU type buy herds, and in the latter case, by an analogous argument, the pU type sell herds (similarly for the contrarian situation).

The next proposition is our main sufficiency result. It follows from the discussion above, concerning (3) and (4), and Lemma 4 (for completeness, we state the result for buy and sell herding and for buy and sell contrarianism).

Proposition 3 (a) *Let S be nU-shaped. If another signal is pU-shaped, assume the third signal has a non negative bias. Then there exists $\mu_{bh} \in (0,1]$ such that S buy herds if $\mu < \mu_{bh}$.*
(b) *Let S be pU-shaped. If another signal is nU-shaped, assume the third signal has a non positive bias. Then there exists $\mu_{sh} \in (0,1]$ such that S sell herds if $\mu < \mu_{sh}$.*
(c) *Let S be nHill-shaped. If another signal is pHill-shaped assume the third signal has a non positive bias. Then there exists $\mu_{bc} \in (0,1]$ such that S is a buy contrarian if $\mu < \mu_{bc}$.*
(d) *Let S be pHill-shaped. If another signal is nHill-shaped, assume the third signal has a non negative bias. Then there exists $\mu_{sc} \in (0,1]$ such that S is a sell contrarian if $\mu < \mu_{sc}$.*

Discussion of the Noise Restriction. For each of our sufficiency results above (Lemma 4 and Proposition 3) we assume that μ is less than some upper bound. We will now discuss whether these bounds are necessary for our results, and whether they are restrictive.

Consider the case of buy herding by an nU-shaped type S as in Lemma 4 (i). For this sufficiency result we assume two restrictions: $\mu < \mu^i$ and $\mu < \mu^s_{bh}$. These conditions respectively ensure that the spread is small enough at the initial history and at history H^t at which there is a switch in behaviour. In the appendix we show that there exists a unique value for the first bound, $\mu^i \in (0,1]$, such that $\mu < \mu^i$ is also necessary for buy herding.

In general we cannot find a unique upper bound for the second value, μ^s_{bh}, such that $\mu < \mu^s_{bh}$ is also necessary for buy herding. The reason is that the upper bound that ensures that the spread is not too large at the history H^t at which S switches to buying may depend on what the types other than S do at H^t. Since there may be more than one history at which S switches to buying, this upper bound may not be unique.

The above difficulty with respect to the necessity of the second noise condition $\mu < \mu^s_{bh}$ does not arise, for instance, when types other than S always take the same action.[24] More generally, we show in the supplementary material (Proposition 3a), that $\mu < \mu^s_{bh}$ is also *necessary* for buy herding provided there is at most one U-shaped signal.

A similar argument applies to contrarian behaviour. The noise restriction with respect to the initial trade is necessary, whereas the noise restriction with respect to the switch of behaviour is necessary as long as there is at most one Hill-shaped type.

[24]For example, this happens when the information structure satisfies MLRP; see the next section.

Finally, note that the bounds on μ in our sufficiency results do not always constitute a restriction. Consider again the case of buy herding for an nU type S. If at H^1, type S has the lowest expectation among all informed types, then his expectation must be less than the bid price at H^1, and thus, there is no need for any restriction on the size of the informed at H^1 (μ^i can be set to 1). Likewise, if at H^t at which buy herding occurs type S has the highest expectation (for example, this happens if no type other than S buys at H^t), then his expectation must be greater than the ask price at H^t and therefore, no restriction on the size of the informed is needed (μ^s_{bh} can be set to 1). Consequently, to obtain our sufficiency results there are restrictions on the value of μ only if at H^1 or at the switch history H^t the expectation of the herding candidate is in between those of the other signal types.[25]

5.3 Proof of Theorem 1

The necessity part in Theorem 1 follows immediately from Proposition 2. The proof of the sufficiency part for case (a) of herding is as follows. Let $\mu_h = \min\{\mu_{sh}, \mu_{bh}\}$, where μ_{bh} and μ_{sh} are the bounds for herding given in Proposition 3. Also, assume that $\mu < \mu_h$ and that there exist a U-shaped signal as in part (a) of Theorem 1. Then there are two possibilities: either there is another U-shaped signal with the opposite bias or there is not. If there is no other U-shaped signal with the opposite bias, then by part (a) of Proposition 3, the U-shaped type buy herds if it has a negative bias and by part (b) of Proposition 3 the U-shaped type sell herds if it has a positive bias. If there is another U-shaped signal with the opposite bias, then by parts (a) and (b) of Proposition 3, one of the U-shaped signals must herd. This is because if the third signal is weakly positive then the U-shaped signal with a negative bias buy herds, and if the third signal is weakly negative then the U-shaped signal with a positive bias sell herds. The reasoning for sufficiency in part (b) of Theorem 1 is analogous and is obtained by setting $\mu_c = \min\{\mu_{sc}, \mu_{bc}\}$, where μ_{sc} and μ_{bc} are the bounds for contrarianism from Proposition 3.[26] □

6 Social Learning with MLRP Information Structure

The literature on asymmetric information often assumes that the information structure is monotonic in the sense that it satisfies the monotone likelihood ratio property (MLRP). Here this means that for any signals $S_l, S_h \in \mathbb{S}$ and values $V_l, V_h \in \mathbb{V}$ such that $S_l < S_h$ and $V_l < V_h$, $\Pr(S_h|V_l)\Pr(S_l|V_h) < \Pr(S_h|V_h)\Pr(S_l|V_l)$.

MLRP is very restrictive (it is stronger than first order stochastic dominance) and at first might seem to be too strong to allow herding or contrarianism. This turns out to be

[25] Romano and Sabourian (2010) extend the present model to the case where traders can trade a continuum of quantities. In this setting, they show that no restrictions on μ are needed because at each history, each quantity is traded by a specific signal type.

[26] The bounds $\mu_{bh}, \mu_{sh}, \mu_{sc}, \mu_{bc}$ in the proof of Proposition 3 are such that $\mu_h = \mu_c$.

false. Not only does MLRP not exclude such possibilities, it actually enables one to derive a sharper sets of results for the existence of herding and contrarianism. Moreover, with MLRP the histories that can generate herding or contrarianism can be easily identified.

MLRP is a set of restrictions on the conditional probabilities for the *entire* signal structure and is equivalent to assuming that all minors of order two of the information matrix $\mathcal{I}$ are positive. Herding or contrarianism, on the other hand, relate to the csd of a signal being U- or Hill-shaped, i.e. to the individual row in matrix $\mathcal{I}$ that corresponds to the signal. Therefore, to analyse the possibility of herding or contrarianism with MLRP, we need to consider the csd of the different signals under MLRP. In the next lemma we describe some useful implications of MLRP.

Lemma 5 *Assume $S_1 < S_2 < S_3$ and the information structure satisfies MLRP. Then*

(i) $E[V|S_1, H^t] < E[V|S_2, H^t] < E[V|S_3, H^t]$ *at any t and any H^t.*

(ii) *In any equilibrium S_1 types always sell and S_3 types always buy.*

(iii) *S_1 is strictly decreasing and S_3 is strictly increasing.*

(iv) *The probability of a buy is uniformly increasing in the liquidation value as specified in (5) and the probability of a sale is uniformly decreasing as specified in (6).*

Part (i) states that MLRP imposes a natural order on the signals in terms of their conditional expectations after any history. Part (iv) implies that with MLRP the probability of a buy is uniformly increasing and the probability of a sell is uniformly decreasing in the liquidation values. Parts (ii) and (iii) state that MLRP restricts the behaviour and the shape of the lowest and the highest signals S_1 and S_3. In particular, these two types do not change behaviour and they are decreasing and increasing respectively.

Lemma 5, however, does not impose any restrictions on the behaviour or the shape of the middle signal S_2. In fact, MLRP is consistent with a middle signal S_2 that is decreasing, increasing, Hill-shaped or U-shaped with a negative or a positive bias — Table 1 describes a robust example of all these possibilities. Thus, with MLRP, S_2 is the only type that can display herding or contrarian behaviour. We can then state the following characterisation result for MLRP information structures (again we omit the analogous sell herding and sell contrarian cases).

Theorem 2 *Assume $S_1 < S_2 < S_3$ and the signal structure satisfies MLRP. (a) If S_2 is nU, there exists $\mu_{bh} \in (0, 1]$ such that S_2 buy herds if and only if $\mu < \mu_{bh}$. (b) If S_2 is nHill, there exists $\mu_{bc} \in (0, 1]$ such that S_2 acts as a buy contrarian if and only if $\mu < \mu_{bc}$.*

Proof: Fix the critical levels $\mu_{bh} \equiv \min\{\mu^i, \mu^s_{bh}\}$ for buy herding and $\mu_{bc} = \min\{\mu^i, \mu^s_{bc}\}$ for buy contrarianism, where μ^i is defined in Lemma 8 and μ^s_{bh} and μ^s_{bc} are respectively defined in (12) and (13) in the appendix. The "if" part follows from Lemma 4: By Lemma 5 (iv), conditions (5) and (6) hold. As outlined in the last section, *both* q^t_1/q^t_2 and q^t_1/q^t_3 can be made

$\mathsf{Pr}(S\|V)$	V_1	V_2	V_3
S_1	$\delta(1-\beta)/(\beta+\delta(1-\beta))$	$\delta(1-\alpha)$	0
S_2	$\beta/(\beta+\delta(1-\beta))$	α	$\beta/(\beta+(1-\delta)(1-\beta))$
S_3	0	$(1-\delta)(1-\alpha)$	$(1-\beta)(1-\delta)/(\beta+(1-\delta)(1-\beta))$

Table 1
An Example of an MLRP Signal Distribution.

For $\alpha, \delta \in (0,1)$, and $\beta \in (0,\alpha)$ the above satisfies MLRP. Moreover, S_2 is nU-shaped if $\beta \in (\alpha\delta/(1-\alpha(1-\delta)),\alpha)$ and $\delta < 1/2$, pU-shaped if $\beta \in (\alpha(1-\delta)/(1-\alpha\delta),\alpha)$ and $\delta > 1/2$, nHill-shaped if $\beta \in (0,\alpha\delta/(1-\alpha(1-\delta)))$ and $\delta < 1/2$, pHill-shaped if $\beta \in (0,\alpha(1-\delta)/(1-\alpha\delta))$ and $\delta > 1/2$, decreasing if $\beta \in (\alpha\delta/(1-\alpha(1-\delta)),\alpha(1-\delta)/(1-\alpha\delta))$ and $\delta < 1/2$, and increasing if $\beta \in (\alpha(1-\delta)/(1-\alpha\delta),\alpha\delta/(1-\alpha(1-\delta)))$ and $\delta > 1/2$.

arbitrarily close to zero by considering histories that involve a sufficiently large number of buys, and *both* q_3^t/q_1^t and q_3^t/q_2^t can be made arbitrarily close to zero by considering histories with a sufficiently large number of sales. Then conditions (3) and (4) hold and by Lemma 4, an nU-shaped S_2 buy herds and an nHill-shaped S_2 acts as a buy contrarian.

The "only if" part follows from Proposition 3a in the supplementary material.[27] □

The "if" part of the above proof demonstrates that with MLRP it is strikingly easy to describe histories that induce herding by a U-shaped type or contrarianism by a Hill-shaped type: an nU-shaped S_2 buy herds after a sufficient number of buys and an nHill-shaped S_2 acts as a buy contrarian after a sufficient number of sales.[28]

7 Resilience, Fragility and Large Price Movements

We now consider the robustness of herding and contrarianism and describe the range of prices for which herding and contrarianism can occur. Throughout this section we assume that signals satisfy the well-behaved case of MLRP (we will return to this later) and perform the analysis for buy herding and buy contrarianism; the other cases are analogous.

We first show that buy herding persists if and only if the number of sales during an episode of buy herding is not too large. This implies in particular that buy herding behaviour persists if the buy herding episode consists of only buys. We also show that during

[27]The "only if" part of the theorem also follows from the discussion in the previous section on noise trading: if types other than S_2 always take the same action, then there is a unique upper bound on the size of the informed trading that ensures that the spreads are sufficiently tight for S_2 to buy herd (or to act as a buy contrarian). By Lemma 5, S_3 always buys and S_1 always sells. Therefore, with MLRP the upper bound μ_{bh} is unique.

[28]Note that the bounds on the size of the informed μ_{bh} and μ_{bc} in Theorem 2 must be strictly below 1. To see this, recall that by part (i) of Lemma 5, the expectation of the herding or contrarian candidate type S_2 is always between those of the other two types at every history. Also, by part (ii) of Lemma 5, S_1 always sells and S_3 always buys. Therefore, if μ is arbitrarily close to 1 then, to ensure zero profits for the market maker, $\mathsf{E}[V|S_1,H^t] < \mathsf{bid}^t < \mathsf{E}[V|S_2,H^t]$ and $\mathsf{E}[V|S_2,H^t] < \mathsf{ask}^t < \mathsf{E}[V|S_3,H^t]$ for every H^t. This implies that S_2 does not trade; a contradiction.

a buy herding episode as the number of buys increases, it takes more sales to break the herd. For buy contrarianism the impact of buys and sales work in reverse: in particular, buy contrarianism persists if and only if the number of buys during an episode of buy contrarianism is not too large. This means that buy contrarianism does not end if the buy contrarianism episode consists of only sales. We also show that during a buy contrarianism episode as the number of sales increase, it takes more buys to end buy contrarianism.

Proposition 4 *Assume MLRP. Consider any history $H^r = (a^1, \ldots, a^{r-1})$ and suppose that H^r is followed by $b \geq 0$ buys and $s \geq 0$ sales in some order; denote this history by $H^t = (a^1, \ldots, a^{r+b+s-1})$.*[29]

(a) *If there is buy herding by S at H^r then there exists an increasing function $\bar{s}(\cdot) > 1$ such that S continues to buy herd at H^t if and only if $s < \bar{s}(b)$.*

(b) *If there is buy contrarianism by S at H^r then there exists an increasing function $\bar{b}(\cdot) > 1$ such that S continues to act as a buy contrarian at H^t if and only if $b < \bar{b}(s)$.*

One implication of the above result is that herding is resilient and contrarianism is self defeating. The reason is that when buy herding or buy contrarianism begins, buys become more likely relative to a situation where the herding or contrarian type does not switch. Thus, in both buy herding and buy contrarianism there is a general bias towards buying (relative to the case of no social learning). By Proposition 4, buy herding behaviour persists if there are not too many sales and buy contrarian ends if there is a sufficiently large number of buys. Thus herding is more likely to persist whereas contrarianism is more likely to end.

To see the intuition for Proposition 4 consider the case of buy herding in part (a). At any history the difference between the expectation of the herding type S and that of the market is determined by the relative likelihood that they attach to each of the three states. Since the herding type S must have an nU-shaped csd it follows that in comparing the expectation of the herding type S with that of the market there are two effects: first, S attaches more weight to V_3 relative to V_2 than the market and, second, S attaches more weight to V_1 relative to both V_2 and V_3 than the market. Since $V_1 < V_2 < V_3$ and at any history H^r with buy herding the expectation of the herding type S exceeds that of the market, it then follows that at H^r the first effect must dominate the second one, i.e. q_1^r/q_2^r and q_1^r/q_3^r are sufficiently small so that the first effect dominates. Also, by Lemma 5 (iv), when MLRP holds, buys reduce the probability of V_1 relative to the other states. Therefore, further buys after H^r make the second effect more insignificant and thereby ensure that the expectation of the herding type S remains above the ask price.

On the other hand, by Lemma 5 (iv) when MLRP holds, sales reduce the probability of V_3 relative to the other states; thus sales after H^r make the first effect less significant.

[29] We will henceforth omit past prices from the history H^t to simplify the exposition.

Therefore, with sufficiently many sales, the expectation of the herding type S will move below the ask price so that type S will no longer buy. This ends herding.

The intuition for the buy contrarian case is analogous except that the effect of further buys and further sales work in the opposite direction.

Next, we show that with MLRP large price movements are consistent with both herding and contrarianism. In fact, the range of price movements in both cases can include (almost) the entire set of feasible prices. Specifically, for buy herding the range of feasible prices is $[V_2, V_3]$ and for buy contrarianism the range is $[V_1, V_2]$.[30] As argued above, with MLRP buys increase prices, and sales decrease prices. Furthermore, by Proposition 4, buy herding persists when there are only buys and buy contrarianism persists when there are only sales. Thus, once buy herding starts, a large number of buys can induce prices to rise to levels arbitrarily close to V_3 without ending buy herding, and once buy contrarianism starts, large numbers of sales can induce prices to fall to levels arbitrarily close to V_1 without ending buy contrarianism.

We complete the analysis by showing that there exists a set of priors on $\mathbb{V}$ such that herding and contrarianism can start when prices are close to the middle value, V_2. Together with the arguments in the last paragraph, we have that herding and contrarian prices can span almost the entire range of feasible prices. Formally, we have the following.

Proposition 5 *Let signals obey MLRP.*

(a) *Consider any history $H^r = (a^1, \ldots, a^{r-1})$ at which there is buy herding (contrarianism). Then for any $\epsilon > 0$, there exists history $H^t = (a^1, \ldots, a^{t-1})$ following H^r such that there is buy herding (contrarianism) at every $H^\tau = (a^1, \ldots, a^{\tau-1})$, $r \le \tau \le t$, and $\mathsf{E}[V|H^{r+\tau}]$ exceeds $V_3 - \epsilon$ (is less than $V_1 + \epsilon$).*

(b) *Suppose the assumptions in Theorem 2 that ensure buy herding (contrarianism) hold. Then for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $\mathsf{Pr}(V_2) > 1 - \delta$ there is a history $H^t = (a^1, \ldots, a^{t-1})$ and a date $r < t$ such that (i) there is buy herding (contrarianism) at every $H^\tau = (a^1, \ldots, a^{\tau-1})$, $r \le \tau \le t$, (ii) $\mathsf{E}[V|H^r] < V_2 + \epsilon$ ($\mathsf{E}[V|H^r] > V_2 - \epsilon$) and ($iii$) $\mathsf{E}[V|H^t] > V_3 - \epsilon$ ($\mathsf{E}[V|H^t] > V_1 + \epsilon$).*

The results of this section (and the ones in the next section on volatility) assume that the information structure satisfies the well behaved case of MLRP. This ensures that the probabilities of buys and the sales are uniformly increasing and decreasing in V (see Lemma 5 (iv)). As a result, we have that the relative probability q_1^t/q_ℓ^t falls with buys and rises with sales for all $\ell = 2, 3$, and the opposite holds for q_3^t/q_ℓ^t for all $\ell = 1, 2$. This monotonicity in the relative probabilities of the extreme states is the feature that allows us to establish our persistence and fragility results. If MLRP were not to hold, then the probability of buys

[30] The reason is, by Lemma 3, at any date t buy herding implies $q_3^t > q_1^t$, buy contrarian implies $q_1^t > q_3^t$.

and sales may not be monotonic in V, and the results of this section may not hold.[31] An example of this is the herding example in Avery and Zemsky (1998); see Section 10.

8 The Impact of Social Learning on Volatility and Liquidity

In this section we are concerned with the impact of social learning on price movements. In particular, we ask the following questions: Do buys move prices more with than without social learning? Will sales move prices more with than without social learning?

To address these questions we compare price movements in our set-up with those in a hypothetical *benchmark* economy in which informed traders do not switch behaviour. This economy is identical to our set-up except that each informed type always takes the same action as the one he chooses at the initial history (before receiving any public information). Consequently, in the hypothetical benchmark economy informed traders act as if they do not observe prices and past actions of others. We thus refer to this world as the *opaque* market and discuss examples for such situations at the end of the section. In contrast, in the standard setting traders observe and learn from the actions of their predecessors. To highlight the difference, in this section we refer to the standard case as the *transparent* market. In both the transparent and the opaque economies, the market maker correctly accounts for traders' behaviour when setting prices.

Volatility. We show that with MLRP signals, at any histories at which either herding or contrarianism occurs, trades move prices more in the transparent market than in the opaque one. We found it interesting that larger price movements in the transparent market occur both after buys *and* after sales. Moreover, the result holds for MLRP information structures which, taken at face value, are "well-behaved."

We present the result for the case of buy herding and buy contrarianism; the results for sell herding and sell contrarianism are identical and will thus be omitted. Specifically, fix any history H^r at which buy herding starts and consider the difference between the most recent transaction price in the transparent market with that in the opaque market at any buy herding history that follows H^r. Assuming MLRP signals, we show (a) that the difference between the two prices is positive if the history since H^r consists of only buys, (b) that the difference is negative if the history since H^r consists of only sales and the number of sales is not too large,[32] and (c) that the difference is positive if the history following H^r is such that the number of buys is arbitrarily large relative to the number of sales. We also show an analogous result for buy contrarianism.

[31] The monotonicity of the probability of buys and sales in V can hold under weaker conditions than MLRP. For example, we could assume existence of a strictly increasing and a strictly decreasing signal. All the results of this paper with MLRP also hold with this weaker assumption.

[32] Note that, by Proposition 5, buy herding cannot persist with an arbitrarily large number of sales.

Formally, for any history H^t let $\mathsf{E}_o[V|H^t]$, $q_{i,o}^t$, $\beta_{i,o}^t$ and $\sigma_{i,o}^t$ be respectively the market expectation, the probability of V_i, the probability of a buy in state V_i and the probability of a sale in state V_i in the opaque market at H^t. Then we can show the following.

Proposition 6 *Assume MLRP. Consider any finite history $H^r = (a^1, \ldots, a^{r-1})$ at which the priors in the two markets coincide: $q_i^r = q_{i,o}^r$ for $i = 1, 2, 3$. Suppose that H^r is followed by $b \geq 0$ buys and $s \geq 0$ sales in some order; denote this history by $H^t = (a^1 \ldots, a^{r+b+s-1})$.*

(1) *Assume that there is buy herding at H^τ, for every $\tau = r, \ldots, r+b+s$.*

- (a) *Suppose $s = 0$. Then $\mathsf{E}[V|H^t] > \mathsf{E}_o[V|H^t]$ for any $b > 0$.*
- (b) *Suppose $b = 0$. Then there exists $\overline{s} \geq 1$ such that $\mathsf{E}[V|H^t] < \mathsf{E}_o[V|H^t]$ for any $s \leq \overline{s}$.*
- (c) *For any s there exists $\overline{b}$ such that $\mathsf{E}[V|H^t] > \mathsf{E}_o[V|H^t]$ for any $b > \overline{b}$.*

(2) *Assume that there is buy contrarianism at H^τ, for every $\tau = r, \ldots, r+b+s$.*

- (a) *Suppose $b = 0$. Then $\mathsf{E}[V|H^t] < \mathsf{E}_o[V|H^t]$ for any $s > 0$.*
- (b) *Suppose $s = 0$. Then there exists $\overline{b} \geq 1$ such that $\mathsf{E}[V|H^t] > \mathsf{E}_o[V|H^t]$ for any $b \leq \overline{b}$.*
- (c) *For any b there exists $\overline{s}$ such that $\mathsf{E}[V|H^t] < \mathsf{E}_o[V|H^t]$ for any $s > \overline{s}$.*

The critical element in demonstrating the result is the U-shaped nature of the herding candidate's signal and the Hill-shaped nature of the contrarian candidate's signal in combination with the public belief once herding/contrarianism starts. To see this consider any buy herding history $H^t = (a^1, \ldots, a^{r+b+s-1})$ satisfying the above proposition for the case described in part (1) — the arguments for a buy contrarian history described in part (2) are analogous. Then the prices in the transparent and opaque markets differ because at any buy herding history in the transparent market the market maker assumes that the buy herding candidate S buys whereas in the opaque market the market maker assumes that S sells.[33] Since the buy herding type must have a U-shaped signal we also have $\mathsf{Pr}(S|V_3) > \mathsf{Pr}(S|V_2)$. Then the following must hold: (i) the market maker upon observing a buy increases his belief about the likelihood of V_3 relative to that of V_2 faster in the transparent market (where S is a buyer) than in the opaque market (where S is a seller) and (ii) the market maker upon observing a sale decreases his belief about the likelihood of V_3 relative to V_2 faster in the transparent market than in the opaque market. Now if it is also the case that the likelihood of V_1 is small relative to that of V_3 in both worlds, then it follows from (i) and (ii), respectively, that the market expectation (which is the most recent transaction price) in the transparent market exceeds that in the opaque market after a buy and it is less after a sale.

At H^r in both markets the likelihoods of each state coincide ($q_i^r = q_{i,o}^r$); moreover the likelihood of V_1 in both markets is small relative to that V_3 (to ensure buy herding). Then the following two conclusions follow from the discussion in the previous paragraph: First, if H^t involves only a single buy after H^r (i.e. if $s = 0$ and $b = 1$) then $\mathsf{E}[V|H^t] > \mathsf{E}_o[V|H^t]$.

[33] If we assume $S_1 < S_2 < S_3$, then with MLRP the buy herding (buy contrarian) candidate must be S_2.

Second, if H^t involves only a single sale after H^r (i.e. if $b = 0$ and $s = 1$) then $\mathsf{E}[V|H^t] < \mathsf{E}_o[V|H^t]$. Part 1($b$) follows from the latter. To complete the intuition for 1(a) and 1(c), note that further buys after H^r reduce the probabilities of V_1 relative to V_3 in both markets (see Lemma 5 (iv)). Thus if either the history after H^r involves no sales (as in part 1(a)) or if the number of buys is large relative to the number of sales (as in part 1(c)) then the first conclusion is reinforced, and $\mathsf{E}[V|H^t]$ remains above $\mathsf{E}_o[V|H^t]$ after any such histories.

Notice that with MLRP, any sale beyond H^r increases the probability of V_1 relative to V_3 (and relative to V_2) both in the transparent and in the opaque market. Furthermore, the increase may be larger in the latter than in the former. As a result, for the buy herding case we cannot show that in general prices in the transparent market fall more than in the opaque market after *any arbitrary* number of sales. However, if the relative likelihood of a sale in state V_1 to V_3 in the transparent market is no less than that in the opaque market, i.e. $(\sigma_1/\sigma_3) \geq (\sigma_{1,o}/\sigma_{3,o})$, then we can extend the conclusion in part 1(b) to show that the price in the transparent market falls more than in the opaque market after *any arbitrary* number of sales (the proof is in the supplementary material).[34]

Proposition 6 of course does not address the likelihood of a buy or a sale after herding or contrarianism begins. It is important to note, however, that once buy herding or buy contrarianism starts there will also be more buys in the transparent market compared to the opaque market because the herding type buys at such histories. Thus, given the conclusions of Proposition 6, price paths must have a stronger upward bias in the transparent market than in the opaque market.

Finally, it is often claimed that herding generates excess volatility whereas contrarianism tends to stabilize markets because the contrarian types act against the crowd. The conclusions of this section are consistent with the former claim but contradict the latter. Both herding and contrarianism increase price movements compared to the opaque market and they do so for similar reasons — namely because of the U-shaped nature of the herding type's csd and the Hill-shaped nature of the contrarian type's csd.

Liquidity. In sequential trading models in the tradition of Glosten and Milgrom (1985), liquidity is measured by the bid-ask-spread as a larger spread implies higher adverse selection costs and thus lower liquidity. Since at any date market expectations after a buy and market expectations after a sale respectively coincide with the ask and the bid price at the previous date, the next corollary to Proposition 6 follows:

Corollary *At any history H^t at which type S engages in buy herding or buy contrarianism,*
(a) *the ask price when the buy herding or buy contrarian candidate S rationally buys exceeds the ask price when he chooses not to buy,*
(b) *the bid price when the buy herding or buy contrarian candidate S rationally buys*

[34] The condition $(\sigma_1/\sigma_3) \geq (\sigma_{1,o}/\sigma_{3,o})$ is satisfied if, e.g., the bias of the herding candidate is close to zero.

is lower than the bid price when he chooses to sell.

Part (a) of the result follows from Proposition 6 ($1a$) and ($2b$) when $b = 1$ and Part (b) follows from Proposition 6 ($1b$) and ($2a$) when $s = 1$.[35] The above corollary implies that in equilibrium liquidity (as measured by the bid-ask spread) is lower when some informed types herd or act as a contrarian than when they do not.

Interpretation of the Opaque Market and the Volatility Result. One can think of the traders in the opaque market as automata that always buy or sell depending on their signals. One justification for such naive behaviour is that traders do not observe or remember the public history of actions and prices (including current prices).

Alternatively, the non-changing behaviour may represent actions of rational traders in a trading mechanism where traders submit their orders through a market maker some time *before* the orders get executed. The market maker would receive these orders in some sequence and he would execute them sequentially at prices which reflect all the information contained in the orders received so far. The actions of other traders and the prices are unknown at the time of the order submission and thus, as in the opaque market, the order of each trader is independent of these variables.[36] As traders effectively commit to a particular trade before any information is revealed, the price sequence in this alternative model would coincide with the price sequence in the opaque market that we depict above. Therefore, Proposition 6 can also be used to claim that volatility is greater in the transparent market than in this alternative set-up in which all orders are submitted before any execution.

A slightly more transparent market than the opaque one is one where each trader with herding/contrarian signal S compares his prior expectations, $\mathsf{E}[V|S]$, with the current price and buys if $\mathsf{E}[V|S]$ exceed the ask price, sells if $\mathsf{E}[V|S]$ is less than the bid price and does not trade otherwise. In this "almost opaque" market there is a different kind of non-transparency in that at each period the traders do not observe or recall past actions and prices but they know the bid and ask prices at that period; furthermore they act semi-rationally by comparing their private expectation with current prices without learning about the liquidation value from the current price (e.g., for cognitive reasons).

For the case of herding, the same excess volatility result as in part (1) of Proposition 6 holds if we compare the transparent market with the above almost opaque market. To see this note that at the initial history H^1 every buy herding type sells. Also, at every buy herding history the prices are higher than at H^1; therefore in an almost opaque market the

[35] We show in the appendix, that the proof of Proposition 6 in these cases does not require MLRP. Thus, the corollary is stated without assuming MLRP information structure; see footnotes 45 and 46.

[36] A possible example of such mechanism is a market after a "circuit breaker" is introduced. The latter triggers a trading halt after "large" movements in stock prices. Before trading recommences, traders submit their orders without knowing others' actions. We thank Markus Brunnermeier for this interpretation.

herding type must also sell at every buy herding history.[37] Since Proposition 6 compares price volatility *only* at histories at which buy herding occurs, it follows that the same excess volatility result holds if we compare the transparent with the almost opaque market.

9 Herding and Contrarianism with Many States

Our results intuitively extend to cases with more signals and more values. In fact, with three states and an arbitrary number of signals our characterization results, in terms of U-shaped signals for herding and Hill-shaped signals for contrarianism, and all our conclusions in the previous two sections with respect to fragility, persistence, large price movements, liquidity and price volatility remain unchanged.[38]

With more than three states, U shape and Hill shape are no longer the only possible signal structures that can lead to herding and contrarianism. The intuition for our results with many states does, however, remain the same: the herding type must distribute probability weight to the tails, the contrarian types must distribute weight to the middle.

Assume there are $N > 2$ states and N signals. Denote the value of the asset in state j by V_j and assume that $V_1 \leq V_2 \leq \ldots \leq V_N$. Signal S is said to have an increasing csd if $\mathsf{Pr}(S|V_i) \leq \mathsf{Pr}(S|V_{i+1})$ for all $i = 1, \ldots, N-1$ and a decreasing csd if $\mathsf{Pr}(S|V_i) \geq \mathsf{Pr}(S|V_{i+1})$ for all $i = 1, \ldots, N-1$.

By the same reasoning as in Lemma 1 one can show that $\mathsf{E}[V|S, H^t] - \mathsf{E}[V|H^t]$ has the same sign as

$$\sum_{j=1}^{N-1} \sum_{i=1}^{N-j} (V_{i+j} - V_i) \cdot q_i q_{i+j} [\mathsf{Pr}(S|V_{i+j}) - \mathsf{Pr}(S|V_i)]. \tag{7}$$

For an increasing csd, (7) is always non-negative since $\mathsf{Pr}(S|V_{i+j}) - \mathsf{Pr}(S|V_i)$ is non-negative for all i, j; similarly, for decreasing csds, (7) is always non-positive since $\mathsf{Pr}(S|V_{i+j}) - \mathsf{Pr}(S|V_i)$ is non-positive for all i, j. Therefore, an increasing or decreasing type cannot switch behaviour and we have the following necessity result which is analogous to Proposition 1.

Lemma 6 *An increasing or decreasing S never switches from buying to selling or vice versa.*

Next, we describe two sufficient conditions that yield herding and contrarianism that have a similar flavour as our sufficiency results in Section 5. We will focus only on buy herding and buy contrarianism; sell herding and sell contrarianism are analogous.

In line with the previous analysis we assume for the remainder of this section that the values of the asset are distinct in each state and that they are on an equal grid.

[37] Since at a buy contrarian history prices are lower than at the initial history, the same claim cannot be made for the contrarian case.

[38] With three states, Hill and U shape are still well-defined, irrespective of the number of signals; even with a continuum of signals these concepts can be defined in terms of conditional densities.

Moreover, we assume that the prior probability distribution is symmetric. Thus we set $\{V_1, V_2, \ldots, V_N\} = \{0, \mathcal{V}, 2\mathcal{V}, \ldots, (N-1)\mathcal{V}\}$ and $\mathsf{Pr}(V_i) = \mathsf{Pr}(V_{N+1-i})$ for all i.

We begin with the analysis of the decision problem of selling at H^1 and generalize the concept of a negative bias as follows. Signal S is said to have a *negative bias* if for any pair of values that are equally far from the middle value, the signal happens more frequently when the true value is the smaller one than when it is the larger one: $\mathsf{Pr}(S|V_i) > \mathsf{Pr}(S|V_{N+1-i})$ for all $i < (N+1)/2$. In the supplementary material we show that this property ensures that $\mathsf{E}[V|S] < \mathsf{E}[V]$. This means, by a similar argument as with the three values case, that a negatively biased S must be selling at H^1 if μ is sufficiently small.

Next we generalize the sufficient conditions for switching to buying at some history. Recall that in the three value case, we considered histories at which the probability of one extreme value was small to the point where it can be effectively ignored. Then the expectation of the informed exceeds that of the market if the informed puts more weight on the larger remaining value than on the smaller remaining one.

The sufficient conditions that we describe for the switches in the general case have a similar intuition and are very simple as we impose restrictions only on the most extreme values. Specifically, to ensure buy herding we assume $\mathsf{Pr}(S|V_{N-1}) < \mathsf{Pr}(S|V_N)$ and consider histories at which the probabilities of all values are small relative to the two largest values V_{N-1} and V_N. Since at such histories all but the two largest values can be ignored it must be that the price must have risen and the expectation of type S must exceed that of the public expectation if $\mathsf{Pr}(S|V_{N-1}) < \mathsf{Pr}(S|V_N)$. If in addition the bid-ask spread is not too large (enough noise trading), the expectation of S will also exceed the ask price at H^t and type S switches from selling to buying after a price rise. Similarly, to ensure buy contrarianism we assume $\mathsf{Pr}(S|V_1) < \mathsf{Pr}(S|V_2)$ and consider histories at which the probabilities of all values are small relative to the two smallest ones V_1 and V_2. Since at such histories all but the two smallest values can be ignored and the price must have fallen, the expectation of type S must exceed that of the public expectation if $\mathsf{Pr}(S|V_1) < \mathsf{Pr}(S|V_2)$. If in addition the bid-ask spread is not too large at such histories then S switches from selling to buying after a price fall. Formally, we can show the following analogous result to Lemma 4.

Lemma 7 (*i*) *Suppose S is negatively biased and satisfies $\mathsf{Pr}(S|V_{N-1}) < \mathsf{Pr}(S|V_N)$. Then there exists a $\mu_{bh} \in (0,1]$ such that S buy herds if $\mu < \mu_{bh}$ and if*

$$\text{For all } \epsilon > 0 \text{ exists } H^t \text{ such that } q_i^t/q_l^t < \epsilon \text{ for all } l = N-1, N \text{ and } i < N-1. \quad (8)$$

(*ii*) *Suppose S is negatively biased and satisfies $\mathsf{Pr}(S|V_1) < \mathsf{Pr}(S|V_2)$. Then there exists a $\mu_{bc} \in (0,1]$ such that S is a buy contrarian if $\mu < \mu_{bc}$ and if*

$$\text{For all } \epsilon > 0 \text{ exists } H^t \text{ such that } q_i^t/q_l^t < \epsilon \text{ for all } l = 1, 2 \text{ and } i > 2. \quad (9)$$

The simplest way of ensuring the existence of histories that satisfy (8) and (9) is to assume MLRP. Then, as in Lemma 5 (*iv*) for the three states case, the probability of a buy is increasing and the probability of a sale is decreasing in V. As a result, with MLRP we can always ensure (8) by considering histories that contain a sufficiently large number of buys and (9) by considering histories that contain a sufficiently large number of sales. Hence, Lemma 7 yields the following for buy herding and buy contrarianism.[39]

Theorem 3 *Assume MLRP and suppose signal S is negatively biased.*
(a) If $\mathsf{Pr}(S|V_{N-1}) < \mathsf{Pr}(S|V_N)$ then there exists $\mu_{bh} \in (0,1]$ such that S buy herds if $\mu < \mu_{bh}$.
(b) If $\mathsf{Pr}(S|V_1) < \mathsf{Pr}(S|V_2)$ there exists $\mu_{bh} \in (0,1]$ such that S is a buy contrarian if $\mu < \mu_{bh}$.

The description in this section has assumed that each state is associated with a unique liquidation value of the underlying security. There can be, however, other uncertainties that do not affect the liquidation value but that do have an impact on the price. One example is a situation in which some agents may have superior information about the distribution of information in the economy (e.g., as in Avery and Zemsky's (1998) "composition uncertainty;" see the next section). In the supplementary appendix we prove all the sufficiency results from this section for such a generalized set-up.

10 Avery and Zemsky (1998)

As mentioned in the literature review, Avery and Zemsky (1998), AZ, argue that herd behaviour with informationally efficient asset prices is not possible unless signals are "non-monotonic" and uncertainty is "multi-dimensional." AZ reach their conclusions by (i) showing that herding is not possible when the information structure satisfies their definition of monotonicity and (ii) providing an example of herding that has "multi-dimensional uncertainty." In this section, we explain why our conclusions differ from theirs. We will also discuss the issue of price movements (or lack thereof) in their examples.

AZ's conclusion with respect to monotonicity arises because their adopted definition is non-standard and excludes herding almost by assumption. Specifically, they define a monotonic information structure as one that satisfies the following:

$$\forall S,\ \exists w \text{ s.t. } \forall H^t,\ \mathsf{E}(V|H^t, S) \text{ is weakly between } w \text{ and } \mathsf{E}(V|H^t). \tag{10}$$

This definition does not imply nor is implied by the standard MLRP definition of monotonicity. Also, in contrast to MLRP, it is not a condition on the primitives of the signal distribution. Instead, it is a requirement on endogenous variables that must hold for *all*

[39] Conditions that ensure sell herding and sell contrarian are defined analogously. In particular, to ensure the initial buy we need to assume a positive bias $\mathsf{Pr}(S|V_i) < \mathsf{Pr}(S|V_{N+1-i})$ for all $i < (N+1)/2$. For the switches we reverse the two conditions that ensure switching for buy herding and buy contrarian: for sell herding we need $\mathsf{Pr}(S|V_1) > \mathsf{Pr}(S|V_2)$, for sell contrarian we need $\mathsf{Pr}(S|V_{N-1}) > \mathsf{Pr}(S|V_N)$.

trading histories.[40] Furthermore, it precludes herding almost by definition.[41]

AZ's example of herding uses *Event Uncertainty*, a concept first introduced by Easley and O'Hara (1992). Specifically, in their example, the value of the asset and the signals can take three values $\{0, \frac{1}{2}, 1\}$ and the information structure can be described by the following:

$\mathsf{Pr}(S\|V)$	$V_1 = 0$	$V_2 = \frac{1}{2}$	$V_3 = 1$
$S_1 = 0$	p	0	$1-p$
$S_2 = \frac{1}{2}$	0	1	0
$S_3 = 1$	$1-p$	0	p

for some $p > 1/2$. The idea behind the notion of event uncertainty as used by AZ is that first, informed agents know *if* something (an event) has happened (they know whether $V = V_2$ or $V \in \{V_1, V_3\}$). Second, they receive noisy information with precision p about how this event has influenced the asset's liquidation value. This two stage information structure makes the uncertainty "multi-dimensional." Thus multi-dimensionality is equivalent to informed traders having a *finer information partition* than the market maker. AZ attribute herding to this feature of their example.

Multi-dimensionality is, however, neither necessary nor sufficient for herding and it is relevant to herding only to the extent that it may generate U-shaped signals. First, since AZ's example has three states and three signals, it is a special case of our main setup, and our characterization results apply. Specifically, the two herding types in AZ's example are S_1 and S_3. In addition to having finer partitions of the state space than the market maker, both types are also U-shaped and so our Proposition 3 explains the possibility of buy-herding by S_1 and sell-herding by S_3.[42] Second, our Proposition 3 demonstrates that there would also be herding if the AZ example is changed in such a way that all signals occur with positive probability in all states, while maintaining the U-shaped nature of signals S_1 and S_3.[43] Such an information structure is no longer multi-dimensional (the informed trader's partition would be the same as the market maker's). Third, consider an information

[40] Condition (10) does not imply that each signal has a increasing or decreasing csd; however, one can to show that if every signal has either a strictly increasing or a strictly decreasing csd then the information structure satisfies (10).

[41] For example, for buy herding by type S to occur at some history H^t we must have $\mathsf{E}[V|S] < \mathsf{E}[V]$ and a subsequent price rise $\mathsf{E}[V] < \mathsf{E}[V|H^t]$; but then (10) implies immediately that $w < \mathsf{E}[V|H^t, S] < \mathsf{E}[V|H^t]$ and hence buy herding by S at H^t is not possible.

[42] In AZ's example not all signals arise in all states (in states V_1 and V_3 only signals S_1 and S_3 arise, in state V_2 only signal S_2 arises). Thus when there is herding (by types S_1 or S_3), all informed types that occur with positive probability act alike. Nevertheless, it is important to note that in AZ's example herding does not constitute an informational cascade since at any history not all types take the same action. The reason is that it is never common knowledge that there is herding. Instead, at any finite history the market maker attaches non-zero probability to *all* three states and thus he always attaches non-zero probability to the state in which some trades are made by type S_2. Moreover, the market expects that when type S_1 buy-herds, type S_2 sells, when type S_3 sell-herds, type S_2 buys. See also our discussion in Footnote 17.

[43] For example, take $\mathsf{Pr}(S_i|V_i) = p(1-\epsilon)$, $\mathsf{Pr}(S_2|V_i) = \mathsf{Pr}(S_i|V_2) = \epsilon$, for all $i = 1, 3$ for $0 < \epsilon < (1-p)/2$.

structure for which signals S_1 and S_3 are such that informed traders know whether or not $V = 0$ has occurred (and the market did not). Such signals are multidimensional (they generate a finer partition), but they are not U-shaped and thus do not admit herding.

Our general analysis with three states also provides us with an appropriate framework to understand the nature of histories that generate herding and contrarianism. For example, as we explained in Section 7, to induce buy herding the trading history must be such that the probabilities of the lowest state V_1 is sufficiently small relative to the two other states. With MLRP, such beliefs arise after very simple histories consisting of a sufficiently large number of buys. In AZ's example, one also needs to generate such beliefs, but the trading histories that generate them are more complicated and involve a large number holds followed by a large number of buys.[44]

Turning to price movements, in AZ's event uncertainty example herding has limited capacity to explain price volatility as herding is fragile and price movements during herding are strictly limited. To allow for price movements during herding, AZ introduce a further level of informational uncertainty to their event uncertainty example. Specifically, they assume additionally that for each signal there are high and low quality informed traders. They also assume that there is uncertainty about the proportion of each type of informed trader. AZ claim that this additional level of uncertainty, which they label *composition uncertainty*, complicates learning and allows for large price movements during a buy herding phase (they do this by simulation).

A state of the world in AZ's example with composition uncertainty refers to both the liquidation value of the asset and the proportion of different types of informed traders in the market (the latter influences the prices). Thus, there is more than one state associated with a given value V of the asset. This example is, therefore, formally a special case of the multi-state version of our model and the possibility of herding follows from Lemma IV in the supplementary material. More specifically, our result establishes that to ensure herding in AZ's example with composition uncertainty we need the analogue of U-shaped signals with the property that the probability of a signal in each state with $V = 1/2$ is less than the probability of the signal in each of the states with $V = 0$ or with $V = 1$. This is indeed the case in the example with composition uncertainty. Therefore, herding there is also due to U-shaped signals. (See the discussion in Section F of the Supplementary Appendix.)

To understand the differences in price movements and persistence, recall our discussion of fragility in Section 7 with regards to buy herding by type S at some history H^t. The probability of the lowest state V_1 relative to those of the other two states, q_1^t/q_2^t and q_1^t/q_3^t, must be sufficiently small to start herding, and these relative probabilities need to remain

[44] The holds bring down the probability of V_1 relative to V_2 and the number of buys is chosen to bring down the probability of V_1 relative to V_3 sufficiently while not increasing the probability of V_1 relative to V_2 by too much. Formally, their construction is similar to Subcase D2 in the proof of our Proposition 3.

small for herding to persist beyond H^t. With MLRP, since $\mathsf{Pr}(\text{buy}|V)$ is increasing in V, further buys reduce both q_1^t/q_2^t and q_1^t/q_3^t; at the same time further buys increase prices. Thus herding can persist and prices can move significantly.

In AZ's example *without* composition uncertainty, while buys result in price increases, during any buy herding phase we have that $\mathsf{Pr}(\text{buy}|V_2) < \mathsf{Pr}(\text{buy}|V_1) = \mathsf{Pr}(\text{buy}|V_3)$. Thus, once buy herding begins, further buys cannot ensure that the relative probabilities of V_1 remain low as buys increase q_1^t/q_2^t, while leaving q_1^t/q_3^t unaffected. Hence, buys during buy herding in AZ's example *without* composition uncertainty are self-defeating. In AZ's example *with* composition uncertainty we have that $\mathsf{Pr}(\text{buy}|V_2) < \mathsf{Pr}(\text{buy}|V_1) < \mathsf{Pr}(\text{buy}|V_3)$ once herding starts. Further buys during herding thus reduce q_1^t/q_3^t while increasing q_1^t/q_2^t. As the former offsets somewhat the effect of the latter, buy herding may persist (and allow price increases) for longer than without composition uncertainty.

In conclusion, what makes herding less fragile and more consistent with significant price movements, are the relative probabilities of a specific trade in the different states of the world and not so much the addition of extra dimensions of uncertainty. In fact, when the probability of a buy is increasing in the value of the asset, buy herding is least fragile and most consistent with large price movements. As we have shown, this can happen with only three states, without different dimensions of uncertainty and with "well behaved" information structures satisfying MLRP.

11 Extensions, Discussion and Conclusion

Herding and contrarian behaviour are examples of history-dependent behaviour that may manifest itself in real market data as momentum or mean-reversion. Understanding the causes for the behaviour that underlie the data can thus help interpret these important non-stationarities. In the first part of this paper we characterized specific circumstances under which herding and contrarian behaviour can and cannot occur in markets with efficient prices. In the second part, we showed that both herding and contrarianism can be consistent with large movement in prices and that they both can reduce liquidity and increase volatility relative to situations where these kinds of social learning are absent.

In the early literature on informational social learning (e.g. Banerjee (1992)) herding was almost a generic outcome and would arise, loosely, with any kinds of signals. Herding as defined in our setup does not arise under all circumstances but only under those that we specify here. Namely, the underlying information generating process must be such that there are some signals that people receive under extreme outcomes more frequently than under moderate outcomes. Therefore, to deter herding, mixed messages predicting extreme outcomes (U-shaped signal) should be avoided.

It is important to note that, depending on the information structure, the prevalence of types who herd (or act as contrarians) can vary, and in some cases they can be very substantial. For example, consider the MLRP information structure in Table 1 when S_2 is U-shaped. Then, in *any* state the likelihood that an informed trader is a herding type S_2 has lower bound α. This bound can range between zero and one. Thus, when α is sufficiently close to 1, the likelihood that an informed trader is a herding type is arbitrarily close to 1 and the impact of herding switches can then be very significant.

In this paper, we have presented the results for which we were able to obtain clear-cut analytical results. In the supplementary material, we also explore other implications with numerical simulations. First, an important implication of our analysis for applied research is that when social learning arises according to our definition, simple summary statistics such as the number of buys and sales are not sufficient statistics for trading behaviour. Instead, as some types of traders change their trading modes during herding or contrarianism, prices become history-dependent. Thus as the entry order of traders is permutated, prices with the same population of traders can be strikingly different, as we illustrate with numerical examples. Second, herding results in price paths that are very sensitive to changes in some key parameters. Specifically, in the case with MLRP, comparing the situation where the proportion of informed agents is just below the critical levels described in Theorem 2 with that where the proportion is just above that threshold (so there is no herding), prices deviate substantially in the two cases. Third, herding slows down the convergence to the true value if the herd moves away from that true value, but it accelerates convergence if the herd moves into the right direction.

A Appendix: Omitted Proofs

A.1 Proof of Proposition 2

To save space we shall prove the result for the case of buy herding and buy contrarian; the proof for the sell cases are analogous. Thus suppose that S buy herds or acts as a buy contrarian at some H^t. The proof proceeds in several steps.

Step 1: *S has a negative bias:* Buy herding and buy contrarian imply $\mathsf{E}[V|S] < \mathsf{bid}_1$. Since $\mathsf{bid}_1 < \mathsf{E}[V]$ we must have $\mathsf{E}[V|S] < \mathsf{E}[V]$. Then by Lemma 2, S has a negative bias.

Step 2: $(\mathsf{Pr}(S|V_1) - \mathsf{Pr}(S|V_2))(q_3^t - q_1^t) > 0$: It follows from the definition of buy herding and buy contrarian that $\mathsf{E}[V|S, H^t] > \mathsf{ask}^t$. Since $\mathsf{E}[V|H^t] < \mathsf{ask}^t$ we must have $\mathsf{E}[V|S, H^t] > \mathsf{E}[V|H^t]$. By Lemma 1, this implies that (2) is positive at H^t. Also, by the negative bias (Step 1), the third term in (2) is negative. Therefore, the sum of the first two terms in (2) is positive: $q_3^t(\mathsf{Pr}(S|V_3) - \mathsf{Pr}(S|V_2)) + q_1^t(\mathsf{Pr}(S|V_2) - \mathsf{Pr}(S|V_1)) > 0$. But this means, by negative bias, that $(\mathsf{Pr}(S|V_1) - \mathsf{Pr}(S|V_2))(q_3^t - q_1^t) > 0$.

Step 3a: *If S buy herds at H^t then S is nU-shaped*: It follows from the definition of buy herding that $\mathsf{E}[V|H^t] > \mathsf{E}[V]$. By Lemma 3, this implies that $q_3^t > q_1^t$. Then it follows from Step 2 that $\mathsf{Pr}(S|V_1) > \mathsf{Pr}(S|V_2)$. Also, since S buy-herds, by Lemma 1, S cannot have a decreasing csd and we must have $\mathsf{Pr}(S|V_2) < \mathsf{Pr}(S|V_3)$. Thus, S is nU-shaped.

Step 3b: *If S acts as a buy contrarian at H^t then S is nHill shaped.* It follows from the definition of buy contrarian that $\mathsf{E}[V|H^t] < \mathsf{E}[V]$. By Lemma 3, this implies that $q_3^t < q_1^t$. But then it follows from Step 2 that $\mathsf{Pr}(S|V_1) < \mathsf{Pr}(S|V_2)$. Since by Step 1 S has a negative bias, we have $\mathsf{Pr}(S|V_2) > \mathsf{Pr}(S|V_1) > \mathsf{Pr}(S|V_3)$. Thus S is nHill-shaped.

A.2 Proof of Lemma 4

First consider the decision problem of type S at H^1. If S has a negative bias, by Lemma 2, $\mathsf{E}[V|S] < \mathsf{E}[V]$. Also, $\mathsf{E}[V] - \mathsf{bid}^1 > 0$ and $\lim_{\mu\to 0} \mathsf{E}[V] - \mathsf{bid}^1 = 0$. We can thus establish:

Lemma 8 *If S has a negative bias, then there exists $\mu^i \in (0,1]$ such that $\mathsf{E}[V|S] - \mathsf{bid}^1 < 0$ if and only if $\mu < \mu^i$.*

Simple calculations (see the supplementary material for details) establish the following useful characterization of the buying decision of type S at any H^t.

Lemma 9 $\mathsf{E}[V|S, H^t] - \mathsf{ask}^t$ *has the same sign as*

$$q_1^t q_2^t[\beta_1^t \mathsf{Pr}(S|V_2) - \beta_2^t \mathsf{Pr}(S|V_1)] + q_2^t q_3^t[\beta_2^t \mathsf{Pr}(S|V_3) - \beta_3^t \mathsf{Pr}(S|V_2)] + 2q_1^t q_3^t[\beta_1^t \mathsf{Pr}(S|V_3) - \beta_3^t \mathsf{Pr}(S|V_1)]. \quad (11)$$

To establish buy herding or buy contrarianism we need to show that (11) is positive at some history H^t. To analyze the sign of the expression in (11) we first show that the signs of the first and the second term in (11) are respectively determined by the signs of the expressions $\mathsf{Pr}(S|V_2) - \mathsf{Pr}(S|V_1)$ and $\mathsf{Pr}(S|V_3) - \mathsf{Pr}(S|V_2)$, if and only if μ is sufficiently small. To establish this, let, for any $i = 1, 2$ and any signal type S', $m^i \equiv \mathsf{Pr}(S|V_{i+1}) - \mathsf{Pr}(S|V_i)$, $M^i(S') \equiv \mathsf{Pr}(S'|V_i)\mathsf{Pr}(S|V_{i+1}) - \mathsf{Pr}(S'|V_{i+1})\mathsf{Pr}(S|V_i)$ and

$$\mu_i(S') \equiv \begin{cases} \frac{m^i}{m^i - 3M^i(S')} & \text{if } m^i \text{ and } M^i(S') \text{ are non-zero and have opposite signs,} \\ 1 & \text{otherwise.} \end{cases}$$

Clearly, $\mu_i(S') \in (0, 1]$. Next lemma shows that for some S', $\mu_1(S')$ and $\mu_2(S')$ are respectively the critical bounds on the value of μ that characterize the signs of the first and the second terms in (11).

Lemma 10 *In any equilibrium the following holds:*
(i) Suppose that $\mathsf{Pr}(S|V_3) > \mathsf{Pr}(S|V_2)$. Then at any H^t at which S' buys and $S'' \neq S, S'$ does not, the second term in (11) is positive if and only if $\mu < \mu_2(S')$.
(ii) Suppose that $\mathsf{Pr}(S|V_2) > \mathsf{Pr}(S|V_1)$. Then at any H^t at which S' buys and $S'' \neq S, S'$ does not, the first term in (11) is positive if and only if $\mu < \mu_1(S')$.

Proof of Lemma 10: First we establish (i). By simple computation, it follows that the second term in (11) equals $\gamma m^2 + \mu M^2(S')$. Also, since in this case $\mathsf{Pr}(S|V_3) > \mathsf{Pr}(S|V_2)$, we have from the definition of m^2 and $M^2(S')$ that $\gamma m^2 + \mu M^2(S') > 0$ if and only if $\mu < \mu_2(S')$. This completes the proof of (i).

The proofs of (ii) is analogous: By simple computation, it follows that the first term in (11) equals $\gamma m^1 + \mu M^1(S')$. Also, since in this case $\mathsf{Pr}(S|V_1) < \mathsf{Pr}(S|V_2)$, we have from the definition of m^1 and $M^1(S')$ that $\gamma m^1 + \mu M^1(S') > 0$ if and only if $\mu < \mu_1(S')$.

Each of the two cases in Lemma 10 provides a set of conditions that determine the sign of one of the terms in (11). If the other terms in (11) are sufficiently small, then these conditions also determine if S is a buyer. Specifically, if q_1^t is arbitrarily small relative to q_2^t and q_3^t then the first and the last terms are close to zero (as they multiplied by q_1^t and the second term is not) and can be ignored; thus, at such a history type S buys if the second term in (11) is positive. Also, if q_3^t is arbitrarily small relative to q_1^t and q_2^t then the last two terms are close to zero (as they multiplied by q_3^t and the first term is not) and can be ignored; thus, at such history type S buys if the first term in (11) is positive.

We can now prove Lemma 4 by appealing to Lemmas 3, 8, and 10 as follows:

Proof of Lemma 4: Consider case (i). Let μ^i be as defined in Lemma 8 and

$$\mu_{bh}^s \equiv \min_{S'} \mu_2(S'). \tag{12}$$

Assume also that $\mu < \mu_{bh} \equiv \min\{\mu^i, \mu_{bh}^s\}$. Since by assumption S has a negative bias and $\mu < \mu^i$, it follows from Lemma 8 that S sells at the initial history. Also, since S is U-shaped we have $\mathsf{Pr}(S|V_3) > \mathsf{Pr}(S|V_2)$. Therefore, by $\mu < \mu_{bh}^s$ and Lemma 10 (i), there exists some $\eta > 0$ such that the second term in (11) always exceeds η.

By condition (3) there exists a history H^t such that $q_1^t/q_3^t < 1$ and $\frac{q_1^t}{q_3^t} + \frac{2q_1^t}{q_2^t} < \eta$. Then by the former inequality and Lemma 3 we have $\mathsf{E}[V|H^t] > \mathsf{E}[V]$. Also, since the sum of the first and the third term in (11) is greater than $-q_2q_3(\frac{q_1^t}{q_3^t} + \frac{2q_1^t}{q_2^t})$, it follows from $\frac{q_1^t}{q_3^t} + \frac{2q_1^t}{q_2^t} < \eta$ that the sum must also be greater than $-\eta$. This, together with the second term in (11) exceeding η, implies that (11) is greater than zero, and hence S must be buying at H^t.

The proof of (ii) is analogous. Let

$$\mu_{bc}^s \equiv \min_{S'} \mu_1(S') \tag{13}$$

and assume that $\mu < \mu_{bc} \equiv \min\{\mu^i, \mu_{bc}^s\}$. Then by the same reasoning as above S sells at H^1. Also, since S has a Hill shape we have $\mathsf{Pr}(S|V_1) > \mathsf{Pr}(S|V_1)$. Therefore, by $\mu < \mu_{bc}^s$ and Lemma 10 (ii), there exists some $\eta > 0$ such that the first term in (11) always exceeds η.

By condition (4) there exists a history H^t such that $q_3^t/q_1^t < 1$ and $\frac{q_3^t}{q_1^t} + \frac{2q_3^t}{q_1^t} < \eta$. Then by the former inequality and Lemma 3 we have $\mathsf{E}[V|H^t] < \mathsf{E}[V]$. Also, since the sum of the second and the third term in (11) is greater than $-q_1q_2(\frac{q_3^t}{q_1^t} + \frac{2q_3^t}{q_2^t})$, it follows from $\frac{q_3^t}{q_1^t} + \frac{2q_3^t}{q_1^t} < \eta$ that the sum must also be greater than $-\eta$. Since the first term in (11) exceeds η, this implies that (11) is greater than zero, and hence S must be buying at H^t.

A.3 Proof of Proposition 3

Below we provide a proof for part (a) of the proposition; the arguments for the other parts are analogous and therefore omitted.

The proof of part (a) is by contradiction. Suppose that S is nU-shaped and that all the other assumptions in part (a) of the proposition hold. Also assume, contrary to the claim in part (a), that S does not buy herd. Then, by Lemma 4 (i), we have a contradiction if it can be shown that (3) holds. This is indeed what we establish in the rest of the proof.

First note that the no buy herding supposition implies that S does not buy at any history H^t. Otherwise, since S has a negative bias, by Step 2 in the proof of Proposition 2, $(\mathsf{Pr}(S|V_1)-\mathsf{Pr}(S|V_2))(q_3^t-q_1^t) > 0$. Because S is U-shaped this implies that $q_3^t > q_1^t$; but then since by assumption $\mu < \mu^i$, it follows from Lemma 8 that S buy herds; a contradiction.

Next, we describe conditions that ensure that q_1^t/q_l^t are decreasing in t for any $l = 2, 3$. Denote an infinite path of actions by $H^\infty = \{a^1, a^2, \ldots\}$. For any date t and any finite history $H^t = \{a^1, \ldots, a^{t-1}\}$, let a_k^t be the action that would be taken by type $S_k \in \mathbb{S}\backslash S$ at H^t; thus if the informed trader at date t receives a signal $S_k \in \mathbb{S}\backslash S$ then a^t, the actual action taken at H^t, equals a_k^t. Also denote the action taken by S at H^t by $a^t(S)$. Then we have the following.

Lemma 11 *Fix any infinite path $H^\infty = \{a^1, a^2, \ldots\}$ and any signal $S_k \in \mathbb{S}\backslash S$. Let $S_{k'} \in \mathbb{S}\backslash S$ be such that $S_{k'} \neq S_k$. Suppose that $a^t = a_k^t$. Then for any date t and $l = 2, 3$ we have:*
A. If $a_k^t = a_{k'}^t$ then q_1^t/q_l^t is decreasing.
B. If $a_k^t = a^t(S)$ and the inequality $\mathsf{Pr}(S_{k'}|V_l) \leq \mathsf{Pr}(S_{k'}|V_1)$ holds then q_1^t/q_l^t is non-increasing; furthermore, if the inequality is strict then q_1^t/q_l^t is decreasing.
C. If $a_k^t \neq a_{k'}^t$ and $a_k^t \neq a^t(S)$ and the inequality $\mathsf{Pr}(S_k|V_l) \geq \mathsf{Pr}(S_k|V_1)$ holds then q_1^t/q_l^t is non-increasing; furthermore, if the inequality is strict then q_1^t/q_l^t is decreasing.

Proof of Lemma 11: Fix any $l = 2, 3$. Since $\frac{q_1^{t+1}}{q_l^{t+1}} = \frac{q_1^t \mathsf{Pr}(a^t|H^t,V_1)}{q_l^t \mathsf{Pr}(a^t|H^t,V_l)}$, to establish that q_1^t/q_l^t is decreasing it suffices to show that $\mathsf{Pr}(a^t|H^t, V_l)$ is (greater) no less than $\mathsf{Pr}(a^t|H^t, V_1)$. Now consider each of the three cases $A. - C.$

A. Since signal S is nU-shaped, the combination of S_k and $S_{k'}$ is pHill-shaped. This together with $a^t = a_k^t = a_{k'}^t$ imply that $\mathsf{Pr}(a^t|H^t, V_l)$ exceeds $\mathsf{Pr}(a^t|H^t, V_1)$.

B. If $\mathsf{Pr}(S_{k'}|V_l) \leq \mathsf{Pr}(S_{k'}|V_1)$ we have $\mathsf{Pr}(S_k|V_l) + \mathsf{Pr}(S|V_l) \geq \mathsf{Pr}(S_k|V_1) + \mathsf{Pr}(S|V_1)$. This, together with $a^t = a_k^t = a^t(S)$ imply that $\mathsf{Pr}(a^t|H^t, V_l) \geq \mathsf{Pr}(a^t|H^t, V_1)$. Furthermore, the latter inequality must be strict if $\mathsf{Pr}(S_{k'}|V_l)$ were less than $\mathsf{Pr}(S_{k'}|V_1)$.

C. If $\mathsf{Pr}(S_k|V_l) \geq \Pr(S_k|V_1)$ and $a_k^t \neq a_{k'}^t$ and $a_k^t \neq a^t(S)$ we have immediately that $\mathsf{Pr}(a^t|H^t, V_l) \geq \mathsf{Pr}(a^t|H^t, V_1)$. Furthermore, the latter inequality is strict if $\mathsf{Pr}(S_k|V_l)$ were less than $\Pr(S_k|V_1)$. This concludes the proof of Lemma 11.

Now we show that (3) holds and thereby obtain the required contradiction. This will be done for each feasible csd combination of signals.

Case A: Either there exists a signal that is decreasing or there are two Hill-shaped signals each with a non-negative bias.

Consider an infinite path of actions consisting of an infinite number of buys. We demonstrate (3) by showing that along this infinite history at any date t both q_1^t/q_2^t and q_1^t/q_3^t are decreasing, and hence converge to zero (note that there are a finite number of states and signals). We show this in several steps.

Step 1: If more than one informed type buys at t then q_1^t/q_2^t and q_1^t/q_3^t are both decreasing at any t: Since S does not buy at any t, this follows immediately from Lemma 11.A.

Step 2: If exactly one informed type buys at period t then (i) q_1^t/q_2^t is decreasing and (ii) q_1^t/q_3^t is decreasing if the informed type that buys has a non-zero bias, and is non-increasing otherwise: Let S_i be the only type that buys at t. This implies that S_i cannot be decreasing; therefore, by assumption, S_i must be pHill-shaped and the step follows from Lemma 11.C.

Step 3: If a type has a zero bias he cannot be a buyer at any date t: Suppose not. Then there exist a type S_i with a zero bias such that $\mathsf{E}[V|H^t, S_i] - \mathsf{E}[V|H^t] > 0$. By Lemma 1 we then have

$$[\mathsf{Pr}(S_i|V_3) - \mathsf{Pr}(S_i|V_2)](q_3^t - q_1^t) > 0. \tag{14}$$

Also, by Steps 1 and 2, q_1^t/q_3^t is non-increasing at every t. Moreover by assumption $q_1^1/q_3^1 = 1$. Therefore, $q_1^t/q_3^t \leq 1$. Since S_i buys at t, S_i must be Hill-shaped, contradicting (14).

Step 4: q_1^t/q_2^t and q_1^t/q_3^t are both decreasing at any t. This follows Steps 1-3.

Case B: There exists an increasing S_i s.t. $\mathsf{Pr}(S_i|V_k) \neq \mathsf{Pr}(S_i|V_{k'})$ for some k and k'.

Let S_j be the third signal other than S and S_i. Now we obtain (3) in two steps.

Step 1: If $\mathsf{Pr}(S_i|V_1) = \mathsf{Pr}(S_i|V_2)$ then for any $\epsilon > 0$ there exists a finite history $H^\tau = \{a^1, \ldots, a^{\tau-1}\}$ such that $q_1^\tau/q_2^\tau < \epsilon$. Consider an infinite path $H^\infty = \{a^1, a^2, \ldots\}$ such that $a^t = a_j^t$ (recall that a_j^t is the action taken by S_j at history $H^t = (a^1, \ldots, a^{t-1})$). Note that S is nU-shaped and $\mathsf{Pr}(S_i|V_1) = \mathsf{Pr}(S_i|V_2) < \mathsf{Pr}(S_i|V_3)$. Therefore, $\mathsf{Pr}(S_j|V_2) > \max\{\mathsf{Pr}(S_j|V_1), \mathsf{Pr}(S_j|V_3)\}$.

Then it follows from Lemma 11 that q_1^t/q_2^t is decreasing if $a^t \neq a^t(S)$ and it is constant if $a^t = a^t(S)$. To establish the claim it suffices to show that $a^t \neq a^t(S)$ infinitely often. Suppose not. Then there exists T such that for all $t > T$, $a_j^t = a^t(S)$. Since type S does not buy at any date and there cannot be more than one informed type holding at any date (there is always a buyer or a seller), we must have S_j (and S) selling at at every $t > T$. Then, by Lemma 1 we have

$$\frac{q_3^t}{q_1^t}[\mathsf{Pr}(S_j|V_3) - \mathsf{Pr}(S_j|V_2)] + [\mathsf{Pr}(S_j|V_2) - \mathsf{Pr}(S_j|V_1)] + 2\frac{q_3^t}{q_2^t}[\mathsf{Pr}(S_j|V_3) - \mathsf{Pr}(S_j|V_1)] < 0. \tag{15}$$

for all $t > T$. Also, by $\mathsf{Pr}(S_i|V_1) = \mathsf{Pr}(S_i|V_2) < \mathsf{Pr}(S_i|V_3)$ we have $\mathsf{Pr}(S_j|V_l) + \mathsf{Pr}(S|V_l) > \mathsf{Pr}(S_j|V_3) + \mathsf{Pr}(S|V_3)$ for $l = 1, 2$. Therefore, $\frac{q_3^t}{q_l^t} \to 0$ as $t \to \infty$ for any $l = 1, 2$. This, together with $\mathsf{Pr}(S_j|V_2) > \mathsf{Pr}(S_j|V_1)$, contradict (15).

Step 2: For any $\epsilon > 0$ there exists a history H^t s.t. $q_1^t/q_l^t < \epsilon$ for any $l = 2, 3$: Fix any $\epsilon > 0$. Let H^τ be such that $q_1^\tau/q_2^\tau < \epsilon$ if $\mathsf{Pr}(S_i|V_1) = \mathsf{Pr}(S_i|V_2)$ (by the previous step such a history exists) and be the empty history H^1, otherwise. Consider any infinite path $H^\infty = \{H^\tau, a^\tau, a^{\tau+1}, \ldots.\}$, where for any $t \geq \tau$, a^t is the action that type S_i takes at history $H^t = \{H^\tau, a^\tau, \ldots, a^{t-1}\}$; i.e. we first have the history H^τ and then we look at a subsequent history that consists only of the actions that type S_i takes.

Since S_i is increasing it follows from Proposition 1 that at any history S_i does not sell. Also, by the supposition S does not buy at any history. Therefore, S_i and S always differ at every history H^t with $t \geq \tau$ (there cannot be more than one type holding). But since a^t is the action that type S_i takes at history H^t, S_i is increasing and $\mathsf{Pr}(S_i|V_k) \neq \mathsf{Pr}(S_i|V_{k'})$ for some k and k', it then follows from part A and C of Lemma 11 that for every $t \geq \tau$ (i) $\frac{q_1^t}{q_3^t}$ is decreasing, (ii) $\frac{q_1^t}{q_2^t}$ is non-increasing. This, together with $q_1^\tau/q_2^\tau < \epsilon$ when $\mathsf{Pr}(S_i|V_1) = \mathsf{Pr}(S_i|V_2)$, establishes that there exists t such that $q_1^t/q_l^t < \epsilon$ for any $l = 2, 3$.

Case C: There are two Hill-shaped signals and one has a negative bias.

Let S_i be the Hill-shaped signal with the negative bias. Also, let S_j be the other Hill-shaped signal. Since both S and S_i have negative biases, S_j must have a positive bias.

Next fix any $\epsilon > 0$ and define y and φ_{lm}, for any $l, m = 1, 2, 3$, as follows:

$$
\begin{aligned}
y &:= \frac{[\mathsf{Pr}(S_i|V_2) - \mathsf{Pr}(S_i|V_1)]}{2[\mathsf{Pr}(S_i|V_1) - \mathsf{Pr}(S_i|V_3)]} > 0 \\
\varphi_{lm} &:= \max\left\{\frac{\gamma + \mu\mathsf{Pr}(S_i|V_l)}{\gamma + \mu\mathsf{Pr}(S_i|V_m)}, \frac{\gamma + \mu(1 - \mathsf{Pr}(S|V_l))}{\gamma + \mu(1 - \mathsf{Pr}(S|V_m))}, \frac{\gamma + \mu\mathsf{Pr}(S_j|V_l)}{\gamma + \mu\mathsf{Pr}(S_j|V_m)}\right\}. \qquad (16)
\end{aligned}
$$

Since both S_i and S_j are Hill-shaped we have $\varphi_{12} < 1$. This implies that there exists an integer $M > 0$ and $\delta \in (0, \epsilon)$ such that $y(\varphi_{12})^M < \epsilon$ and $\delta(\varphi_{13})^M < \epsilon$.

Consider the infinite path $H^\infty = \{a^1, a^2, \ldots\}$ where $a^t = a_j^t$ at every t. Then we have:

Claim 1: q_1^t/q_3^t is decreasing at every t: As S_i and S_j have a negative and a positive bias respectively, by Lemma 11, q_1^t/q_3^t is decreasing at every t.

Claim 2: q_1^t/q_2^t converge to zero if there exists T such that $a_i^t \neq a_j^t$ for all $t > T$: Since S_j is Hill-shaped this follows immediately from parts A and C of Lemma 11.

Claim 3: There exists a history H^τ s.t. $q_1^\tau/q_3^\tau < \delta$ and $q_1^\tau/q_2^\tau < y$: Suppose not; then by Claims 1 and 2 there exists a date τ such that $q_1^\tau/q_3^\tau < \delta$ and $a_i^\tau = a_j^\tau$. Since S does not buy at any history, it follows that S_i and S_j must be buying at τ (there is always at least one buyer and seller; thus S_i and S_j cannot both be holding at τ). Then, $\mathsf{E}[V|S_i, H^t] - \mathsf{E}[V|H^t] > 0$. By Proposition 2, this implies

$$
[\mathsf{Pr}(S_i|V_3) - \mathsf{Pr}(S_i|V_2)] + \frac{q_1^\tau}{q_3^\tau}[\mathsf{Pr}(S_i|V_2) - \mathsf{Pr}(S_i|V_1)] + \frac{2q_1^\tau}{q_2^\tau}[\mathsf{Pr}(S_i|V_3) - \mathsf{Pr}(S_i|V_1)] > 0.
$$

Since S_i is nHill-shaped, it follows from the last inequality that

$$\frac{q_1^\tau}{q_2^\tau} < \frac{[\mathsf{Pr}(S_i|V_3) - \mathsf{Pr}(S_i|V_2)] + \frac{q_1^\tau}{q_3^\tau}[\mathsf{Pr}(S_i|V_2) - \mathsf{Pr}(S_i|V_1)]}{2[\mathsf{Pr}(S_i|V_1) - \mathsf{Pr}(S_i|V_3)]} < \frac{\frac{q_1^\tau}{q_3^\tau}[\mathsf{Pr}(S_i|V_2) - \mathsf{Pr}(S_i|V_1)]}{2[\mathsf{Pr}(S_i|V_1) - \mathsf{Pr}(S_i|V_3)]}. \tag{17}$$

As $q_1^\tau/q_3^\tau < \delta$ and $\delta < 1$, we have $q_1^\tau/q_2^\tau < y$. This contradicts the supposition.

To complete the proof for this case, fix any τ and H^τ such that $q_1^\tau/q_3^\tau < \delta$ and $q_1^\tau/q_2^\tau < y$ (by Claim 3 such a history exists). Consider a history $\overline{H}^t$ that consists of path $H^\tau = (a^1, \ldots, a^{\tau-1})$ followed by M periods of buys. Thus $t = \tau + M$ and $H^t = \{H^\tau, \overline{a}^1, \ldots, \overline{a}^M\}$, where for any $m \leq M$, $\overline{a}^m =$ buy. Since a buy must be either from S_j or S_i or both, it then follows from the definitions of φ_{13}, M and δ, and from $q_1^\tau/q_3^\tau < \delta$ that

$$q_1^t/q_3^t \leq (\varphi_{13})^M(q_1^\tau/q_3^\tau) < (\varphi_{13})^M\delta < \epsilon. \tag{18}$$

Also, since $q_1^\tau/q_2^\tau < y$ we have

$$q_1^t/q_2^t < (\varphi_{12})^M(q_1^\tau/q_2^\tau) < (\varphi_{12})^M y < \epsilon. \tag{19}$$

Since the initial choice of ϵ was arbitrary, (3) follows immediately from (18) and (19).

Case D: There exists a U shaped signal $S_i \in \mathbb{S}\backslash S$.

Since both S and S_i are U shaped it follows that the third signal S_j is Hill shaped. Moreover, by assumption S_j must have a non-negative bias. To establish (3) fix any $\epsilon > 0$ and consider the two possible subcases that may arise.

Subcase D1: S_i has a zero bias. We establish the result in two claims.

Claim 1: There exists a history H^τ such that $q_1^\tau/q_3^\tau < \epsilon$. Consider the infinite path $H^\infty = \{a^1, a^2, \ldots\}$ such that $a^t =$ buy for each t. Since a buy must be either from S_j or S_i or both, and S_i has a zero bias, it follows from parts A and C of Lemma 11 that q_1^t/q_3^t is non-increasing at every t. Furthermore, q_1^t/q_3^t is decreasing if $a^t = a_j^t$. Therefore, the claim follows if S_j buys infinitely often along the path H^∞. To show that the latter is true suppose it is not; then there exists T such that for all $t \geq T$, $a_j^t \neq a_i^t =$buy. Then for all $t > T$, by Lemma 9,

$$\frac{q_2^t}{q_1^t}[\beta_2^t\mathsf{Pr}(S_j|V_3) - \beta_3^t\mathsf{Pr}(S_j|V_2)] + \frac{q_2^t}{q_3^t}[\beta_1^t\mathsf{Pr}(S_j|V_2) - \beta_2^t\mathsf{Pr}(S_j|V_1)] + [\beta_1^t\mathsf{Pr}(S_j|V_3) - \beta_3^t\mathsf{Pr}(S_j|V_1)] < 0, \tag{20}$$

Also, since S_i is U shaped both $\frac{q_2^t}{q_1^t}$ and $\frac{q_2^t}{q_3^t}$ must be decreasing at every $t > T$. But this is a contradiction because at every $t > T$, the last term in (20) is positive: $\beta_1^t\mathsf{Pr}(S_j|V_3) - \beta_3^t\mathsf{Pr}(S_j|V_1) = \gamma(\mathsf{Pr}(S_j|V_3) - \mathsf{Pr}(S_j|V_1)) > 0$ (the equality follows from S_i's zero bias).

Claim 2: There exists a history H^t such that $q_1^t/q_l^t < \epsilon$ for all $l = 2, 3$: By the previous claim there exists a history H^τ such that $q_1^\tau/q_3^\tau < \epsilon$. Next, consider a history $H^\infty = \{H^\tau, a^\tau, a^{\tau+1}, \ldots\}$ that consists of path H^τ followed by a sequence of actions $\{a^\tau, a^{\tau+1}, \ldots\}$ such that $a^t = a_j^t$ at every history $H^t = \{H^\tau, a^\tau, \ldots, a^{t-1}\}$. Since S_i has a zero bias, it follows from Lemma 11 that at every $t > \tau$, q_1^t/q_3^t is non-increasing. Also, we have $q_1^\tau/q_3^\tau < \epsilon$; therefore we have that at every $t > \tau$, $q_1^t/q_3^t < \epsilon$. Furthermore, since S and

S_i are U shaped, and S_j is Hill shaped, by Lemma 11, q_1^t/q_2^t is decreasing at every $t > \tau$; hence there must exists $t > \tau$ such that $q_1^t/q_2^t < \epsilon$.

Since the initial choice of ϵ was arbitrary, (3) follows from Claim 2.

Subcase D2: Both S_i and S_j have non-zero bias.

Consider first the infinite path $H^\infty = \{a^1, a^2, \ldots\}$ such that $a^t = a_j^t$ at every history $H^t = \{a^1, \ldots, a^{t-1}\}$. Then the following claims must hold.

Claim 1: q_1^t/q_2^t is decreasing at every t : Since S_j and S_i are respectively Hill shaped and U shaped, it follows from Lemma 11 that q_1^t/q_2^t is decreasing.

Claim 2: If S_i has a negative bias then q_1^t/q_3^t is decreasing at every t: Since S_j has a positive bias and S_i has a negative bias, by Lemma 11, q_1^t/q_3^t is decreasing at every t.

Claim 3: If S_j has a positive bias and there exists a period T such that, for all $t > T$, $a_j^t = buy$ then q_1^t/q_3^t is decreasing at every $t > T$: Since S_j has a positive bias and S does not buy at any date, by Lemma 11, q_1^t/q_3^t must be decreasing at every $t > T$.

Before stating the next claim, consider φ_{ml} defined in (16). If both S_i and S_j have positive biases, $\varphi_{13} < 1$. Thus, if S_i has a positive bias there exist an integer M such that

$$\text{(i) } (\phi_{ml})^M < \epsilon \text{ if } S_j \text{ has a positive bias and (ii) } \left(\frac{\gamma + \mu \mathsf{Pr}(S_i|V_l)}{\gamma + \mu \mathsf{Pr}(S_i|V_m)}\right)^M < \epsilon. \tag{21}$$

Fix any such M. Then there also exists $\delta \in (0, \epsilon)$ such that

$$\delta(\varphi_{12})^M < \epsilon. \tag{22}$$

Claim 4: If S_j has a zero bias, then there exists a history H^τ s.t. $q_1^\tau/q_2^\tau < \delta$ and $q_1^\tau/q_3^\tau = 1$: Since $q_1^1/q_3^1 = 1$ it follows that at date 1, S_j holds. By recursion it follows that at every history $H^t = \{a^1, \ldots, a^{t-1}\}$ we have $q_1^t/q_3^t = 1$ and the claim follows from Claim 1.

Claim 5: If both S_i and S_j have positive biases, then there exists a history H^τ s.t. $q_1^\tau/q_2^\tau < \delta$ and $q_1^\tau/q_3^\tau < x$, where δ satisfies (21) and

$$x \equiv \frac{[\mathsf{Pr}(S_i|V_3) - \mathsf{Pr}(S_i|V_2)] + 2\epsilon[\mathsf{Pr}(S_i|V_3) - \mathsf{Pr}(S_i|V_1)]}{[\mathsf{Pr}(S_i|V_1) - \mathsf{Pr}(S_i|V_2)]}.$$

Suppose not. Then by Claims 1 and 3 there exists date τ such that $q_1^\tau/q_2^\tau < \delta$ and $a_j^\tau \neq$ buy. Since S also does not buy at H^τ, it follows that only S_i buys at τ. Then $\mathsf{E}[V|S_i, H^\tau] > \mathsf{E}[V|H^\tau]$. By Proposition 2, this implies

$$[\mathsf{Pr}(S_i|V_3) - \mathsf{Pr}(S_i|V_2)] + \frac{q_1^\tau}{q_3^\tau}[\mathsf{Pr}(S_i|V_2) - \mathsf{Pr}(S_i|V_1)] + \frac{2q_1^\tau}{q_2^\tau}[\mathsf{Pr}(S_i|V_3) - \mathsf{Pr}(S_i|V_1)] > 0.$$

Since $q_1^\tau/q_2^\tau < \delta < \epsilon$ and S_i is pU shaped, we can rearrange the above to show that

$$\frac{q_1^\tau}{q_3^\tau} \le \frac{[\mathsf{Pr}(S_i|V_3) - \mathsf{Pr}(S_i|V_2)] + \frac{2q_1^\tau}{q_2^\tau}[\mathsf{Pr}(S_i|V_3) - \mathsf{Pr}(S_i|V_1)]}{\mathsf{Pr}(S_i|V_1) - \mathsf{Pr}(S_i|V_2)} < \frac{[\mathsf{Pr}(S_i|V_3) - \mathsf{Pr}(S_i|V_2)] + 2\epsilon[\mathsf{Pr}(S_i|V_3) - \mathsf{Pr}(S_i|V_1)]}{[\mathsf{Pr}(S_i|V_1) - \mathsf{Pr}(S_i|V_2)]} = x.$$

Claim 6: If S_i has a positive bias and then there exists a history $\overline{H}^t$ s.t. $q_1^t/q_l^t < \epsilon$ for *any* $l = 2, 3$. Fix any history $H^\tau = (a^1, \ldots, a^{\tau-1})$ s.t. $q_1^\tau / q_2^\tau < \delta$ and $q_1^\tau / q_3^\tau = 1$ if S_j has a zero bias and $q_1^\tau / q_2^\tau < \delta$ and $q_1^\tau / q_3^\tau < x$ if S_j has a positive bias (by Claims 4 and 5 such histories exist). Next, consider a history $\overline{H}^t$ that consists of path H^τ followed by M periods of buys. Thus $t = \tau + M$ and $H^t = \{h^\tau, \overline{a}^1, \ldots, \overline{a}^M\}$, where for any $m \leq M$, $\overline{a}^m =$ buy. Since a buy must be either from S_j or S_i or both, it then follows from the definitions of φ_{12} in (16), from (22) and from $q_1^\tau/q_2^\tau < \delta$ that $\frac{q_1^t}{q_2^t} \leq \frac{q_1^\tau}{q_2^\tau} (\varphi_{12})^M < \delta(\varphi_{12})^M < \epsilon$. To show that $q_1^\tau / q_3^\tau < \epsilon$ consider the two cases of S_j having a zero bias and S_j having a positive bias separately. In the latter case, we have $q_1^\tau / q_3^\tau < x$. Then, by (i) in (21), we have $\frac{q_1^t}{q_3^t} < \frac{q_1^\tau}{q_3^\tau} \, (\varphi_{13})^M < x(\varphi_{13})^M < \epsilon$. In the former case, since $q_1^\tau / q_3^\tau = 1$ it must be that $\overline{a}^m = a_i^{t+1} \neq a_j^{\tau+m}$. Hence, $\frac{q_1^t}{q_3^t} < 1$. Recursively, it then follows that $\overline{a}^{m'} = a_i^{t+1} \neq a_j^{\tau+m'}$ for all $m' \leq m$. Thus, by (ii) in (21), $\frac{q_1^t}{q_3^t} < \frac{q_1^\tau}{q_3^\tau} \, (\frac{\gamma + \mu \mathsf{Pr}(S_i|V_l)}{\gamma + \mu \mathsf{Pr}(S_i|V_m)})^M < \epsilon$.

Since the initial choice of ϵ was arbitrary, (3) follows from Claims 1,2 and 5.

A.4 Proof of Proposition 4

(a) At H^t buy herding occurs if and only if $\mathsf{E}[V|S, H^t] - \mathsf{ask}^t > 0$ and $\mathsf{E}[V|H^t] - \mathsf{E}[V] > 0$. Thus, to demonstrate the existence of the function $\overline{s}$, we need to characterize the expressions $\mathsf{E}[V|S, H^t] - \mathsf{ask}^t$ and $\mathsf{E}[V|H^t] - \mathsf{E}[V]$ for different values of b and s.

Let $\beta_i = \mathsf{Pr}(\text{buy}|V_i)$ and $\sigma_i = \mathsf{Pr}(\text{sale}|V_3)$ at every buy herding history (these probabilities are are always the same at every history at which S buy herds). Note that, by Lemma 9, $\mathsf{E}[V|S, H^t] - \mathsf{ask}^t$ has the same sign as

$$\left(\frac{\beta_1}{\beta_3}\right)^b \left(\frac{\sigma_1}{\sigma_3}\right)^s q_2^r q_1^r [\beta_1 \mathsf{Pr}(S|V_2) - \beta_2 \mathsf{Pr}(S|V_1)] + q_3^r q_2^r [\beta_2 \mathsf{Pr}(S|V_3) - \beta_3 \mathsf{Pr}(S|V_2)] + 2\left(\frac{\beta_1}{\beta_2}\right)^b \left(\frac{\sigma_1}{\sigma_2}\right)^s q_3^r q_1^r [\beta_1 \mathsf{Pr}(S|V_3) - \beta_3 \mathsf{Pr}(S|V_1)]. \tag{23}$$

Also, by MLRP and Lemma 5 (iv) we have

$$\beta_1 < \beta_2 < \beta_3 \text{ and } \sigma_3 < \sigma_2 < \sigma_1. \tag{24}$$

Since by Proposition 2, S must have an nU-shaped csd, it then follows that

$$\beta_1 \mathsf{Pr}(S|V_2) - \beta_2 \mathsf{Pr}(S|V_1) < 0, \quad \beta_1 \mathsf{Pr}(S|V_3) - \beta_3 \mathsf{Pr}(S|V_1) < 0, \quad \beta_2 \mathsf{Pr}(S|V_3) - \beta_3 \mathsf{Pr}(S|V_2) > 0. \tag{25}$$

(The last inequality in (25) follows from the first two and from (11) being positive at H^r.) Thus, the first and the third terms in (23) are negative, the second is positive. Hence it follows from (24) that the expression in (23) satisfies the following three properties: (i) it increases in b, (ii) it decreases in s and (iii) for any b it is negative for sufficiently large s. By (24), the expression $\mathsf{E}(V|H^t) - \mathsf{E}(V)$ must also satisfy (i)-(iii) (note that q_3^t/q_1^t is increasing in b and decreasing in s). Since (23) and $\mathsf{E}(V|H^t) - \mathsf{E}(V)$ are both increasing in b and since by assumption there is buy herding at H^r, it must be that for any b both (23) and $\mathsf{E}(V|H^t) - \mathsf{E}(V)$ are positive when $s = 0$. Thus, it follows from (ii) and (iii) that for any b there exists an integer $\overline{s} > 1$ such that both (23) and $\mathsf{E}(V|H^t) - \mathsf{E}(V)$ are positive for any integer $s < \overline{s}$, and either (23) or $\mathsf{E}(V|H^t) - \mathsf{E}(V)$ are non-positive for any integer $s \geq \overline{s}$.

To complete the proof of this part we need to show that $\overline{s}$ is increasing in b. To show this suppose otherwise; then there exists b' and b'' such that $b' < b''$ and $\overline{s}' > \overline{s}''$ where $\overline{s}'$ and $\overline{s}''$ are respectively the critical values of sales corresponding to b' and b'' described in the previous paragraph. Now since $\overline{s}' > \overline{s}''$ it follows that both (23) and $\mathsf{E}(V|H^t) - \mathsf{E}(V)$ are positive if $b = b'$ and $s = \overline{s}''$. But since both (23) and $\mathsf{E}(V|H^t) - \mathsf{E}(V)$ are increasing in b, we must then have that both (23) and $\mathsf{E}(V|H^t) - \mathsf{E}(V)$ are positive if $b = b''$ and $s = \overline{s}''$. By the definition of $\overline{s}''$ this is a contradiction.

(b) At H^t buy contrarianism occurs if and only if $\mathsf{E}[V|S, H^t] - \mathsf{ask}^t > 0$ and $\mathsf{E}[V] - \mathsf{E}[V|H^t] > 0$. By Lemma 9, $\mathsf{E}[V|S, H^t] - \mathsf{ask}^t$ has the same sign as

$$q_2^r q_1^r[\beta_1 \mathsf{Pr}(S|V_2) - \beta_2 \mathsf{Pr}(S|V_1)] + \left(\frac{\beta_3}{\beta_1}\right)^b \left(\frac{\sigma_3}{\sigma_1}\right)^s q_3^r q_2^r[\beta_2 \mathsf{Pr}(S|V_3) - \beta_3 \mathsf{Pr}(S|V_2)] + 2\left(\frac{\beta_3}{\beta_2}\right)^b \left(\frac{\sigma_3}{\sigma_2}\right)^s q_3^r q_1^r[\beta_1 \mathsf{Pr}(S|V_3) - \beta_3 \mathsf{Pr}(S|V_1)]. \tag{26}$$

Also, with buy contrarianism S must have an nHill-shaped csd and therefore $\beta_1 \mathsf{Pr}(S|V_2) - \beta_2 \mathsf{Pr}(S|V_1) > 0$, $\beta_1 \mathsf{Pr}(S|V_3) - \beta_3 \mathsf{Pr}(S|V_1) < 0$, and $\beta_2 \mathsf{Pr}(S|V_3) - \beta_3 \mathsf{Pr}(S|V_2) < 0$. Thus, the second and the third terms in (26) are negative, and the first is positive. Hence, by (24), the expression in (26) satisfies the following: (i) it increases in s, (ii) it decreases in b and (iii) for each s, it is negative for sufficiently large b.

The expression $\mathsf{E}(V) - \mathsf{E}(V|H^t)$ also satisfies the same three properties. The existence of the function $\overline{b}$ is now analogous to that for part (a), with reversed roles for buys and sales.

A.5 Proof of Proposition 5

We show the proof for buy herding; the proof for buy contrarianism is analogous.

(a) In the proof of Proposition 4 we have shown for the case of buy herding that if the history following H^r consists only of buys, then type S herds at any point during that history. What remains to be shown is that for an arbitrary number of buys after herding has started, the price will approach V_3. Observe that $\mathsf{E}[V|H^t] = \sum_i V_i q_i^t = q_3^t \left(\frac{q_2^t}{q_3^t} V_2 + V_3\right)$. Also, q_2^t/q_3^t is arbitrarily small at any history H^t that includes a sufficiently large number of buys as outlined following conditions (5) and (6). Consequently, for every $\epsilon > 0$, there exists a history H^t consisting of H^r followed by sufficiently many buys such that $\mathsf{E}[V|H^t] > V_3 - \epsilon$.

(b) Since the assumptions of Theorem 2 that ensure buy herding hold, S_2 must be selling initially and also $\mu < \mu_{bh}^s \leq \mu_2(S_3)$. The latter implies that here exists $\eta > 0$ such that

$$[\beta_2^t \mathsf{Pr}(S_2|V_3) - \beta_3^t \mathsf{Pr}(S_2|V_2)] > \eta, \text{ for every } t. \tag{27}$$

By MLRP, type S_1 does not buy at any history. Therefore, for any history H^r consisting only of $r-1$ buys it must be that $q_1^r/q_3^r \leq \left(\max\{\frac{\gamma+\mu \mathsf{Pr}(S_3|V_1)}{\gamma+\mu \mathsf{Pr}(S_3|V_3)}, \frac{\gamma+\mu(1-\mathsf{Pr}(S_1|V_1))}{\gamma+\mu(1-\mathsf{Pr}(S_1|V_3))}\right)^{r-1}$. Also, by MLRP, S_3 is strictly increasing and S_1 is strictly decreasing. Thus, there must exist $r > 1$

such that $q_1^r/q_3^r < \eta/2$. Fix one such r. Then it follows from (27) that

$$[\beta_2^r \mathsf{Pr}(S_2|V_3) - \beta_3^r \mathsf{Pr}(S_2|V_2)] + \frac{q_1^r}{q_3^r}[\beta_1^r \mathsf{Pr}(S_2|V_2) - \beta_2^r \mathsf{Pr}(S_2|V_1)] > \eta/2. \tag{28}$$

Next, fix any $\epsilon > 0$. Note that there exists $\delta > 0$ such that if $q_2^1 > 1 - \delta$ then $\mathsf{ask}^r = \mathsf{E}[V|H^r, \text{buy}] = q_2^r V_2 + q_3^r V_3 \quad \in (V_2, V_2 + \epsilon)$ and

$$2\,\frac{q_1^r}{q_2^r}[\beta_3^r \mathsf{Pr}(S|V_1) - \beta_1^r \mathsf{Pr}(S|V_3)] < \eta/2. \tag{29}$$

Fix any such δ. Then it follows from (28) and (29) that

$$q_2^r q_3^r[\beta_2^r \mathsf{Pr}(S_2|V_3) - \beta_3^r \mathsf{Pr}(S_2|V_2)] + q_1^r q_2^r[\beta_1^r \mathsf{Pr}(S_2|V_2) - \beta_2^r \mathsf{Pr}(S_2|V_1)] + 2\,q_1^r q_3^r[\beta_1^r \mathsf{Pr}(S_2|V_3) - \beta_3^r \mathsf{Pr}(S_2|V_1)] > 0.$$

Since S_2 sells initially and $q_1^r/q_3^r < \eta/2 < 1$, it follows from the last inequality that S_2 is buy herding at H^r at an ask price that belongs to the interval $(V_2, V_2 + \epsilon)$.

Next, as shown in part (a), there must also exist a history H^t with $t = r + b$ following H^r such that there is buy herding at any history H^τ, $r \leq \tau \leq t$, and $\mathsf{E}[V|H^t] > V_3 - \epsilon$.

A.6 Proof of Proposition 6

We shall prove the two results for the case of buy herding; the proof for the buy contrarian case is analogous and will be omitted.

Proof of part 1(a) of Proposition 6. Let β_i and σ_i be respectively the probability of a buy and the probability of a sale in the transparent world at any date $\tau = r, \ldots, r + b + s$. Also, let $\beta_{i,o}$ and $\sigma_{i,o}$ be the analogous probabilities in the opaque world. Then $\mathsf{E}[V|H^t] - \mathsf{E}_o[V|H^t] = \mathcal{V}\{(q_2^t - q_{2,o}^t) + 2(q_3^t - q_{3,o}^t)\}$

$$= \mathcal{V}\left\{ q_2^r \left(\frac{\beta_2^b \sigma_2^s}{\sum_i q_i^r \beta_i^b \sigma_i^s} - \frac{\beta_{2,o}^b \sigma_{2,o}^s}{\sum_i q_i^r \beta_{i,o}^b \sigma_{i,o}^s} \right) + 2q_3^r \left(\frac{\beta_3^b \sigma_3^s}{\sum_i q_i^r \beta_i^b \sigma_i^s} - \frac{\beta_{3,o}^b \sigma_{3,o}^s}{\sum_i q_i^r \beta_{i,o}^b \sigma_{i,o}^s} \right) \right\}.$$

Therefore, $\mathsf{E}[V|H^t] - \mathsf{E}_o[V|H^t]$ has the same sign as

$$\begin{aligned} &q_2^r q_1^r[(\beta_2\beta_{1,o})^b(\sigma_2\sigma_{1,o})^s - (\beta_{2,o}\beta_1)^b(\sigma_{2,o}\sigma_1)^s] + q_3^r q_2^r[(\beta_3\beta_{2,o})^b(\sigma_3\sigma_{2n})^s - (\beta_{3,o}\beta_2)^b(\sigma_{3,o}\sigma_2)^s] \\ &+ 2q_3^r q_1^r[(\beta_3\beta_{1,o})^b(\sigma_3\sigma_{1,o})^s - (\beta_{3,o}\beta_1)^b(\sigma_{3,o}\sigma_1)^s]. \end{aligned} \tag{30}$$

Suppose that S buy herds at H^r. Then, by Lemma 9, we have

$$\begin{aligned} q_2^r q_1^r[\beta_1 \mathsf{Pr}(S|V_2) - \beta_2 \mathsf{Pr}(S|V_1)] + q_3^r q_2^r[\beta_2 \mathsf{Pr}(S|V_3) - \beta_3 \mathsf{Pr}(S|V_2)] \\ + 2\,q_3^r q_1^r[\beta_1 \mathsf{Pr}(S|V_3) - \beta_3 \mathsf{Pr}(S|V_1)] > 0. \end{aligned} \tag{31}$$

By simple computation we also have

$$\begin{aligned} \beta_2\beta_{1,o} - \beta_{2,o}\beta_1 &= \mu[\beta_1 \mathsf{Pr}(S|V_2) - \beta_2 \mathsf{Pr}(S|V_1)], \\ \beta_3\beta_{1,o} - \beta_{3,o}\beta_1 &= \mu[\beta_1 \mathsf{Pr}(S|V_3) - \beta_3 \mathsf{Pr}(S|V_1)], \\ \beta_3\beta_{2,o} - \beta_{3,o}\beta_2 &= \mu[\beta_2 \mathsf{Pr}(S|V_3) - \beta_3 \mathsf{Pr}(S|V_2)]. \end{aligned} \tag{32}$$

Therefore, it follows from (31) that

$$q_2^r q_1^r[\beta_2\beta_{1,o} - \beta_{2,o}\beta_1] + q_3^r q_2^r[\beta_3\beta_{2,o} - \beta_{3,o}\beta_2] + 2q_3^r q_1^r[\beta_3\beta_{1,o} - \beta_{3,o}\beta_1] > 0. \tag{33}$$

To prove $1(a)$ in Proposition 6 suppose that $s=0$ (thus $t=b$). Then by expanding (30) it must be that $\mathsf{E}[V|H^t]-\mathsf{E}_o[V|H^t]$ has the same sign as

$$q_2^r q_1^r \left\{(\beta_2\beta_{1,o}-\beta_{2,o}\beta_1)\sum\nolimits_{\tau=0}^{b-1}(\beta_2\beta_{1,o})^{b-1-\tau}(\beta_{2,o}\beta_1)^{\tau}\right\} \tag{34}$$
$$+q_3^r q_2^r[(\beta_3\beta_{2,o})-(\beta_{3,o}\beta_2)]\sum\nolimits_{\tau=0}^{b-1}(\beta_3\beta_{2,o})^{b-1-\tau}(\beta_{3,o}\beta_2)^{\tau}$$
$$+2q_3^r q_1^r\left\{(\beta_3\beta_{1,o}-\beta_{3,o}\beta_1)\sum\nolimits_{\tau=0}^{b-1}(\beta_3\beta_{1,o})^{b-1-\tau}(\beta_{3,o}\beta_1)^{\tau}\right\}.$$

Also, by MLRP $\beta_3>\beta_2>\beta_1$ and $\beta_{3,o}>\beta_{2,o}>\beta_{1,o}$. Therefore,

$$\sum_{\tau=0}^{b-1}(\beta_3\beta_{2,o})^{b-1-\tau}(\beta_{3,o}\beta_2)^{\tau} \;>\; \sum_{\tau=0}^{b-1}(\beta_2\beta_{1,o})^{b-1-\tau}(\beta_{2,o}\beta_1)^{\tau}, \tag{35}$$
$$\sum_{\tau=0}^{b-1}(\beta_3\beta_{2,o})^{b-1-\tau}(\beta_{3,o}\beta_2)^{\tau} \;>\; \sum_{\tau=0}^{b-1}(\beta_3\beta_{1,o})^{b-1-\tau}(\beta_{3,o}\beta_1)^{\tau}. \tag{36}$$

Also, by (25) and (32) the first and the third terms in (34) are negative and the second is positive. Therefore, by (33), (35), and (36), $\mathsf{E}[V|H^t]-\mathsf{E}_o[V|H^t]>0$ for $s=0$. This completes the proof of part $1(a)$ of Proposition 6.[45]

Proof of part 1(b) of Proposition 6. Suppose that $b=0$ and $s=1$ ($t=r+1$). Since S buys in the transparent world, $\mathsf{E}[V|S,H^r]-\mathsf{bid}^r>0$. Simple computations analogous to the proof of Lemma 9 show that this is equivalent to

$$q_2^r q_1^r[\sigma_1\mathsf{Pr}(S|V_2)-\sigma_2\mathsf{Pr}(S|V_1)]+q_3^r q_2^r[\sigma_2\mathsf{Pr}(S|V_3)-\sigma_3\mathsf{Pr}(S|V_2)]+2q_3^r q_1^r[\sigma_1\mathsf{Pr}(S|V_3)-\sigma_3\mathsf{Pr}(S|V_1)]>0 \tag{37}$$

Also, by the definition of σ_i and σ_i we have

$$\begin{aligned}\sigma_3\sigma_{2,o}-\sigma_{3,o}\sigma_2 &= -\mu[\sigma_2\mathsf{Pr}(S|V_3)-\sigma_3\mathsf{Pr}(S|V_2)],\\ \sigma_3\sigma_{1,o}-\sigma_{3,o}\sigma_1 &= -\mu[\sigma_1\mathsf{Pr}(S|V_3)-\sigma_3\mathsf{Pr}(S|V_1)],\\ \sigma_2\sigma_{1,o}-\sigma_{2,o}\sigma_1 &= -\mu[\sigma_1\mathsf{Pr}(S|V_2)-\sigma_2\mathsf{Pr}(S|V_1)].\end{aligned} \tag{38}$$

Therefore, (37) is equivalent to

$$q_2^r q_1^r[\sigma_2\sigma_{1,o}-\sigma_{2,o}\sigma_1]+q_3^r q_2^r[\sigma_3\sigma_{2,o}-\sigma_{3,o}\sigma_2]+\;2q_3^r q_1^r[\sigma_3\sigma_{1,o}-\sigma_{3,o}\sigma_1]<0. \tag{39}$$

Since the LHS of (39) is the same as the expression in (30) when $b=0$ and $s=1$, it follows that in this case $\mathsf{E}[V|H^t]-\mathsf{E}_o[V|H^t]<0$.[46] This completes the proof of this part.

Proof of $1(c)$ in Proposition 6. First, note that (30) can be written as:

$$q_2^r q_1^r\tfrac{(\beta_{2,o}\beta_1)^b}{(\beta_3\beta_{2,o})^b}\left[\tfrac{(\beta_2\beta_{1,o})^b}{(\beta_{2,o}\beta_1)^b}(\sigma_2\sigma_{1,o})^s-(\sigma_{2,o}\sigma_1)^s\right]+q_3^r q_2^r\left[(\sigma_3\sigma_{2n})^s-\tfrac{(\beta_{3,o}\beta_2)^b}{(\beta_3\beta_{2,o})^b}(\sigma_{3,o}\sigma_2)^s\right]$$
$$+2q_3^r q_1^r\tfrac{(\beta_{3,o}\beta_1)^b}{(\beta_3\beta_{2,o})^b}\left[\tfrac{(\beta_3\beta_{1,o})^b}{(\beta_{3,o}\beta_1)^b}(\sigma_3\sigma_{1,o})^s-(\sigma_{3,o}\sigma_1)^s\right]. \tag{40}$$

[45]Note that MLRP is assumed in the above proof in order to establish conditions (35), and (36). When $b=1$ and $s=0$ these conditions, and hence MLRP, is not needed as $\mathsf{E}[V|H^t]-\mathsf{E}_o[V|H^t]>0$ follows immediately from (30) and (33).

[46]This claim also does not require the assumption of MLRP.

Fix s and let $b \to \infty$. Then since by (25) and (32) $\beta_3\beta_{2,o} > \beta_{3,o}\beta_2$ we have that the second term in (40) converges to $q_3^r q_2^r(\sigma_3\sigma_{2n})^s$ as $b \to \infty$. Also, since $\beta_3 > \beta_2 > \beta_1$ it follows that $\beta_{2,o}\beta_1 < \beta_{2,o}\beta_3$ and $\beta_{3,o}\beta_2 > \beta_{3,0}\beta_1$. The former, together with (25) and (32), imply that the first term in (40) vanishes as $b \to \infty$. The latter, together with (25) and (32), imply that $\beta_3\beta_{2,o} > \beta_{3,o}\beta_1$; therefore, using (25) and (32) again, the last term in (40) also vanishes. Consequently, as $b \to \infty$ the expression in (40) converges to $q_3^r q_2^r(\sigma_3\sigma_{2n})^s$. Since $(\sigma_3\sigma_{2n})^s > 0$ and $\mathsf{E}[V|H^t] - \mathsf{E}_o[V|H^t]$ has the same sign as the expression in (40), the claim in $1(c)$ of the proposition is established.

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Herding and Contrarian Behaviour in Financial Markets

Andreas Park and Hamid Sabourian

– Supplementary Material –

There are results in the paper that were not fully discussed or proven fully. This supplementary material contains what was omitted or mentioned. We organize this appendix in the same way as the sections in the main paper.

B Supplementary Material for Section 5

B.1 Proof of Lemma 1

Observe first that

$$\mathsf{E}[V|S,H^t]-\mathsf{E}[V|H^t]=\mathcal{V}q_2^t\left(\frac{\mathsf{Pr}(S|V_2)}{\mathsf{Pr}(S)}-1\right)+2\mathcal{V}q_3^t\left(\frac{\mathsf{Pr}(S|V_3)}{\mathsf{Pr}(S)}-1\right).$$

The RHS of the the above equality has the same sign as

$$\begin{aligned}
& q_2^t\left(\mathsf{Pr}(S|V_2)\sum_j q_j^t-\sum_j \mathsf{Pr}(S|V_j)q_j^t\right)+2\ q_3^t\left(\mathsf{Pr}(S|V_3)\sum_j q_j^t-\sum_j \mathsf{Pr}(S|V_j)q_j^t\right)\\
= \ & q_1^tq_2^t\left(\mathsf{Pr}(S|V_2)-\mathsf{Pr}(S|V_1)\right)+q_2^tq_3^t\left(\mathsf{Pr}(S|V_2)-\mathsf{Pr}(S|V_3)\right)\\
& +2\ q_3^t\left(q_1^t(\mathsf{Pr}(S|V_3)-\mathsf{Pr}(S|V_1)+q_2^t\left(\mathsf{Pr}(S|V_3)-\mathsf{Pr}(S|V_2)\right)\right).
\end{aligned}$$

B.2 Proof of Lemma 2

This follows by Lemma 1: By the symmetry assumption on the priors ($q_1^1=q_3^1$), the (2) is negative (positive) at $t=1$ if and only if $(\mathsf{Pr}(S|V_3)-\mathsf{Pr}(S|V_1))(q_2^1+2q_1^1)q_3^1$ is less (greater) than 0; the latter is equivalent to S having a negative (positive) bias.

B.3 Proof of Lemma 3

The claim follows from $\mathsf{E}[V|H^t]-\mathsf{E}[V]=\mathcal{V}[(1-q_1^t-q_3^t)+2q_3^t]-\mathcal{V}=\mathcal{V}(q_3^t-q_1^t)$.

B.4 Proof of Lemma 9

The proof is analogous to the derivation in the proof of Lemma 1. To show (i) note that

$$\mathsf{E}[V|S,H^t]-\mathsf{ask}^t=\mathcal{V}q_2\left(\frac{\mathsf{Pr}(S|V_2)}{\mathsf{Pr}(S)}-\frac{\beta_2}{\mathsf{Pr}(\mathsf{buy}|H^t)}\right)+2\mathcal{V}q_3\left(\frac{\mathsf{Pr}(S|V_3)}{\mathsf{Pr}(S)}-\frac{\beta_3}{\mathsf{Pr}(\mathsf{buy}|H^t)}\right).$$

The RHS of the above has the same sign as

$$\begin{aligned}
& q_2\left(\mathsf{Pr}(S|V_2)\sum_j\beta_jq_j-\beta_2\sum_j\mathsf{Pr}(S|V_j)q_j\right)+2\,q_3\left(\mathsf{Pr}(S|V_3)\sum_j\beta_jq_j-\beta_3\sum_j\mathsf{Pr}(S|V_j)q_j\right)\\
= \; & q_1q_2\left(\beta_1\mathsf{Pr}(S|V_2)-\beta_2\mathsf{Pr}(S|V_1)\right)+q_2q_3\left(\beta_3\mathsf{Pr}(S|V_2)-\beta_2\mathsf{Pr}(S|V_3)\right)\\
& +2\,q_3\left(q_1\left(\beta_1\mathsf{Pr}(S|V_3)-\beta_3\mathsf{Pr}(S|V_1)\right)+q_2\left(\beta_2\mathsf{Pr}(S|V_3)-\beta_3\mathsf{Pr}(S|V_2)\right)\right).
\end{aligned}$$

B.5 Necessity of the Bounds on Noise Trading

In the paper we claimed that the bounds on noise trading are also necessary for some cases. The following proposition outlines these scenarios.

Proposition 3a

(i) *Suppose that S buy herds and that there is at most one U-shaped signal. Then $\mu<\min\{\mu^i,\mu^s_{bh}\}$, where μ^i is defined in Lemma 8 and μ^s_{bh} is defined in (12).*

(ii) *Suppose that S acts as a buy contrarian and there is at most one Hill-shaped signal. Then $\mu<\min\{\mu^i,\mu^s_{bc}\}$, where μ^i is defined in Lemma 8 and μ^s_{bc} is defined in (13).*

Proof: We shall prove (i); the proof of (ii) is analogous. Assume S buy herds. Then S sells initially. It follows from Lemma 5 that $\mu<\mu^i$.

To show that $\mu<\mu^s_{bh}$ first note that by Proposition 2, S must be nU-shaped. Next consider the different possibilities separately.

Case A. There is no signal $S'\neq S$ such that $\mathsf{Pr}(S'|V_3)>\mathsf{Pr}(S'|V_2)$. Then it must be that $\mu_2(S')=1$ for all S' and therefore it must be that $\mu<\mu^s_{bh}=1$.

Case B. There is a signal $S'\neq S$ such that $\mathsf{Pr}(S'|V_3)>\mathsf{Pr}(S'|V_2)$. Since S is U-shaped it must be that $\mathsf{Pr}(S|V_3)>\mathsf{Pr}(S|V_2)$ and $\mathsf{Pr}(S''|V_3)\leq\mathsf{Pr}(S''|V_2)$ for $S''\neq S,S'$. This implies that $\mu_2(S'')=1$ and hence, $\mu_2(S')=\mu^s_{bh}$.

Now there are two cases. First, if $\mu_2(S')$ also equals 1 then clearly $\mu^s_{bh}=1$ and the claim is trivially true.

Second, assume that $\mu_2(S')=\mu^s_{bh}<1$. Since S buy herds at H^t, to show that $\mu<\min\{\mu^i,\mu^s_{bh}\}$ it suffices to show that S' also buys whenever S buys (the alternative is that S' does not buy so that $\mu_2(S')=1>\mu^s_{bh}$). When S' buys, $\mathsf{E}[V|S',H^t]-\mathsf{ask}^t>0$. Suppose S' does not buy. As the sign of $\mathsf{E}[V|S',H^t]-\mathsf{ask}^t$ is given by equation (3), it must then hold that

$$q_1 q_2 \left[\beta_1 \mathsf{Pr}(S'|V_2) - \beta_2 \mathsf{Pr}(S'|V_1)\right] + q_2 q_3 [\beta_2 \mathsf{Pr}(S'|V_3) - \beta_3 \mathsf{Pr}(S'|V_2)] \\ +2\ q_1 q_3 \left[\beta_1 \mathsf{Pr}(S'|V_3) - \beta_3 \mathsf{Pr}(S'|V_1)\right] \quad \leq 0. \tag{B-1}$$

Also, since there is at most one U-shaped signal it must be that

$$\mathsf{Pr}(S'|V_3) > \mathsf{Pr}(S'|V_2) \geq \mathsf{Pr}(S'|V_1). \tag{B-2}$$

By Proposition 1 this implies that S' does not sell. By supposition S' does not buy and therefore S is the only buyer at H^t (S'' is selling). Since S is nU-shaped we must also have $\beta_1^t > \beta_3^t \geq \beta_2^t$. This, together with (B-2) imply that the first and the third term in (B-1) are positive. Furthermore, the second term has the same sign as

$$\gamma(\mathsf{Pr}(S'|V_3) - \mathsf{Pr}(S'|V_2)) + \mu(\mathsf{Pr}(S|V_2)\mathsf{Pr}(S'|V_3) - \mathsf{Pr}(S|V_3)\mathsf{Pr}(S'|V_2)). \tag{B-3}$$

By (B-2) the first term in the last expression is positive; furthermore, since S is nU, we have $m^2 = \mathsf{Pr}(S|V_3) > \mathsf{Pr}(S|V_2)$. Since $\mu_2(S') < 1$ we must have that $M^2(S') < 1$ is negative. But $-\mu M^2(S')$ is the second term in the last expression and it is thus positive. Consequently, (B-3) is positive. Therefore, the second term in (B-1) must also be positive. Therefore, S' must be buying at any H^t at which S buys and thus $\mu_{bh}^s < 1$ is unique.

C Supplementary Material for Section 6

Proof of Lemma 5: (i) By standard results on MLRP and stochastic dominance it must be that $\mathsf{E}[V|S_l] < \mathsf{E}[V|S_h]$. By a similar reasoning, at any history H^t, $\mathsf{E}[V|S_l, H^t] < \mathsf{E}[V|S_h, H^t]$ if the following MLRP condition holds at H^t: for any $S_l < S_h$ and any $V_l < V_h$

$$\frac{\mathsf{Pr}(S_h|V_h, H^t)}{\mathsf{Pr}(S_l|V_h, H^t)} > \frac{\mathsf{Pr}(S_h|V_l, H^t)}{\mathsf{Pr}(S_l|V_l, H^t)}. \tag{C-4}$$

To show this note first that $\mathsf{Pr}(V|H^t, S) = \mathsf{Pr}(V|S)\mathsf{Pr}(H^t|V)/\sum_{V'\in\mathbb{V}} \mathsf{Pr}(V'|S)\mathsf{Pr}(H^t|V')$. Then we have by the following manipulations that the MLRP condition $\frac{\mathsf{Pr}(S_h|V_h)}{\mathsf{Pr}(S_l|V_h)} > \frac{\mathsf{Pr}(S_h|V_l)}{\mathsf{Pr}(S_l|V_l)}$ implies the MLRP condition (C-4) at any H^t:

$$\begin{aligned}
\mathsf{Pr}(S_l|V_l)\mathsf{Pr}(S_h|V_h) &> \mathsf{Pr}(S_l|V_h)\mathsf{Pr}(S_h|V_l) \\
\Leftrightarrow \mathsf{Pr}(V_l|S_l)\mathsf{Pr}(V_h|S_h) &> \mathsf{Pr}(V_h|S_l)\mathsf{Pr}(V_l|S_h) \\
\Leftrightarrow \frac{\mathsf{Pr}(V_l|S_l)\mathsf{Pr}(H^t|V_l)}{\sum_{\mathbb{V}} \mathsf{Pr}(V|S_l)\mathsf{Pr}(H^t|V)} \frac{\mathsf{Pr}(V_h|S_h)\mathsf{Pr}(H^t|V_h)}{\sum_{\mathbb{V}} \mathsf{Pr}(V|S_h)\mathsf{Pr}(H^t|V)} &> \frac{\mathsf{Pr}(V_h|S_l)\mathsf{Pr}(H^t|V_h)}{\sum_{\mathbb{V}} \mathsf{Pr}(V|S_l)\mathsf{Pr}(H^t|V)} \frac{\mathsf{Pr}(V_l|S_h)\mathsf{Pr}(H^t|V_l)}{\sum_{\mathbb{V}} \mathsf{Pr}(V|S_h)\mathsf{Pr}(H^t|V)} \\
\Leftrightarrow \mathsf{Pr}(V_l|H^t, S_l)\mathsf{Pr}(V_h|H^t, S_h) &> \mathsf{Pr}(V_h|H^t, S_l)\mathsf{Pr}(V_l|H^t, S_h).
\end{aligned}$$

(ii) Suppose contrary to the claim, that an informed trader with signal S_1 does not sell at some history H^t. Then by part (i) no informed trader sells at H^t. This implies that at history H^t, $\mathsf{bid}^t = \mathsf{E}[V|H^t]$. But since, by part (i), $\mathsf{E}[V|H^t]$ exceeds $\mathsf{E}[V|S_1, H^t]$, we have $\mathsf{bid}^t > \mathsf{E}[V|S_1, H^t]$. Hence, an informed trader with signal S_1 sells at H^t. This is a contradiction.

The proof that informed traders with signal S_3 always buy is analogous.

(iii) First we show that $\mathsf{Pr}(S_1|V_1) > \mathsf{Pr}(S_1|V_3)$. Suppose otherwise; thus $\mathsf{Pr}(S_1|V_1) \leq \mathsf{Pr}(S_1|V_3)$. Then the two MLRP conditions $\mathsf{Pr}(S_1|V_1)\mathsf{Pr}(S_2|V_3) > \mathsf{Pr}(S_2|V_1)\mathsf{Pr}(S_1|V_3)$ and $\mathsf{Pr}(S_1|V_1)\mathsf{Pr}(S_3|V_3) > \mathsf{Pr}(S_3|V_1)\mathsf{Pr}(S_1|V_3)$ imply respectively that $\mathsf{Pr}(S_2|V_1) < \mathsf{Pr}(S_2|V_3)$ and $\mathsf{Pr}(S_3|V_1) < \mathsf{Pr}(S_3|V_3)$. Hence, since $\mathsf{Pr}(S_1|V_1) \leq \mathsf{Pr}(S_1|V_3)$ we have $\sum_{i=1}^3 \mathsf{Pr}(S_i|V_3) > \sum_{i=1}^3 \mathsf{Pr}(S_i|V_1)$. But this contradicts $\sum_{i=1}^3 \mathsf{Pr}(S_i|V_j) = 1$ for every j.

The same argument can be applied to show that $\mathsf{Pr}(S_1|V_1) > \mathsf{Pr}(S_1|V_2)$ and $\mathsf{Pr}(S_1|V_2) > \mathsf{Pr}(S_1|V_3)$, and also in the reverse direction for $\mathsf{Pr}(S_3|V_1) < \mathsf{Pr}(S_3|V_2) < \mathsf{Pr}(S_3|V_3)$.

(iv) Consider any arbitrary history H^t and any two values $V_l < V_h$. By (ii) type S_1 always sells, type S_3 always buys. There are thus two cases for a buy at H^t: either only S_3 types buy or S_2 *and* S_3 types buy. In the former case, $\beta_i^t = \gamma + \mu \mathsf{Pr}(S_3|V_i)$. As S_3 is strictly increasing, there exits $\epsilon > 0$ such that $\beta_h^t - \beta_l^t > \epsilon$. In the latter case,

$$\begin{aligned}\beta_h^t - \beta_l^t &= \mu\left(\mathsf{Pr}(S_3|V_h) + \mathsf{Pr}(S_2|V_h) - \mathsf{Pr}(S_3|V_l) - \mathsf{Pr}(S_2|V_l)\right) \\ &= \mu\left(1 - \mathsf{Pr}(S_1|V_h) - (1 - \mathsf{Pr}(S_1|V_l))\right) = \mu\left(\mathsf{Pr}(S_1|V_l) - \mathsf{Pr}(S_1|V_h)\right).\end{aligned}$$

Since S_1 is strictly decreasing, there exists an $\epsilon > 0$ such that $\beta_h^t - \beta_l^t > \epsilon$.

By a similar reasoning it can be shown that there must exist $\epsilon > 0$ so that $\sigma_l^t - \sigma_h^t > \epsilon$.

D Supplementary Material for Section 8

Proposition 6a *Assume MLRP. Consider any finite history $H^r = (a^1, \ldots, a^{r-1})$ at which the priors in the two markets coincide: $q_i^r = q_{i,o}^r$ for $i = 1, 2, 3$. Suppose that H^r is followed by $s \geq 0$ sales; denote this history by $H^t = (a^1, \ldots, a^{r+s-1})$. If $\sigma_1/\sigma_3 \geq \sigma_{1,o}/\sigma_{3,o}$ then $\mathsf{E}[V|H^t] < \mathsf{E}_o[V|H^t]$.*

Proof: First, note that, by (33) in the proof of Proposition 7, we have

$$\begin{aligned}\sigma_3\sigma_{2,o} - \sigma_{3,o}\sigma_2 &= -\mu^2\rho_{12}^{23} + \mu\gamma(\mathsf{Pr}(S|V_2) - \mathsf{Pr}(S|V_3)) < 0 \\ \sigma_2\sigma_{1,o} - \sigma_{2,o}\sigma_1 &> \sigma_3\sigma_{1,o} - \sigma_{3,o}\sigma_1\end{aligned} \qquad \text{(D-5)}$$

Also, since for herding we require $\mathsf{E}[V|S, H^1] < \mathsf{bid}^1$, it follows from Lemma 6 and (32) that

$$q_2^1 q_1^1[\sigma_2\sigma_{1,o} - \sigma_{2,o}\sigma_1] + q_3^1 q_2^1[\sigma_3\sigma_{2,o} - \sigma_{3,o}\sigma_2] + 2q_3^1 q_1^1[\sigma_3\sigma_{1,o} - \sigma_{3,o}\sigma_1] > 0.$$

But then by (D-5) we have

$$\sigma_2\sigma_{1,o} - \sigma_{2,o}\sigma_1 > 0. \qquad \text{(D-6)}$$

Since $\mathsf{E}[V|H^t] - \mathsf{E}_o[V|H^t]$ has the same sign as the expression in (29), by simple expansion of this expression we have that if $b = 0$ then $\mathsf{E}[V|H^t] - \mathsf{E}_o[V|H^t]$ has the same sign as

$$q_2^r q_1^r \left\{ (\sigma_2\sigma_{1,o} - \sigma_{2,o}\sigma_1) \sum_{\tau=0}^{s-1} (\sigma_2\sigma_{1,o})^{s-1-\tau}(\sigma_{2,o}\sigma_1)^\tau \right\}$$
$$+ q_3^r q_2^r \left\{ [(\sigma_3\sigma_{2,o}) - (\sigma_{3,o}\sigma_2)] \sum_{\tau=0}^{s-1} (\sigma_3\sigma_{2,o})^{s-1-\tau}(\sigma_{3,o}\sigma_2)^\tau \right\}$$
$$+ 2q_3^r q_1^r \left\{ (\sigma_3\sigma_{1,o} - \sigma_{3,o}\sigma_1) \sum_{\tau=0}^{s-1} (\sigma_3\sigma_{1,o})^{s-1-\tau}(\sigma_3\sigma_1)^\tau \right\}.$$

Rearranging, $\mathsf{E}[V|H^t] - \mathsf{E}_o[V|H^t]$ has the same sign as

$$q_2^r q_1^r \frac{\sum_{\tau=0}^{s-1}(\sigma_2\sigma_{1,o})^{s-1-\tau}(\sigma_{2,o}\sigma_1)^\tau}{\sum_{\tau=0}^{s-1}(\sigma_3\sigma_{2,o})^{s-1-\tau}\ (\sigma_{3,o}\sigma_2)^\tau}[\sigma_2\sigma_{1,o} - \sigma_{2,o}\sigma_1] + q_3^r q_1^r[\sigma_3\sigma_{2,o} - \sigma_{3,o}\sigma_2]$$
$$+ \quad 2q_3^r q_1^r \frac{\sum_{\tau=0}^{s-1}(\sigma_3\sigma_{1,o})^{s-1-\tau}(\sigma_3\sigma_1)^\tau}{\sum_{\tau=0}^{s-1}(\sigma_3\sigma_{2,o})^{s-1-\tau}(\sigma_{3,o}\sigma_2)^\tau}[\sigma_3\sigma_{1,o} - \sigma_{3,o}\sigma_1]. \tag{D-7}$$

Further manipulations show that

$$\left(\frac{\sigma_1}{\sigma_3}\right)^s > \frac{\sum_{\tau=0}^{s-1}(\sigma_2\sigma_{1,o})^{s-1-\tau}(\sigma_{2,o}\sigma_1)^\tau}{\sum_{\tau=0}^{s-1}(\sigma_3\sigma_{2,o})^{s-1-\tau}(\sigma_{3,o}\sigma_2)^\tau} \Leftrightarrow \sum_{\tau=0}^{s-1} {\sigma_2}^{s-1-\tau}{\sigma_{2,o}}^\tau(\sigma_1\sigma_3)^\tau \left((\sigma_1\sigma_{3,o})^{s-1-\tau} - (\sigma_3\sigma_{1,o})^{s-1-\tau}\right) > 0.$$

Also, by assumption we have $\frac{\sigma_1}{\sigma_3} > \frac{\sigma_{1,o}}{\sigma_{3,o}}$. Therefore, we must have

$$\left(\frac{\sigma_1}{\sigma_3}\right)^s > \frac{\sum_{\tau=0}^{s-1}(\sigma_2\sigma_{1,o})^{s-1-\tau}(\sigma_{2,o}\sigma_1)^\tau}{\sum_{\tau=0}^{s-1}(\sigma_3\sigma_{2,o})^{s-1-\tau}(\sigma_{3,o}\sigma_2)^\tau}. \tag{D-8}$$

Similar manipulations show that

$$\left(\frac{\sigma_1}{\sigma_2}\right)^s < \frac{\sum_{\tau=0}^{s-1}(\sigma_3\sigma_{1,o})^{s-1-\tau}(\sigma_3\sigma_1)^\tau}{\sum_{\tau=0}^{s-1}(\sigma_3\sigma_{2,o})^{s-1-\tau}(\sigma_{3,o}\sigma_2)^\tau} \Leftrightarrow \sum_{\tau=0}^{s-1} {\sigma_3}^{s-1-\tau}{\sigma_{3,o}}^\tau(\sigma_1\sigma_2)^\tau \left((\sigma_2\sigma_{1,o})^{s-1-\tau} - (\sigma_1\sigma_{2,o})^{s-1-\tau}\right) > 0.$$

This together with (D-6), implies that

$$\left(\frac{\sigma_1}{\sigma_2}\right)^s < \frac{\sum_{\tau=0}^{s-1}(\sigma_3\sigma_{1,o})^{s-1-\tau}(\sigma_3\sigma_1)^\tau}{\sum_{\tau=0}^{s-1}(\sigma_3\sigma_{2,o})^{s-1-\tau}(\sigma_{3,o}\sigma_2)^\tau}. \tag{D-9}$$

Also, since $\mathsf{E}[V|S, H^t] - \mathsf{bid}^t > 0$, by Lemma 6

$$q_2^r q_1^r \left(\frac{\sigma_1}{\sigma_3}\right)^s [\sigma_2\sigma_{1,o} - \sigma_{2,o}\sigma_1] + q_3^r q_2^r[\sigma_3\sigma_{2,o} - \sigma_{3,o}\sigma_2]$$
$$+2q_3^r q_1^r \left(\frac{\sigma_1}{\sigma_2}\right)^s [\sigma_3\sigma_{1,o} - \sigma_{3,o}\sigma_1] \quad < \quad 0. \tag{D-10}$$

Then it follows from (D-10), together with $\frac{\sigma_1}{\sigma_3} > \frac{\sigma_{1,o}}{\sigma_{3,o}}$, (D-6), (D-8) and (D-9), that the expression in (D-7) is negative. Thus $\mathsf{E}[V|H^t] - \mathsf{E}_o[V|H^t] < 0$ and the result follows.

E Supplementary Material for Section 9

We prove Lemma 7 and Theorem 3 of Section 9 in a more general set-up than that described in the main body of the paper. This more general set-up is of independent interest as it allows for uncertainties other than those relating to the value of of the asset.

Specifically, suppose that there are $N \geq 3$ states, where each state represents all exogenous variables that might influence the prices, and assume that there are N signals. Without any loss of generality, order the states such that $V_1 \leq V_2 \leq \ldots \leq V_N$, where V_j denotes the value of the asset in state $j = 1, \ldots, N$. Note that here, in contrast to the model in the text, we allow for the possibility that the asset has the same values in different states to reflect the idea that there may be factors, other than the value of the asset, that may influence prices. In particular, we assume that the asset can have at most $I \leq N$ different values. We denote the (public) probability of state j at date t by q_j^t and the likelihood of signal S in state j by $\mathsf{Pr}(S|j)$.

We also restrict ourselves to a symmetric structure with respect to the values and the initial beliefs on the distribution of values of the asset, as in the three states model of the paper. Formally, we assume that the values are distributed on a symmetrical grid; thus $V_j \in \{0, \mathcal{V}, \ldots, (I-1)\mathcal{V}\}$ for all $j = 1, \ldots, N$ and $\mathcal{V} > 0$. Further, for any $r = 1, \ldots, I$, let $C_r := \{j | V_j = (r-1)\mathcal{V}\}$ be the set of states with valuations $(r-1)\mathcal{V}$ and $c_r := |C_r|$ be the number of states with valuation $(r-1)\mathcal{V}$. Assume (i) $q_j^1 = q_{N+1-j}^1$ for every $j \leq N/2$ and (ii) $c_r = c_{I+1-r}$ for every $r \leq I/2$.

We say that signal S is *negatively biased* if for all $j \leq \frac{N}{2}$ we have $\mathsf{Pr}(S|j) < \mathsf{Pr}(S|N{+}1{-}j)$.

Notice that when $c_r = 1$ for all r (hence $I = N$), this setup is identical to the one in the main text.

We first prove that any informed type buys initially if it has a negative bias and if there are enough noise traders.

Lemma II *Let S be negatively biased. Then $\mathsf{E}[V|S] < \mathsf{E}[V]$. Hence, there exists $\mu^i \in (0,1]$ such that S sells at the initial history if $\mu < \mu^i$.*

Proof of Lemma II: Without loss of generality, we present the proof only for the case when the number of value classes I is even so that $I = 2k$ for some integer k. Then by the symmetry of the prior $\mathsf{E}[V] = \mathcal{V}(2k-1)/2$. Also, $\mathsf{E}[V|S] = \mathcal{V}\sum_{r=1}^{2k}(r-1)\mathsf{Pr}(r|S)$, where $\mathsf{Pr}(r|S) = \sum_{j \in C_r} \mathsf{Pr}(j|S)$. Thus, we need to show

$$\sum_{r=1}^{2k}(r-1)\mathsf{Pr}(r|S) < \frac{2k-1}{2}. \tag{E-11}$$

Next, since (a) $\mathsf{Pr}(S|j) > \mathsf{Pr}(S|N+1-j)$, (b) $q_j^1 = q_{N+1-j}^1$ and (c) $c_r = c_{I+1-r}$, we have $\mathsf{Pr}(r|S) > \mathsf{Pr}(2k+1-r|S)$ for all $r < (2k+1)/2$. Using this and $\sum_{r=1}^{R}\mathsf{Pr}(r|S) = 1$, we have $\sum_{r=1}^{k} \mathsf{Pr}(r|S) > \frac{1}{2} > \sum_{r=k+1}^{2k} \mathsf{Pr}(r|S)$. Therefore

$$(k-1) + \sum_{r=k+1}^{2k} \mathsf{Pr}(r|S) < (k-1) + \frac{1}{2} = \frac{2k-1}{2}. \tag{E-12}$$

Then by (E-11) it is sufficient to show that

$$\sum_{r=1}^{k}(r-1)\mathsf{Pr}(r|S) + \sum_{r=k+1}^{2k}(r-1)\mathsf{Pr}(r|S) < (k-1) + \sum_{r=k+1}^{2k}\mathsf{Pr}(r|S). \tag{E-13}$$

But the second term on the left hand side of (E-13) satisfies the following:

$$\begin{aligned}
& \sum_{r=k+1}^{2k}(r-1)\mathsf{Pr}(r|S) = \sum_{r=k+1}^{2k}\mathsf{Pr}(r|S) + \sum_{r=k+1}^{2k}(r-2)\mathsf{Pr}(r|S) \\
< & \sum_{r=k+1}^{2k}\mathsf{Pr}(r|S) + (k-1)\mathsf{Pr}(k+1|S) + [(k-1)\mathsf{Pr}(k+2|S) + \mathsf{Pr}(k-1|S)] \\
& +[(k-1)\mathsf{Pr}(k+3|S) + 2\mathsf{Pr}(k-2|S)] + \ldots + [(k-1)\mathsf{Pr}(2k|S) + (k-1)\mathsf{Pr}(1|S)] \\
= & \sum_{r=k+1}^{2k}\mathsf{Pr}(r|S) + (k-1)\sum_{r=k+1}^{2k}\mathsf{Pr}(r|S) + \sum_{r=1}^{k}(k-r)\mathsf{Pr}(r|S).
\end{aligned}$$

Therefore, the LHS of (E-13) is less than

$$\sum_{r=1}^{k}(r-1)\mathsf{Pr}(r|S) + \sum_{r=k+1}^{2k}\mathsf{Pr}(r|S) + (k-1)\sum_{r=k+1}^{2k}\mathsf{Pr}(r|S) + \sum_{r=1}^{k}(k-r)\mathsf{Pr}(r|S) = (k-1) + \sum_{r=k+1}^{2k}\mathsf{Pr}(r|S).$$

This demonstrates that (E-13) holds. Hence we must have that $\mathsf{E}[V|S] < \mathsf{E}[V]$. To complete the proof of the lemma we also need to show that there exists $\mu^i \in (0,1]$ such that $\mathsf{E}[V|S] < \mathsf{bid}^1$ if $\mu < \mu^i$. As in Lemma 5 this follows immediately from $\mathsf{E}[V|S] < \mathsf{E}[V]$ and from $\lim_{\mu\to 0}\mathsf{E}[V] - \mathsf{bid}^1 = 0$. This completes the proof of Lemma II.

Next, we turn to the switching of behaviour. In our main characterization results for the three state – three signal case to obtain switching by a herding type we assumed that the signal is more likely when the value is highest than when the asset has the middle value, and for the switching by a contrarian type we assumed that the signal is more likely when the value is lowest than when the asset has the middle value. The analogue of those conditions to the current setting with N states and I liquidation values are the following:

$$\mathsf{Pr}(S|j) > \mathsf{Pr}(S|i) \text{ for all } j \in C_I,\ i \in C_{I-1} \tag{E-14}$$

$$\mathsf{Pr}(S|j) < \mathsf{Pr}(S|i) \text{ for all } j \in C_1,\ i \in C_2 \tag{E-15}$$

In the three state case, a negatively biased signal that satisfies (E-14) is nU-shaped a negatively biased signal that satisfies (E-15) is nHill-shaped.

Next we show that if the probability of informed trading is sufficiently small, then conditions (E-14) and (E-15) can be used to establish the following.

Lemma III
(i) Let S satisfy (E-14). Then there exists $\mu^s_{bh} \in (0,1]$ such that $\beta^t_i \mathsf{Pr}(S|j) - \beta^t_j \mathsf{Pr}(S|i) > 0$, for all $i \in C_{I-1}, j \in C_I$, t and H^t.
(ii) Let S satisfy (E-15). Then there exists $\mu^s_{bc} \in (0,1]$ such that $\beta^t_i \mathsf{Pr}(S|j) - \beta^t_j \mathsf{Pr}(S|i) > 0$, for all $i \in C_1, j \in C_2$, t and H^t.

Proof of Lemma III: We show (i); the argument for (ii) follows analogously. Fix any $j \in C_I$ and $i \in C_{I-1}$. For any date t and history H^t, let $\mathcal{S}^t$ be a set of signal types that buy at history H^t. Since $\beta^t_i \mathsf{Pr}(S|j) - \beta^t_j \mathsf{Pr}(S|i) = (\gamma + \mu \sum_{S' \in \mathcal{S}^t} \mathsf{Pr}(S'|i))\mathsf{Pr}(S|j) - (\gamma + \mu \sum_{S' \in \mathcal{S}^t} \mathsf{Pr}(S'|j))\mathsf{Pr}(S|i)$, it follows that $\beta^t_i \mathsf{Pr}(S|j) - \beta^t_j \mathsf{Pr}(S|i) > 0$ is equivalent to

$$\mathsf{Pr}(S|j) - \mathsf{Pr}(S|i) > \frac{\mu}{\gamma}\left(\sum_{S' \in \mathcal{S}^t} \mathsf{Pr}(S'|j)\mathsf{Pr}(S|i) - \sum_{S' \in \mathcal{S}^t} \mathsf{Pr}(S'|i)\mathsf{Pr}(S|j)\right) \tag{E-16}$$

By (E-14), the left hand side of (E-16) is positive. Also, since there is a finite number of signals, the expression in parentheses on right hand side of (E-16) is uniformly bounded in t. Therefore, there must exist $\mu^s_{bh} \in (0,1]$ such that for any $\mu < \mu^s_{bh}$ (E-16) holds for all t and H^t.

Next, we state out first characterization result in this general set-up (it is equivalent to Lemma 7 when $I = N$).

Lemma IV

(i) *Suppose S is negatively biased, satisfies (E-14) and let the following condition hold*

$$\forall \epsilon > 0\ \exists\ H^t \text{ such that } q^t_i / q^t_j < \epsilon \text{ for all } j \in C_{I-1} \cup C_I \text{ and } i \notin C_{I-1} \cup C_I. \tag{E-17}$$

Then there exists a $\mu_{bh} \in (0,1]$ such that S buy herds if $\mu < \mu_{bh}$.

(ii) *Suppose S is negatively biased, satisfies (E-15) and let the following condition hold*

$$\forall \epsilon > 0\ \exists\ H^t \text{ such that } q^t_i / q^t_j < \epsilon \text{ for all } j \in C_1 \cup C_2 \text{ and } i \notin C_1 \cup C_2. \tag{E-18}$$

Then there exists a $\mu_{bc} \in (0,1]$ such that S is a buy contrarian if $\mu < \mu_{bc}$.

Proof of Lemma IV: We show part (i); part (ii) follows analogously. Assume that $\mu < \mu_{bh} \equiv \min\{\mu^i_{bh}, \mu^s_{bh}\}$, where μ^i_{bh} and μ^s_{bh} are respectively the bounds on the size of the informed traders given in Lemmas II and III. Since S is negatively biased and $\mu < \mu_{bh}$, by Lemma II, S sells at the initial history.

Analogously to Lemma 9, by simple calculations, it can be shown that for any history H_t, $\mathsf{E}[V|S,H^t]-\mathsf{ask}^t$ has the same sign as

$$\sum_{i<j}(V_j-V_i)\frac{q_i^t q_j^t}{\rho_I^t\rho_{I-1}^t}(\beta_i^t\mathsf{Pr}(S|j)-\beta_i^t\mathsf{Pr}(S|i)), \tag{E-19}$$

where $\rho_r=\sum_{j\in C_r}q_j^t$ is the probability that the valuation is $(r-1)\mathcal{V}$, for all r. Consider now all terms in (E-19) that pertain to both C_I and C_{I-1}. These are

$$\sum_{j\in C_I}\sum_{i\in C_{I-1}}\frac{q_i^t q_j^t}{\rho_I^t\rho_{I-1}^t}(\beta_i^t\mathsf{Pr}(S|j)-\beta_j^t\mathsf{Pr}(S|i)). \tag{E-20}$$

Since $\mu<\mu_{bh}^s$, by Lemma III, there exists an $\eta>0$ such that $\beta_i^t\mathsf{Pr}(S|j)-\beta_j^t\mathsf{Pr}(S|i)>\eta$ for all $i\in C_{I-1}, j\in C_I$. Thus

$$\text{(E-20)}>\eta\cdot\sum_{j\in C_I}\sum_{i\in C_{I-1}}\frac{q_i^t q_j^t}{\rho_I^t\rho_{I-1}^t}=\eta. \tag{E-21}$$

Furthermore, note that $\mathsf{E}[V|H^t]>\rho_{I-1}^t\mathcal{V}(I-2)+\rho_I^t\mathcal{V}(I-1)$. Also, by the symmetries assumed it must be that $\mathsf{E}[V]\leq\mathcal{V}(I-2)$. Therefore, $\mathsf{E}[V|H^t]-\mathsf{E}[V]>\rho_I^t\mathcal{V}-(1-\rho_I^t-\rho_{I-1}^t)\mathcal{V}(I-2)=\mathcal{V}\sum_{j\in C_I}q_j-\mathcal{V}(I-2)\sum_{j\notin C_I\cup C_{I-1}}q_j$. This together with (E-17) and finiteness of the state space imply that there exists a history H^t such that the following two conditions hold:

$$\mathsf{E}[V|H^t] > E[V] \tag{E-22}$$

$$\sum_{\substack{i<j,\\ i,j\notin C_{I-1}\cup C_I}}(V_j-V_i)\frac{q_i^t q_j^t}{\rho_I^t\rho_{I-1}^t}(\beta_i^t\mathsf{Pr}(S|j)-\beta_i^t\mathsf{Pr}(S|i)) > -\eta. \tag{E-23}$$

The latter, together with (E-21), imply that at such a history (E-19)>0. Thus, by (E-22), type S buy herds at H^t. This completes the proof of Lemma IV.

Notice that the above result is the analogue of Lemma 4 for our current set-up with N and I values. Also, properties (E-17) and (E-18) are respectively analogous to (3) and (4) for our set-up.

As with (3) and (4), conditions (E-17) and (E-18) are assumptions on endogenous variables. One restriction on the information structure that ensures these properties is MLRP. In particular, with MLRP one can show (as in the three states case) that the probability of a buy is increasing in the liquidation values and the probability of a sale is decreasing in the liquidation values; these relationship in turn ensure (E-17) and (E-18).

Lemma V *Suppose that the signals satisfy MLRP and assume that* $S_1 < \ldots < S_N$*. Then there exists* $\delta < 1$ *such that for all* i*,*j *with* $i < j$ *and all* t*, we have* $\beta_i^t/\beta_j^t < \delta$ *and* $\sigma_j^t/\sigma_i^t < \delta$*.*

Proof of Lemma V: We will show only $\beta_1^t < \beta_2^t < \ldots < \beta_N^t$; the result for σ_i follows analogously. To show the former, observe that with MLRP signals, expectations are ordered in signals: $\mathsf{E}[V|H^t, S_i] > \mathsf{E}[V|H^t, S_j]$ if $i > j$. Thus, if signal type S_i buys, so will all $S_l > S_i$ and for any t and any $i < j$, $\beta_i^t - \beta_j^t$ has the same sign as

$$\begin{aligned}\sum_{l=k}^{N} \mathsf{Pr}(S_l|j) - \sum_{l=k}^{N} \mathsf{Pr}(S_l|i) &= 1 - \sum_{l=1}^{k-1} \mathsf{Pr}(S_l|i) - \left(1 - \sum_{l=1}^{k-1} \mathsf{Pr}(S_l|j)\right) \\ &= \sum_{l=1}^{k-1} \mathsf{Pr}(S_l|i) - \sum_{l=1}^{k-1} \mathsf{Pr}(S_l|j), \text{ for some } k \leq N.\end{aligned}$$

Since MLRP implies First Order Stochastic Dominance and there are a finite number of signals it follows that there exists $\epsilon > 0$ such that $\sum_{l=1}^{k-1} \mathsf{Pr}(S_l|i) - \sum_{l=1}^{k-1} \mathsf{Pr}(S_l|j) > \epsilon$ for all k and $i < j$. But then $\beta_i^t/\beta_j^t < 1 - \epsilon$. This completes the proof of Lemma V.

Theorem 3a *Assume that signals satisfy MLRP and let signal* S *be negatively biased.*
(a) *If* $\mathsf{Pr}(S|V_{N-1}) < \mathsf{Pr}(S|V_N)$ *then there exists* $\mu_{bh} \in (0, 1]$ *such that* S *buy herds if* $\mu < \mu_{bh}$*.*
(b) *If* $\mathsf{Pr}(S|V_1) < \mathsf{Pr}(S|V_2)$ *there exists* $\mu_{bh} \in (0, 1]$ *such that* S *is a buy contrarian if* $\mu < \mu_{bh}$*.*

Note that Theorem 3a is more general than Theorem 3 as it applies to the general setup depicted in this supplementary material.

Proof of Theorem 3a. Part (a): It remains to be shown that histories exists such that (E-17) holds. Consider the infinite path consisting of only buys at every date. By MLRP and Lemma V, there must exist $\delta \in (0, 1)$ such that for every H^t and for any i and j with $i < j$, we have $\beta_i^t/\beta_j^t < \delta$. Since $\frac{q_i^{t+1}}{q_k^{t+1}} = \frac{\beta_i^t}{\beta_k^t}\frac{q_i^t}{q_k^t}$, it then follows that q_i^t/q_j^t converges to zero along this infinite path of buys, for all $i \notin C_{I-1} \cup C_I$ and $j \in C_{I-1} \cup C_I$. This together with Lemma IV (i) concludes the proof for the existence of buy herding.

Part (b) follows analogously.

F Supplementary Material for Section 10

In this section we show how AZ's *composition uncertainty* can be accommodated within our N-state framework of the last section and why the types that herd in this set-up also have U-shaped signals.

In AZ's set-up there are three liquidation values $0, 1/2$ and 1, as in their basic example. When the liquidation values are 0 and 1, there are two levels of informativeness of the market W and P. Thus, there are five states $(0, W), (0, P), 1/2, (1, P)$ and $(1, W)$, and we enumerate them by $1, 2, 3, 4$ and 5, respectively. Thus, in terms of the notation from the previous section of this supplementary material, $I = 3, \mathcal{V} = 1/2$ and $N = 5$. We also denote the states with valuation i by C_i. Therefore, $C_0 = \{(0, W), (0, P)\}$, $C_{1/2} = \{1/2\}$, and $C_1 = \{(1, W), (1, P)\}$.

In terms of the private information of the traders, AZ's description is as follows. There are two kinds of informed traders. All have a common partition of the liquidation values given by $\{(0, 1), 1/2\}$. When values 0 or 1 are realized, the two types have different precisions with respect to the two valuations. Specifically, high "quality" type h has precision p_h and low "quality" type l has precision p_l, with $1 \geq p_h > p_l > 1/2$. Thus, in this model there five signals. We denote the signal that confirms valuation 1/2 by S_3, and let S_1 and S_5 be the signals that high quality traders receive and S_2 and S_4 those that low quality types receive. Finally, the proportion of different kind of traders depends on the informativeness of the market; in particular, the likelihood of quality type $i = h, l$ occurring in market j is given by μ_i^j, $i \in \{h, l\}$ and $j \in \{W, P\}$, with $\mu_h^W + \mu_l^W = \mu_h^P + \mu_l^P = \mu$.

Using our notation, we can then describe the information structure as follows:

$P(S \mid i)$	$(0, W)$ $i = 1$	$(0, P)$ $i = 2$	$1/2$ $i = 3$	$(1, P)$ $i = 4$	$(1, W)$ $i = 5$
S_1	$\frac{\mu_h^W}{\mu} p_h$	$\frac{\mu_h^P}{\mu} p_h$	0	$\frac{\mu_h^P}{\mu}(1 - p_h)$	$\frac{\mu_h^W}{\mu}(1 - p_h)$
S_2	$\frac{\mu_l^W}{\mu} p_l$	$\frac{\mu_l^P}{\mu} p_l$	0	$\frac{\mu_l^P}{\mu}(1 - p_l)$	$\frac{\mu_l^W}{\mu}(1 - p_l)$
S_3	0	0	1	0	0
S_4	$\frac{\mu_l^W}{\mu}(1 - p_l)$	$\frac{\mu_l^P}{\mu}(1 - p_l)$	0	$\frac{\mu_l^P}{\mu} p_l$	$\frac{\mu_l^W}{\mu} p_l$
S_5	$\frac{\mu_h^W}{\mu}(1 - p_h)$	$\frac{\mu_h^W}{\mu}(1 - p_h)$	0	$\frac{\mu_h^W}{\mu} p_h$	$\frac{\mu_h^W}{\mu} p_h$

To demonstrate buy herding consider signals S_1 and S_2 (sell-herding arguments are analogous and involve considering S_4 and S_5). These signals are U-shaped in the sense that $\mathsf{Pr}(S_i|j) > \mathsf{Pr}(S_i|k)$ for every $i = 1, 2$ and $j \in C_0 \cup C_1$ and $k \in C_{1/2}$. By appealing to the arguments of the previous section, the U-shaped nature of signals S_1 and S_2 allows us to establish buy herding as follows.

Take the case of S_1 and suppose that S_1 does not buy herd. Note also that S_1 is negatively biased (in the generalized sense defined in the last section: $\mathsf{Pr}(S_1|j) > \mathsf{Pr}(S_1|6-j)$ for each $j = 1, 2$) and satisfies (E-14). Therefore, if (E-17) holds, Lemma IV applies and types S_1 buy herd when there are a sufficient amount of noise traders; a contradiction. The last step is to construct histories that satisfy (E-17).

The proof of (E-17) is as in Proposition 3. (Since S_3 is a Hill shaped signal, it corresponds to the case D2 of that proof.[47]) Specifically, one constructs a two-stage history.

[47]This sub-case proves the existence of the histories that yield (E-17) for the case with two U/Hill shaped signals with opposing biases and a Hill/U shaped signal with a zero bias and it is the case for which our results formally subsume AZ's example of event uncertainty.

During the first stage, the actions are such that q_i^t/q_j^t for $i \in C_0$, $j \in C_{1/2}$ decrease while q_i^t/q_j^t, for $i \in C_0$, $j \in C_1$ do not change (during this stage the actions correspond to those that S_3 will take). The second stage involves buys only. During this stage q_i^t/q_j^t for each $i \in C_0$, $j \in C_1$ decrease while q_i^t/q_j^t for $i \in C_0$, $j \in C_{1/2}$ may increase. Finally, the length of the two stages are chosen appropriately so that (E-17) holds (if the second stage is long enough then q_i^t/q_j^t for each $i \in C_0$, $j \in C_1$ is sufficiently small and if the length of the first stage is sufficiently long relative to the second stage then q_i^t/q_j^t for each $i \in C_0$, $j \in C_{1/2}$ is sufficiently small).

G Supplementary Material for Section 11

Simple History Dependence. The order of trades and traders does not affect the price path as long as the model primitives do not allow any type of trader to change behaviour. Clearly, herding or contrarian behaviour involve such a change of behaviour; changes from buying to holding or selling to holding also qualify as a change of behaviour.

Without changes in behaviour, it suffices to study the order imbalance (number of buys minus number of sales) to determine prices, but with changes, the order of arrival matters a great deal. Consider the following numerical example[48] of an MLRP signal structure with an nU-shaped S_2

$\mathsf{Pr}(S\|V)$	V_1	V_2	V_3
S_1	$\frac{40}{49}$	$\frac{4}{49}$	0
S_2	$\frac{9}{49}$	$\frac{9}{490}$	$\frac{243}{12250}$
S_3	0	$\frac{9}{10}$	$\frac{12007}{12250}$

$\mu = \frac{1209}{1600}$,
$\mathbb{V} = (0, 10, 20)$, and
$\mathsf{Pr}(V) = (1/6, 2/3, 1/6)$.

For illustrative purposes, assume that the first fifteen traders are all informed and each signal S_i, $i = 1, 2, 3$, is received by five of the first fifteen traders. Next, we compare the price paths for different arrival orders of these traders.

Series 1: The arrival order is $5{\times}S_1$–$5\ {\times}S_2$–$5{\times}S_3$ (meaning the first five receive S_1, the next five S_2 and the last five S_3). The S_1 types, who move first, all sell and thus the price drops. The S_2 types also sell and the S_3 types buy. Computations show that after these 15 trades the public expectation will drop from 10 to .15.

Series 2: $5{\times}S_1$–$5{\times}S_3$–$5{\times}S_2$. Here the outcome is the same as in the previous series with S_1 traders selling, S_3 types' buying and finally the S_2 types selling. The public expectation also drops from 10 to .15.

Series 3: $5{\times}S_3$–$5{\times}S_2$–$5{\times}S_1$. The S_3 traders move first and buy. The S_2 types will now behave differently from the previous two series and will be buy-herding. The public

[48]We chose the numbers so that there can be herding after a small number of trades.

expectation now rises to about 13.5. Finally, the five S_1 type sell, and then the public expectation drops to 10.31.

The difference between the outcome for Series 3 with those of Series 1 and 2 illustrates how the arrival order of traders matters: since there are S_2 types who trade, this type's change in trading-mode (from selling to buying) strongly affects the price-path.

Note, however, that even if there are no S_2-types directly involved in trading, the market maker has to consider the possibility that this type trades and thus has to account for this type's change of trading mode. To illustrate this, we next compare the outcome when the same number of buys and sales occurs, but in different orders.

SERIES 4: 20 BUYS FOLLOWED BY 20 SALES. After 20 buys, the public expectation is 15.36, after 20 subsequent sales it is 3.12.

SERIES 4: 20 SALES FOLLOWED BY 20 BUYS. After 20 sales, the public expectation is 1.16×10^{-13}, after 20 subsequent buys it is 10.0064.

In summary, the S_2-type can change trading modes in response to observing the order flow; thus the order flow affects prices and the frequency of different types of future trades. In the short run, the fluctuations may thus be influenced by the precise order of trades.

Price Sensitivity. To further elaborate on the price sensitivity induced by herding, we simulate price paths (Figure 1) using the following MLRP specification:

$$\begin{array}{l} \mu^s_{bh} = 0.7656 \equiv \mu_{bh} \\ \mu^i = 0.9215 \\ \mathbb{V} = (0, 10, 20), \\ \mathsf{Pr}(V) = (1/10, 4/5, 1/10), \text{ and} \end{array} \qquad \begin{array}{c|ccc} \mathsf{Pr}(S|V) & V_1 & V_2 & V_3 \\ \hline S_1 & \frac{40049}{49000} & \frac{4}{49} & 0 \\ S_2 & \frac{8951}{49000} & \frac{9}{490} & \frac{243}{12250} \\ S_3 & 0 & \frac{9}{10} & \frac{12007}{12250} \end{array} \tag{G-24}$$

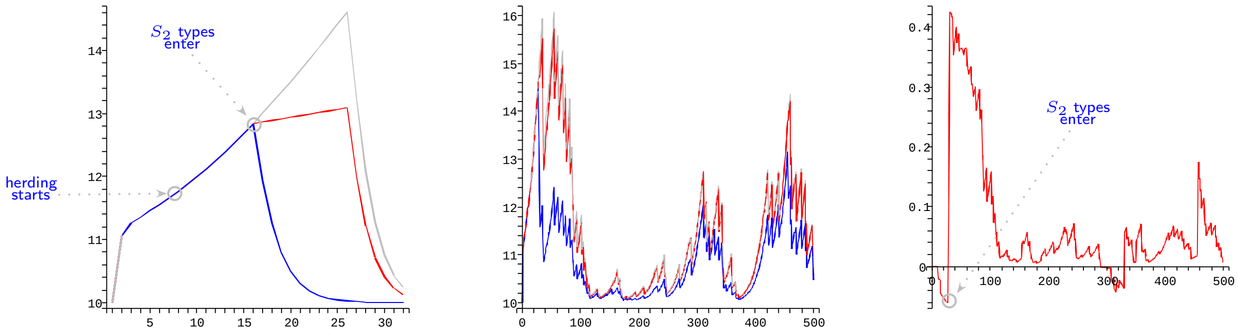
In the left panel, there are two relevant price paths: the first (in gray) is for a setting with $\mu = \mu_{bh} - \epsilon$, $\epsilon = 1/10,000$; in other words, there is just enough noise so that herding is possible. The second price path (in red) is for $\mu = \mu_{bh}$ so that there cannot be herding.[49] The entry series for the graph is as follows: first, there is a long series of S_3 types, who all buy; this is followed by a group of S_2 types and eventually by some S_1 types. The point when S_2 types start entering is clearly marked; the S_1 types enter at the point when both curves peak. The point at which herding starts is marked too.

The series is constructed so that there are S_3 types who enter during herding. When the S_2 types enter, in the herding case, they buy, in the no-herding case, they hold. Even with holds, however, prices increase (this is due to the U-shaped csd).[50]

[49]The third price path (in blue) is for the case of the opaque economy as described in the Section 8. For the opaque case the differences in prices for the two levels of μ are negligible.

[50]The same simulation for the case of the opaque economy as described in the Section 8 results in S_2 types selling and prices falling for both levels of μ.

Figure 1
Illustrations of the Sensitivity in Prices Paths with and without Herding.



In the middle panel we plot prices for the same specifications, this time for a random sequence of traders; both series have the same sequence of traders but due to herding their actions may differ.[51] In the right panel we plot the difference of the two rational price-series from the middle panel. As the series with herding-prices has more noise (because $\mu < \mu_{bh}$), initially, the price for the no-herding series is above the price of the herd series. Once herding starts (here after 8 trades), and once an S_2 type enters, this relation flips, illustrating that due to herding prices move stronger in the direction of the herd than in the no-herding case.

Does Herding Hamper Learning? To explore this issue, we use Monte Carlo simulations and compare the two scenarios outlined when discussing price sensitivity. That is, for the first series, there is just enough noise so that buy-herding can be triggered, $\mu = \mu_{bh} - \epsilon$, $\epsilon \approx 1/10,000$. In the second series, herding cannot occur, because there is too much informed trading, $\mu = \mu^s_{bh}$. We will refer to prices in the first setting as herding-prices, irrespective of whether or not herding actually occurred; we refer to prices in the second setting as no-herding prices. Comparing the speeds of convergence for our two sets of simulations we note the following two observations:

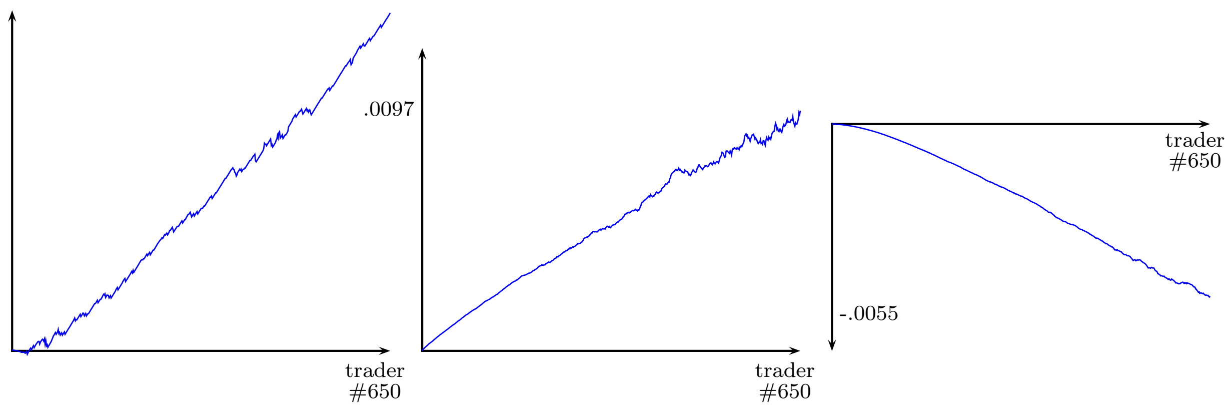
1. if the true value is V_1 or V_2, then herding-prices converge slower;
2. if the true value is V_3, then convergence with herding is faster.

These observations are based on the following: For the simulations we again used the specification of the parameters given by (G-24). Fixing the true liquidation values, we then drew 650 traders at random (noise and informed) assuming that $\mu_{bh} \approx .766$. Since the proportion of the informed agents μ is large — approximately three quarters for both simulations — the 650 trades are almost always sufficient to obtain convergence to the true

[51]There is also a series for the the opaque economy (Section 8) which, not surprisingly, is entirely below both rational series. Again, the opaque economy price series for $\mu = \mu_{bh}$ and $\mu = \mu_{bh} - \epsilon$ are almost identical.

Figure 2
The Difference in Speeds of Convergence.

Each graph plots the difference of the negative of the average log-distance of the transaction prices of herding and no-herding case. An up-sloping line thus indicates that for any t herding-prices are further from the true value than no-herding prices. All graphs are scaled to fit the page. The underlying signal distribution is listed in Appendix G-24.



value. Next, we computed the time series of the transaction prices for both the herding and the no-herding case, and then recorded for each t and for both cases the absolute distance of the transaction price from the true value (which we know). We then repeated this procedure a large number of times, and calculated for each t and for each case the average distance from the true value. Since prices converge to the true value, these average distances decline in t. In the simulations, this distance declines approximately exponentially to zero. Thus the slope of the logarithm of the average distance measures the speed of convergence.

As the final step, we subtract at each t the log-averages for the no-herding from the herding series. A positive number indicates that the herding series is slower, i.e. that the average herding price is further away from the true value. Figure 2 plots these differences and the graphs are striking; they confirm our two observations mentioned above.[52]

To see the intuition for the these observations compare the effects of buy-herding on the herding and no-herding prices. First, when buy-herding occurs, S_2 types buy in the herding case and thus there are more buys with herding than in the no-herding case. Second, in the case of a buy, prices in the herding case tend to be higher than in the no-herding case. Since the no-herding prices here are the similar or higher than the ones that arise in the opaque economy of Section 8 (only S_3 types buy in both cases), this second effect follows from the same reasoning used in the previous simulation to explain why, in the case of a

[52] We have also made a formal analysis by regressing the log-distance on time and, using the Chow test, checking if one slope is steeper than the other. The results were highly significant.

buy, prices in the rational world, when herding starts, exceed those in the opaque economy (see Proposition 6(1a)). Third, when there is a sale, prices in the herding and no-herding cases are almost identical and unaffected by buy-herding. This is because in both cases only S_1 types sell: in the herding case this is so by definition and in the no-herding case, the S_2 type's expectation is almost equal to the ask-price (expression (11) is almost zero) and thus larger than the bid-price.[53]

Now it follows from the above that if the true value is V_1 or V_2, herding prices converge slower: during herding, herd-buys move prices *away* from the true value by a larger magnitude and there are more such buys than in the no-herding case (sales have a similar effect in both cases). If, however, the true value is V_3 then once herding starts, prices in the herding-case move up more strongly because of the first two effects and thus they move faster *towards* the true value. This leads to a higher speed of convergence in the herding case. Figure 2 documents these three cases.

The Probability of the Fastest Herd. The shortest sequence of trades that leads to buy-herding is one with only buys; this is the "fastest" herd. We now want get a sense of how likely this sequence is. Keeping the csd and the prior distribution fixed but varying the proportion of informed trading, we compute first how many buys are needed for buy-herding to begin, and then we determine how likely this sequence of buys is. The same type of analysis clearly applies to sell-herding.

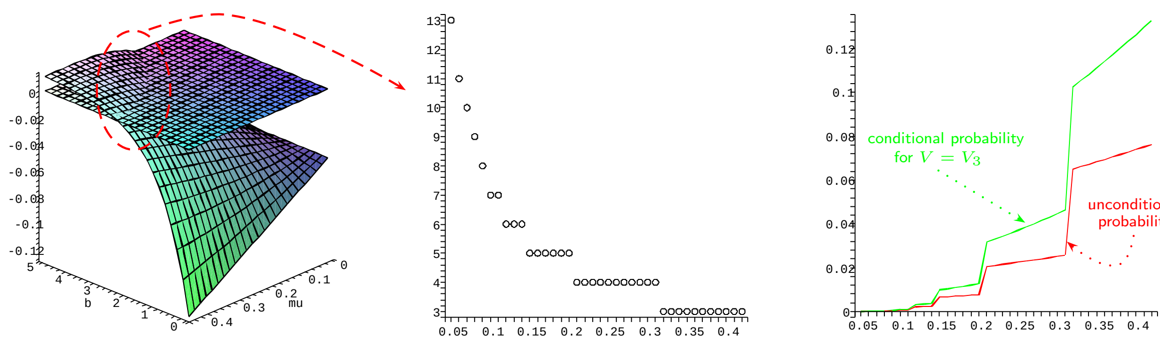
As was explained before, S_2 types buy at any history H_t if the expression in (11) is positive. As the amount of informed trading increases from 0 to μ_{bh}, there are then two opposing effects. First, as noise decreases, the positive term in expression (11) (the first term) becomes smaller. This implies that for any history, the difference between the market maker's and the S_2 type's expectation becomes smaller; thus to get buy-herding one needs more buys. Second, as noise decreases, the informational content of past behaviour (public information) improves and this makes herding more likely. Formally, the second and third terms in (11), the negative terms, decline as μ increases. This is because for any $i = 2, 3$, $\frac{\beta_1}{\beta_i} = \frac{\mu \mathsf{Pr}(S_3|V_1)+\gamma}{\mu \mathsf{Pr}(S_3|V_i)+\gamma}$, $\frac{\partial(\beta_1/\beta_i)}{\partial\mu} = (\mathsf{Pr}(S_3|V_1) - \mathsf{Pr}(S_3|V_i))/{\beta_i}^2$ and thus, since S_1's csd is decreasing, $\frac{\partial(\beta_1/\beta i)}{\partial\mu} < 0$.

While we do not have an analytical result on the net effect of increasing μ from 0 to μ_b, in all numerical examples that we computed the second effect dominates. Thus

[53]The herding and no-herding price paths may also differ even if no buy-herding occurs (if S_2 types behave the same way in the two cases) because the proportions of informed trading μ are different for the two cases. In particular, when S_2 types do not buy-herd, since μ is smaller in the herding case, each price-movement in the herding-price series is smaller than than in the no-herding case, and as a result speed of convergence is slower in the former series. However, since for the simulations the difference between the values of μ is small ($\epsilon = 1/10,000$), the consequence of this effect is small relative to the first two effects mentioned above.

Figure 3
Trades needed for Herding the Probabilities for these trades.

The left panel plots the value of expression (11) as a function of μ, with $\mu \in (0, \mu_{bh})$, and of no-herd buys b. Whenever the bend curve crosses the 0-surface from below, herding is triggered. The middle panel computes the minimum integer number of no-herd buys that would trigger herding as a function of noise level μ. The right panel computes two probabilities: the first is the probability of having exactly the threshold number of buys at the beginning of trade (the thresholds are taken from the middle panel) conditional on the true state being V_3. The second probability is the unconditional likelihood of this threshold number. The plots in the right panel are functions of the μ. The signal distribution that underlies these plots is listed in line (G-24).



Equation (11) — Minimum b given μ — Probability of b given μ

as noise trading declines (μ increases to μ_{bh}) it takes *fewer* buys to trigger buy-herding. Figure 3 plots the minimum number of such consecutive time-zero buys needed to trigger buy-herding for our simulations. As the amount of noise decreases, ex ante it gets more likely that these consecutive buy-trades occur. (Figure 3's right panel illustrates these probabilities.)