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HEREDITARILY NON-SENSITIVE DYNAMICAL SYSTEMS AND LINEAR REPRESENTATIONS

BY

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Abstract. For an arbitrary topological group G any compact G-dynamical system (G, X) can be linearly G-represented as a weak*-compact subset of a dual Banach space V^*. As was shown in [45] the Banach space V can be chosen to be reflexive iff the metric system (G, X) is weakly almost periodic (WAP). In the present paper we study the wider class of compact G-systems which can be linearly represented as a weak*-compact subset of a dual Banach space with the Radon–Nikodým property. We call such a system a Radon–Nikodým (RN) system. One of our main results is to show that for metrizable compact G-systems the three classes: RN, HNS (hereditarily non-sensitive) and HAE (hereditarily almost equicontinuous) coincide. We investigate these classes and their relation to previously studied classes of G-systems such as WAP and LE (locally equicontinuous). We show that the Glasner–Weiss examples of recurrent-transitive locally equicontinuous but not weakly almost periodic cascades are actually RN. Using fragmentability and Namioka's theorem we give an enveloping semigroup characterization of HNS systems and show that the enveloping semigroup $E(X)$ of a compact metrizable HNS G-system is a separable Rosenthal compact, hence of cardinality $\leq 2^{\aleph_0}$. We investigate a dynamical version of the Bourgain–Fremlin–Talagrand dichotomy and a dynamical version of the Todorčević dichotomy concerning Rosenthal compacts.

Contents

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Introduction. The main goal of this paper is to exhibit new and perhaps unexpected connections between the (lack of) chaotic behavior of a dynamical system and the existence of linear representations of the system on certain Banach spaces. The property of *sensitive dependence on initial conditions* appears as a basic constituent in several definitions of "chaos" (see, for example, [9, 16, 25, 11] and references therein). In the present paper we introduce the classes of *hereditarily not sensitive* (HNS for short; intuitively these are the non-chaotic systems) and *hereditarily almost equicontinuous systems* (HAE). It turns out that these classes of dynamical systems are well behaved with respect to the standard operations on dynamical systems and they admit elegant characterizations in terms of Banach space representations.

For an arbitrary topological group G any compact G-system X can be linearly G-represented as a weak*-compact subset of a dual Banach space V^*. As was shown in [45] the Banach space V can be chosen to be reflexive iff the metric G-system X is weakly almost periodic (WAP). We say that a dynamical system (G, X) is a *Radon–Nikodým system* (RN) if V^* can be chosen as a Banach space with the Radon–Nikodým property. One of our main results is to show that for metrizable compact G-systems the three classes of RN, HNS and HAE dynamical systems coincide. For general compact G-systems X we prove that X is in the class HNS iff X is RN-approximable. In other words: a compact system is non-chaotic if and only if it admits sufficiently many G-representations in RN dual Banach spaces. The link between the various topological dynamics aspects of almost equicontinuity on the one hand and the Banach space RN properties on the other is the versatile notion of *fragmentability*. It played a central role in the works on RN compacta (see e.g. Namioka [48]) and their dynamical analogues (see Megrelishvili [42, 43, 45]). It also serves as an important tool in the present work.

The following brief historical review will hopefully help the reader to get a clearer perspective on the context of our results. The theory of weakly almost periodic (WAP) functions on topological groups was developed by W. F. Eberlein [17], A. Grothendieck [28] and I. Glicksberg and K. de Leeuw [15]. About thirty years ago, W. A. Veech [58], in an attempt to unify and generalize the classical theory of weakly almost periodic functions on a discrete group G, introduced a class of functions in $\ell^\infty(G)$ which he denoted by $K(G)$. He showed that $K(G)$ is a uniformly closed left and right G-invariant subalgebra of $\ell^\infty(G)$ containing the algebra of weakly almost periodic functions WAP(G) and shares with WAP(G) the property that every minimal function in $K(G)$ is actually almost periodic.

In [51] Shtern has shown that for any compact Hausdorff semitopological semigroup S there exists a reflexive Banach space V such that S is topologically isomorphic to a closed subsemigroup of $\mathbf{B} = \{s \in \mathcal{L}(V) : \|s\| \leq 1\}$. Here $\mathcal{L}(V)$ is the Banach space of bounded linear operators from V to itself and $\mathbf{B}$ is equipped with the weak operator topology. Megrelishvili provided an alternative proof for this theorem in [43] and has shown in [45] that WAP dynamical systems are characterized as those systems that have sufficiently many linear G-representations on weakly compact subsets of reflexive Banach spaces. In particular, if V is a reflexive Banach space then for every topological subgroup G of the linear isometry group Iso(V) the natural action of G on the weak*-compact unit ball V_1^* of V^* is WAP. Moreover, every WAP metric compact G-space X is a G-subsystem of V_1^* for a suitable reflexive Banach space V.

A seemingly independent development is the new theory of *almost equicontinuous dynamical systems* (AE). This was developed in a series of papers: Glasner & Weiss [25], Akin, Auslander & Berg [1, 2] and Glasner & Weiss [26]. In the latter the class of *locally equicontinuous dynamical systems* (LE) was introduced and studied. It was shown there that the collection LE(G) of locally equicontinuous functions forms a uniformly closed G-invariant subalgebra of $\ell^\infty(G)$ containing WAP(G) and having the property that each minimal function in LE(G) is almost periodic.

Of course the classical theory of WAP functions is valid for a general topological group G and it is not hard to see that the AE theory, as well as the theory of $K(G)$-functions—which we call *Veech functions*—extend to such groups as well.

Let V be a Banach space, V^* its dual. A compact dynamical G-system X is *V^*-representable* if there exist a weakly continuous co-homomorphism $G \to \mathrm{Iso}(V)$, where Iso(V) is the group of linear isometries of a Banach space V onto itself, and a G-embedding $\phi : X \to V_1^*$, where V_1^* is the weak*-compact unit ball of the dual Banach space V^* and the G-action is the dual action induced on V_1^* from the G-action on V. An old observation (due to Teleman [53]) is that every compact dynamical G-system X is $C(X)^*$-representable.

The notion of an *Eberlein compact* (Eb) space in the sense of Amir and Lindenstrauss [4] is well studied and it is known that such spaces are characterized by being homeomorphic to a weakly compact subset of a Banach (equivalently: reflexive Banach) space. Later the notion of *Radon–Nikodým* (RN) compact topological spaces was introduced. These can be characterized as weak*-compact sets in the duals V^* with the RN property. A Banach space V whose dual has the Radon–Nikodým property is called an *Asplund* *space* (see, for example, [22, 48] and Remark 6.2.3). We refer to the excellent 1987 paper of I. Namioka [48] where the theory of RN compacts is expounded.

One of the main objects of [45] was the investigation of RN systems (a dynamical analog of RN compacta) and the related class of functions called "Asplund functions". More precisely, call a dynamical system which is linearly representable as a weak*-compact subset of a dual Banach space with the Radon–Nikodým property a *Radon–Nikodým system* (RN for short). The class of *RN-approximable systems*, that is, subsystems of products of RN systems, will be denoted by $\mathrm{RN}_{\mathrm{app}}$. It was shown in [45] that $\mathrm{WAP} \subset \mathrm{RN}_{\mathrm{app}} \subset \mathrm{LE}$.

Given a compact dynamical G-system X, a subgroup $H < G$ and a function $f \in C(X)$, define a pseudometric $\varrho_{H,f}$ on X as follows:

$$\varrho_{H,f}(x, x') = \sup_{h \in H} |f(hx) - f(hx')|.$$

We say that f is an *Asplund function* (notation: $f \in \mathrm{Asp}(X)$) if the pseudometric space $(X, \varrho_{H,f})$ is separable for every countable subgroup $H < G$. These are exactly the functions which come from linear G-representations of X on V^* with V Asplund. By [45], a compact G-system X is $\mathrm{RN}_{\mathrm{app}}$ iff $C(X) = \mathrm{Asp}(X)$ and always $\mathrm{WAP}(X) \subset \mathrm{Asp}(X)$.

The first section of the paper is a brief review of some known aspects of abstract topological dynamics which provide a convenient framework for our results. In the second section we discuss enveloping semigroups and semigroup compactifications. Our treatment differs slightly from the traditional approach and terminology and contains some new observations. For more details we refer to the books [19, 23, 24, 60, 10, 6]. See also [8, 38, 59].

In [37] Köhler shows that the well known Bourgain–Fremlin–Talagrand dichotomy, when applied to the family $\{f^n : n \in \mathbb{N}\}$ of iterates of a continuous interval map $f : I \to I$, yields a corresponding dichotomy for the enveloping semigroups. In the third section we generalize this and obtain a Bourgain–Fremlin–Talagrand dichotomy for enveloping semigroups of metric dynamical systems.

Section 4 treats the property of *m-approximability*, i.e. of being approximable by metric systems. For many groups G every dynamical G-system is m-approximable and we characterize such groups as being exactly the *uniformly Lindelöf* groups.

In Section 5 we recall some important notions like almost equicontinuity, WAP and LE and relate them to universal systems. We also study the related notion of *lightness* of a function $f \in \mathrm{RUC}(G)$, i.e. the coincidence of the pointwise and the norm topologies on its G-orbit.

Section 6 is devoted to some results concerning fragmentability. These will be crucial at many points in the rest of the paper. In Section 7 we investigate

Asplund functions and their relations to fragmentability. In Section 8 we deal with the related class of Veech functions. As already mentioned the latter class $K(G)$ is a generalization of Veech's definition [58]. We show that every Asplund function is a Veech function and that for separable groups these two classes coincide.

In Section 9 we introduce the dynamical properties of HAE and HNS and show that they are intimately related to the linear representation condition of being an RN system. In particular for metrizable compact systems we establish the following equalities and inclusions:

$$\mathrm{Eb} = \mathrm{WAP} \subset \mathrm{RN} = \mathrm{HAE} = \mathrm{HNS} = \mathrm{RN}_{\mathrm{app}} \subset \mathrm{LE}.$$

Here Eb stands for Eberlein systems—a dynamical version of Eberlein compacts (see Definition 7.5). Section 10 is devoted to various examples and applications. We show that for symbolic systems the RN property is equivalent to having a countable phase space; and that any $\mathbb{Z}$-dynamical system (f, X), where X is either the unit interval or the unit circle and $f : X \to X$ is a homeomorphism, is an RN system.

In Section 11 we show that the Glasner–Weiss examples of recurrent-transitive LE but not WAP metric cascades are actually HAE. In Section 12 we investigate the mincenter of an HAE system, and in Section 13 we use a modified construction to produce an example of a recurrent-transitive, LE but not HAE system. This example exhibits the sharp distinction between the possible mincenters of LE and HAE systems.

In Section 14, using *fragmented families* of functions and *Namioka's joint continuity theorem*, we establish an enveloping semigroup characterization of Asplund functions and HNS systems. Our results imply that the Ellis semigroup $E(X)$ of a compact metrizable HNS system (G, X) is a Rosenthal compact. In particular, by a result of Bourgain–Fremlin–Talagrand [12], we deduce that $E(X)$ is *angelic* (hence, it cannot contain a subspace homeomorphic to $\beta\mathbb{N}$). Finally in Section 15 we show how a theorem of Todorčević implies that for a metric RN system, $E(X)$ either contains an uncountable discrete subspace or admits an at most two-to-one metric G-factor.

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1. Topological dynamics background. Usually all the topological spaces we deal with are assumed to be Hausdorff and completely regular. However, occasionally we will consider a pseudometric on a space, in which case of course the resulting topology need not be even T_0. Let $G \times X \to X$ be a continuous (left) action of the topological group G on the topological space X. As usual, we say that (G, X), or X (when the group is understood), is a *G-space* or a *G-action*. Every G-invariant subset $Y \subset X$ defines a *G-subspace* of X. Recall that every topological group G can be treated as a G-space under the left regular action of G on itself. If X is a compact G-space then sometimes we call it also a *G-system* or just a *system*. We say that a G-space X is a *subdirect product* of a class Γ of G-spaces if X is a G-subspace of a G-product of some members of Γ.

The notations (X, τ) and (X, μ) are used for a topological and a uniform space respectively. When the acting group is the group $\mathbb{Z}$ of integers, we sometimes write (T, X) instead of $(\mathbb{Z}, X)$, where $T : X \to X$ is the homeomorphism which corresponds to the element $1 \in \mathbb{Z}$ (such systems are sometimes called *cascades*). We write gx for the image of $x \in X$ under the homeomorphism $\breve{g} : X \to X$ which corresponds to $g \in G$. As usual, $Gx = \mathcal{O}_G(x) = \{gx : g \in G\}$ is the *orbit* of x and $\overline{\mathcal{O}}_G(x) = \mathrm{cls}(Gx)$ is the closure in X of $\mathcal{O}_G(x)$. If (G, Y) is another G-system then a surjective continuous *G-map* $\pi : X \to Y$ (that is, $g\pi(x) = \pi(gx)$ for all $(g, x) \in G \times X$) is called a *homomorphism*. We also say that Y is a *G-factor* of X. When (G, X) is a dynamical system and $Y \subset X$ is a non-empty closed G-invariant subset, we say that the dynamical system (G, Y), obtained by restriction to Y, is a *subsystem* of (G, X).

Denote by $C(X)$ the Banach algebra of all real-valued bounded functions on a topological space X under the supremum norm. Let G be a topological group. We write $\mathrm{RUC}(G)$ for the Banach subalgebra of $C(G)$ of *right uniformly continuous* ([1]) real-valued bounded functions on G. These are the functions which are uniformly continuous with respect to the *right uniform structure* on G. Thus, $f \in \mathrm{RUC}(G)$ iff for every $\varepsilon > 0$ there exists a neighborhood V of the identity element $e \in G$ such that $\sup_{g \in G} |f(vg) - f(g)| < \varepsilon$ for every $v \in V$. It is equivalent to say that the orbit map $G \to C(G)$, $g \mapsto {}_gf$, is norm continuous where ${}_gf$ is the *left translation* of f defined by ${}_gf(x) = L_g(f)(x) := f(gx)$. Analogously can be defined the algebra $\mathrm{LUC}(G)$ of *left uniformly continuous* functions and the *right translations* $f_g(x) = R_g(f)(x) := f(xg)$. It is easy to see that $\mathrm{UC}(G) := \mathrm{RUC}(G) \cap \mathrm{LUC}(G)$ is a left and right G-invariant closed subalgebra of $\mathrm{RUC}(G)$.

More generally: for a given (not necessarily compact) G-space X a function $f \in C(X)$ will be called *right uniformly continuous* if the orbit map $G \to C(X)$, $g \mapsto {}_gf := L_g(f)$, is norm continuous, where $L_g(f)(x) := f(gx)$. The map $C(X) \times G \to C(X)$, $(f, g) \mapsto {}_gf$, defines a right action. The set

([1]) Some authors call these functions *left uniformly continuous*.

RUC(X) of all right uniformly continuous functions on X is a uniformly closed G-invariant subalgebra of $C(X)$.

A *G-compactification* of a G-space X is a dense continuous G-map $\nu : X \to Y$ into a compact G-system Y. A compactification $\nu : X \to Y$ is *proper* when ν is a topological embedding. We say that a G-compactification $\nu : G \to S$ of $X := G$ (the left regular action) is a *right topological semigroup compactification* of G if S is a *right topological semigroup* (that is, for every $x \in S$ the map $\varrho_s : S \to S$, $\varrho_s(x) = xs$, is continuous) and ν is a homomorphism of semigroups. There exists a canonical 1-1 correspondence (see for example [59]) between the G-compactifications of X and uniformly closed G-subalgebras ("subalgebra" will always mean a subalgebra *containing the constants*) of RUC(X). The G-compactification $\nu : X \to Y$ induces an isometric G-embedding of G-algebras

$$j_\nu : C(Y) \to \mathrm{RUC}(X), \quad \phi \mapsto \phi \circ \nu,$$

and the algebra $\mathcal{A}_\nu$ (corresponding to ν) is defined as the image $j_\nu(C(Y))$. Conversely, if $\mathcal{A}$ is a uniformly closed G-subalgebra of RUC(X), then its Gelfand space $|\mathcal{A}| \subset (\mathcal{A}^*, \mathrm{weak}^*)$ has a structure of a dynamical system $(G, |\mathcal{A}|)$ and the map $\nu_{\mathcal{A}} : X \to Y := |\mathcal{A}|$, $x \mapsto \mathrm{eva}_x$, where $\mathrm{eva}_x(\varphi) := \varphi(x)$ is the multiplicative functional of evaluation at x, defines a G-compactification. If $\nu_1 : X \to Y_1$ and $\nu_2 : X \to Y_2$ are two G-compactifications then $\mathcal{A}_{\nu_1} \subset \mathcal{A}_{\nu_2}$ iff $\nu_1 = \alpha \circ \nu_2$ for some G-homomorphism $\alpha : Y_2 \to Y_1$. The algebra $\mathcal{A}_\nu$ determines the compactification ν uniquely, up to equivalence of G-compactifications.

The G-algebra RUC(X) defines the corresponding Gelfand space $|\mathrm{RUC}(X)|$ (which we denote by $\beta_G X$) and the *maximal G-compactification* $i_\beta : X \to \beta_G X$. Note that this map may not be an embedding even for Polish X and G (see [40]); it follows that there is no proper G-compactification for such X. If X is a compact G-system then $\beta_G X$ can be identified with X and $C(X) = \mathrm{RUC}(X)$.

A point $x_0 \in X$ is a *transitive point* (notation: $x_0 \in \mathrm{Trans}(X)$) if $\overline{\mathcal{O}}_G(x_0) = X$, and the G-space X is called *point-transitive* (or just *transitive*) if $\mathrm{Trans}(X) \neq \emptyset$. It is *topologically transitive* if for any two non-empty open subsets $U, V \subset X$ there exists $g \in G$ with $gU \cap V \neq \emptyset$. Every point-transitive G-space is topologically transitive. When X is a metrizable system, topological transitivity is equivalent to point-transitivity and, in fact, to the existence of a dense G_δ set of transitive points. For a G-space (G, X) with G locally compact we say that a point $x \in X$ is a *recurrent point* if there is a net $G \ni g_i \to \infty$ with $x = \lim_{i\to\infty} g_i x$. A system (G, X) with a recurrent transitive point is called a *recurrent-transitive system.* Note that a transitive infinite $\mathbb{Z}$-system is recurrent-transitive iff X has no isolated points.

A system (G,X) is called *weakly mixing* if the product system $(G, X\times X)$ (where $g(x,x') = (gx, gx'))$ is topologically transitive. A system (G,X) is called *minimal* if every point of X is transitive.

A triple (G,X,x_0) with X compact and a distinguished transitive point x_0 is called a *pointed dynamical system* (or sometimes an *ambit*). For homomorphisms $\pi : (X,x_0) \to (Y,y_0)$ of pointed systems we require that $\pi(x_0) = y_0$. When such a homomorphism exists it is unique. A pointed dynamical system (G,X,x_0) can be treated as a G-compactification $\nu_{x_0} : G \to X$, $\nu_{x_0}(g) = gx_0$. We associate with every $F \in C(X)$ the function $j_{x_0}(F) = f \in \mathrm{RUC}(G)$ defined by $f(g) = F(gx_0)$. Then the map j_{x_0} is actually the above-mentioned isometric embedding $j_{\nu_{x_0}} : C(X) \to \mathrm{RUC}(G)$. Let us denote its image by $j_{x_0}(C(X)) = \mathcal{A}(X,x_0)$. We have ${}_gf = {}_g(j_{x_0}(F)) = j_{x_0}(F\circ g)$. The Gelfand space $|\mathcal{A}(X,x_0)|$ of the algebra $\mathcal{A}(X,x_0)$ is naturally identified with X and in particular the multiplicative functional $\mathrm{eva}_e : f \mapsto f(e)$ is identified with the point x_0. Moreover the action of G on $\mathcal{A}(X,x_0)$ by left translations induces an action of G on $|\mathcal{A}(X,x_0)|$ and under this identification the pointed systems (X,x_0) and $(|\mathcal{A}(X,x_0)|, \mathrm{eva}_e)$ are isomorphic.

Conversely, if $\mathcal{A}$ is a G-invariant uniformly closed subalgebra of $\mathrm{RUC}(G)$ (here and in what follows, when we say that a subalgebra of $\mathrm{RUC}(G)$ is G-invariant we mean *left* G-invariant, that is, invariant with respect to the action $\mathcal{A}\times G \to \mathcal{A}, (f,g) \mapsto {}_gf)$, then its Gelfand space $|\mathcal{A}|$ has a structure of a pointed dynamical system $(G, |\mathcal{A}|, \mathrm{eva}_e)$. In particular, we have, corresponding to the algebra $\mathrm{RUC}(G)$, the *universal ambit* $(G, G^{\mathrm{R}}, \mathrm{eva}_e)$ where we denote the Gelfand space $|\mathrm{RUC}(G)| = \beta_G G$ by G^{R}. (See for example [19] or [60] for more details.)

It is easy to check that for any collection $\{(G, X_\theta, x_\theta) : \theta \in \Theta\}$ of pointed systems we have

$$\mathcal{A}\Big(\bigvee\{(X_\theta, x_\theta) : \theta\in\Theta\}\Big) = \bigvee\{\mathcal{A}(X_\theta, x_\theta) : \theta\in\Theta\},$$

where $\bigvee\{(X_\theta, x_\theta) : \theta\in\Theta\}$ is the orbit closure of the point x in the product space $\prod_{\theta\in\Theta} X_\theta$ whose θ-coordinate is x_θ, and the algebra on the right hand side is the closed subalgebra of $\mathrm{RUC}(G)$ generated by the union of the subalgebras $\mathcal{A}(X_\theta, x_\theta)$.

Definition 1.1. 1. We say that a function $f \in C(X)$ on a G-space X *comes from* a *G-system* Y if there exist a G-compactification $\nu : X \to Y$ (so, ν is onto if X is compact) and a function $F \in C(Y)$ such that $f = \nu \circ F$ (equivalently, $f \in \mathcal{A}_\nu$). Then necessarily $f \in \mathrm{RUC}(X)$. Only the maximal G-compactification $i_\beta : X \to \beta_G X$ has the property that every $f \in \mathrm{RUC}(X)$ comes from i_β.

2. A function $f \in C(G)$ *comes from a pointed system* (Y, y_0) (and then necessarily $f \in \mathrm{RUC}(G)$) if for some continuous function $F \in C(Y)$ we have $f(g) = F(gy_0)$ for all $g \in G$, i.e. $f = j_{y_0}(F)$ (equivalently, if $f \in \mathcal{A}(Y, y_0)$). Defining $\nu : X = G \to Y$ by $\nu(g) = gy_0$ we observe that this is indeed a particular case of 1.1.1.
3. A function $f \in \mathrm{RUC}(X)$ is called *minimal* if it comes from a minimal system.

2. The enveloping semigroup. The *enveloping* (or *Ellis*) *semigroup* $E = E(G, X) = E(X)$ of a dynamical system (G, X) is defined as the closure in X^X (with its compact, usually non-metrizable, pointwise convergence topology) of the set $\breve{G} = \{\breve{g} : X \to X\}_{g \in G}$ considered as a subset of X^X. With the operation of composition of maps this is a right topological semigroup. Moreover, the map $i : G \to E(X)$, $g \mapsto \breve{g}$, is a right topological semigroup compactification of G.

PROPOSITION 2.1. *The enveloping semigroup of a dynamical system* (G, X) *is isomorphic* (*as a dynamical system*) *to the pointed product*

$$(E', \omega_0) = \bigvee \{(\overline{\mathcal{O}}_G(x), x) : x \in X\} \subset X^X.$$

Proof. It is easy to see that the map $p \mapsto p\omega_0$, $(G, E, i(e)) \to (G, E', \omega_0)$, is an isomorphism of pointed systems. ∎

Let X be a (not necessarily compact) G-space. Given $f \in \mathrm{RUC}(X)$ let $I = [-\|f\|, \|f\|] \subset \mathbb{R}$ and $\Omega = I^G$, the product space equipped with the compact product topology. We let G act on Ω by $g\omega(h) = \omega(hg)$, $g, h \in G$.

Define the continuous map

$$f_\sharp : X \to \Omega, \quad f_\sharp(x)(g) = f(gx),$$

and the closure $X_f := \mathrm{cls}(f_\sharp(X))$ in Ω. Note that $X_f = f_\sharp(X)$ whenever X is compact.

Denoting the unique continuous extension of f to $\beta_G X$ by $\widetilde{f}$ we now define a map

$$\psi : \beta_G X \to X_f, \quad \psi(y)(g) = \widetilde{f}(gy), \quad y \in \beta_G X,\ g \in G.$$

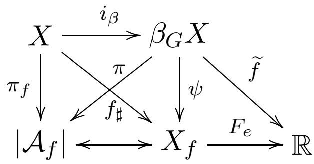
Let $\mathrm{pr}_e : \Omega \to \mathbb{R}$ denote the projection of $\Omega = I^G$ onto the e-coordinate and let $F_e := \mathrm{pr}_e\restriction_{X_f} : X_f \to \mathbb{R}$ be its restriction to X_f. Thus, $F_e(\omega) := \omega(e)$ for every $\omega \in X_f$.

For every $f \in \mathrm{RUC}(X)$ denote by $\mathcal{A}_f$ the smallest closed G-invariant subalgebra of $\mathrm{RUC}(X)$ which contains f. There is then a naturally defined G-action on the Gelfand space $|\mathcal{A}_f|$ and a G-compactification (homomorphism of dynamical systems if X is compact) $\pi_f : X \to |\mathcal{A}_f|$. Next consider the map $\pi : \beta_G X \to |\mathcal{A}_f|$, the canonical extension of π_f.

The action of G on Ω is not in general continuous. However, the restricted action on X_f is continuous for every $f \in \mathrm{RUC}(X)$. This follows from the second assertion of the next proposition.

PROPOSITION 2.2. 1. *Each* $\omega \in X_f$ *is an element of* $\mathrm{RUC}(G)$.

2. *The map* $\psi : \beta_G X \to X_f$ *is a continuous homomorphism of* G*-systems. The dynamical system* $(G, |\mathcal{A}_f|)$ *is isomorphic to* (G, X_f) *and the diagram*



commutes.

3. $f = F_e \circ f_\sharp$. *Thus every* $f \in \mathrm{RUC}(X)$ *comes from the system* X_f. *Moreover, if* f *comes from a system* Y *and a* G*-compactification* $\nu : X \to Y$ *then there exists a homomorphism* $\alpha : Y \to X_f$ *such that* $f_\sharp = \alpha \circ \nu$. *In particular,* $f \in \mathcal{A}_f \subset \mathcal{A}_\nu$.

Proof. 1. $f \in \mathrm{RUC}(X)$ implies that $f_\sharp(X)$ is a uniformly equicontinuous subset of I^G (endowing G with its right uniform structure). Thus, the pointwise closure $\mathrm{cls}(f_\sharp(X)) = X_f$ is also uniformly equicontinuous. In particular, for every $\omega \in X_f$ the function $\omega : G \to I$ is right uniformly continuous.

2. Suppose $i_\beta(x_\nu) \in i_\beta(X)$ is a net converging to $y \in \beta_G X$. Then $\psi(y)(g) = \widetilde{f}(gy) = \lim_\nu f(gx_\nu) = \lim_\nu f_\sharp(x_\nu)(g)$. Thus $\psi(y) = \lim_\nu f_\sharp(x_\nu)$ is indeed an element of X_f and it is easy to see that ψ is a continuous G-homomorphism. In particular, we see that X_f, being a G-factor of $\beta_G X$, is indeed a G-system (i.e. the G-action on X_f is jointly continuous).

Now we use the map $\pi : \beta_G X \to |\mathcal{A}_f|$. By definition, the elements of $\beta_G X$ are continuous multiplicative linear functionals on the algebra $\mathrm{RUC}(X)$, and for $y \in \beta_G X$ its value $\pi(y) \in |\mathcal{A}_f|$ is the restriction $y\restriction_{\mathcal{A}_f}$ to the subalgebra $\mathcal{A}_f \subset \mathrm{RUC}(X)$. For $g \in G$, as above, let ${}_g f \in \mathcal{A}_f \subset \mathrm{RUC}(X)$ be defined by ${}_g f(x) = f(gx)$. Then $\pi(y_1) = \pi(y_2)$ implies $y_1({}_g f) = \widetilde{f}(gy_1) = \widetilde{f}(gy_2) = y_2({}_g f)$ for every $g \in G$.

Conversely, assuming $\widetilde{f}(gy_1) = \widetilde{f}(gy_2)$ for every $g \in G$, we observe that, as y_1 and y_2 are multiplicative functionals, we also have $y_1(h) = y_2(h)$ for every h in the subalgebra $\mathcal{A}_0$ generated by the family $\{{}_g f : g \in G\}$. Since $\mathcal{A}_0$ is dense in $\mathcal{A}_f$ and as y_1 and y_2 are continuous we deduce that $\pi(y_1) = y_1\restriction_{\mathcal{A}_f} = y_2\restriction_{\mathcal{A}_f} = \pi(y_2)$.

We clearly have $\psi(y_1) = \psi(y_2) \Leftrightarrow \widetilde{f}(gy_1) = \widetilde{f}(gy_2)$ for every $g \in G$. Thus for $y_1, y_2 \in \beta_G X$ we have $\pi(y_1) = \pi(y_2) \Leftrightarrow \psi(y_1) = \psi(y_2) \Leftrightarrow \widetilde{f}(gy_1) =$

$\widetilde{f}(gy_2)$ for every $g \in G$, and we find that indeed $|\mathcal{A}_f|$ and X_f are isomorphic G-systems.

The verification of the commutativity of the diagram is straightforward.

3. Clearly, $F_e(f_\sharp(x)) = f_\sharp(x)(e) = f(ex) = f(x)$ for every $x \in X$. For the rest use the G-isomorphism $|\mathcal{A}_f| \leftrightarrow X_f$ (assertion 2). If $f = F \circ \nu$ for some $F \in C(Y)$ then $f \in \mathcal{A}_\nu$. This implies the inclusion of G-subalgebras $\mathcal{A}_f \subset \mathcal{A}_\nu$, which leads to the desired G-homomorphism $\alpha : Y \to X_f$. ∎

REMARK 2.3. 1. Below we use the map $f_\sharp : X \to X_f$ and Proposition 2.2 in two particular cases. First, for a compact G-space X when clearly $\beta_G X$ can be replaced by X. We also frequently consider the case of the left regular action of G on $X := G$ (see Proposition 2.4). Here the canonical maximal G-compactification $i_\beta : X \to \beta_G X$ is actually the compactification $G \to G^{\mathrm{R}}$ and the orbit $Gf = \{R_g(f)\}_{g\in G} = f_\sharp(G)$ of $f \in \mathrm{RUC}(G)$ is pointwise dense in $X_f = \mathrm{cls}(f_\sharp(G)) \subset \Omega = I^G$.

2. $\beta_G X$ is a subdirect product of the G-systems X_f where $f \in \mathrm{RUC}(X)$. This follows easily from Proposition 2.2 and the fact that elements of $C(\beta_G X) = \{\widetilde{f} : f \in \mathrm{RUC}(X)\}$ separate points and closed subsets of $\beta_G X$.

3. Proposition 2.2.3 actually says that the compactification $f_\sharp : X \to X_f$ is *minimal* (in fact, *the smallest*) among all G-compactifications $\nu : X \to Y$ such that $f \in \mathrm{RUC}(X)$ comes from ν. The *maximal* compactification in the same setting is clearly $i_\beta : X \to \beta_G X$.

PROPOSITION 2.4. 1. *Consider the left regular action of G on $X := G$. For every $f \in \mathrm{RUC}(G)$ we have $Gf \subset X_f = \overline{\mathcal{O}}_G(f) \subset \Omega$, $f_\sharp(e) = f$ and $F_e(gf) = f(g)$ for every $g \in G$.*

2. *The pointed G-system $(|\mathcal{A}_f|, \mathrm{eva}_e)$ is isomorphic to (X_f, f) (hence $\mathcal{A}_f = \mathcal{A}(X_f, f)$).*

3. *$f = F_e \circ f_\sharp$. Thus every $f \in \mathrm{RUC}(G)$ comes from the pointed system (X_f, f). Moreover, if f comes from a pointed system (Y, y_0) and $\nu : (G, e) \to (Y, y_0)$ is the corresponding G-compactification then there exists a homomorphism $\alpha : (Y, y_0) \to (X_f, f)$ such that $f_\sharp = \alpha \circ \nu$. In particular, $f \in \mathcal{A}_f \subset \mathcal{A}(Y, y_0)$.*

4. *Denote by $X_f^H \subset I^H$ the dynamical system constructed for the subgroup $H < G$ and the restriction $f\restriction_H$ (e.g., $X_f^G = X_f$). If $H < G$ is a dense subgroup then, for every $f \in \mathrm{RUC}(G)$, the dynamical systems (H, X_f) and (H, X_f^H) are canonically isomorphic.*

Proof. For assertions 1, 2 and 3 use Proposition 2.2 and Remark 2.3.1.

4. Let $j : X_f \to X_f^H$ be the restriction of the natural projection $I^G \to I^H$. Clearly, $j : (H, X_f) \to (H, X_f^H)$ is a surjective homomorphism. If $j(\omega) = j(\omega')$ then $\omega(h) = \omega'(h)$ for every $h \in H$. Since by Proposition 2.2.1

every $\omega \in X_f$ is a continuous function on G and since we assume that H is dense in G, we conclude that $\omega = \omega'$ so that j is an isomorphism. ∎

Definition 2.5. We say that a pointed dynamical system (G, X, x_0) is *point-universal* if for every $x \in X$ there is a homomorphism $\pi_x : (X, x_0) \to (\overline{\mathcal{O}}_G(x), x)$. A closed G-invariant subalgebra $\mathcal{A} \subset \mathrm{RUC}(G)$ is called *point-universal* if the corresponding Gelfand system $(G, |\mathcal{A}|, \mathrm{eva_e})$ is point-universal.

Proposition 2.6. *The following conditions on the pointed dynamical system (G, X, x_0) are equivalent*:

1. (X, x_0) *is point-universal.*
2. $\mathcal{A}(X, x_0) = \bigcup_{x \in X} \mathcal{A}(\overline{\mathcal{O}}_G(x), x)$.
3. (X, x_0) *is isomorphic to its enveloping semigroup* $(E(X), i(e))$.

Proof. 1 ⇒ 2: Clearly, $\mathcal{A}(X, x_0) = \mathcal{A}(\overline{\mathcal{O}}_G(x_0), x_0) \subset \bigcup_{x \in X} \mathcal{A}(\overline{\mathcal{O}}_G(x), x)$. Suppose $f(g) = F(gx)$ for all $g \in G$ and for some $F \in C(\overline{\mathcal{O}}_G(x))$ and $x \in X$. Since (X, x_0) is point-universal there exists a homomorphism $\pi_x : (X, x_0) \to (\overline{\mathcal{O}}_G(x), x)$. Hence $f(g) = F(gx) = F(g\pi_x(x_0)) = F(\pi_x(gx_0)) = (F \circ \pi_x)(gx_0) = j_{x_0}(F \circ \pi_x)(g)$ and we conclude that $f = j_{x_0}(F \circ \pi_x) \in \mathcal{A}(X, x_0)$.

2 ⇒ 3: Proposition 2.1 guarantees the existence of a pointed isomorphism between the systems $(E(X), i(e))$ and $\bigvee_{x \in X}(\overline{\mathcal{O}}_G(x), x)$. Now, using our assumption we have

$$\mathcal{A}(E(X), i(e)) = \mathcal{A}\Big(\bigvee_{x \in X} (\overline{\mathcal{O}}_G(x), x)\Big) = \bigvee_{x \in X} \mathcal{A}(\overline{\mathcal{O}}_G(x), x) = \mathcal{A}(X, x_0),$$

whence the isomorphism of (X, x_0) and $(E(X), i(e))$.

3 ⇒ 1: For any fixed $x \in X$ the map $\pi_x : E(X) \to X$ defined by $\pi_x(p) = px$ is a G-homomorphism with $\pi_x(i(e)) = x$. Our assumption that (X, x_0) and $(E(X), i(e))$ are isomorphic now implies the point-universality of (X, x_0). ∎

Proposition 2.7. *A transitive system (G, X, x_0) is point-universal iff the map $G \to X$, $g \mapsto gx_0$, is a right topological semigroup compactification of G.*

Proof. The necessity of the condition follows directly from Proposition 2.6. Suppose now that the map $G \to X$, $g \mapsto gx_0$, is a right topological semigroup compactification of G. Given $x \in X$ we observe that the map $\varrho_x : (X, x_0) \to (X, x)$, $\varrho_x(z) = zx$, is a homomorphism of pointed systems, so that (G, X, x_0) is point-universal. ∎

In particular, for every G-system X the enveloping semigroup $(E(X), i(e))$, as a pointed G-system, is point-universal. Here, as before, $i : G \to E(X)$, $g \mapsto \breve{g}$, is the canonical enveloping semigroup compactification.

PROPOSITION 2.8. *Let* (G, X, x_0) *be a pointed compact system and* $\mathcal{A} = \mathcal{A}(X, x_0)$ *the corresponding* (*always left* G*-invariant*) *subalgebra of* $\mathrm{RUC}(G)$. *The following conditions are equivalent*:

1. (G, X, x_0) *is point-universal.*
2. $X_f \subset \mathcal{A}$ *for every* $f \in \mathcal{A}$ (*in particular*, $\mathcal{A}$ *is also right* G*-invariant*).

Proof. $1 \Rightarrow 2$: Let $f : G \to \mathbb{R}$ belong to $\mathcal{A}$. Consider the G-compactification $f_\sharp : G \to X_f := \mathrm{cls}(Gf)$ as defined by Proposition 2.4. We have to show that $\varphi \in \mathcal{A}$ for every $\varphi \in X_f$. Consider the orbit closure $X_\varphi = \mathrm{cls}(G\varphi)$ in X_f. By Definition 1.1.2 there exists a continuous function $F : X \to \mathbb{R}$ such that $f(g) = F(gx_0)$ for every $g \in G$. That is, f comes from the pointed system (X, x_0). For some net $g_i \in G$ we have $\varphi(g) = \lim_i f(gg_i)$ for every $g \in G$ and with no loss in generality we have $x_1 = \lim_i g_i x_0 \in X$. Then

$$\varphi(g) = \lim_i f(gg_i) = \lim_i F(gg_i x_0) = F(gx_1).$$

Thus φ comes from the pointed system $(\overline{\mathcal{O}}_G(x_1), x_1)$ and in view of Proposition 2.6 we conclude that indeed $\varphi \in \mathcal{A}$.

$2 \Rightarrow 1$: Define the G-ambit

$$(Y, y_0) := \bigvee \{(X_f, f) : f \in \mathcal{A}\}.$$

First we show that $\mathcal{A}(X, x_0) = \mathcal{A}(Y, y_0)$. Indeed, as we know,

$$\mathcal{A}(Y, y_0) = \bigvee \{\mathcal{A}(X_f, f) : f \in \mathcal{A}\}.$$

Proposition 2.4 implies that $f \in \mathcal{A}_f = \mathcal{A}(X_f, f)$ for every $f \in \mathcal{A}(X, x_0)$. Thus

$$f \in \mathcal{A}_f = \mathcal{A}(X_f, f) \subset \mathcal{A}(Y, y_0) \quad \forall f \in \mathcal{A}(X, x_0).$$

Therefore, $\mathcal{A}(X, x_0) \subset \mathcal{A}(Y, y_0)$. On the other hand, $\mathcal{A}_f = \mathcal{A}(X_f, f) \subset \mathcal{A}(X, x_0)$ (for every $f \in \mathcal{A}(X, x_0)$) because $\mathcal{A}(X, x_0)$ is left G-invariant and $\mathcal{A}_f$ is the smallest closed left G-invariant subalgebra of $\mathrm{RUC}(G)$ which contains f. This implies that $\mathcal{A}(Y, y_0) \subset \mathcal{A}(X, x_0)$. Thus, $\mathcal{A}(X, x_0) = \mathcal{A}(Y, y_0)$. Denote this algebra simply by $\mathcal{A}$.

Suppose $py_0 = qy_0$ for $p, q \in E(Y)$ (the enveloping semigroup of (G, Y)). By our assumption, $X_f \subset \mathcal{A}$ for every $f \in \mathcal{A}$. Then every $y \in Y$, considered as an element of the product space $\prod_{f \in \mathcal{A}} X_f$, has the property that its f-coordinate, say y_f, is again an element of $\mathcal{A}$ and it follows that y_f appears as a coordinate of y_0 as well. Therefore also $py_f = qy_f$ and it follows that $py = qy$. Thus the map $p \mapsto py_0$ from $(E(Y), i(e))$ to (Y, y_0) is an isomorphism. By Proposition 2.6, (Y, y_0) (and hence also (X, x_0)) is point-universal.

(Observe that $Gf = \{R_g(f)\}_{g \in G} \subset X_f := \mathrm{cls}(Gf)$. Therefore, the condition $X_f \subset \mathcal{A}$ for all $f \in \mathcal{A}$ trivially implies that $\mathcal{A}$ is right invariant.) ∎

PROPOSITION 2.9. *Let* P *be a property of compact* G*-dynamical systems which is preserved by products*, *subsystems and* G*-isomorphisms.*

1. *Let X be a (not necessarily compact) G-space and let $\mathcal{P}_X \subset C(X)$ be the collection of functions coming from systems having property P. Then there exists a maximal G-compactification $X^{\mathcal{P}}$ of X with property P. Moreover, $j(C(X^{\mathcal{P}})) = \mathcal{P}_X$. In particular, $\mathcal{P}_X$ is a uniformly closed, G-invariant subalgebra of* $\mathrm{RUC}(X)$.
2. *Let $\mathcal{P} \subset C(G)$ be the set of functions coming from systems with property P. Then $(G^{\mathcal{P}}, \mathrm{eva_e})$ is the universal point-transitive compact G-system having property P. Moreover $\mathcal{P}$ is a point-universal subalgebra of* $\mathrm{RUC}(G)$. (*Thus, $\mathcal{P}$ is uniformly closed, right and left G-invariant, and $X_f \subset \mathcal{P}$ for every $f \in \mathcal{P}$.*)
3. *If in addition P is preserved by factors then $f \in \mathcal{P}$ iff X_f has property P.*

Proof. 1. We only give an outline of the rather standard procedure. There is a complete *set* $\{\nu_i : X \to Y_i\}_{i\in I}$ of equivalence classes of G-compactifications of X such that each Y_i has property P. Define the desired compactification $\nu : X \to Y = \mathrm{cls}(\nu(X)) \subset \prod_{i\in I} Y_i$ via the diagonal product. Then we get the suprema of our class of G-compactifications. In fact, Y has property P because the given class is closed under subdirect products. $f \in \mathcal{P}$ means that it comes from some Y_i via the compactification $\nu_i : X \to Y_i$. Denote Y by $X^{\mathcal{P}}$. Now using the natural projection of Y on Y_i it follows that f comes from $Y = X^{\mathcal{P}}$. This implies $j(C(X^{\mathcal{P}})) = \mathcal{P}_X$.

2. The construction of the maximal ambit $(G^{\mathcal{P}}, \mathrm{eva_e})$ with property P is similar. In fact it is a particular case of the first assertion identifying G-ambits (Y, y_0) and G-compactifications $\nu_{y_0} : G \to Y$, $\nu_{y_0}(g) = gy_0$, of $X := G$. As to the point-transitivity of $\mathcal{P}$ note that according to the definition the uniformly closed subalgebra $\mathcal{P} \subset \mathrm{RUC}(G)$ is the set of functions coming from systems with property P. Every subsystem of $G^{\mathcal{P}}$ has property P. In particular, $(\overline{\mathcal{O}}_G(x), x)$ has property P. Therefore, $\mathcal{P}$ contains the algebra $\mathcal{A}(\overline{\mathcal{O}}_G(x), x)$ for every $x \in X$. By Proposition 2.6 it follows that $\mathcal{P}$ is point-universal. Thus Proposition 2.8 guarantees that $X_f \subset \mathcal{P}$ for every $f \in \mathcal{P}$ (and that $\mathcal{P}$ is right and left G-invariant).

3. Use Proposition 2.2.3. ■

3. A dynamical version of the Bourgain–Fremlin–Talagrand theorem. Let $E = E(X)$ be the enveloping semigroup of a G-system X. For every $f \in C(X)$ define

$$E^f := \{p_f : X \to \mathbb{R}\}_{p\in E} = \{f \circ p : p \in E\}, \quad p_f(x) = f(px).$$

Then E^f is a pointwise compact subset of $\mathbb{R}^X$, being a continuous image of E under the map $q_f : E \to E^f$, $p \mapsto p_f$.

Recall that a topological space K is *Rosenthal compact* [27] if it is homeomorphic to a pointwise compact subset of the space $B_1(X)$ of functions of the first Baire class on a Polish space X. All metric compact spaces are Rosenthal. An example of a separable non-metrizable Rosenthal compact is the *Helly compact* of all (not only strictly) increasing selfmaps of $[0,1]$ in the pointwise topology. Another is the "two arrows" space of Aleksandrov and Urysohn (see Example 14.10 below). A topological space K is *angelic* if the closure of every subset $A \subset K$ is the set of limits of sequences from A and every relatively countably compact set in K is relatively compact. Note that the second condition is superfluous if K is compact. Clearly, $\beta\mathbb{N}$, the Stone–Čech compactification of the natural numbers $\mathbb{N}$, is not angelic, and hence it cannot be embedded into a Rosenthal compact space.

The following theorem is due to Bourgain–Fremlin–Talagrand [12, Theorem 3F], generalizing a result of Rosenthal. The second assertion (BFT dichotomy) is presented as in the book of Todorčević [54] (see Proposition 1 of Section 13).

THEOREM 3.1. 1. *Every Rosenthal compact space K is angelic.*

2. (*BFT dichotomy*) *Let X be a Polish space and let $\{f_n\}_{n=1}^{\infty} \subset C(X)$ be a sequence of real-valued functions which is pointwise bounded* (*i.e. for each $x \in X$ the sequence $\{f_n(x)\}_{n=1}^{\infty}$ is bounded in $\mathbb{R}$*)*. Let K be the pointwise closure of $\{f_n\}_{n=1}^{\infty}$ in $\mathbb{R}^X$. Then either $K \subset B_1(X)$* (*i.e. K is Rosenthal compact*) *or K contains a homeomorphic copy of $\beta\mathbb{N}$.*

Next we will show how the BFT dichotomy leads to a corresponding dynamical dichotomy (see also [37]). In the proof we will use the following observation. Let G be an arbitrary topological group. For every compact G-space X, denote by $j : G \to \mathrm{Homeo}(X)$, $g \mapsto \breve{g}$, the associated (always *continuous*) homomorphism into the group of all selfhomeomorphisms of X. Then the topological group $\breve{G} = j(G)$ (we will call it the *natural restriction*) naturally acts on X. If X is a compact metric space then $\mathrm{Homeo}(X)$, equipped with the topology of uniform convergence, is a Polish group. Hence, the subgroup $\breve{G} = j(G)$ is second countable. In particular one can always find a countable dense subgroup G_0 of $\breve{G}$.

THEOREM 3.2 (A dynamical BFT dichotomy). *Let (G,X) be a metric dynamical system and let $E = E(X)$ be its enveloping semigroup. We have the following alternative*: *either*

1. *E is a separable Rosenthal compact* (*hence* $\mathrm{card}\, E \le 2^{\aleph_0}$), *or*
2. *the compact space E contains a homeomorphic copy of $\beta\mathbb{N}$, hence* $\mathrm{card}\, E = 2^{2^{\aleph_0}}$.

The first possibility holds iff E^f is a Rosenthal compact for every $f \in C(X)$.

Proof. Since X is compact and metrizable, one can choose a sequence $\{f_n\}_{n\in\mathbb{N}}$ in $C(X)$ which separates the points of X. For every pair s,t of distinct elements of E there exist a point $x_0 \in X$ and a function f_{n_0} from our sequence such that $f_{n_0}(sx_0) \neq f_{n_0}(tx_0)$. It follows that the continuous diagonal map

$$\Phi : E \to \prod_{n\in\mathbb{N}} E^{f_n}, \quad p \mapsto (f_1 \circ p, f_2 \circ p, \dots),$$

separates the points of E and hence is a topological embedding.

Now if for each n the space E^{f_n} is a Rosenthal compact then so is $E \cong \Phi(E) \subset \prod_{n=1}^{\infty} E^{f_n}$, because the class of Rosenthal compacts is closed under countable products and closed subspaces. On the other hand the map $q_f : E \to E^f$, $p \mapsto f \circ p$, is a continuous surjection for each $f \in C(X)$. Therefore, $E^f = \mathrm{cls}(q_f(G_0)) = \mathrm{cls}\{f \circ g : g \in G_0\}$, where G_0 is a countable dense subgroup of $\breve{G}$. By Theorem 3.1 (BFT dichotomy), if at least one E^{f_n} is not Rosenthal then it contains a homeomorphic copy of $\beta\mathbb{N}$ and it is easy to see that so does its preimage E. (In fact if $\beta\mathbb{N} \cong Z \subset E^{f_n}$ then any closed subset Y of E which projects onto Z and is minimal with respect to these properties is also homeomorphic to $\beta\mathbb{N}$.)

Again an application of the BFT dichotomy yields the fact that in the first case E is angelic. Clearly, the cardinality of every separable angelic space is at most $2^{\aleph_0}$. Now in order to complete the proof observe that for every compact metric G-system X the space E, being the pointwise closure of $\breve{G}$ in X^X, is separable, hence $\mathrm{card}\, E \leq 2^{2^{\aleph_0}}$.

The last assertion clearly follows from the above proof. ■

4. Metric approximation of dynamical systems. Let (X, μ) be a uniform space and let $\varepsilon \in \mu$. We say that X is *ε-Lindelöf* if the uniform cover $\{\varepsilon(x) : x \in X\}$, where $\varepsilon(x) = \{y \in X : (x, y) \in \varepsilon\}$, has a countable subcover. If X is ε-Lindelöf for each $\varepsilon \in \mu$, then it is called *uniformly Lindelöf* [42]. We note that (X, μ) is uniformly Lindelöf iff it is $\aleph_0$-*precompact* in the sense of Isbell [30]. If X, as a topological space, is either separable, Lindelöf or ccc (see [30, p. 24]), then (X, μ) is uniformly Lindelöf. For a metrizable uniform structure μ, (X, μ) is uniformly Lindelöf iff X is separable. Uniformly continuous maps send uniformly Lindelöf subspaces onto uniformly Lindelöf subspaces.

A topological group G is $\aleph_0$-*bounded* (in the sense of Guran [29]) if for every neighborhood U of e there exists a countable subset $C \subset G$ such that $G = CU$. Clearly, G being $\aleph_0$-bounded means exactly that G is uniformly Lindëlof with respect to its right (or left) uniform structure. By [29] a group G is $\aleph_0$-bounded iff G is a topological subgroup of a product of second countable topological groups. If G is either separable or Lindelöf (σ-compact, for instance) then G is uniformly Lindelöf.

Recall our notation for the "natural restriction" $\breve{G} = j(G)$, where for a compact G-system (G, X), the map $j : G \to \mathrm{Homeo}(X)$ is the associated continuous homomorphism of G into the group of all selfhomeomorphisms of X (see Section 3).

We say that a compact G-system X is *m-approximable* if it is a subdirect product of *metric* compact G-systems (see also the notion of *quasi-separablity* in the sense of [36, 60]). By Keynes [36], every transitive system X with σ-compact acting group G is m-approximable. The following generalization provides a simple criterion for m-approximability.

PROPOSITION 4.1. *Let X be a compact G-system. The following conditions are equivalent*:

1. *X is an inverse limit of metrizable compact G-systems* (*of dimension* $\leq \dim X$).
2. *(G, X) is m-approximable.*
3. *$\breve{G}$ is uniformly Lindelöf.*

Proof. $1 \Rightarrow 2$ is trivial.

For $2 \Rightarrow 3$ observe that for every metric compact G-factor X_i of X the corresponding natural restriction $G_i \subset \mathrm{Homeo}(X_i)$ of G is second countable with respect to the compact open topology. By our assumption it follows that the group $\breve{G} \subset \mathrm{Homeo}(X)$ can be topologically embedded into the product $\prod_i G_i$ of second countable groups. Hence $\breve{G}$ is uniformly Lindelöf by the theorem of Guran mentioned above.

The implication $3 \Rightarrow 1$ has been proved (one can assume that $G = \breve{G}$) in [39, p. 82] and [41, Theorem 2.19] (see also [56, Lemma 10]). ■

PROPOSITION 4.2. *Let G be a topological group. The following conditions are equivalent*:

1. *G is uniformly Lindelöf.*
2. *The greatest ambit G^{R} is m-approximable.*
3. *Every compact G-system is m-approximable.*
4. *For every G-space X and each $f \in \mathrm{RUC}(X)$ the G-system X_f is metrizable.*

Proof. $1 \Rightarrow 4$: Given $f \in \mathrm{RUC}(X)$ the orbit map $G \to \mathrm{RUC}(X)$, $g \mapsto {}_gf$, is uniformly continuous, where G is endowed with its right uniform structure. Since G is uniformly Lindelöf the orbit $fG = \{{}_gf\}_{g\in G}$ is also uniformly Lindelöf, hence separable in the Banach space $\mathrm{RUC}(X)$ (inspired by [56, Lemma 10]). It follows that the Banach G-algebra $\mathcal{A}_f$ generated by fG is also separable. By Proposition 2.2.2, X_f is metrizable.

$4 \Rightarrow 2$: Consider the G-space $X := G$. Assuming that each X_f is metrizable, we see, by Remark 2.3.2, that $G^{\mathrm{R}} = \beta_G X$ is an m-approximable G-system.

$2 \Rightarrow 1$: Since G naturally embeds as an orbit into G^{R}, we see that the map $j : G \to \breve{G} \subset \mathrm{Homeo}(G^{\mathrm{R}})$ is a homeomorphism. If G^{R} is m-approximable then by Proposition 4.1, $\breve{G}$ (and hence G) is uniformly Lindelöf.

$1 \Rightarrow 3$: Immediate by Proposition 4.1.

$3 \Rightarrow 2$: Trivial. ■

5. Almost equicontinuity, local equicontinuity and variations. By a *uniform* G-space (X, μ) we mean a G-space (X, τ) where τ is a (completely regular Hausdorff) topology, with a *compatible* uniform structure μ, so that the topology $\mathrm{top}(\mu)$ defined by μ is τ.

DEFINITION 5.1. Let (X, μ) be a uniform G-space.

1. A point $x_0 \in X$ is a *point of equicontinuity* (notation: $x_0 \in \mathrm{Eq}(X)$) if for every entourage $\varepsilon \in \mu$, there is a neighborhood U of x_0 such that $(gx_0, gx) \in \varepsilon$ for every $x \in U$ and $g \in G$. The G-space X is *equicontinuous* if $\mathrm{Eq}(X) = X$. As usual, X is *uniformly equicontinuous* if for every $\varepsilon \in \mu$ there is $\delta \in \mu$ such that $(gx, gy) \in \varepsilon$ for every $g \in G$ and $(x, y) \in \delta$. For X compact, equicontinuity and uniform equicontinuity coincide.
2. The G-space X is *almost equicontinuous* (AE for short) if $\mathrm{Eq}(X)$ is dense in X.
3. We say that the G-space X is *hereditarily almost equicontinuous* (HAE for short) if every closed uniform G-subspace of X is AE.

The following fact is well known at least for metric compact G-spaces. See for example [2, Proposition 3.4]. Note that neither metrizability nor compactness of (X, μ) are needed in the proof.

LEMMA 5.2. *If (X, μ) is a point-transitive* (^{2}) *uniform G-space and $\mathrm{Eq}(X)$ is not empty then $\mathrm{Eq}(X) = \mathrm{Trans}(X)$.*

Let $\pi : G \times X \to X$ be a separately continuous (at least) action on a uniform space (X, μ). Following [3, Ch. 4] define the injective map

$$\pi_\sharp : X \to C(G, X), \qquad \pi_\sharp(x)(g) = gx,$$

where $C(G, X)$ is the collection of continuous maps from G into X. Given a subgroup $H < G$ endow $C(H, X)$ with the uniform structure of *uniform convergence* whose basis consists of the sets of the form

$$\widetilde{\varepsilon} = \{(f, f') \in C(H, X) : (f(h), f'(h)) \in \varepsilon \text{ for all } h \in H\} \qquad (\varepsilon \in \mu).$$

We use the map $\pi_\sharp : X \to C(H, X)$ to define a uniform structure μ_H on X, as follows. For $\varepsilon \in \mu$ set

$$[\varepsilon]_H := \{(x, y) \in X \times X : (hx, hy) \in \varepsilon \text{ for all } h \in H\}.$$

The collection $\{[\varepsilon]_H : \varepsilon \in \mu\}$ is a basis for μ_H.

(^{2}) By Lemma 9.2.5 one can assume that X is only topologically transitive.

Always $\mu \subset \mu_H$ and equality occurs iff the action of H on (X,μ) is uniformly equicontinuous. If (X,μ) is metrizable and d denotes some compatible metric on X, then the corresponding μ_H is uniformly equivalent to the following metric:

$$d_H(x,x') = \sup_{g\in H} d(gx,gx').$$

REMARK 5.3. 1. It is easy to characterize μ_G for G-subsets of $\mathrm{RUC}(G)$ (e.g., for $X_f = \mathrm{cls}(f_\sharp(X)) \subset \mathrm{RUC}(G)$), where μ is the pointwise uniform structure on $\mathrm{RUC}(G)$. The corresponding μ_G is the metric uniform structure inherited from the norm of $\mathrm{RUC}(G)$.

2. The arguments of [1, Theorem 2.6] show that the uniform space (X,μ_G) is complete for every compact (not necessarily metric) G-system (X,μ).

LEMMA 5.4. *Let (X,μ) be a uniform G-space. The following conditions are equivalent*:

1. *x_0 is a point of equicontinuity of the G-space (X,μ).*
2. *x_0 is a point of continuity of the map $\pi_\sharp : X \to C(G,X)$.*
3. *x_0 is a point of continuity of the map $\mathrm{id}_X : (X,\mu) \to (X,\mu_G)$.*

Proof. Straightforward. ■

COROLLARY 5.5. *Given a compact system $(G,(X,\mu))$ (with the unique compatible uniform structure μ) the following conditions are equivalent*:

1. *$(G,(X,\mu))$ is (uniformly) equicontinuous.*
2. *$\mu_G = \mu$.*
3. *$\pi_\sharp : X \to C(G,X)$ is continuous.*
4. *μ_G is precompact.*

Proof. By Remark 5.3.2 the uniform space (X,μ_G) is complete. Thus precompact implies compact. This establishes $4 \Rightarrow 1$.

The implications $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ are trivial upon taking into account Lemma 5.4. ■

LEMMA 5.6. *The uniform structure μ_G defined above is compatible with subdirect products. More precisely*:

1. *Let G act on the uniform space (X,μ) and let Y be a G-invariant subset. Then $(\mu_G)\restriction_Y = (\mu\restriction_Y)_G$.*
2. *Let $\{(X_i,\mu_i) : i \in I\}$ be a family of uniform G-spaces. Then $(\prod_i \mu_i)_G = \prod_i (\mu_i)_G$.*

Proof. Straightforward. ■

DEFINITION 5.7. 1. Let us say that a subset K of a uniform G-space (X,μ) is *light* if the topologies induced by the uniformities μ and μ_G coincide on K. We say that X is *orbitwise light* if all orbits are light in X.

2. (X, μ) is said to be *locally equicontinuous* (LE for short) if every point $x_0 \in X$ is a point of equicontinuity of the uniform G-subspace $\mathrm{cls}(Gx_0)$. That is, for every $x_0 \in X$ and every element ε of the uniform structure μ there exists a neighborhood O of x_0 in X such that $(gx, gx_0) \in \varepsilon$ for every $g \in G$ and every $x \in O \cap \mathrm{cls}(Gx_0)$ (see [26]). It is easy to see that the latter condition, equivalently, can be replaced by the weaker condition: $x \in O \cap Gx_0$ (this explains Lemma 5.8.1 below). It follows by Lemma 5.2 that X is LE iff every point-transitive closed G-subspace of X is AE.

LEMMA 5.8. 1. *$x_0 \in X$ is a point of equicontinuity of* $\mathrm{cls}(Gx_0)$ *iff Gx_0 is light in X.*
2. *X is LE iff X is orbitwise light.*
3. *A pointed system (X, x_0) is AE iff the orbit Gx_0 is light in X.*
4. *Let $f \in \mathrm{RUC}(X)$. A subset $K \subset X_f = \mathrm{cls}(f_\sharp(X))$ is light iff the pointwise and norm topologies coincide on $K \subset \mathrm{RUC}(G)$.*

Proof. 1. Straightforward.

2. Follows directly from assertion 1.

3. X is point-transitive and AE. Therefore the nonempty set $\mathrm{Eq}(X)$ coincides with the set of transitive points (Lemma 5.2). In particular, $x_0 \in \mathrm{Eq}(X)$. Thus, Gx_0 is light in $X = \mathrm{cls}(Gx_0)$ by assertion 1.

Conversely, let Gx_0 be a light subset and x_0 be a transitive point. Then again by the first assertion $x_0 \in \mathrm{Eq}(X)$. Hence $\mathrm{Eq}(X)$ (containing Gx_0) is dense in X.

4. For the last assertion see Remark 5.3.1. ∎

Given a G-space X the collection $\mathrm{AP}(X)$ of functions in $\mathrm{RUC}(X)$ coming from equicontinuous systems is the G-invariant uniformly closed algebra of almost periodic functions, where a function $f \in C(X)$ is *almost periodic* iff the set of translates $\{L_g(f) : g \in G\}$, where $L_g(f)(x) = f(gx)$, forms a precompact subset of the Banach space $C(X)$. This happens iff X_f is norm compact iff (G, X_f) is an AP system.

A function $f \in C(X)$ is called *weakly almost periodic* (WAP for short, notation: $f \in \mathrm{WAP}(X)$) if the set of translates $\{L_g(f) : g \in G\}$ forms a weakly precompact subset of $C(X)$. We say that a dynamical system (G, X) is *weakly almost periodic* if $C(X) = \mathrm{WAP}(X)$. The classical theory shows that $\mathrm{WAP}(G)$ is a left and right G-invariant, uniformly closed, point-universal algebra containing $\mathrm{AP}(G)$ and that every minimal function in $\mathrm{WAP}(G)$ is in $\mathrm{AP}(G)$. In fact $f \in \mathrm{WAP}(X)$ iff X_f is weakly compact iff (G, X_f) is a WAP system.

The following characterization of WAP dynamical systems is due to Ellis [18] (see also Ellis and Nerurkar [20]) and is based on a result of

Grothendieck [28] (namely: pointwise compact bounded subsets in $C(X)$ are weakly compact for every compact X).

THEOREM 5.9. *Let (G, X) be a dynamical system. The following conditions are equivalent*:

1. (G, X) *is WAP.*
2. *The enveloping semigroup $E(X)$ consists of continuous maps.*

REMARK 5.10. When (G, X) is WAP the enveloping semigroup $E(X)$ is a semitopological semigroup; i.e. for each $p \in E$ both $\varrho_p : q \mapsto qp$ and $\lambda_p : q \mapsto pq$ are continuous maps. The converse holds if in addition we assume that (G, X) is point-transitive. As one can verify, the enveloping semigroup of the dynamical system described in Example 10.7 below is isomorphic to the Bohr compactification of the integers (use Proposition 2.1). In particular it is a topological group; however, the original system is not even AE and therefore not WAP as we will shortly see.

The next characterization, of AE metric systems, is due to Akin, Auslander and Berg [2].

THEOREM 5.11. *Let (G, X) be a compact metrizable system. The following conditions are equivalent*:

1. (G, X) *is almost equicontinuous.*
2. *There exists a dense G_δ subset $X_0 \subset X$ such that every member of the enveloping semigroup E is continuous on X_0.*

Combining these results Akin, Auslander and Berg [2] deduce that every compact metric WAP system is AE. Since every subsystem of a WAP system is WAP it follows from Theorems 5.9 and 5.11 that every metrizable WAP system is both AE and LE. This result is retrieved, and generalized, in [45] for all compact $\mathrm{RN}_{\mathrm{app}}$ G-systems using linear representation methods.

Note that a point-transitive LE system is of course AE but there are nontransitive LE systems which are not AE (e.g., see Remark 10.9.1 below). It was shown in [26] that the LE property is preserved under products, under passage to a subsystem and under factors $X \to Y$ provided that X is metrizable (for arbitrary systems X see Proposition 5.14 below).

Let $\mathrm{LE}(X)$ be the set of functions on a G-space X coming from LE dynamical systems. It then follows from Proposition 2.9 that $\mathrm{LE}(G)$ is a uniformly closed point-universal left and right G-invariant subalgebra of $\mathrm{RUC}(G)$ and that $\mathrm{LE}(X)$, for compact X, is the G-subalgebra of $C(X)$ that corresponds to the unique maximal LE factor of (G, X). The results and methods of [26] show that $\mathrm{WAP}(X) \subset \mathrm{LE}(X)$ and that a minimal function in $\mathrm{LE}(X)$ is almost periodic (see also Corollary 5.15.2 below).

REMARK 5.12. In contrast to the well behaved classes of WAP and LE systems, it is well known that the class of AE systems is closed neither under passage to subsystems nor under taking factors; see [25, 1] and Remark 10.9.1 below.

By Proposition 2.9 we see that for every G-space X the classes $\mathrm{AP}(X)$, $\mathrm{WAP}(X)$, $\mathrm{LE}(X)$ form G-invariant Banach subalgebras of $\mathrm{RUC}(X)$. Recall that for a topological group G we denote the greatest ambit of G by $G^{\mathrm{RUC}(G)} = G^{\mathrm{R}} = |\mathrm{RUC}(G)|$. It is well known that the maximal compactification $u_{\mathrm{R}} : G \to G^{\mathrm{R}}$ is a right topological semigroup compactification of G. We adopt the following notation. For a G-invariant closed subalgebra $\mathcal{A}$ of $\mathrm{RUC}(G)$ let $G^{\mathcal{A}}$ denote the corresponding factor $G^{\mathrm{R}} \to G^{\mathcal{A}}$, and for a G-space X and a closed G-subalgebra $\mathcal{A} \subset \mathrm{RUC}(X)$, let $X^{\mathcal{A}} = |\mathcal{A}|$ denote the corresponding factor $\beta_G X \to X^{\mathcal{A}}$.

In the next proposition we sum up some old and new observations concerning some subalgebras of $\mathrm{RUC}(X)$ and $\mathrm{RUC}(G)$.

PROPOSITION 5.13. *Let G be a topological group.*

1. *For every G-space X we have the inclusions*
$$\mathrm{RUC}(X) \supset \mathrm{LE}(X) \supset \mathrm{Asp}(X) \supset \mathrm{WAP}(X) \supset \mathrm{AP}(X),$$
and the corresponding G-factors
$$\beta_G X \to X^{\mathrm{LE}} \to X^{\mathrm{Asp}} \to X^{\mathrm{WAP}} \to X^{\mathrm{AP}}.$$
2. *For every topological group G we have the inclusions*
$$\mathrm{RUC}(G) \supset \mathrm{UC}(G) \supset \mathrm{LE}(G) \supset \mathrm{Asp}(G) \supset \mathrm{WAP}(G) \supset \mathrm{AP}(G),$$
and the corresponding G-factors
$$G^{\mathrm{R}} \to G^{\mathrm{UC}} \to G^{\mathrm{LE}} \to G^{\mathrm{Asp}} \to G^{\mathrm{WAP}} \to G^{\mathrm{AP}}.$$
3. *The compactifications G^{AP} and G^{WAP} of G are respectively a topological group and a semitopological semigroup; G^{R} and G^{Asp} are right topological semigroup compactifications of G.*

Proof. For the properties of $\mathrm{Asp}(X)$ we refer to Section 7, Theorem 7.6.6 and Lemma 9.8.2.

In order to show that $\mathrm{UC}(G) \supset \mathrm{LE}(G)$ we only have to check that $\mathrm{LUC}(G) \supset \mathrm{LE}(G)$. Let $f \in \mathrm{LE}(G)$. By the definition f comes from a point-transitive LE system (X, x_0). Therefore for some continuous function $F : X \to \mathbb{R}$ we have $f(g) = F(gx_0)$. Let μ be the natural uniform structure on X. For a given $\varepsilon > 0$ choose an entourage $\delta \in \mu$ such that $|F(x) - F(y)| < \varepsilon$ for every $(x, y) \in \delta$. Since x_0 is a point of equicontinuity we can choose a neighborhood O of x_0 such that $(gx, gx_0) \in \delta$ for every $(g, x) \in G \times O$. Now pick a neighborhood U of $e \in G$ such that $Ux_0 \subset O$. Then clearly

$|F(gux_0) - F(gx_0)| < \varepsilon$ for every $(g, u) \in G \times U$; equivalently, $|f(gu) - f(g)| < \varepsilon$. This means that $f \in \mathrm{LUC}(G)$. ■

Now we show the hereditariness of LE under factors.

PROPOSITION 5.14. *Let X be a compact LE G-system. If $\pi : X \to Y$ is a G-homomorphism then (G, Y) is LE.*

Proof. We have to show that each point y_0 in the space Y is an equicontinuity point of the subsystem $\overline{\mathcal{O}}_G(y_0)$. Fix $y_0 \in Y$ and assume, with no loss in generality, that $\overline{\mathcal{O}}_G(y_0) = Y$. Furthermore, since by Zorn's Lemma there is a subsystem of X which is minimal with the property that it projects onto Y, we may and do assume that X itself is minimal with respect to this property. Denoting by Y_0 the subset of transitive points in Y it then follows that the set $X_0 = \pi^{-1}(Y_0)$ coincides with the set of transitive points in X. Let ε be an element of the uniform structure of Y (i.e. a neighborhood of the identity in $Y \times Y$). Then the preimage $\delta := \pi^{-1}(\varepsilon)$ is an element of the uniform structure of X. Let q be a preimage of y_0. Then $q \in \mathrm{Eq}(X)$ since q is transitive and X is LE (see Lemma 5.2). Thus there exists an open neighborhood U_q of q such that $(gx, gq) \in \delta$ for all $g \in G$ and $x \in U_q$. Let V be the union of all such U_q's for q running over the preimages of y_0. Then V is an open neighborhood of $\pi^{-1}(y_0)$. Set W to be $Y \setminus \pi(X \setminus V)$. Then W is an open neighborhood of y_0 and $W \subset \pi(V)$. For any $y \in W$ we can find some preimage q of y_0 and some point $x \in U_q$ such that $\pi(x) = y$. Then $(gx, gq) \in \delta$ for all $g \in G$, which means that $(gy, gy_0) \in \varepsilon$ for all $g \in G$. Therefore $y_0 \in \mathrm{Eq}(Y)$. ■

COROLLARY 5.15. *Let G be a topological group, X a G-space and $f \in \mathrm{RUC}(X)$. Then*

1. $f \in \mathrm{LE}(X) \Leftrightarrow X_f$ *is LE.*
2. *If $f \in \mathrm{LE}(X)$ is a minimal function then $f \in \mathrm{AP}(X)$.*

Proof. 1. Use Propositions 5.14 and 2.9.3.

2. Observe that every minimal LE system is AP. ■

Our next result is an intrinsic characterization of the LE property of a function.

First recall that for the left regular action of G on $X := G$, the space X_f can be defined as the pointwise closure of the orbit Gf (Remark 2.3.1) in $\mathrm{RUC}(G)$.

DEFINITION 5.16. We say that a function $f \in \mathrm{RUC}(G)$ is

1. *light* (notation: $f \in \mathrm{light}(G)$) if the pointwise and norm topologies coincide on the orbit $Gf = \{R_g(f)\}_{g \in G} = \{f_g\}_{g \in G} \subset X_f$ (with $X := G$) as a subset of $\mathrm{RUC}(G)$;
2. *hereditarily light* (notation: $f \in \mathrm{hlight}(G)$) if the pointwise and norm topologies coincide on the orbit Gh for every $h \in X_f$.

By Lemma 5.8.4 and Definition 5.7.1, $f \in \mathrm{light}(G)$ (resp. $f \in \mathrm{hlight}(G)$) iff Gf is a light subset of the G-system X_f (resp. iff X_f is orbitwise light).

PROPOSITION 5.17. *For every topological group G and $f \in \mathrm{RUC}(G)$ we have:*

1. $\mathrm{UC}(G) \supset \mathrm{light}(G)$.
2. $f \in \mathrm{light}(G) \Leftrightarrow X_f$ *is AE.*
3. $f \in \mathrm{hlight}(G) \Leftrightarrow X_f$ *is LE.*

Proof. 1. $f \in \mathrm{light}(G)$ means that the pointwise and norm topologies coincide on Gf. It follows that the orbit map $G \to \mathrm{RUC}(G)$, $g \mapsto f_g$, is norm continuous. This means that f is also left uniformly continuous.

2. Since f is a transitive point of $X_f = \mathrm{cls}(Gf)$ we can use Lemma 5.8.3.

3. Use Lemma 5.8.2. ■

THEOREM 5.18. $\mathrm{LE}(G) = \mathrm{hlight}(G)$ *for every topological group G.*

Proof. Follows from Proposition 5.17.3 and Corollary 5.15.1. ■

REMARK 5.19.
1. By [45, Theorem 8.5], for every topological group G and every $f \in \mathrm{WAP}(G)$ the pointwise and norm topologies coincide on $fG = \{L_g(f)\}_{g\in G} = \{{}_gf\}_{g\in G}$. Using the involution
$$\mathrm{UC}(G) \to \mathrm{UC}(G), \quad f \mapsto f^* \quad (f^*(g) := f(g^{-1}))$$
(observe that $Gf^* = (fG)^*$) we get the coincidence of the above-mentioned topologies also on Gf^*. Since $(\mathrm{WAP}(G))^* = \mathrm{WAP}(G)$ we can conclude that $\mathrm{WAP}(G) \subset \mathrm{light}(G)$ for every topological group G. Theorem 5.18 provides a stronger inclusion $\mathrm{LE}(G) \subset \mathrm{light}(G)$ (since $\mathrm{WAP}(G) \subset \mathrm{LE}(G)$ by Proposition 5.13.2).
2. In view of Proposition 5.17.2 a minimal function is light iff it is AP. Thus, for example, the function $f(n) = \cos(n^2)$ on the integers, which comes from a minimal distal but not equicontinuous $\mathbb{Z}$-system on the 2-torus, is not light.

6. Fragmented maps and families. The following definition is a generalized version of *fragmentability* (implicitly it appears in a paper of Namioka and Phelps [49]) in the sense of Jayne and Rogers [33].

DEFINITION 6.1 ([42]). Let (X, τ) be a topological space and (Y, μ) a uniform space.

1. We say that X is *(τ, μ)-fragmented* by a (not necessarily continuous) function $f : X \to Y$ if for every non-empty subset A of X and every $\varepsilon \in \mu$ there exists an open subset O of X such that $O \cap A$ is non-empty and the set $f(O \cap A)$ is ε-small in Y. We also say in that case that the function f is *fragmented*. Note that it is enough to check the condition

above only for closed subsets $A \subset X$ and for $\varepsilon \in \mu$ from a *subbase* γ of μ (that is, the finite intersections of elements of γ form a base of the uniform structure μ).

2. If the condition holds only for every non-empty *open subset* A of X then we say that f is *locally fragmented*.
3. When the inclusion map $i : X \subset Y$ is (locally) fragmented we say that X is (*locally*) (τ, μ)-fragmented, or more simply, (*locally*) μ*-fragmented*.

REMARK 6.2. 1. Note that in Definition 6.1.1 when $Y = X$, $f = \mathrm{id}_X$ and μ is a metric uniform structure, we get the usual definition of fragmentability [33]. For the case of functions see also [32].

2. Namioka's *joint continuity theorem* [47] (see also Theorem 14.1 below) implies that every weakly compact subset K of a Banach space is (weak, norm)-fragmented (that is, $\mathrm{id}_K : (K, \mathrm{weak}) \to (K, \mathrm{norm})$ is fragmented).
3. Recall that a Banach space V is an *Asplund space* if the dual of every separable Banach subspace is separable, iff every bounded subset A of the dual V^* is (weak*, norm)-fragmented, iff V^* has the Radon–Nikodým property. Reflexive spaces and spaces of the type $c_0(\Gamma)$ are Asplund. For more details cf. [13, 22, 48].
4. A topological space (X, τ) is *scattered* (i.e., every non-empty subspace has an isolated point) iff X is (τ, ϱ)-fragmented, where $\varrho(x, y) = 1$ iff $x \neq y$.

Following [46] we say that $f : X \to Y$ is *barely continuous* if for every non-empty closed subset $A \subset X$, the restricted map $f\restriction_A$ has at least one point of continuity.

LEMMA 6.3. 1. *If f is (τ, μ)-continuous then X is (τ, μ)-fragmented by f.*

2. *Suppose that there exists a dense subset of (τ, μ)-continuity points of f. Then X is locally (τ, μ)-fragmented by f.*
3. *X is (τ, μ)-fragmented by f iff X is hereditarily locally fragmented by f (that is, for every closed subset $A \subset X$ the restricted function $f\restriction_A$ is (relatively) locally (τ, μ)-fragmented).*
4. *Every barely continuous f is fragmented.*
5. *Fragmentability is preserved under products. More precisely, if $f_i : (X_i, \tau) \to (Y_i, \mu_i)$ is fragmented for every $i \in I$ then the product map*
$$f := \prod_{i \in I} f_i : \prod_{i \in I} X_i \to \prod_{i \in I} Y_i$$
is (τ, μ)-fragmented with respect to the product topology τ and the product uniform structure μ.
6. *Let $\alpha : X \to Y$ be a continuous map. If $f : Y \to (Z, \mu)$ is a fragmented map then the composition $f \circ \alpha : X \to (Z, \mu)$ is also fragmented.*

Proof. Assertions 1, 2 and 6 are straightforward.

For 3 and 4 use the fact that it is enough to check the fragmentability condition only for closed subsets $A \subset X$.

The verification of 5 is straightforward if we take into account that it is enough to check the fragmentability (see Definition 6.1.1) for $\varepsilon \in \gamma$, where γ is a subbase of μ. ∎

Fragmentability has good stability properties, being closed under passage to subspaces (trivial), products (Lemma 6.3.5) and quotients. Here we include the details for quotients. The following lemma is a generalized version of [42, Lemma 4.8], which in turn was inspired by Lemma 2.1 of Namioka's paper [48].

LEMMA 6.4. *Let (X_1, τ_1) and (X_2, τ_2) be compact (Hausdorff) spaces, and let (Y_1, μ_1) and (Y_2, μ_2) be uniform spaces. Suppose that $F : X_1 \to X_2$ is a continuous surjection, $f : (Y_1, \mu_1) \to (Y_2, \mu_2)$ is uniformly continuous, and $\phi_1 : X_1 \to Y_1$ and $\phi_2 : X_2 \to Y_2$ are maps such that the diagram*

$$\begin{array}{ccc} (X_1,\tau_1) & \xrightarrow{\phi_1} & (Y_1,\mu_1) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle f} \\ (X_2,\tau_2) & \xrightarrow{\phi_2} & (Y_2,\mu_2) \end{array}$$

commutes. If X_1 is fragmented by ϕ_1 then X_2 is fragmented by ϕ_2.

Proof. We modify the proof of [42, Lemma 4.8]. In the definition of fragmentability it suffices to check the condition for *closed subsets*. So, let $\varepsilon \in \mu_2$ and let A be a non-empty closed, and hence compact, subset of X_2. Choose $\delta \in \mu_1$ such that $(f \times f)(\delta) \subset \varepsilon$. By Zorn's Lemma, there exists a minimal compact subset M of X_1 such that $F(M) = A$. Since X_1 is fragmented by ϕ_1, there exists $V \in \tau_1$ such that $V \cap M \neq \emptyset$ and $\phi_1(V \cap M)$ is δ-small. Then the set $f\phi_1(V \cap M)$ is ε-small. Consider the set $W := A \setminus F(M \setminus V)$. Then

(a) $\phi_2(W)$ is ε-small, being a subset of $f\phi_1(V \cap M) = \phi_2 F(V \cap M)$;
(b) W is relatively open in A;
(c) W is non-empty (otherwise, $M \setminus V$ is a *proper* compact subset of M such that $F(M \setminus V) = A$). ∎

The next lemma provides a key to understanding the connection between fragmentability and separability properties.

LEMMA 6.5. *Let (X, τ) be a separable metrizable space and (Y, ϱ) a pseudometric space. Suppose that X is (τ, ϱ)-fragmented by a surjective map $f : X \to Y$. Then Y is separable.*

Proof. Assume (to the contrary) that the pseudometric space (Y, ϱ) is not separable. Then there exist an $\varepsilon > 0$ and an uncountable subset H

of Y such that $\varrho(h_1, h_2) > \varepsilon$ for all distinct $h_1, h_2 \in H$. Choose a subset A of X such that $f(A) = H$ and f is bijective on A. Since X is second countable, the uncountable subspace A of X (in its relative topology) is a disjoint union of a countable set and a non-empty closed perfect set M comprising the condensation points of A (this follows from the proof of the Cantor–Bendixson theorem; see e.g. [35]). By fragmentability there exists an open subset O of X such that $O \cap M$ is non-empty and $f(O \cap M)$ is ε-small. By the property of H the intersection $O \cap M$ must be a singleton, contradicting the fact that no point of M is isolated. ■

PROPOSITION 6.6. *If X is locally fragmented by $f : X \to Y$, where (X, τ) is a Baire space and (Y, ϱ) is a pseudometric space, then f is continuous at the points of a dense G_δ subset of X.*

Proof. For a fixed $\varepsilon > 0$ consider

$$O_\varepsilon := \{\text{union of all } \tau\text{-open subsets } O \text{ of } X \text{ with } \operatorname{diam}_\varrho f(O) \le \varepsilon\}.$$

The local fragmentability implies that O_ε is dense in X. Clearly, $\bigcap\{O_{1/n} : n \in \mathbb{N}\}$ is the required dense G_δ subset of X. ■

A topological space X is *hereditarily Baire* if every closed subspace of X is a Baire space. Recall that for metrizable spaces X and Y a function $f : X \to Y$ is of *Baire class* 1 if $f^{-1}(U) \subset X$ is an F_σ subset for every open $U \subset Y$. If X is separable then a real-valued function $f : X \to \mathbb{R}$ is of Baire class 1 iff f is the pointwise limit of a sequence of continuous functions (see e.g. [35]).

PROPOSITION 6.7. *Let (X, τ) be a hereditarily Baire* (*e.g., Polish or compact*) *space, and (Y, ϱ) a pseudometric space. Consider the following assertions*:

(a) *X is (τ, ϱ)-fragmented by $f : X \to Y$*;
(b) *f is barely continuous*;
(c) *f is of Baire class* 1.

Then:

1. (a)⇔(b).
2. *If X is Polish and Y is a separable metric space then* (a)⇔(b)⇔(c).

Proof. For (a)⇔(b) combine Lemma 6.3 and Proposition 6.6.

The equivalence (b)⇔(c) for Polish X and separable Y is well known (see [35, Theorem 24.15]) and actually goes back to Baire. ■

The following new definition will play a crucial role in Section 14.

DEFINITION 6.8. 1. We say that a family of functions $\mathcal{F} = \{f : (X, \tau) \to (Y, \mu)\}$ is *fragmented* if the condition of Definition 6.1.1 holds simultaneously for all $f \in \mathcal{F}$. That is, $f(O \cap A)$ is ε-small for every $f \in \mathcal{F}$. It is equivalent to say that the mapping

$$\pi_\sharp : X \to Y^{\mathcal{F}}, \quad \pi_\sharp(x)(f) = f(x),$$

is (τ, μ_{U})-fragmented, where μ_{U} is the uniform structure of uniform convergence on the set $Y^{\mathcal{F}}$ of all mappings from $\mathcal{F}$ into (Y, μ).

2. Analogously one can define the notions of a *locally fragmented family* and a *barely continuous family*. The latter means that every closed non-empty subset $A \subset X$ contains a point $a \in A$ such that $\mathcal{F}_A = \{f\restriction_A : f \in \mathcal{F}\}$ is equicontinuous at a. If μ is pseudometrizable then so is μ_{U}. Therefore if in addition (X, τ) is hereditarily Baire then it follows by Proposition 6.7.1 that $\mathcal{F}$ is fragmented iff $\mathcal{F}$ is barely continuous.

Fragmented families, like equicontinuous families, are stable under pointwise closures as the following lemma shows.

LEMMA 6.9. *Let $\mathcal{F} = \{f : (X, \tau) \to (Y, \mu)\}$ be a fragmented family of functions. Then the pointwise closure $\overline{\mathcal{F}}$ of $\mathcal{F}$ in Y^X is also a (τ, μ)-fragmented family.*

Proof. Use a straightforward “3ε-trick” argument. ■

7. Asplund functions and RN systems. Let H be a subgroup of G. Recall that we denote by μ_H the uniform structure on the uniform G-space (X, μ) inherited by the inclusion $\pi_\sharp : X \to C(H, X)$. Precisely, μ_H is generated by the basis $\{[\varepsilon]_H : \varepsilon \in \mu\}$, where

$$[\varepsilon]_H := \{(x, y) \in X \times X : (hx, hy) \in \varepsilon \text{ for all } h \in H\}.$$

For every $f \in C(X)$ and $H < G$ denote by $\varrho_{H,f}$ the pseudometric on X defined by

$$\varrho_{H,f}(x, y) = \sup_{h \in H} |f(hx) - f(hy)|.$$

Then $\mu_{\mathrm{cls}(H)} = \mu_H$ and $\varrho_{\mathrm{cls}(H),f} = \varrho_{H,f}$.

DEFINITION 7.1. 1. A continuous function $f : X \to \mathbb{R}$ on the compact G-space X is an *Asplund function* [45] if for every countable subgroup $H \subset G$ the pseudometric space $(X, \varrho_{H,f})$ is separable. It is an *s-Asplund function* (notation: $f \in \mathrm{Asp_s}(X)$) when $(X, \varrho_{G,f})$ is separable. A pseudometric d on a set X is called *Asplund* (respectively, *s-Asplund*) if for every countable subgroup $H < G$ (respectively, for $H = G$) the pseudometric space (X, d_H) is separable, where

$$d_H(x, y) = \sup_{h \in H} d(hx, hy).$$

2. More generally, we say that a function $f \in \mathrm{RUC}(X)$ on a (not necessarily compact) G-space X is an *Asplund function* (notation: $f \in \mathrm{Asp}(X)$) if f comes (in the sense of Definition 1.1) from an Asplund function F on a G-system Y and a G-compactification $\nu : X \to Y$. By Remark 7.2.2 below, equivalently, one can take each of the following G-compactifications (see Remark 2.3.3): $f_\sharp : X \to X_f$ (minimal possible) or $i_\beta : X \to \beta_G X$ (maximal). Analogously we define the class $\mathrm{Asp_s}(X)$ of s-Asplund functions on a G-space X.
3. In particular, a function $f \in \mathrm{RUC}(G)$ is an *Asplund function* (*s-Asplund function*) if it is Asplund (s-Asplund) for the G-space $X := G$ with respect to the regular left action. Notation: $f \in \mathrm{Asp}(G)$ (resp. $f \in \mathrm{Asp_s}(G)$).

REMARK 7.2. 1. Note that in the definition of Asplund functions $F : X \to \mathbb{R}$, equivalently, H can run over all uniformly Lindelöf subgroups of G. Indeed, as in the proof of Proposition 4.2, the orbit $FH = \{_hF\}_{h\in H}$ is norm separable. Let $K < G$ be a countable subgroup of H such that FK is dense in FH. Then $\varrho_{H,F} = \varrho_{K,F}$.

2. Let $q : Y_1 \to Y_2$ be a G-homomorphism of compact G-spaces. It is straightforward to show that a continuous bounded function $F : Y_2 \to \mathbb{R}$ is Asplund (resp. s-Asplund) iff the function $f = F \circ q : Y_1 \to \mathbb{R}$ is Asplund (resp. s-Asplund).
3. Of course every s-Asplund function is Asplund. If G, or the natural restriction $\breve{G}$, is uniformly Lindelöf (e.g. $\breve{G}$ is second countable if X is compact and metrizable) then clearly the converse is also true. Thus in this case $\mathrm{Asp}(X) = \mathrm{Asp_s}(X)$.
4. Let (G, X) be a dynamical system and d a pseudometric on X. Suppose $F : X \to \mathbb{R}$ is d-uniformly continuous. If d is Asplund or s-Asplund then so is F.

Let X be a G-space. By Proposition 2.2.1, $X_f := \mathrm{cls}(f_\sharp(X))$ is a subset of $\mathrm{RUC}(G)$ for every $f \in \mathrm{RUC}(X)$. Let $r_G : X_f \hookrightarrow \mathrm{RUC}(G)$ be the inclusion map. For every subgroup $H < G$ we can define the natural restriction operator $q_H : \mathrm{RUC}(G) \to \mathrm{RUC}(H)$. Denote by $r_H := q_H \circ r_G : X_f \to \mathrm{RUC}(H)$ the composition and let $\xi_{H,f}$ be the corresponding pseudometric induced on X_f by the norm of $\mathrm{RUC}(H)$. Precisely,

$$\xi_{H,f}(\omega, \omega') = \sup_{h\in H} |\omega(h) - \omega'(h)|.$$

Finally, define the composition $f_\sharp^H := r_H \circ f_\sharp : X \to \mathrm{RUC}(H)$. The corresponding pseudometric induced by $f_\sharp^H$ on X is just $\varrho_{H,f}$.

LEMMA 7.3. *Let X be a G-space and $f \in \mathrm{RUC}(X)$. Let $F_e : X_f \to \mathbb{R}$ be the map $F_e(\omega) = \omega(e)$ (defined before Proposition* 2.2). *The following are equivalent*:

1. $f \in \mathrm{Asp}(X)$.
2. $F_e \in \mathrm{Asp}(X_f)$.
3. $(X_f, \xi_{H,f})$ *is separable for every countable* (*uniformly Lindelöf*) *subgroup $H < G$.*
4. $r_H(X_f)$ *is norm separable in* $\mathrm{RUC}(H)$ *for every countable* (*uniformly Lindelöf*) *subgroup $H < G$.*

Proof. $1 \Leftrightarrow 2$ follows by Definition 7.1.2, Remark 7.2.2 and Proposition 2.2.3.

$3 \Leftrightarrow 4$ is clear by the definitions of $\xi_{H,f}$ and r_H.

$2 \Leftrightarrow 3$: $F_e \in \mathrm{Asp}(X_f)$ means, by Definition 7.1.1, that for every countable (uniformly Lindelöf) subgroup $H < G$ the pseudometric space (X_f, ϱ_{H,F_e}) is separable, where

$$\varrho_{H,F_e}(\omega,\omega') = \sup_{h\in H} |F_e(h\omega) - F_e(h\omega')|.$$

Recall that by the definition of $F_e : X_f \to \mathbb{R}$ we have $F_e(h\omega) = (h\omega)(e) = \omega(h)$. Hence

$$\xi_{H,f}(\omega,\omega') = \sup_{h\in H} |\omega(h) - \omega'(h)| = \sup_{h\in H} |F_e(h\omega) - F_e(h\omega')| = \varrho_{H,F_e}(\omega,\omega').$$

Therefore the pseudometrics $\xi_{H,f}$ and ϱ_{H,F_e} coincide on X_f. This clearly completes the proof. ∎

COROLLARY 7.4. *Let X be a G-space and $f \in \mathrm{RUC}(X)$. The following are equivalent*:

1. $f \in \mathrm{Asp_s}(X)$.
2. $F_e \in \mathrm{Asp_s}(X_f)$.
3. X_f *is norm separable in* $\mathrm{RUC}(G)$.

Proof. The proof of Lemma 7.3 shows that in fact $\xi_{H,f}$ and ϱ_{H,F_e} coincide on X_f for every $H < G$. Consider the particular case of $H := G$ taking into account that $r_G(X_f) = X_f$. ∎

The following definition of RN dynamical systems (a natural generalization of RN compacta in the sense of Namioka [48]) and Eberlein systems (a natural generalization of Eberlein compacta in the sense of Amir–Lindenstrauss [4]) were introduced in [45]. For the definition and properties of Asplund spaces see Remark 6.2.3 and [13, 48, 22].

DEFINITION 7.5. Let (G, X) be a compact dynamical system.

1. A continuous (*proper*) *representation* of (G, X) on a Banach space V is a pair (h, α), where $h : G \to \mathrm{Iso}(V)$ is a strongly continuous co-homomorphism of topological groups and $\alpha : X \to V^*$ is a weak*-

continuous bounded G-mapping (resp. embedding) (with respect to the dual action $G \times V^* \to V^*$, $(g\varphi)(v) := \varphi(h(g)(v))$).

2. (G, X) is a *Radon–Nikodým system* (RN for short) if there exists a proper representation of (G, X) on an Asplund Banach space V. If we can choose V to be reflexive, then (G, X) is called an *Eberlein system*. The classes of Radon–Nikodým and Eberlein compact systems will be denoted by RN and Eb respectively.
3. (G, X) is called an RN-*approximable* system ($\mathrm{RN}_{\mathrm{app}}$) if it can be represented as a subdirect product (or equivalently, as an inverse limit) of RN systems.

Note that compact spaces which are not RN are necessarily non-metrizable, while there are many natural *metric* compact G-systems which are not RN.

The next theorem collects some useful properties which were obtained recently in [45].

THEOREM 7.6. *Let (G, X) be a compact G-system.*

1. *X is WAP iff X is a subdirect product of Eberlein G-systems. A metric system X is WAP iff X is Eberlein.*
2. *The system (G, X) is RN iff there exists a representation (h, α) of (G, X) on a Banach space V such that: $h : G \to \mathrm{Iso}(V)$ is a co-homomorphism (no continuity assumptions on h), $\alpha : X \to V^*$ is a bounded weak* G-embedding and $\alpha(X)$ is (weak*, norm)-fragmented.*
3. *$f : X \to \mathbb{R}$ is an Asplund function iff f arises from an* Asplund representation (*that is, there exists a continuous representation (h, α) of (G, X) on an Asplund space V, such that $f(x) = \alpha(x)(v)$ for some $v \in V$), or equivalently, iff f comes from an RN (or* $\mathrm{RN}_{\mathrm{app}}$) *$G$-factor Y of X.*
4. *The system (G, X) is* $\mathrm{RN}_{\mathrm{app}}$ *iff* $\mathrm{Asp}(X) = C(X)$.
5. RN *is closed under countable products and* $\mathrm{RN}_{\mathrm{app}}$ *is closed under quotients. For metric compact systems* $\mathrm{RN}_{\mathrm{app}} = \mathrm{RN}$ *holds.*
6. $\mathrm{Asp}(X)$ *is a closed G-invariant subalgebra of $C(X)$ containing* $\mathrm{WAP}(X)$. *The canonical compactification $u_{\mathrm{A}} : G \to G^{\mathrm{Asp}}$ is the universal* $\mathrm{RN}_{\mathrm{app}}$ *compactification of G. Moreover, u_{A} is a right topological semigroup compactification of G.*
7. *(G, X) is RN iff $(G, (C(X)^*_1, weak^*))$ is RN iff $(G, P(X))$ is RN, where $P(X)$ denotes the space of all probability measures on X (with the induced action of G).*

The proofs of assertions 1, 2 and 3 use several ideas from Banach space theory; mainly the notion of Asplund sets and Stegall's generalization of a factorization construction by Davis, Figiel, Johnson and Pełczyński [14, 13, 48, 52, 22].

PROPOSITION 7.7. *Let G be an arbitrary topological group. Then $(G^{\mathrm{Asp}}, u_{\mathrm{A}}(e))$ is point-universal (hence $X_f \subset \mathrm{Asp}(G)$ for every $f \in \mathrm{Asp}(G)$).*

Proof. $\mathcal{P} := \mathrm{Asp}(G)$ is an algebra of functions coming from $\mathrm{RN}_{\mathrm{app}}$ systems. Since the class $\mathrm{RN}_{\mathrm{app}}$ is preserved by products and subsystems we can apply Proposition 2.9.2. ■

Let (X, τ) be a topological space. As usual, a metric ϱ on the set X is said to be *lower semicontinuous* if the set $\{(x, y) : \varrho(x, y) \leq t\}$ is closed in $X \times X$ for each $t > 0$. A typical example is any subset $X \subset V^*$ of a dual Banach space equipped with the weak* topology and the norm metric. It turns out that every lower semicontinuous metric on a compact Hausdorff space X arises in this way (Lemma 7.8.1). This important result has been established in [31] using ideas of Ghoussoub and Maurey.

LEMMA 7.8. 1 ([31]). *Let (X, τ) be a compact space and let $\varrho \leq 1$ be a lower semicontinuous metric on (X, τ). Then there is a dual Banach space V^* and a homeomorphic embedding $\alpha : (X, \tau) \to (V_1^*, weak^*)$ such that*

$$\|\alpha(x) - \alpha(y)\| = \varrho(x, y)$$

for all $x, y \in X$.

2. *If in addition X is a G-space and ϱ is G-invariant, then assertion* 1 *admits a G-generalization. More precisely, there is a linear isometric (not necessarily jointly continuous) right action $V \times G \to V$ such that $\alpha : X \to V_1^*$ is a G-map.*

Proof. 2. As in the proof of [31, Theorem 2.1] the required Banach space V is defined as the space of all continuous real-valued functions f on (X, τ) which satisfy a uniform Lipschitz condition of order 1 with respect to ϱ, endowed with the norm

$$p(f) = \max\{\|f\|_{\mathrm{Lip}}, \|f\|\},$$

where $\|f\| = \sup\{|f(x)| : x \in X\}$ and the seminorm $\|f\|_{\mathrm{Lip}}$ is defined to be the least constant K such that $|f(x_1) - f(x_2)| \leq K\varrho(x_1, x_2)$ for all $x_1, x_2 \in X$. Then $\alpha : (X, \tau) \to (V_1^*, \mathrm{weak}^*)$ is defined by $\alpha(x)(f) = f(x)$.

Define now the natural right action $\pi : V \times G \to V$ by $\pi(f, g) = fg = {}_g f$, where ${}_g f(x) := f(gx)$. Then clearly $p(fg) = p(f)$ and $\alpha : X \to V_1^*$ is a G-map. ■

THEOREM 7.9. *Let (G, X) be a compact dynamical system. The following conditions are equivalent*:

1. *(G, X) is RN.*
2. *X is fragmented with respect to some bounded lower semicontinuous G-invariant metric ϱ.*

Proof. $1 \Rightarrow 2$: Our G-system X, being RN, is a G-subsystem of the ball $V_1^* = (V_1^*, \text{weak}^*)$ for some Asplund space V. By a well known characterization of Asplund spaces, V_1^* is $(\text{weak}^*, \text{norm})$-fragmented. Hence, X is also fragmented by the lower semicontinuous G-invariant metric $\varrho(x_1, x_2) = \|x_1 - x_2\|$ on X, inherited from the norm of V^*.

$2 \Rightarrow 1$: We can suppose that $\varrho \leq 1$. Using Lemma 7.8.1 we can find a Banach space V and a weak* embedding $\alpha : (X, \tau) \to V_1^*$ such that α is (ϱ, norm)-isometric. Since X is (τ, ϱ)-fragmented, $\alpha(X) \subset V_1^*$ is $(\text{weak}^*, \text{norm})$-fragmented. Moreover, by Lemma 7.8.2, there exists a co-homomorphism (without continuity assumptions) $h : G \to \text{Iso}(V)$ (the right action $V \times G \to V$ leads to the co-homomorphism h) such that the map $\alpha : X \to V_1^*$ is G-equivariant with respect to the dual action of G on V^* defined by $(g\varphi)(v) := \varphi(h(g)(v))$. Therefore we get a representation (h, α) of (G, X) on V such that $\alpha(X) \subset V_1^*$ is $(\text{weak}^*, \text{norm})$-fragmented. By Theorem 7.6.2 we deduce that the G-system (X, τ) is RN. ∎

8. Veech functions. The algebra $K(G)$ was defined by Veech in [58], for a discrete group G, as the algebra of functions $f \in \ell^\infty(G)$ such that for every countable subgroup $H < G$ the collection $X_{f\restriction_H} = \overline{\mathcal{O}}_H(\eta_0) \subset \Omega_H = [-\|f\|, \|f\|]^H$, with $\eta_0 = f\restriction_H$, considered as a subspace of the Banach space $\ell^\infty(H)$, is norm separable. Replacing $\ell^\infty(G)$ and $\ell^\infty(H)$ by $\text{RUC}(G)$ and $\text{RUC}(H)$, respectively, we define, for any topological group G, the algebra $K(G) \subset \text{RUC}(G)$ as follows.

Definition 8.1. Let G be a topological group. We say that a function $f \in \text{RUC}(G)$ is a *Veech function* if for every countable (equivalently: separable) subgroup $H < G$ the corresponding H-dynamical system $(H, X_{f\restriction_H}, \eta_0)$, when considered as a subspace of the Banach space $\text{RUC}(H)$ (see Proposition 2.4.4), is norm separable (that is, $r_H(X_{f\restriction_H}) \subset \text{RUC}(H)$ is separable; see the definitions before Lemma 7.3). We denote by $K(G)$ the collection of Veech functions in $\text{RUC}(G)$.

Theorem 8.2. *For any topological group G we have*:

1. *$K(G)$ is a closed left G-invariant subalgebra of* $\text{RUC}(G)$.
2. *The algebra $K(G)$ is point-universal.*
3. $\text{Asp}(G) \subset K(G)$.
4. $K(G) = \text{Asp}(G) = \text{Asp}_s(G)$ *for every separable G.*

Proof. 1. For every $f \in K(G)$ let (G, X_f, f) be the corresponding pointed dynamical system as constructed in Proposition 2.4. If f_i, $i = 1, 2$, are in $K(G)$ and $H < G$ is a countable subgroup then the subsets $X_{f_i\restriction_H}$, $i = 1, 2$, are norm separable in $\text{RUC}(H)$ and therefore so is $X = \{\omega + \eta : \omega \in X_{f_1\restriction_H},\ \eta \in X_{f_2\restriction_H}\}$. Since $X_{(f_1+f_2)\restriction_H} \subset X$ it follows that $f_1 + f_2 \in K(G)$. Likewise $f_1 \cdot f_2 \in K(G)$, and we conclude that $K(G)$ is a subalgebra. Uniformly convergent countable sums are treated similarly and it follows that $K(G)$ is uniformly closed. The left G-invariance is clear.

2. Given $f \in K(G)$ one shows, as in [58, Lemma 3.4], that every element $\omega \in X_f$ is also in $K(G)$. Now use Proposition 2.8.

3. By Lemma 7.3, a function $f \in \mathrm{RUC}(G)$ is Asplund iff $r_H(X_f)$ is norm separable in $\mathrm{RUC}(H)$ for every countable subgroup $H < G$. Consider $\mathrm{cls}(Hf)$, the H-orbit closure in X_f (for $f \in X_f = \mathrm{cls}(Gf)$). Then $r_H(\mathrm{cls}(Hf))$ is also separable in $\mathrm{RUC}(H)$. On the other hand, it is easy to see that the set $r_H(X_{f\restriction_H})$ coincides with $r_H(\mathrm{cls}(Hf))$. Hence, $r_H(X_{f\restriction_H})$ is also separable in $\mathrm{RUC}(H)$. This exactly means that $f \in K(G)$.

4: Let $f \in K(G)$. Then the collection $X_{f\restriction_H}$ is norm separable for every separable subgroup $H < G$. In particular, X_f (for $H := G$) is norm separable. Now by Corollary 7.4 we can conclude that $f \in \mathrm{Asp_s}(G)$. ■

9. Hereditary AE and NS systems. We begin with a generalized version of sensitivity. The *functional version* (Definition 9.1.3) will be convenient in the proof of Theorem 14.2.

DEFINITION 9.1. 1. The uniform G-space (X,μ) has *sensitive dependence on initial conditions* (or simply, is *sensitive*) if there exists an $\varepsilon \in \mu$ such that for every $x \in X$ and any neighborhood U of x there exists $y \in U$ and $g \in G$ such that $(gx, gy) \notin \varepsilon$ (for metric cascades see for example [9, 16, 25]). Thus a (metric) G-space (X,μ) is *non-sensitive*, NS for short, if for every $(\varepsilon > 0)$ $\varepsilon \in \mu$ there exists an open non-empty subset O of X such that gO is ε-small in (X,μ) for all $g \in G$, or equivalently, O is $[\varepsilon]_G$-small in (X,μ_G) (respectively: whose d_G-diameter is less than ε, where d is the metric on X and as usual $d_G(x,x') = \sup_{g\in G} d(gx, gx')$).

2. We say that (G,X) is *hereditarily non-sensitive* (HNS for short) if every non-empty closed G-subspace A of X is not sensitive.

3. More generally, we say that a map $f : (X,\tau) \to (Y,\mu)$ is *not sensitive* if there exists an open non-empty subset O of X such that $f(gO)$ is ε-small in (Y,μ) for every $g \in G$. The function f is *hereditarily non-sensitive* if for every closed G-subspace A of X the restricted function $f\restriction_A : A \to (Y,\mu)$ is not sensitive. Using these notions we can define the classes of NS and HNS functions. Observe that (X,μ) is NS iff the map $\mathrm{id}_X : (X, \mathrm{top}(\mu)) \to (X,\mu)$ is NS.

Let (X,μ) be a uniform G-space and $\varepsilon \in \mu$. Define Eq_ε as the union of all non-empty $\mathrm{top}(\mu)$-open $[\varepsilon]_G$-small subsets in X. More precisely,

$$\mathrm{Eq}_\varepsilon := \bigcup\{U \in \mathrm{top}(\mu) : (gx, gx') \in \varepsilon \text{ for all } (x,x',g) \in U \times U \times G\}.$$

Then Eq_ε is an open G-invariant subset of X and $\mathrm{Eq}(X) = \bigcap\{\mathrm{Eq}_\varepsilon : \varepsilon \in \mu\}$.

LEMMA 9.2. *Let (X, μ) be a uniform G-space.*

1. *X is NS if and only if* $\mathrm{Eq}_\varepsilon \neq \emptyset$ *for every* $\varepsilon \in \mu$. *Therefore, if* $\mathrm{Eq}(X) \neq \emptyset$ *then* (X, μ) *is NS.*
2. *X is locally μ_G-fragmented iff* Eq_ε *is dense in X for every* $\varepsilon \in \mu$. *Thus, if X is locally μ_G-fragmented then X is NS.*
3. *If X is NS then* $\mathrm{Eq}(X) \supset \mathrm{Trans}(X)$.
4. *If X is NS and topologically transitive then* $\mathrm{Eq}(X) = \mathrm{Trans}(X)$ *and so X is point-transitive iff* $\mathrm{Eq}(X) \neq \emptyset$.
5. *If* $\mathrm{Eq}(X) \neq \emptyset$ *and X is topologically transitive then* $\mathrm{Eq}(X) = \mathrm{Trans}(X)$.

Proof. The first two assertions are trivial.

3. If X is NS then Eq_ε is not empty for every $\varepsilon \in \mu$. Any transitive point is contained in any non-empty invariant open subset of X. In particular, $\mathrm{Trans}(X) \subset \mathrm{Eq}_\varepsilon$. Hence, $\mathrm{Trans}(X) \subset \bigcap\{\mathrm{Eq}_\varepsilon : \varepsilon \in \mu\} = \mathrm{Eq}(X)$.

4. By assertion 3 it now suffices to show that if X is topologically transitive then $\mathrm{Eq}(X) \subset \mathrm{Trans}(X)$. Let $x_0 \in \mathrm{Eq}(X)$, $y \in X$ and let $\varepsilon \in \mu$. We have to show that the orbit Gx_0 intersects the ε-neighborhood $\varepsilon(y) := \{x \in X : (x, y) \in \varepsilon\}$ of y. Choose $\delta \in \mu$ such that $\delta \circ \delta \subset \varepsilon$. Since $x_0 \in \mathrm{Eq}(X)$ there exists a neighborhood U of x_0 such that $(gx_0, gx) \in \delta$ for every $(x, g) \in U \times G$. Since X is topologically transitive we can choose $g_0 \in G$ such that $g_0U \cap \delta(y) \neq \emptyset$. This implies that $(g_0x, y) \in \delta$ for some $x \in U$. Then $(g_0x_0, y) \in \delta \circ \delta \subset \varepsilon$.

5. Combine assertions 1 and 4. ∎

COROLLARY 9.3. *A weakly mixing NS system is trivial.*

Proof. Let (G, X) be a weakly mixing NS system. Let ε be a neighborhood of the diagonal and choose a symmetric neighborhood of the diagonal δ with $\delta \circ \delta \circ \delta \subset \varepsilon$. By the NS property and Lemma 9.2.1, Eq_δ is non-empty. Thus there exists a non-empty open subset $U \subset X$ such that $W = \bigcup_{g \in G} gU \times gU \subset \delta$. By weak mixing the open invariant set W is dense in $X \times X$ and hence $X \times X \subset \varepsilon$. Since ε is arbitrary we conclude that X is trivial. ∎

Next we provide some useful results which link our dynamical and topological definitions (and involve fragmentability and sensitivity).

LEMMA 9.4. 1. *Let $f : X \to Y$ be a G-map from a topological G-space (X, τ) into a uniform G-space (Y, μ). Then the following are equivalent:*

(a) $f : (X, \tau) \to (Y, \mu)$ *is HNS.*
(b) $f : (X, \tau) \to (Y, \mu_G)$ *is fragmented.*
(c) $f : (A, \tau\restriction_A) \to (Y, \mu_G)$ *is locally fragmented for every closed non-empty G-subset A of X.*

2. *(X,μ) is HNS iff* $\mathrm{id}_X : (X,\tau) \to (X,\mu_G)$ *is fragmented.*
3. HAE $\subset$ HNS.

Proof. 1. (a)$\Rightarrow$(b): Suppose that $f : (X,\tau) \to (Y,\mu)$ is HNS. We have to show that f is (τ,μ_G)-fragmented. Let A be a non-empty subset of X and $[\varepsilon]_G \in \mu_G$. Consider the closed G-subspace $Z := \mathrm{cls}(GA)$ of X. Then by our assumption $f\restriction_Z : Z \to (Y,\mu)$ is NS. Hence there exists a relatively open non-empty subset $W \subset Z$ such that $(f(gx), f(gy)) = (gf(x), gf(y)) \in \varepsilon$ for every $(g,x,y) \in G \times W \times W$. Therefore, $f(W)$ is $[\varepsilon]_G$-small. Since GA is dense in Z, the intersection $W \cap GA$ is non-empty. There exists $g_0 \in G$ such that $g_0^{-1}W \cap A \neq \emptyset$. On the other hand, clearly, $f(g_0^{-1}W)$ is also $[\varepsilon]_G$-small. Thus the same is true for $f(g_0^{-1}W \cap A)$.

(b)$\Rightarrow$(c): This is trivial by Definition 6.1.

(c)$\Rightarrow$(a): Let A be a closed non-empty G-subspace of X and $\varepsilon \in \mu$. Take a non-empty open subset O of the space A (say, $O = A$). Since $f : A \to (Y,\mu_G)$ is locally fragmented one can choose a non-empty open subset $U \subset O$ such that $f(U)$ is $[\varepsilon]_G$-small in Y. This means, in particular, that $f\restriction_A : A \to (Y,\mu)$ is NS for every closed G-subspace A. Hence, f is HNS.

2. This is a particular case of the first assertion for $f = \mathrm{id}_X : (X,\mu) \to (X,\mu)$.

3. Let (G,X) be HAE. For every closed non-empty G-subsystem A there exists a point of equicontinuity of (G,A). By Lemma 9.2.1, (G,A) is NS. Therefore, (G,X) is HNS. ∎

PROPOSITION 9.5. *Let X be a compact G-system with its unique uniform structure μ. Consider the following conditions*:

(a) *X is AE.*
(b) *X is locally μ_G-fragmented.*
(c) *X is NS.*

Then we have:

1. *Always*, (a)$\Rightarrow$(b)$\Rightarrow$(c).
2. *If μ_G is metrizable (e.g., if μ is metrizable) then* (a)$\Leftrightarrow$(b)$\Rightarrow$(c).
3. *If X is point-transitive then* (a)$\Leftrightarrow$(b)$\Leftrightarrow$(c).
4. *If X is topologically transitive then* (a)$\Rightarrow$(b)$\Leftrightarrow$(c).

Proof. 1. (a)$\Rightarrow$(b): Let U be a non-empty open subset of X and $\varepsilon \in \mu$. Since X is AE we can choose a point $x_0 \in \mathrm{Eq}(X) \cap U$. Now we can pick an open neighborhood $O \subset U$ of x_0 such that $(gx, gx') \in \varepsilon$ for every $g \in G$ and $x, x' \in O$. Therefore, $(x,x') \in [\varepsilon]_G$. This proves that X is locally μ_G-fragmented.

(b)$\Rightarrow$(c): Trivial by Lemma 9.2.2.

2. (a)⇐(b): If μ_G is metrizable then Proposition 6.6 guarantees that $\mathrm{id}_X : (X, \mu) \to (X, \mu_G)$ is continuous at the points of a dense G_δ subset (say, Y) of X. By Lemma 5.4, $Y \subset \mathrm{Eq}(X)$. Hence, $\mathrm{Eq}(X)$ is also dense in X. Therefore, X is AE.

3. (c)⇒(a): Observe that $\mathrm{Trans}(X) \subset \mathrm{Eq}(X)$ by Lemma 9.2.3.

4. (b)⇐(c): Since X is NS the subset Eq_ε is non-empty for every $\varepsilon \in \mu$ (Lemma 9.2.1). Since the open set Eq_ε is invariant and X is topologically transitive we see that Eq_ε is dense for every $\varepsilon \in \mu$. By Lemma 9.2.2 this means that X is locally μ_G-fragmented. ∎

The equivalence of AE and NS for transitive metric systems is shown in [25, 1]. The referee proposed the following problem. Does there exist a topologically transitive NS system which is not point-transitive? That is, can it happen for a topologically transitive system that every Eq_ε is dense but the intersection Eq is empty?

COROLLARY 9.6. *For every topological group G and $f \in \mathrm{RUC}(G)$ the following are equivalent*:

1. $f \in \mathrm{light}(G)$.
2. X_f *is AE.*
3. X_f *is locally norm fragmented* (*with respect to the norm of* $\mathrm{RUC}(G)$).
4. X_f *is NS.*

Proof. Use Propositions 9.5.3 and 5.17.2. It should be noted here that if μ is the natural pointwise uniform structure on $X_f = \mathrm{cls}(Gf) \subset \mathrm{RUC}(G)$ then the norm of $\mathrm{RUC}(G)$ induces on X_f the uniform structure μ_G (Remark 5.3.1). ∎

LEMMA 9.7. HNS *is closed under quotients of compact G-systems.*

Proof. Let $f : X \to Y$ be a G-quotient. Denote by μ_X and μ_Y the original uniform structures on X and Y respectively. Assume that X is HNS, or equivalently (see Lemma 9.4.2), that X is $(\mu_X)_G$-fragmented. Since $f : (X, \mu_X) \to (Y, \mu_Y)$ is uniformly continuous, it is easy to see that so is the G-map $f : (X, (\mu_X)_G) \to (Y, (\mu_Y)_G)$. We can now apply Lemma 6.4. It follows that Y is $(\mu_Y)_G$-fragmented. Hence, Y is HNS (use again Lemma 9.4.2). ∎

Note that the class NS is not closed under quotients (see [25]).

LEMMA 9.8. 1. *Every RN compact G-system X is HAE. In particular, such a system is always LE and HNS.*

2. $\mathrm{Asp}(X) \subset \mathrm{LE}(X)$ *for every G-space X.*

Proof. 1. By Definition 7.5 there exists a representation (h, α) of (G, X) on an Asplund space V such that $h : G \to \mathrm{Iso}(V)$ is a co-homomorphism and $\alpha : (X, \tau) \to (V^*, \mathrm{weak}^*)$ is a bounded weak* G-embedding. Since V is Asplund, it follows that $\alpha(X)$ is (weak*, norm)-fragmented. The map id_X :

$(X, \tau) \to (X, \text{norm})$ has a dense subset of points of continuity by Proposition 6.6. The norm induces on X the metric uniform structure which majorizes the original uniform structure μ on X. On the other hand the norm is G-invariant. It follows that every point of continuity of $\text{id}_X : (X, \mu) \to (X, \text{norm})$ is a point of equicontinuity for the system (G, X). Clearly, the same is true for every restriction on a closed G-invariant non-empty subset Y of X. Hence X is HAE. Then clearly X is LE (see Definition 5.7.2). Lemma 9.4.3 implies that X is also HNS.

2. Use the first assertion and Theorem 7.6.3 (taking into account Definition 7.1.2). ∎

In the following theorem we show that the classes HNS and RN_{app} coincide. Loosely speaking, we can rephrase this by saying that a compact G-system X admits sufficiently many good (namely: Asplund) representations if and only if X is "non-chaotic".

THEOREM 9.9. *For a compact G-space X (with its unique compatible uniform structure μ) the following are equivalent*:

1. X *is* RN_{app}.
2. X *is HNS.*
3. $\pi_\sharp : X \to C(G, X)$ *is a fragmented map.*
4. $\breve{G} = \{\breve{g} : X \to X\}_{g \in G}$ *is a fragmented family.*
5. (X, μ_H) *is uniformly Lindelöf for every countable* (*equivalently*, *uniformly Lindelöf*) *subgroup* $H < G$.

Proof. $1 \Rightarrow 2$: The first assertion means that (X, μ) is a subdirect product of a collection X_i of RN G-systems (with the uniform structure μ_i). By Lemma 9.8.1 every X_i is HNS. Lemma 9.4.2 guarantees that each X_i is $(\mu_i)_G$-fragmented. Then X is μ_G-fragmented. Indeed, this follows by Lemma 5.6 and the fact that fragmentability is closed under passage to products (Lemma 6.3.5) and subspaces. Now, by Lemma 9.4.2, X is HNS.

$2 \Leftrightarrow 3$: $\pi_\sharp : X \to C(G, X)$ is fragmented iff X is μ_G-fragmented. Hence, we can use Lemma 9.4.2.

$3 \Leftrightarrow 4$: See Definition 6.8.1.

$2 \Rightarrow 5$: Let $X \in$ HNS and $H < G$ be a uniformly Lindelöf subgroup. We have to show that (X, μ_H) is uniformly Lindelöf. The system (H, X) (being m-approximable by Proposition 4.1) is a subdirect product of a family of compact metric H-systems $\{X_i : i \in I\}$. Uniform product of uniformly Lindelöf spaces is uniformly Lindelöf. Therefore by Lemma 5.6 it suffices to establish that every $(X_i, (\mu_i)_H)$ is uniformly Lindelöf. Since μ_i and $(\mu_i)_H$ are metrizable, this is equivalent to showing that $(\mu_i)_H$ is separable. Since (H, X) is HNS, Lemma 9.7 shows that the H-quotient (H, X_i) is also HNS.

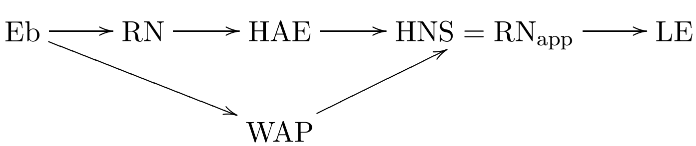
Hence, $\mathrm{id}_{X_i} : (X_i, \mu_i) \to (X_i, (\mu_i)_H)$ is fragmented by Lemma 9.4.2. Now, Lemma 6.5 guarantees that $(X_i, (\mu_i)_H)$ is separable.

$5 \Rightarrow 1$: We have to show that X is $\mathrm{RN}_{\mathrm{app}}$. Equivalently, by Theorem 7.6.4 we need to check that $C(X) = \mathrm{Asp}(X)$. Let $F \in C(X)$ and $H < G$ be a countable subgroup. By our assumption, (X, μ_H) is uniformly Lindelöf. Since $F : (X, \mu) \to \mathbb{R}$ is uniformly continuous, so is $\mathrm{id}_X : (X, \mu_H) \to (X, \varrho_{H,F})$. Therefore, $(X, \varrho_{H,F})$ is uniformly Lindelöf as well. Since $\varrho_{H,F}$ is a pseudometric, we conclude that $(X, \varrho_{H,F})$ is separable. This proves that $F \in \mathrm{Asp}(X)$. ■

Remark 9.10. 1. Every precompact uniform space is uniformly Lindelöf. Note here that (X, μ_G) is precompact iff (G, X) is equicontinuous (cf. Corollary 5.5). Therefore, $\mathrm{RN}_{\mathrm{app}}$, and its equivalent concept HNS, can be viewed as a natural generalization of equicontinuity.

2. Theorem 9.9 implies that $\mathrm{RN}_{\mathrm{app}}$ (or HNS) is "countably determined". That is, (G, X) is $\mathrm{RN}_{\mathrm{app}}$ iff (H, X) is $\mathrm{RN}_{\mathrm{app}}$ for every *countable* subgroup $H < G$.
3. Let $H < G$ be a *syndetic* subgroup (that is, there exists a compact subset $K \subset G$ such that $G = KH$) of a uniformly Lindelöf group G. Then a system (G, X) is $\mathrm{RN}_{\mathrm{app}}$ iff (H, X) is $\mathrm{RN}_{\mathrm{app}}$. Indeed, K acts μ-uniformly equicontinuously on X. Thus if (X, μ_H) is uniformly Lindelöf then so is (X, μ_{KH}).
4. $\mathrm{RN}_{\mathrm{app}} \subset \mathrm{LE}$ by Lemma 9.8.1 (or by [45, Theorem 6.10]).

We now have the following diagram for compact G-systems:



Remark 9.11. 1. We do not know (even for cascades) if HAE $\neq$ HNS for non-metrizable systems. All other implications, in general, are proper:

2. RN $\neq$ HAE, Eb $\neq$ WAP. Indeed, take a system (G, X) with trivial G and a compact X which is not RN in the sense of Namioka, and hence not Eberlein, as a compact space (e.g. $X := \beta\mathbb{N}$). Such a G-system, however, is trivially WAP and also HAE.
3. Eb $\neq$ RN. Take a trivial action on a compact RN space which is not Eberlein.
4. $\mathrm{RN}_{\mathrm{app}} \neq$ LE even for transitive metric systems (cf. Remark 10.9.1 and Theorem 11.1).
5. WAP $\neq$ HNS. See again Theorem 11.1.

THEOREM 9.12. *For a compact G-system X the following are equivalent:*

1. $f \in \mathrm{Asp}(X)$.
2. $f_\sharp^G : X \to \mathrm{RUC}(G)$ *is fragmented.*
3. $f_\sharp : X \to X_f$ *is HNS.*
4. $f : X \to \mathbb{R}$ *is HNS.*
5. $\breve{G}^f := \{\breve{g}_f : X \to \mathbb{R}\}_{g \in G}$ *(where $\breve{g}_f(x) = f(gx)$) is a fragmented family.*
6. $X_f \subset \mathrm{RUC}(G)$ *is norm fragmented.*
7. *The G-system X_f is RN.*

Proof. $1 \Rightarrow 2$: By Theorem 7.6.3 there exist a G-quotient $\alpha : (X, \mu_X) \to (Y, \mu_Y)$ with $Y \in$ RN and $F \in C(Y)$ such that $f = F \circ \alpha$. Then $f_\sharp = F_\sharp \circ \alpha$. Therefore, by Lemma 6.3.6, it is enough to show that $F_\sharp : Y \to \mathrm{RUC}(G)$ is fragmented, or equivalently, that Y is $\varrho_{G,F}$-fragmented (see remarks before Lemma 7.3). By our assumption (Y, μ_Y) is RN. Therefore, Theorem 9.9 guarantees that Y is $(\mu_Y)_G$-fragmented. Since $\mathrm{id}_Y : (Y, (\mu_Y)_G) \to (Y, \varrho_{G,F})$ is uniformly continuous, it follows that Y is $\varrho_{G,F}$-fragmented, as required.

$2 \Leftrightarrow 3$: Use Lemma 9.4.1 taking into account Remark 5.3.1.

$3 \Leftrightarrow 4$: Let $f_\sharp : X \to X_f$ be HNS. Then $f_\sharp \restriction_A : A \to X_f$ is NS for every non-empty invariant closed subset of $A \subset X$. Therefore by Definition 9.1 (observe that the uniform structure of $X_f \subset \mathbb{R}^G$ is the pointwise uniform structure inherited from $\mathbb{R}^G$) for every $\varepsilon > 0$ and every finite subset $S \subset G$ there exists a relatively open non-empty subset $O \subset A$ such that

$$|f_\sharp(gx)(s) - f_\sharp(gx')(s)| < \varepsilon \quad \text{for all } (s,g) \in S \times G \text{ and all } (x,x') \in O \times O.$$

Now since $|f_\sharp(gx)(s) - f_\sharp(gx')(s)| = |f(sgx) - f(sgx')|$ and g runs over all elements of G our condition is equivalent to the inequality

$$|f(gx) - f(gx')| < \varepsilon \quad \text{for all } g \in G.$$

The latter means that $f(gO)$ is ε-small for every $g \in G$. Equivalently, $f : X \to \mathbb{R}$ is HNS.

$2 \Leftrightarrow 5$: See Definition 6.8.1.

$2 \Rightarrow 6$: Let $f_\sharp : X \to X_f$ be the canonical G-quotient. Then by Lemma 6.4 (with $Y_1 = Y_2 = \mathrm{RUC}(G)$) the fragmentability of $f_\sharp^G : X \to \mathrm{RUC}(G)$ guarantees the fragmentability of $r_G : X_f \to \mathrm{RUC}(G)$. This means that X_f is norm fragmented.

$6 \Rightarrow 7$: The norm on $\mathrm{RUC}(G)$ is lower semicontinuous with respect to the pointwise topology. Hence, Theorem 7.9 ensures that the G-system X_f is RN.

$7 \Rightarrow 1$: Since X_f is RN, by Theorem 7.6.3 and Proposition 2.2.3 we see that $f \in \mathrm{Asp}(X)$. ∎

REMARK 9.13. Note in the following list how, for a G-space X, topological properties of X_f correspond to dynamical properties of $f \in \mathrm{RUC}(X)$ and provide an interesting dynamical hierarchy:

$$X_f \text{ is norm compact} \Leftrightarrow f \text{ is AP},$$
$$X_f \text{ is weakly compact} \Leftrightarrow f \text{ is WAP},$$
$$X_f \text{ is norm fragmented} \Leftrightarrow f \text{ is Asplund},$$
$$X_f \text{ is orbitwise light} \Leftrightarrow f \text{ is LE}.$$

In the domain of compact metric systems NS and AE are distinct properties. In contrast to this fact, if these conditions hold hereditarily then they are equivalent.

THEOREM 9.14. *Let (X,d) be a compact metric G-space. The following properties are equivalent*:

1. *X is RN.*
2. *X is HAE.*
3. *Every closed G-subsystem Y of X has a point of equicontinuity.*
4. *X is HNS.*
5. *X is d_G-fragmented (recall that $d_G(x,x') = \sup_{g\in G} d(gx,gx')$).*
6. *(X,d_G) is separable (that is, d is an s-Asplund metric).*
7. *Every continuous function $F: X \to \mathbb{R}$ is s-Asplund.*

Proof. Since X is metric, $\breve{G} \subset \mathrm{Homeo}(X)$ is second countable. So we can and do assume, for simplicity, that G is second countable.

By Theorem 7.6.5, $\mathrm{RN} = \mathrm{RN}_{\mathrm{app}}$ in the domain of compact metric systems. Hence, it follows by our diagram above that $1 \Leftrightarrow 2 \Leftrightarrow 4$.

$2 \Rightarrow 3$: Trivial.

$3 \Rightarrow 4$: By the assumption $\mathrm{Eq}(Y) \neq \emptyset$ for every subsystem (G,Y). Thus, Y is NS by Lemma 9.2.1. It follows that X is HNS.

$4 \Leftrightarrow 5$: By Lemma 9.4.2.

$5 \Rightarrow 6$: Apply Lemma 6.5 to the map $\mathrm{id}_X : (X,d) \to (X,d_G)$.

$6 \Rightarrow 7$: By our assumption (X,d_G) is separable. Since $\mathrm{id}_X : (X,d_G) \to (X,\varrho_{G,F})$ is uniformly continuous, we deduce that $(X,\varrho_{G,F})$ is also separable. Hence, $f \in \mathrm{Asp_s}(X)$.

$7 \Rightarrow 1$: Every s-Asplund function is Asplund. Hence, $C(X) = \mathrm{Asp}(X)$. By assertions 4 and 5 of Theorem 7.6 we can conclude that X is RN. ∎

Summing up we have the following simple diagram (with two proper inclusions) for *metric* compact systems:

$$\mathrm{Eb} = \mathrm{WAP} \to \mathrm{RN} = \mathrm{HAE} = \mathrm{HNS} = \mathrm{RN_{app}} \to \mathrm{LE}.$$

10. Some examples

COROLLARY 10.1. *The class of compact metrizable HNS* (*hence also RN*, *HAE*) *systems is closed under factors and countable products.*

Proof. RN = HAE = HNS by Theorem 9.14. Now use Lemma 9.7 and Theorem 7.6.5. ∎

COROLLARY 10.2. *Every scattered* (*e.g.*, *countable*) *compact G-space X is RN* (*see also* [45]).

Proof. Apply Theorem 7.9 using Remark 6.2.4. ∎

A metric G-space (X, d) is called *expansive* if there exists a constant $c > 0$ such that $d_G(x, y) := \sup_{g\in G} d(gx, gy) > c$ for any distinct $x, y \in X$.

COROLLARY 10.3. *An expansive compact metric G-space* (X, d) *is RN iff X is countable.*

Proof. If X is RN then by Theorem 9.14, (X, d_G) is separable. On the other hand, (X, d_G) is discrete for every expansive system (X, d). Thus, X is countable. ∎

For a countable discrete group G and a finite alphabet S the compact space S^G is a G-space under left translations $(g\omega)(h) = \omega(g^{-1}h)$, $\omega \in S^G$, $g, h \in G$. A closed invariant subset $X \subset S^G$ defines a subsystem (G, X). Such systems are called *subshifts* or *symbolic dynamical systems*.

COROLLARY 10.4. *For a countable discrete group G and a finite alphabet S let* $X \subset S^G$ *be a subshift. The following properties are equivalent*:

1. *X is RN.*
2. *X is countable.*

Moreover if $X \subset S^G$ *is an RN subshift and* $x \in X$ *is a recurrent point then it is periodic* (*i.e.* Gx *is a finite set*).

Proof. It is easy to see (and well known) that every subshift is expansive.

For the last assertion recall that if x is a recurrent point with an infinite orbit then its orbit closure contains a homeomorphic copy of the Cantor set. ∎

For some (one-dimensional) compact spaces every selfhomeomorphism will produce an RN system.

PROPOSITION 10.5. 1. *For each element* $f \in \mathrm{Homeo}(I)$, *the homeomorphism group of the unit interval* $I = [0, 1]$, *the corresponding dynamical system* (f, I) *is HNS.*

2. *For each element* $f \in \mathrm{Homeo}(S^1)$, *the homeomorphism group of the circle* $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, *the corresponding dynamical system* (f, S^1) *is HNS.*

Proof. 1. Fix an element $f \in \mathrm{Homeo}(I)$, which with no loss of generality we assume is orientation preserving. Consider the dynamical system (f, I) and for a set $A \subset I$ define $O_f(A) = \bigcup_{n\in\mathbb{Z}} f^n(A)$. Let us note first that for every $x \in [0,1]$ the sequence $\dots, f^{-2}(x), f^{-1}(x), x, f(x), f^2(x), \dots$ is increasing, hence the orbit closure of x is just the orbit together with the points $\lim_{n\to\infty} f^{-n}(x)$ and $\lim_{n\to\infty} f^n(x)$. In particular the dynamical system (f, I) is LE.

Next we show that (f, I) is NS. If this is not the case then there exists an $\varepsilon > 0$ such that for every non-empty open set $U \subset I$ there exists $n \in \mathbb{Z}$ such that $\mathrm{diam}(f^n U) \geq \varepsilon$. Let $(a, b) \subset I$ be an open interval and let $\{U_k\}_{k\in\mathbb{N}}$ be a countable basis for open sets in (a, b). If for every k the set $(a, b) \cap O_f(U_k)$ is dense in (a, b) then the orbit of any point $x \in (a, b) \cap \bigcap_{k=1}^{\infty} O_f(U_k)$ will be dense in (a, b), which is impossible.

We conclude that for every interval (a, b) and every proper subinterval J_1 there is another subinterval $J_2 \subset (a, b)$ which is disjoint from $O_f(J_1)$. By induction we can find an infinite sequence of disjoint intervals J_j in (a, b) such that for every j the set J_{j+1}, and hence also $O_f(J_{j+1})$, is disjoint from $\bigcup_{i\leq j} O_f(J_i)$. Since for each j the set $O_f(J_j)$ contains an interval of length at least ε we arrive at a contradiction. This concludes the proof that (f, I) is NS.

Next consider any non-empty closed invariant subset $Y \subset I$. If Y contains an isolated point then clearly the system (f, Y) is NS. Thus we now assume that Y is a perfect set. We can then repeat the argument that showed that (f, I) is NS for the system (f, Y) and arrive at the same kind of contradiction since again an orbit of a single point in Y cannot be everywhere dense in a non-empty set of the form $(a, b) \cap Y$.

2. We will use Poincaré's classification of the systems (S^1, f) whose nature is well understood (see for example [34, Section 11.2]). Again we can assume with no loss of generality that our homeomorphism f preserves the orientation on S^1. Let $r(f) \in \mathbb{R}$ denote the rotation number of f. If $r(f)$ is rational then some power of f has a fixed point and we are reduced to the case of a homeomorphism of $I = [0, 1]$. Thus we can assume that $r(f)$ is irrational. There are two cases to consider.

The first case is when the system (S^1, f) is minimal; then f is conjugate to an irrational rotation and is therefore equicontinuous.

In the second case, when (S^1, f) is not minimal, there exists a unique minimal subset $K \subset S^1$ with K a Cantor set and there are wandering intervals $J \subset S^1$. For such an interval, given an $\varepsilon > 0$ there exists an N such that for every $n \in \mathbb{Z}$ with $|n| \geq N$, $\mathrm{diam}(f^n(J)) < \varepsilon$; hence the NS property of (S^1, f) follows.

For the HNS property consider an arbitrary subsystem (Y, f) with $Y \subset S^1$. Again distinguish between the cases when Y has an isolated point and when it is a perfect set. The presence of an isolated point ensures NS. Finally, when Y is perfect it is either equal to K, hence equicontinuous, or we can still use the existence of the wandering intervals in (S^1, f) to obtain a non-empty set $J \cap Y$ with the property that the diameter of its images under the iterates of f tends to zero. ∎

EXAMPLES 10.6. Of course it is easy to find non-RN metric systems. Here are some "random" examples.

1. The cascades on the torus $\mathbb{T}^2$ defined by a hyperbolic automorphism, or the horocycle flows, being weakly mixing (see Corollary 9.3), are not RN. Likewise Anosov diffeomorphisms on a compact manifold, being expansive (see [5]), are not RN by Corollary 10.3.
2. Systems which contain non-equicontinuous minimal subsystems fail to be RN.
3. Let X be compact metric and uncountable and set $G = \mathrm{Homeo}(X)$. Then in many cases (like $X = [0,1]$) the action is expansive, hence not RN (Corollary 10.3).
4. As we have seen, any uncountable subshift is not RN. Thus, for example, the well known "generator of the Morse cascade"

$$w = \dots 0110100110010110\dot{0}110100110010110\dots$$

considered as a function $w : \mathbb{Z} \to \mathbb{R}$ is not an Asplund function on the group $\mathbb{Z}$.

A point-transitive LE system is, by definition, AE but there are non-transitive LE systems which are not AE.

EXAMPLE 10.7. As can be easily seen, the $\mathbb{Z}$-system (T, D), where $D = \{z \in \mathbb{C} : |z| \le 1\}$ is the unit disk in the complex plane and $T : D \to D$ is the homeomorphism given by the formula $Tz = z\exp(2\pi i|z|)$, is an LE system which is not AE.

There exist many compact metrizable transitive AE systems which fail to be HAE. This follows, for example, from the lemma below. We will use the following construction which is due to Takens. For a metric cascade (T, X) define an *asymptotic pseudo-orbit* to be a bi-infinite sequence $\{x_n\}$ such that $\lim_{|n|\to\infty} d(Tx_n, x_{n+1}) = 0$. Note that (T, X) is chain transitive iff it admits an asymptotic pseudo-orbit with alpha and omega limit point sets the whole space.

LEMMA 10.8. *Let (T, X) be a metric cascade.*

1. *If (T, X) is a chain recurrent $\mathbb{Z}$-space then X is isomorphic to a subsystem of a compact metric transitive AE cascade (T, Y).*
2. *If (T, X) is transitive-recurrent then X is also a retract of the ambient transitive AE system (T, Y).*

Proof. Let $\{t_n\}$ be a bi-infinite monotonic sequence in $(0,1)$ with $\lim_{n\to\infty} t_n = 1$, $\lim_{n\to\infty} t_{-n} = 0$. Let S be the circle represented as the interval $[0,1]$ with 0 identified with 1. Let $\{x_n\}$ be an asymptotic pseudo-orbit in X. Identify X with the subset $X\times\{0\} \subset X\times S$ and let $Y = X\cup\{(x_n,t_n) : n \in \mathbb{Z}\}$. Extend T to Y by $T(x_n,t_n) = (x_{n+1},t_{n+1})$. This completes the proof of part 1. For part 2 note that if the pseudo-orbit is actually an orbit then the first coordinate projection from Y to X is a $\mathbb{Z}$-retraction. ■

REMARK 10.9. 1. If we apply the construction of Lemma 10.8 to the (clearly chain recurrent) system $(T,X) = (T,D)$ of Example 10.7, we obtain a transitive (but not recurrent-transitive) metric LE system (T,Y) which is not HAE (nor $\mathrm{RN}_{\mathrm{app}}$). Applying Lemma 10.8 to a transitive non-AE system (T,X) (e.g. a minimal weakly mixing system), we obtain an example of an AE system with both a subsystem and a factor which are not AE (see [25]).

2. As noted above, HAE is preserved under both passage to subsystems and the operation of taking factors. In the next section we will show that the Glasner–Weiss family of recurrent-transitive LE but not WAP systems consists, in fact, of HAE systems. On the other hand, in Section 13 we will modify these examples so that the resulting dynamical system will still be recurrent-transitive, LE, but no longer HAE. Thus even among metric recurrent-transitive $\mathbb{Z}$-systems we have the proper inclusions
$$\mathrm{WAP} \subset \mathrm{HAE} \subset \mathrm{LE}.$$
Then we can conclude that the following inclusions are also proper:
$$\mathrm{WAP}(\mathbb{Z}) \subset \mathrm{Asp}(\mathbb{Z}) \subset \mathrm{LE}(\mathbb{Z}).$$
3. It is interesting to compare some of the current definitions of chaos and the corresponding classes of dynamical systems (see, for example, [16, 25, 11]) with the class of G-systems X such that $\mathrm{Asp}(X) = \{\text{constants}\}$. The latter are the systems which admit only trivial representations on Asplund Banach spaces. Every weakly mixing compact system belongs to this class because by Corollary 9.3 every Asplund function (in fact, every continuous NS function) on such a system is constant.
4. By Theorem 1.3 of [25] and the variational principle, an LE (e.g., RN) cascade has topological entropy zero. This probably holds for a much broader class of acting groups but we have not investigated this direction.

11. The G-W examples are HAE. In this section we assume that the reader is familiar with the details of the paper [26]. In particular we use the notations of that paper with no further comments.

THEOREM 11.1. *The G-W examples of recurrent-transitive LE but not WAP systems are actually HAE.*

Proof. Recall that Ω is the space of continuous maps $x : \mathbb{R} \to 2^I$, where $I = [0, 1]$ and 2^I is the compact metric space of closed subsets of I equipped with the Hausdorff metric d. (In fact, the values x assumes are either intervals or points.) The topology on Ω is that of uniform convergence on compact sets: $x_n \to x$ if for every $\varepsilon > 0$ and every $M > 0$ there exists $N > 0$ such that for all $n > N$, $\sup_{|t|\le M} d(x_n(t), x(t)) < \varepsilon$. On Ω there is a natural $\mathbb{R}$-action defined by translations: $(T^t x)(s) = x(s + t)$. The compact metrizable dynamical system (T, X), where $T = T^1$, is obtained as the orbit closure $X = \mathrm{cls}\{T^n\omega : n \in \mathbb{Z}\}$ for a carefully constructed (kite-like) element $\omega \in \Omega$ (see also the figure in Section 13). The fact that $\omega : \mathbb{R} \to 2^I$ is a Lipschitz function implies that each member of X is Lipschitz as well with the same constant, so that X as a family of functions is equicontinuous. The compactness of X follows from the Arzelà–Ascoli theorem. We next sum up some of the salient facts we have about (T, X):

(a) For every $x \in X$ there is a unique interval $[a, b] \subset [0, 1]$ such that:

  (i) $x(t) \subset [a, b], \ \forall t \in \mathbb{R}$,

  (ii) there exists a sequence $t_l \in \mathbb{R}$ with $\lim x(t_l) = [a, b]$.

  We set
$$\mathbf{N}(x) = [a, b].$$

(b) The function $x \mapsto \mathbf{N}(x)$ is lower semicontinuous, that is, $\lim_\nu x_\nu = x \Rightarrow \liminf_\nu \mathbf{N}(x_\nu) \supset \mathbf{N}(x)$.

(c) Call intervals $[a, b] \subset [0, 1]$ of the form $\mathbf{N}(x)$, $x \in X$, *admissible*. Then for every admissible $[a, b] \subset [0, 1]$ there exists a unique element $\omega_{ab} \in X$ with $\mathbf{N}(\omega_{ab}) = \omega_{ab}(0) = [a, b]$. (In particular $\omega_{01} = \omega$.)

(d) Let $J = \{\omega_{ab} \in X : 0 \le a \le b \le 1\}$. Then J is a closed subset of X and $\mathbf{N} : J \to \{(a, b) : 0 \le a \le b \le 1\} \subset [0, 1] \times [0, 1]$ is a homeomorphism onto the set of admissible intervals. (Not every subinterval of $[0, 1]$ is admissible. For example neither $[0, 9/10]$ nor any degenerate interval with $9/10 < a = b \le 1$ is attained.)

(e) Defining $X_{ab} = \bar{\mathcal{O}}_T(\omega_{ab})$ we have $x \in X_{ab}$ iff $\mathbf{N}(x) \subset [a, b]$.

(f) For each admissible interval $[a, b] \subset [0, 1]$ the subsystem (T, X_{ab}) is AE, with $\mathrm{Eq}(X_{ab}) = \{x \in X : \mathbf{N}(x) = [a, b]\}$.

These facts, perhaps except (b), are either stated explicitly and proved in [26] or can be easily deduced from the results in that paper. For completeness we provide a proof for (b).

Proof of (b). With no loss in generality we assume $\liminf_\nu \mathbf{N}(x_\nu) = \lim_\nu \mathbf{N}(x_\nu) = [a, b]$ and we then have to show that $[a, b] \supset \mathbf{N}(x)$. There exists a sequence m_i such that $\lim_i T^{m_i}x(0) = \mathbf{N}(x)$. Therefore, given $\varepsilon > 0$, there exists an i with

$$d(T^{m_i}x(0), \mathbf{N}(x)) < \varepsilon. \tag{11.1}$$

Next choose ν such that

$$d(T^{m_i}x_\nu(0), T^{m_i}x(0)) < \varepsilon \tag{11.2}$$

and

$$d(\mathbf{N}(x_\nu), [a, b]) < \varepsilon. \tag{11.3}$$

Now, by (11.3) we have

$$[a - \varepsilon, b + \varepsilon] \supset \mathbf{N}(x_\nu) \supset T^{m_i}x_\nu(0),$$

hence by (11.1) and (11.2),

$$[a - 3\varepsilon, b + 3\varepsilon] \supset \mathbf{N}(x).$$

Since $\varepsilon > 0$ is arbitrary we conclude that indeed $[a, b] \supset \mathbf{N}(x)$. ∎

Of course this list implies the LE property of (T, X). However, we are after the stronger property HAE. For this purpose consider now an arbitrary closed invariant non-empty subset Y of X. Let J_Y be the subset of Y which consists of those elements $y \in J \cap Y$ for which $\mathbf{N}(y) = y(0)$ is maximal; that is, if $z \in Y$ and $\mathbf{N}(z) \supset \mathbf{N}(y)$ then $\mathbf{N}(z) = \mathbf{N}(y)$.

Claim 1. *The restriction* $\mathbf{N}\restriction_Y : Y \to [0, 1] \times [0, 1]$ *is continuous at points of* J_Y.

Proof. Suppose $Y \ni y_n \to y \in J_Y$. By the lower semicontinuity of $\mathbf{N}$,

$$[a, b] = \liminf_n \mathbf{N}(y_n) \supset \mathbf{N}(y).$$

Choose a subsequence n_i such that $\mathbf{N}(y_{n_i}) \to [a, b]$. Then for some sequence m_i we have $T^{m_i}y_{n_i}(0) \to [a, b]$. By compactness we can assume with no loss in generality that $T^{m_i}y_{n_i} \to z$ for some $z \in Y$. Now, $T^{m_i}y_{n_i}(0) \to z(0) = [a, b] \supset \mathbf{N}(y)$, whence $\mathbf{N}(y) = [a, b]$. It follows easily that $\lim_n \mathbf{N}(y_n) = \mathbf{N}(y)$. ∎

In item (d) of the above list we noted that J is a closed subset of X and $\mathbf{N} : J \to [0, 1] \times [0, 1]$ is a homeomorphism into. Set $K = \mathbf{N}(J \cap Y)$ and let $K_0 \subset K$ be the subset of maximal elements in K; i.e. $[a, b] \in K_0$ iff $[a, b] \in K$ and $K \ni [c, d] \supset [a, b]$ implies $[c, d] = [a, b]$. Clearly K_0 is a closed subset of the closed set K and for every $[c, d] \in K$ there exists some $[a, b] \in K_0$ with $[c, d] \subset [a, b]$.

Claim 2. $K_0 = \mathbf{N}(J_Y)$.

Proof. Let $[a, b]$ be an element of K_0; then $[a, b] = \mathbf{N}(y)$ for some $y \in J \cap Y$. If $[c, d] = \mathbf{N}(z) \supset [a, b]$ for some $z \in Y$, then for some $z' \in \overline{\mathcal{O}}_T(z) \subset Y$ we have $z'(0) = [c, d] = \mathbf{N}(z')$. In particular $z' \in J \cap Y$ and $\mathbf{N}(z') = [c, d] \in K$. Hence $[c, d] = [a, b]$ and it follows that $y \in J_Y$.

Conversely, if $y \in J_Y$ with $y(0) = [a, b] = \mathbf{N}(y)$ and $\mathbf{N}(z) = z(0) = [c, d] \supset [a, b]$ for $z \in Y$, then $[c, d] = [a, b]$ and $[a, b] \in K_0$. ■

Claim 3. *J_Y is closed and non-empty; in fact* $Y = \mathrm{cls}\{T^n J_Y : n \in \mathbb{Z}\}$.

Proof. The fact that J_Y is closed and non-empty is a direct consequence of Claim 2. Clearly $\mathbf{N}(Y) = \mathbf{N}(J \cap Y) = K$ and it follows that every $[a, b] = \mathbf{N}(y) \in \mathbf{N}(Y)$ is a subset of some $[c, d] = \mathbf{N}(\omega_{ab}) \in K_0$. By item (e) we have $y \in X_{ab} = \overline{\mathcal{O}}_T(\omega_{ab})$ and our claim follows. ■

Claim 4. *Every $\omega_{ab} \in J_Y$ with $a < b$ is in* $\mathrm{Eq}(Y)$.

Proof. The key fact in proving the inclusion $J_Y \setminus \{\text{constant functions}\} \subset \mathrm{Eq}(Y)$ is a certain uniformity of the function $\varepsilon' = \varepsilon'(\varepsilon, b - a)$ provided by Lemma 3.5 of [26]. In essence, as can be seen by combining Lemmas 3.5, 3.6 and 1.1 of [26], this function is the equicontinuity modulus function for $D(z, w) = \sup_{n\in\mathbb{Z}} d(T^n z, T^n w)$ on orbit closures in (T, X); i.e. given a point $x \in X$ with $\mathbf{N}(x) = [a, b]$ and $\varepsilon > 0$, the ε'-neighborhood of x, $B_{\varepsilon'}(x) \cap \overline{\mathcal{O}}_T(x)$, in $\overline{\mathcal{O}}_T(x)$ is (ε, D)-small. The point is that the $\varepsilon' = \varepsilon'(\varepsilon, b-a)$ provided by Lemma 3.5 of [26] is uniform in x as long as $b - a$ is bounded away from zero.

Therefore, given a point $\omega_{ab} \in J_Y$ with $a < b$, and $\varepsilon > 0$, we can choose a point $\omega_{a'b'} \in J$ with $a' < a < b < b'$ so that $a - a', b' - b$ are sufficiently small to ensure that $\omega_{ab} \in B_{\varepsilon'}(\omega_{a'b'})$. Of course by (e) we have $\omega_{ab} \in \overline{\mathcal{O}}_T(\omega_{a'b'})$.

By Claim 1, ω_{ab} is a continuity point for the restriction of the map $\mathbf{N}$ to Y and it follows that there exists a neighborhood V of ω_{ab} such that $\mathbf{N}(y) \subset [a', b']$ for every $y \in V$, hence $y \in \overline{\mathcal{O}}_T(\omega_{a'b'})$. We now conclude that $B_{\varepsilon'}(\omega_{a'b'}) \cap V$ is an (ε, D)-small neighborhood of ω_{ab} *in the subsystem* Y, and the proof that ω_{ab} is an equicontinuity point of the system (T, Y) is complete. ■

We next observe that T acts as the identity on the open subset

$$U = Y \setminus \mathrm{cls}\{T^n \omega_{ab} : \omega_{ab} \in J_Y,\ a < b,\ n \in \mathbb{Z}\}$$

(when non-empty) and thus every point in U is an equicontinuity point. This observation together with Claims 3 and 4 shows that the set $\mathrm{Eq}(Y)$ of equicontinuity points is dense in Y. That is, (T, Y) is an AE system, and our proof of the HAE property of (T, X) is complete. ■

12. The mincenter of an RN system. Unlike the case of transitive WAP systems, where the *mincenter* (i.e. the closure of the union of the minimal subsets of X) consists of a single minimal equicontinuous subsystem, the mincenter of a transitive RN system need not be minimal. In the G-W examples the mincenter consists of a continuum of fixed points; moreover, as was shown in [26], a slight modification of the construction there will yield examples of HAE systems whose mincenter consists of uncountably many non-trivial minimal equicontinuous subsystems all isomorphic to a single circle rotation. However, in Section 13 we will present a more sophisticated modification which produces an example of an LE system with a mincenter containing uncountably many non-isomorphic rotations. In the present section we obtain some information about the mincenter of RN systems. This will be used in the next section to draw a sharp distinction between LE and HAE systems. For simplicity we deal with metrizable systems. Recall that for such systems RN is the same as HAE.

The *prolongation relation* $\mathrm{Prol}(X) \subset X \times X$ of a compact dynamical system (G, X) is defined as follows:

$$\mathrm{Prol}(X) = \{(x, x') : \text{there exist nets } g_\nu \in G \text{ and } x_\nu \in X \text{ such that } \lim_\nu x_\nu = x \text{ and } \lim_\nu g_\nu x_\nu = x'\}.$$

It is easy to verify that $\mathrm{Prol}(X)$ is a closed symmetric and G-invariant relation. For $x_0 \in X$ we let

$$\mathrm{Prol}[x_0] = \{x \in X : (x_0, x) \in \mathrm{Prol}(X)\}.$$

Note that always $\overline{\mathcal{O}}_G(x) \subset \mathrm{Prol}[x]$, and if $x_0 \in \overline{\mathcal{O}}_G(x)$ then $x \in \mathrm{Prol}[x_0]$. For closed invariant sets $A \subset B \subset X$ we say that A is *capturing in* B if $x \in B$ and $\overline{\mathcal{O}}_G(x) \cap A \neq \emptyset$ imply $x \in A$ (see [7]).

LEMMA 12.1. 1. *Let* (X, d) *be a metric* G*-system*, $x_0 \in \mathrm{Eq}(X)$ *and* $x \in \mathrm{Prol}[x_0]$. *Then* $x \in \overline{\mathcal{O}}_G(x_0)$. *Hence*,

$$\mathrm{Prol}[x_0] = \overline{\mathcal{O}}_G(x_0).$$

2. *If* $x_0 \in \mathrm{Eq}(X)$ *and* $x_0 \in \overline{\mathcal{O}}_G(x)$, *then* $x \in \mathrm{Eq}(X)$ *and* $x \in \overline{\mathcal{O}}_G(x_0)$; *that is*, $\mathrm{Eq}(X)$ *is a capturing subset of* X.

Proof. 1. Given $\varepsilon > 0$ there exists $\delta > 0$ such that $z \in B_\delta(x_0)$ implies $d_G(x_0, z) < \varepsilon$. There are nets $g_\nu \in G$ and $x_\nu \in X$ such that $\lim_\nu x_\nu = x_0$ and $\lim_\nu g_\nu x_\nu = x$. For sufficiently large ν we have $x_\nu \in B_\delta(x_0)$ and $d(g_\nu x_\nu, x) < \varepsilon$, hence

$$d(g_\nu x_0, x) \leq d(g_\nu x_0, g_\nu x_\nu) + d(g_\nu x_\nu, x) < 2\varepsilon,$$

hence $x \in \overline{\mathcal{O}}_G(x_0)$. Thus $\mathrm{Prol}[x_0] \subset \overline{\mathcal{O}}_G(x_0)$. The inclusion $\mathrm{Prol}[x_0] \supset \overline{\mathcal{O}}_G(x_0)$ is always true.

2. Given $\varepsilon > 0$ there exists a $\delta > 0$ such that $d_G(x_0, z) < \varepsilon$ for every $z \in B_\delta(x_0)$. There exists $g \in G$ with $gx \in B_\delta(x_0)$ and therefore an $\eta > 0$ with $gB_\eta(x) \subset B_\delta(x_0)$. Now for every $h \in G$ and $w \in B_\eta(x)$ we have

$$d(hgx, hgw) < d(hgx, hx_0) + d(hgw, hx_0) < 2\varepsilon.$$

Thus also $x \in \mathrm{Eq}(X)$. By assumption $x_0 \in \overline{\mathcal{O}}_G(x)$ hence $x \in \mathrm{Prol}[x_0]$ and by part 1, $x \in \overline{\mathcal{O}}_G(x_0)$. ∎

PROPOSITION 12.2. *Let* (X, d) *be a metrizable RN G-system, and M its mincenter. Then* $\mathrm{Eq}(M)$ *is a disjoint union of minimal equicontinuous systems, each a capturing subset of M.*

Proof. Our system X is HAE by Theorem 9.14. Therefore the subsystem (G, M) is AE. Let $x_0 \in M$ be an equicontinuity point of M. Given $\varepsilon > 0$ there exists a $0 < \delta < \varepsilon$ such that $x \in B_\delta(x_0) \cap M$ implies $d(gx_0, gx) < \varepsilon$ for every $g \in G$. Let $x' \in B_\delta(x_0)$ be a minimal point. It then follows that $S = \{g \in G : gx' \in B_\delta(x_0)\}$ is a syndetic subset of G (i.e. $FS = G$ for some finite subset F of G). Collecting these estimates we get, for every $g \in S$,

$$d(gx_0, x_0) \le d(gx_0, gx') + d(gx', x_0) \le 2\varepsilon.$$

Thus for each $\varepsilon > 0$ the set $N(x_0, B_\varepsilon(x_0)) = \{g \in G : d(gx_0, x_0) \le \varepsilon\}$ is syndetic, whence x_0 is minimal.

Thus every equicontinuity point x_0 of M is minimal and we apply Lemma 12.1 to conclude that $\mathrm{Eq}(M)$ is a capturing subset of M. ∎

COROLLARY 12.3. *The mincenter Z of a metrizable RN system* (G, X) *is transitive iff Z is minimal and equicontinuous.*

REMARK 12.4. The *Birkhoff center* Y of a compact metrizable $\mathbb{Z}$-dynamical system (T, X) can be defined as the closure of its recurrent points. A non-empty open set $U \subset X$ such that $T^j U \cap U = \emptyset$ for all $j \in \mathbb{Z} \setminus \{0\}$ is called a *wandering set*. The complement of the union of all wandering sets is a closed invariant subsystem $Z_1 \subset X$ which contains Y. Repeating this process (countably many times) we get by transfinite induction a countable ordinal η such that $Z_\eta = Y$. Since an isolated transitive point of any compact metric system is always an equicontinuity point it follows easily that the system (T, X) is LE iff its Birkhoff center (T, Y) is LE. The same statement does not hold for RN systems. An example of a compact sensitive system (T, X) whose Birkhoff center consists of fixed points was shown to us by E. Akin (private communication).

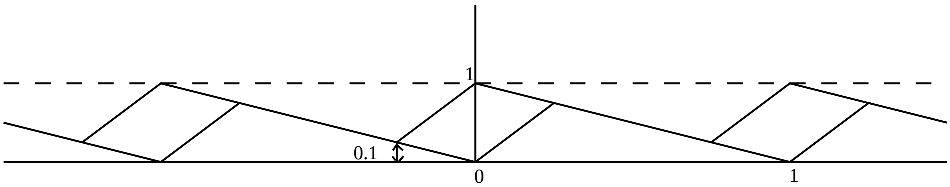
13. A recurrent-transitive LE but not HAE system. As promised in Section 10 we will sketch in the present section a modification of the G-W construction that will yield a recurrent-transitive system which is LE but not HAE. The possibility of introducing such a modification (in order to achieve another goal) occurred to the authors of [26] already at the time when that paper was written. The first author (E.G.) would like to thank B. Weiss for his help in checking the details of the modified construction.

THEOREM 13.1. *There exists a recurrent-transitive LE but not HAE system.*

Proof. In the original construction the basic "frames" α_n were defined by the formula

$$\alpha_n(t) = \alpha_0(t/p_n), \quad n = 1, 2, \ldots,$$

where α_0 is the original periodic kite-like function:



The kite-like function α_0

and the sequence p_k is defined by $p_0 = 1$ and $p_{n+1} = 10k_n p_n$ for a sequence of integers $k_n \nearrow \infty$ such that

$$\sum_{n=1}^{\infty} \frac{p_n}{p_{n+1}} = \sum_{n=1}^{\infty} \frac{1}{10k_n} < \infty.$$

In the modified construction the kite-like parts of α_n will not be changed but the lines between consecutive kites will contain larger and larger segments in which the original straight line will be replaced by graphs of functions of the form

$$f_\theta : t \mapsto \sin(2\pi\theta t), \tag{13.1}$$

properly scaled so that they fit into our strip $\mathbb{R} \times [0, 1]$. At the outset the sequence k_n will be chosen to grow sufficiently fast in order to leave room for the insertion of the sine functions. The parameters θ will be constructed inductively as a binary tree of irrational numbers $\{\theta_\varepsilon : \varepsilon \in \{0, 1\}^n\}$, $n = 1, 2, \ldots,$ where at the $n + 1$ stage $\theta_{\varepsilon 0} = \theta_\varepsilon$ and $\theta_{\varepsilon 1}$ is a new point in $[0, 1]$. The numbers θ_ε will satisfy inequalities of the form

$$\|p_n\theta_\varepsilon\| \ll 1/n^n \quad \text{for all } \varepsilon \in \bigcup_{k=1}^{\infty} \{0, 1\}^k, \tag{13.2}$$

where $\|\lambda\|$ denotes the distance of the real number λ from the closest integer. The points on the circle which satisfy the inequality (13.2) at stage $n + 1$ form a union of finitely many disjoint open intervals, and the "neighbor" $\theta_{\varepsilon 1}$ of $\theta_{\varepsilon 0} = \theta_\varepsilon$ will be chosen in that same interval which already contains $\theta_{\varepsilon 0}$. When the construction is finished we end up with a Cantor set $\Lambda \subset \mathbb{T}$ consisting of the closure of the set $\{\theta_\varepsilon : \varepsilon \in \bigcup_{k=1}^{\infty}\{0, 1\}^k\}$. At stage n there will be finitely many functions f_θ with parameters θ_ε, $\varepsilon \in \bigcup_{k=1}^{n}\{0, 1\}^k$, and they will replace segments of the straight lines connecting the kites of α_n. Each of these functions will grow in amplitude very gradually from zero to say $1/100$ and then after running for a long time with maximal amplitude $1/100$ will symmetrically diminish in amplitude till it becomes again a straight line. Each function will appear once and their occurrences will be separated by very long stretches of the straight line. Of course this picture will be repeated periodically between any two consecutive kites of α_n. Apart from these changes the construction of the functions β_n will be repeated unmodified as in [26].

We claim that the construction sketched above, when carefully carried out, will yield an element $\omega \in \Omega$ whose orbit closure $X = \mathrm{cls}\{T^n\omega : n \in \mathbb{Z}\}$ will be, like the original system, a recurrent-transitive LE system. However, unlike the old system, whose minimal sets were all fixed points, our new system will have, for each $\theta \in \Lambda$, a minimal subset isomorphic to the irrational rotation $(R_\theta, \mathbb{T})$. We will not verify these claims, whose proofs parallel the proofs of the original construction in [26]. We will though demonstrate that (T, X) is not HAE. Indeed, this is a direct consequence of the following proposition. (A second proof will be given in Remark 14.9.)

PROPOSITION 13.2. *Let (T, X) be a compact metric cascade and suppose that there exists an uncountable subset $\Lambda \subset \mathbb{T}$ with the property that for each $\lambda \in \Lambda$ there exists a subsystem $Y_\lambda \subset X$ such that the system (T, Y_λ) is isomorphic to the rotation $(R_\lambda, \mathbb{T})$ on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Then (T, X) is not HAE.*

Proof. Suppose to the contrary that (T, X) is HAE and let $Y = \mathrm{cls}(\bigcup\{Y_\lambda : \lambda \in \Lambda\})$. By assumption the system (T, Y) is also HAE and clearly Y coincides with its mincenter: $Y = M(Y)$. Let A_0 be a subset of Y such that for each $\lambda \in \Lambda$ there is exactly one point in the intersection $A_0 \cap Y_\lambda$, and let $A = \bigcup\{T^n A_0 : n \in \mathbb{Z}\}$. If $\{U_m\}_{m=1}^\infty$ is a countable basis for open sets in Y then the set $O = \bigcup\{U_m : \mathrm{card}(U_m \cap A) \le \aleph_0\}$ is open and it meets at most countably many Y_λ's. Omitting, at the outset, this countable set from Λ we can and do assume that $U_m \cap A$ is uncountable for every m. By the AE property the set $Y_0 = \mathrm{Eq}(Y)$ of equicontinuity points is a dense G_δ subset of Y, and by Proposition 12.2 each point of Y_0 belongs to a minimal set. Since the set $\mathrm{fix}(Y)$ of fixed points in Y is closed, it has an empty interior and it follows that the set $Y_1 = Y_0 \setminus \mathrm{fix}(Y)$ is also a dense G_δ subset of Y.

Choose a point $z_0 \in Y_1$; then $z_0 \in Z$ for some non-trivial minimal set Z. Now the system Z can admit at most a countable set of eigenvalues and therefore can be *not disjoint* from at most countably many of the systems Y_λ. We can therefore choose an infinite sequence $\{\lambda_n\} \subset \Lambda$ and a sequence of points $y_n \in Y_{\lambda_n}$ such that (i) $\lim_{n\to\infty} y_n = z_0$, (ii) the set $\{\lambda_n : n = 1, 2, \dots\}$ is independent over the rational numbers $\mathbb{Q}$, and (iii) Z is disjoint from the minimal system $\prod_{n=1}^\infty (R_{\lambda_n}, \mathbb{T})$. Thus the dynamical system

$$(T, \Omega) = (T, Z) \times \prod_{n=1}^{\infty} (R_{\lambda_n}, \mathbb{T})$$

is minimal and in particular for some sequence m_i we have

$$\lim_{i\to\infty} T^{m_i} y_n = y_n \quad \text{for } n = 1, 2, \ldots, \quad \text{while} \quad \lim_{i\to\infty} T^{m_i} z_0 = z_1 \neq z_0.$$

Since $\lim_{n\to\infty} y_n = z_0$, this contradicts the fact that z_0 is an equicontinuity point and the proof of the proposition is complete. ■

This also concludes the proof of Theorem 13.1. ■

14. An enveloping semigroup characterization of HNS. In this section we give an enveloping semigroup characterization of Asplund functions and HNS systems in terms of fragmented families (Definition 6.8). In addition to fragmentability, our approach essentially uses Namioka's theorem. First we recall this fundamental result and an auxiliary definition. A topological space X is said to be *Čech-complete* if X is a G_δ subset in some compact Hausdorff space. If X is either a locally compact Hausdorff space or a complete metric space then X is Čech-complete. We need the following version of Namioka's theorem.

THEOREM 14.1 (Namioka's joint continuity theorem, [47]). *Let $w : K \times X \to M$ be a separately continuous function where M is a metric space, K is compact and X is Čech-complete. Then there exists a dense G_δ set X_0 in X such that w is jointly continuous at every point of $K \times X_0$.*

Let $E = E(X)$ be the enveloping semigroup of a compact G-system X. Recall that

$$E^f := \{p_f : X \to \mathbb{R}\}_{p\in E}, \qquad p_f(x) = f(px),$$

is a pointwise compact subset of $\mathbb{R}^X$, being a continuous image of E under the map

$$q_f : E \to E^f, \qquad q_f(p) = p_f$$

(see Section 3).

For every $f \in C(X)$ define the map

$$w_f : E \times X \to \mathbb{R}, \qquad w_f(p, x) := f(px).$$

In turn w_f induces the mapping $E^f \times X_f \to \mathbb{R}$, $(p_f, f_\sharp(x)) \mapsto f(px)$. Observe that by the proof of Proposition 2.2.2 (with $f_\sharp = \psi : \beta_G(X) = X \to X_f$) we have $\psi(x_1) = \psi(x_2)$ iff $f(gx_1) = f(gx_2)$ for all $g \in G$. It follows that $\psi(x_1) = \psi(x_2)$ iff $f(px_1) = f(px_2)$ for all $p \in E$. Hence, $E^f \times X_f \to \mathbb{R}$ and the following commutative diagram is well defined:

$$\begin{array}{ccc} E \times X & \longrightarrow & X \\ {\scriptstyle q_f}\downarrow \quad \downarrow{\scriptstyle f_\sharp} & & \downarrow{\scriptstyle f} \\ E^f \times X_f & \longrightarrow & \mathbb{R} \end{array}$$

We are now ready to prove the following result.

THEOREM 14.2. *Let X be a compact G-system. The following are equivalent*:

1. $f \in \mathrm{Asp}(X)$.
2. E^f *is a fragmented family.*
3. E^f *is a barely continuous family.*
4. *For every closed* (G*-invariant*) *subset* $Y \subset X$ *there exists a dense* G_δ *subset* Y_0 *of* Y *such that the induced map* $p_f : Y_0 \to \mathbb{R}$, $p_f(y) = f(py)$, *is continuous for every member* p *of the enveloping semigroup* E.

Proof. $1 \Rightarrow 2$: By Theorem 9.12 the family $\breve{G}^f := \{\breve{g}_f : X \to \mathbb{R}\}_{g\in G}$ is fragmented. Then so is the family E^f, being the pointwise closure of $\breve{G}^f$ (Lemma 6.9).

$2 \Leftrightarrow 3$: See Definition 6.8.2.

$2 \Rightarrow 4$: Since E^f is a fragmented family, for every closed non-empty subset $Y \subset X$ the family of restrictions $E^f_Y := \{p_f\restriction_Y : Y \to \mathbb{R}\}$ is (locally) fragmented. Now by Proposition 6.6 (see also Definition 6.8.1) there exists a dense G_δ subset $Y_0 \subset Y$ such that every $y_0 \in Y_0$ is a point of equicontinuity of the family E^f_Y. Clearly this implies that $p_f : Y_0 \to \mathbb{R}$ is continuous for every $p \in E$.

$4 \Rightarrow 1$: We have to show by Theorem 9.12 that the G-map $f_\sharp : X \to \mathrm{RUC}(G)$ is norm fragmented. The action of G on $\mathrm{RUC}(G)$ preserves the norm. Therefore, in this case $\mu_G = \mu$ holds, where μ is the uniform structure generated by the norm. By Lemma 9.4.1 it suffices to check that $f_\sharp\restriction_Y : Y \to (\mathrm{RUC}(G), \mu)$ is locally fragmented for every closed non-empty G-subset Y in X.

By our assumption we can pick a dense G_δ subset Y_0 of Y such that the induced map $p_f : Y_0 \to \mathbb{R}$, $p_f(y) = f(py)$, is continuous for every $p \in E(X)$. It follows that

$$w_f\restriction_{E\times Y_0} : E \times Y_0 \to \mathbb{R}, \qquad w_f(p, y) = f(py),$$

is separately continuous. Since Y_0 is Čech-complete, by Namioka's theorem there exists a dense subset Y_1 of Y_0 such that $w_f\restriction_{E\times Y_0}$ is jointly continuous at every $(p, y_1) \in E \times Y_1$. Our aim is to prove that $f_\sharp\restriction_Y : Y \to \mathrm{RUC}(G)$ is continuous at every $y_1 \in Y_1$. In fact we have to show that every $y_1 \in Y_1$ is a point of equicontinuity of the family of maps $\{{}_gf\restriction_Y : Y \to \mathbb{R}\}_{g\in G}$. By the compactness of E and the inclusion $\breve{G} \subset E$ it is sufficient to check that the map

$$w_f\restriction_{E\times Y} : E \times Y \to \mathbb{R}$$

is continuous at each $(p, y_1) \in E \times Y_1$. In order to check the latter condition fix $\varepsilon > 0$. By the joint continuity of $w_f\restriction_{E\times Y_1} : E \times Y_1 \to \mathbb{R}$, one can choose an open neighborhood U of p in E and an open neighborhood O of y_1 in the space Y such that

$$|f(py_1) - f(qy)| < \varepsilon/3$$

for every $q \in U$ and $y \in O \cap Y_1$. We claim that $|f(py_1) - f(qz)| < \varepsilon$ for every $(q, z) \in U \times O$. Fix such a pair (q, z) and choose $g := g_{q,z} \in G$ such that the corresponding g-translation $\breve{g} : X \to X$ belongs to U and satisfies

$$|f(gz) - f(qz)| < \varepsilon/3.$$

Since Y_1 is dense in Y and $\breve{g} : X \to X$ is continuous, one can pick $a \in Y_1 \cap O$ such that

$$|f(ga) - f(gz)| < \varepsilon/3.$$

Putting these estimates together we obtain the desired inequality $|f(py_1) - f(qz)| < \varepsilon$. Thus, we have shown that $f_\sharp\restriction_Y : Y \to \mathrm{RUC}(G)$ is continuous at every $y_1 \in Y_1$. Since Y_1 is dense in Y, we can conclude by Lemma 6.3.2 that $f_\sharp\restriction_Y$ is locally fragmented. ∎

As a corollary we obtain the following enveloping semigroup characterization of metric RN systems. It certainly can also be derived from Theorem 9.14 and the result of Akin–Auslander–Berg mentioned earlier (see Theorem 5.11).

COROLLARY 14.3. *Let X be a compact metric G-system. The following are equivalent*:

1. *(G, X) is RN.*
2. *For every closed (G-invariant) subspace $Y \subset X$ there exists a dense G_δ subset Y_0 of Y such that for every $p \in E$ the induced map $p : Y_0 \to X$, $p(y) := py$, is continuous.*

Proof. (2)⇒(1) follows by Theorem 14.2. Now we prove (1)⇒(2). Since X is a metric compact space we can choose a countable dense subset $\{f_n : n \in \mathbb{N}\}$ in $C(X)$. By Theorem 7.6.4, $C(X) = \mathrm{Asp}(X)$. By Theorem 14.2 for a given closed (G-invariant) subset $Y \subset X$ and every $n \in \mathbb{N}$ there exists a dense G_δ subset Y_n of Y such that for $p \in E$ the induced map $p_{f_n} : Y_n \to \mathbb{R}$ is continuous. Then it is easy to see that $Y_0 := \bigcap_{n\in\mathbb{N}} Y_n$ is the desired subset of Y. ∎

DEFINITION 14.4. We say that a compact right topological semigroup S is an *$\mathcal{F}$-semigroup* if the family of maps $\{\lambda_p : S \to S\}_{p\in S}$, where $\lambda_p(s) = ps$, is a fragmented family. By Definitions 6.8.1 and 6.1.1 it is equivalent to say that $S^f := \{p_f : S \to \mathbb{R}\}_{p\in S}$ (where $p_f(x) = f(px)$) is a fragmented family for every $f \in C(S)$. Yet another way to formulate the definition is to require that for every non-empty closed subset $A \subset S$, every $f \in C(S)$ and $\varepsilon > 0$ there exists an open subset $O \subset S$ such that $A \cap O$ is non-empty and the subset $f(p(A \cap O))$ is ε-small in $\mathbb{R}$ for every $p \in S$.

Every compact semitopological semigroup is an $\mathcal{F}$-semigroup. The verification is easy applying Namioka's theorem to the map $S \times A \to \mathbb{R}$, $(s,a) \mapsto f(sa)$, where A is a closed non-empty subset of S.

THEOREM 14.5. *Let X be a compact G-system. Consider the following conditions*:

(a) *X is HNS* (*equivalently*, $\mathrm{RN_{app}}$).
(b) *$\breve{G} := \{\breve{g} : X \to X\}_{g \in G}$ is a fragmented family.*
(c) *$E(X) = \{p : X \to X\}_{p \in E(X)}$ is a fragmented family.*
(d) *$(G, E(X))$ is HNS* (*equivalently*, $\mathrm{RN_{app}}$).
(e) *$E(X)$ is an $\mathcal{F}$-semigroup.*

Then we have:

1. *Always*, (a)⇔(b)⇔(c)⇒(d)⇔(e).
2. *If X is point-transitive then* (a)⇔(b)⇔(c)⇔(d)⇔(e).

Proof. 1. (a)⇔(b): The proof follows from Theorem 9.9.

(b)⇔(c): Use Lemma 6.9.

(a)⇒(d): By the definition (G, E) is a G-subsystem of X^X. Since $\mathrm{RN_{app}}$ is closed under subdirect products we deduce that E is also in $\mathrm{RN_{app}}$.

(d)⇔(e): $E(X)$ is an $\mathcal{F}$-semigroup iff $\{\lambda_p : E \to E\}_{p \in E}$ is a fragmented family iff the subfamily $\{\lambda_g : E \to E\}_{g \in G}$ is a fragmented family (use once again Lemma 6.9). The latter condition is equivalent to assertion (d) as follows by the equivalence (a)⇔(b) (applied to the system (G, E)).

2. (d)⇒(a): If x_0 is a transitive point of X then the map $E \to X$, $p \mapsto px_0$, is a continuous onto G-map. Since $\mathrm{RN_{app}}$ is closed under quotients we find that X also belongs to $\mathrm{RN_{app}}$. ∎

COROLLARY 14.6. *G^{Asp} is an $\mathcal{F}$-semigroup for every topological group G.*

Proof. The compact G-system $X := G^{\mathrm{Asp}}$ is $\mathrm{RN_{app}}$ by Theorem 7.6.6. Therefore, Theorem 14.5 implies that the enveloping semigroup $E(G^{\mathrm{Asp}})$ is an $\mathcal{F}$-semigroup. Since $(G^{\mathrm{Asp}}, u_{\mathrm{A}}(e))$ is point-universal (Proposition 7.7), by Proposition 2.6 there exists a G-isomorphism $\phi : (E(G^{\mathrm{Asp}}), i(e)) \to (G^{\mathrm{Asp}}, u_{\mathrm{A}}(e))$ of pointed G-systems. In fact this map is an isomorphism of (right topological) semigroups because $u_{\mathrm{A}}(G)$ is dense in G^{Asp} and $i(G)$ is dense in $E(G^{\mathrm{Asp}})$. ∎

COROLLARY 14.7. *Let (G, X) be a compact HNS system. Then $p : X \to X$ is fragmented* (*equivalently*, *Baire class* 1, *when X is metric*) *for every $p \in E(X)$.*

Proof. Use Theorem 14.5 (and Proposition 6.7.2). ∎

For the definition of Rosenthal compacts see Section 3.

THEOREM 14.8. 1. *Let X be a compact metrizable G-space. For every $f \in \mathrm{Asp}(X)$ the compact space $E^f \subset \mathbb{R}^X$ is a Rosenthal compact.*

2. *Let X be a metrizable compact RN G-space. Then the enveloping semigroup E is a (separable) Rosenthal compact with cardinality $\leq 2^{\aleph_0}$ (in particular, no subspace of E can be homeomorphic to $\beta\mathbb{N}$).*

Proof. 1. Since $f \in \mathrm{Asp}(X)$, by Theorem 14.2, $E^f = \{p_f : X \to \mathbb{R}\}_{p \in E}$ is a fragmented family. In particular, each map $p_f : X \to \mathbb{R}$ is fragmented. Since X is compact and metrizable we can apply Proposition 6.7. Hence, each function $p_f \in E^f$ is of Baire class 1 (on the Polish space X). Therefore, E^f is a Rosenthal compact.

2. $C(X) = \mathrm{Asp}(X)$ by Theorem 7.6.4. It follows by the first assertion that E^f is a Rosenthal compact for every $f \in C(X)$. An application of the dynamical version of the BFT theorem, Theorem 3.2, concludes the proof. ∎

REMARK 14.9. Theorem 14.8.2 can be used to obtain an alternative proof of Proposition 13.2. In fact, as can be seen from Proposition 2.1, the enveloping semigroup of the system (T, X) in Proposition 13.2 has cardinality $2^{2^{\aleph_0}}$.

Our next example is of a metric minimal cascade (T, X) which is not RN yet its enveloping semigroup $E = E(T, X)$: (a) is a separable Rosenthal compact of cardinality $2^{\aleph_0}$, and (b) has the property that each $p \in E$ is of Baire class 1. Thus this example shows that the converse of Theorem 14.8.2 does not hold and neither does that of Corollary 14.7.

EXAMPLE 14.10. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the one-dimensional torus, and let $\alpha \in \mathbb{R}$ be a fixed irrational number and $T_\alpha : \mathbb{T} \to \mathbb{T}$ the rotation by α, $T_\alpha \beta = \beta + \alpha \pmod 1$. We define a topological space X and a continuous map $\pi : X \to \mathbb{T}$ as follows. For $\beta \in \mathbb{T} \setminus \{n\alpha : n \in \mathbb{Z}\}$ the preimage $\pi^{-1}(\beta)$ will be a singleton x_β. On the other hand, for each $n \in \mathbb{Z}$, $\pi^{-1}(n\alpha)$ will consist of exactly two points $x_{n\alpha}^-$ and $x_{n\alpha}^+$. For convenience we will use the notation $\beta^\pm$ ($\beta \in \mathbb{T}$) for points of X, where $(n\alpha)^- = x_{n\alpha}^-$, $(n\alpha)^+ = x_{n\alpha}^+$ and $\beta^- = \beta^+ = x_\beta$ for $\beta \in \mathbb{T} \setminus \{n\alpha : n \in \mathbb{Z}\}$. A basis for the topology at a point of the form x_β, $\beta \in \mathbb{T} \setminus \{n\alpha : n \in \mathbb{Z}\}$, is the collection of sets $\pi^{-1}(\beta - \varepsilon, \beta + \varepsilon)$, $\varepsilon > 0$. For $(n\alpha)^-$ a basis will be the collection of sets of the form $\{(n\alpha)^-\} \cup \pi^{-1}(n\alpha - \varepsilon, n\alpha)$, where $\varepsilon > 0$. Finally, for $(n\alpha)^+$ a basis will be the collection of sets of the form $\{(n\alpha)^+\} \cup \pi^{-1}(n\alpha, n\alpha + \varepsilon)$. It is not hard to check that this defines a compact metrizable zero-dimensional topology on X (in fact X is homeomorphic to the Cantor set) with respect to which π is continuous. Next define $T : X \to X$ by the formula $T\beta^\pm = (\beta + \alpha)^\pm$. Again it is not hard to see that $\pi : (T, X) \to (R_\alpha, \mathbb{T})$ is a homomorphism of dynamical systems and that (T, X) is minimal and not equicontinuous (in fact it is almost-automorphic; see e.g. Veech [57]). In particular (T, X) is not RN.

We now define for each $\gamma \in \mathbb{T}$ two distinct maps $p_\gamma^\pm : X \to X$ by the formulas

$$p_\gamma^+(\beta^\pm) = (\beta+\gamma)^+, \qquad p_\gamma^-(\beta^\pm) = (\beta+\gamma)^-.$$

We leave the verification of the following claims as an exercise.

1. For every $\gamma \in \mathbb{T}$ and every *sequence* $n_i \nearrow \infty$ with $\lim_{i\to\infty} n_i\alpha = \gamma$ and $n_i\alpha < \gamma$ for all i, we have $\lim_{i\to\infty} T^{n_i} = p_\gamma^-$ in $E(T, X)$. An analogous statement holds for p_γ^+.
2. $E(T, X) = \{T^n : n \in \mathbb{Z}\} \cup \{p_\gamma^\pm : \gamma \in \mathbb{T}\}$.
3. The subspace $\{T^n : n \in \mathbb{Z}\}$ inherits from E the discrete topology.
4. The subspace $E(T, X)\setminus\{T^n : n \in \mathbb{Z}\} = \{p_\gamma^\pm : \gamma \in \mathbb{T}\}$ is homeomorphic to the "two arrows" space of Aleksandrov and Urysohn (see [21, p. 212], and also Ellis' example [19, Example 5.29]). It thus follows that E is a separable Rosenthal compact of cardinality $2^{\aleph_0}$.
5. For each $\gamma \in \mathbb{T}$ the complement of the set $C(p_\gamma^\pm)$ of continuity points of $p_\gamma^\pm$ is the countable set $\{\beta^\pm : \beta+\gamma = n\alpha \text{ for some } n \in \mathbb{Z}\}$. In particular each element of E is of Baire class 1.

15. A dynamical version of Todorčević's theorem. A surprising result of Todorčević asserts that a Rosenthal compact X which is not metrizable obeys the following alternative: either X contains an uncountable discrete subspace or it is an at most two-to-one continuous preimage of a compact metric space ([55, Theorem 3]). We present here the following dynamical version.

PROPOSITION 15.1 (A dynamical Todorčević dichotomy). *Let G be a uniformly Lindelöf group and (G, X) a compact system with the property that X is a Rosenthal compact. Then either X contains an uncountable discrete subspace or there exists a metric dynamical system (G, Y) and a G-factor $\pi : (G, X) \to (G, Y)$ such that $|\pi^{-1}(y)| \le 2$ for every $y \in Y$.*

Proof. If we rule out the first alternative in Todorčević's theorem then it follows by that theorem that there exists a compact metric space Z and a continuous map $\phi : X \to Z$ with $|\phi^{-1}(z)| \le 2$ for every $z \in Z$. By [41, Theorem 2.11] there exist a compact metric G-space Y, a continuous onto G-map $f_1 : X \to Y$ and a continuous map $f_2 : Y \to Z$ such that $\phi = f_2 \circ f_1$. Clearly, $|f_1^{-1}(y)| \le 2$ for every $y \in Y$. ∎

We do not know whether Theorem 14.8.2 can be strengthened to the statement that the enveloping semigroup of any compact metric RN system is in fact metric. However, Proposition 15.1 yields the following.

COROLLARY 15.2. *Let X be a metric RN G-system, where G is an arbitrary topological group. Then either the enveloping semigroup $E = E(X)$ contains an uncountable discrete subspace, or it admits a metric G-factor $\pi : (G,E) \to (G,Y)$ such that $|\pi^{-1}(y)| \leq 2$ for every $y \in Y$.*

Proof. This follows directly from Theorem 14.8.2 and Proposition 15.1 because the natural restriction $\breve{G}$ (see Section 3) is second countable (and hence, uniformly Lindelöf). ∎

PROBLEM 15.3. By Theorem 14.8.2 the enveloping semigroup of the G-W example is a separable Rosenthal compact (of cardinality $2^{\aleph_0}$). We do not have a concrete description of this enveloping semigroup and do not even know whether it is metrizable or if it contains an uncountable discrete subspace.

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