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Article

Herglotz Variational Problems Involving Distributed-Order Fractional Derivatives with Arbitrary Smooth Kernels

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Abstract: In this paper, we consider Herglotz-type variational problems dealing with fractional derivatives of distributed-order with respect to another function. We prove necessary optimality conditions for the Herglotz fractional variational problem with and without time delay, with higher-order derivatives, and with several independent variables. Since the Herglotz-type variational problem is a generalization of the classical variational problem, our main results generalize several results from the fractional calculus of variations. To illustrate the theoretical developments included in this paper, we provide some examples.

Keywords: fractional calculus; calculus of variations; Euler–Lagrange equations; Herglotz fractional variational problems; higher-order derivatives; time delay

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1. Introduction

Fractional calculus is as old as ordinary calculus, but it was only at the end of the 20th Century that it managed to attract the attention of many researchers, who in their studies showed that this theory is an important tool to model problems, not only in mathematics, but also in other areas, such as physics, engineering, chemistry, biology, epidemiology, and control theory, among others (see [1–6]). Many important mathematicians such as Euler, Lagrange, Fourier, Abel, Liouville, and Riemann worked in fractional calculus. Fractional derivatives and fractional integrals are generalizations of the notions of integer-order derivatives and integrals and include n-th derivatives and n-fold integrals as special cases. Several different fractional derivatives have been defined, such as Riemann–Liouville, Caputo, Riesz, Erdelyi–Kober, and Hadamard, just to mention a few [7,8]. We note that each definition has its own properties and that many of them are not equivalent to each other. In this paper, we deal with the general notions of distributed-order fractional derivatives with respect to an arbitrary kernel in the Riemann–Liouville and Caputo sense, recently introduced in [9]. One of the advantages of fractional derivatives is that these operators are non-local, thus conserving system memories, as opposed to integer-order derivatives, which are local operators.

The calculus of variations deals with the optimization of functionals involving an integral in which the Lagrangian depends on the independent variable, an unknown function, and its derivative (or derivatives). The classical problem of the calculus of variations was generalized by G. Herglotz in 1930 [10], who presented a new problem involving a first-order initial-value problem that defines a function $z(\cdot)$ in a given interval $[a, b]$ and consists of finding trajectories x and z that extremize the $z(b)$ value. This problem is also known as a generalized variational problem. One of the advantages of this problem is that it allows giving a variational description of non-conservative and dissipative processes, even when the Lagrangian is autonomous [11], which is not possible using the classical variational calculus. Herglotz's problem only attracted the attention of the scientific community in the late Twentieth Century. Thereafter, many important results of the classical calculus of variations were then generalized to Herglotz problems [11–16].

The fractional calculus of variations generalizes the classical variational calculus, replacing the integer-order derivatives by fractional derivatives in the Lagrangian function given in the integral of the functional to be extremized. It was in 1996 that the fractional calculus of variations had a considerable development, being relevant to better describe non-conservative systems in mechanics. Furthermore, this theory provides a more realistic approach to physics, allowing it to consider non-conservative systems in a more natural way [6,17]. Since then, this theory has attracted much attention from a large number of researchers, with several articles published [12,18–23].

In [9], the authors introduced a new fractional operator, combining two fractional operators: fractional derivatives of distributed-order and fractional derivatives with respect to another function. The order of this new fractional derivative is not constant, and this operator is defined using a function of probability, which acts as a distribution of orders of differentiation, multiplied by a fractional derivative. Our objective in this paper is to study several Herglotz-type problems involving this new fractional derivative.

This paper is organized as follows. In Section 2, we present the classical Herglotz variational problem and some necessary background on fractional calculus. In Section 3, we study the fractional Herglotz problem, in four different cases, using distributed-order fractional derivatives with arbitrary kernels. Namely, we study fractional variational problems of the Herglotz-type for the case where the orders of differentiation belong to the interval $[0,1]$, for the higher-order case, for problems involving time delay, and with several independent variables. Finally, we present some examples to illustrate our main results.

2. Preliminaries

2.1. Herglotz's Variational Problem

We begin this section presenting the Herglotz variational problem.

Problem $(\mathcal{P}_{\mathcal{H}})$: Determine trajectories $x \in C^2([a,b],\mathbb{R})$ and $z \in C^1([a,b],\mathbb{R})$ that extremize

$$z(b),$$

where the pair (x,z) satisfies the differential equation:

$$z'(t) = L(t, x(t), x'(t), z(t)), \quad t \in [a,b],$$

with initial condition:

$$z(a) = \gamma \in \mathbb{R},$$

where it is assumed that the Lagrangian L satisfies the following conditions:

(1) $L \in C^1([a,b] \times \mathbb{R}^3, \mathbb{R})$;

(2) $t \mapsto \dfrac{\partial L}{\partial x}(t, x(t), x'(t), z(t))$, $t \mapsto \dfrac{\partial L}{\partial x'}(t, x(t), x'(t), z(t))$, and $t \mapsto \dfrac{\partial L}{\partial z}(t, x(t), x'(t), z(t))$

are differentiable functions for any admissible trajectories (x,z).

It is clear that, if the Lagrangian function L does not depend on the variable z, Problem $(\mathcal{P}_{\mathcal{H}})$ reduces to the classical problem of the calculus of variations. Herglotz proved that a necessary optimality condition for a pair (x,z) to be a local extremizer of Problem $(\mathcal{P}_{\mathcal{H}})$ is given by the following equation, known as the generalized Euler–Lagrange equation [10]:

$$\begin{aligned} &\frac{\partial L}{\partial x}(t, x(t), x'(t), z(t)) - \frac{d}{dt}\frac{\partial L}{\partial x'}(t, x(t), x'(t), z(t)) \\ &\qquad + \frac{\partial L}{\partial z}(t, x(t), x'(t), z(t))\frac{\partial L}{\partial x'}(t, x(t), x'(t), z(t)) = 0, \quad t \in [a,b]. \end{aligned}$$

Obviously, if the Lagrangian function L does not depend on z, then we obtain the famous Euler–Lagrange equation (see [24]):

$$\frac{\partial L}{\partial x}(t, x(t), x'(t)) - \frac{d}{dt}\frac{\partial L}{\partial x'}(t, x(t), x'(t)) = 0, \quad t \in [a, b].$$

2.2. Distributed-Order Fractional Calculus with Respect to an Arbitrary Smooth Kernel

For the notions of distributed-order fractional derivatives with respect to an arbitrary smooth kernel in the Riemann–Liouville and Caputo senses, in the case where $\alpha \in [0, 1]$, we refer the reader to [9].

Next, we recall the extensions of these two derivatives for the case of higher-order derivatives.

Let $n \in \mathbb{N}$ and $\phi : [n-1, n] \to [0, 1]$ be a continuous function such that

$$\int_{n-1}^{n} \phi(\alpha) d\alpha > 0.$$

We start with some needed definitions (see e.g., [25]).

Definition 1. *Let $x : [a, b] \to \mathbb{R}$ be an integrable function and $\psi \in C^n([a, b], \mathbb{R})$ be an increasing function such that $\psi'(t) \neq 0$, for all $t \in [a, b]$. The left and right Riemann–Liouville distributed-order fractional derivatives of a function x with respect to the kernel ψ are defined by:*

$$\mathrm{D}_{a^+}^{\phi(\alpha),\psi} x(t) := \int_{n-1}^{n} \phi(\alpha) \mathrm{D}_{a^+}^{\alpha,\psi} x(t) d\alpha \quad \text{and} \quad \mathrm{D}_{b^-}^{\phi(\alpha),\psi} x(t) := \int_{n-1}^{n} \phi(\alpha) \mathrm{D}_{b^-}^{\alpha,\psi} x(t) d\alpha,$$

where $\mathrm{D}_{a^+}^{\alpha,\psi}$ and $\mathrm{D}_{b^-}^{\alpha,\psi}$ are the left and right ψ-Riemann–Liouville fractional derivatives of order $\alpha \in [n-1, n]$, respectively.

Definition 2. *Let $x, \psi \in C^n([a, b], \mathbb{R})$ be two functions such that ψ is increasing and $\psi'(t) \neq 0$, for all $t \in [a, b]$. The left and right Caputo distributed-order fractional derivatives of x with respect to ψ are defined by:*

$${}^{C}\mathrm{D}_{a^+}^{\phi(\alpha),\psi} x(t) := \int_{n-1}^{n} \phi(\alpha) {}^{C}\mathrm{D}_{a^+}^{\alpha,\psi} x(t) d\alpha \quad \text{and} \quad {}^{C}\mathrm{D}_{b^-}^{\phi(\alpha),\psi} x(t) := \int_{n-1}^{n} \phi(\alpha) {}^{C}\mathrm{D}_{b^-}^{\alpha,\psi} x(t) d\alpha,$$

where ${}^{C}\mathrm{D}_{a^+}^{\alpha,\psi}$ and ${}^{C}\mathrm{D}_{b^-}^{\alpha,\psi}$ are the left and right ψ-Caputo fractional derivatives of order $\alpha \in [n-1, n]$, respectively.

In the following, we denote

$$\mathrm{I}_{a^+}^{n-\phi(\alpha),\psi} x(t) := \int_{n-1}^{n} \phi(\alpha) \mathrm{I}_{a^+}^{n-\alpha,\psi} x(t) d\alpha \quad \text{and} \quad \mathrm{I}_{b^-}^{n-\phi(\alpha),\psi} x(t) := \int_{n-1}^{n} \phi(\alpha) \mathrm{I}_{b^-}^{n-\alpha,\psi} x(t) d\alpha,$$

where $\mathrm{I}_{a^+}^{n-\alpha,\psi}$ and $\mathrm{I}_{b^-}^{n-\alpha,\psi}$ are, respectively, the left and right Riemann–Liouville fractional integrals of order $n - \alpha$ with respect to the kernel ψ. For brevity's sake, we will use the following notation:

$$y_{\psi}^{[m]}(t) := \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^m y(t).$$

To finalize this section, we present the following result, which is fundamental in the proof of our results (cf. [25]).

Theorem 1 (Integration by parts formulas)**.** *Given $x : [a,b] \to \mathbb{R}$ a continuous function and $y \in C^n([a,b],\mathbb{R})$, then*

$$\begin{aligned}\int_a^b x(t)^C\mathrm{D}_{a^+}^{\phi(\alpha),\psi}y(t)dt &= \int_a^b \left(\mathrm{D}_{b^-}^{\phi(\alpha),\psi}\frac{x(t)}{\psi'(t)}\right)y(t)\psi'(t)dt \\ &+ \left[\sum_{k=0}^{n-1}\left(\frac{-1}{\psi'(t)}\frac{d}{dt}\right)^k\left(\mathrm{I}_{b^-}^{n-\phi(\alpha),\psi}\frac{x(t)}{\psi'(t)}\right)y_\psi^{[n-k-1]}(t)\right]_{t=a}^{t=b}\end{aligned}$$

and

$$\begin{aligned}\int_a^b x(t)^C\mathrm{D}_{b^-}^{\phi(\alpha),\psi}y(t)dt &= \int_a^b \left(\mathrm{D}_{a^+}^{\phi(\alpha),\psi}\frac{x(t)}{\psi'(t)}\right)y(t)\psi'(t)dt \\ &+ \left[\sum_{k=0}^{n-1}(-1)^{n-k}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^k\left(\mathrm{I}_{a^+}^{n-\phi(\alpha),\psi}\frac{x(t)}{\psi'(t)}\right)y_\psi^{[n-k-1]}(t)\right]_{t=a}^{t=b}.\end{aligned}$$

3. Main Results

In this section, we study four different types of variational problems of the Herglotz type involving distributed-order fractional derivatives with arbitrary smooth kernels.

3.1. Herglotz Fractional Variational Problem—Case 1

For this problem, we restrict ourselves to the case where $\alpha \in [0,1]$, that is considering the definitions introduced in [9].

Consider two continuous functions $\phi, \varphi : [0,1] \to [0,1]$ satisfying the following conditions:

$$\int_0^1 \phi(\alpha)d\alpha > 0 \qquad \text{and} \qquad \int_0^1 \varphi(\alpha)d\alpha > 0.$$

In what follows, we use the notation:

$$[x,z](t) := \left(t, x(t), {}^C\mathrm{D}_{a^+}^{\phi(\alpha),\psi}x(t), {}^C\mathrm{D}_{b^-}^{\varphi(\alpha),\psi}x(t), z(t)\right)$$

and we denote the partial derivative of L with respect to its ith-coordinate by $\partial_i L$.

We can formulate the problem as follows:

Problem ($\mathcal{P}_{\mathcal{H}1}$): Determine trajectories $x \in C^1([a,b],\mathbb{R})$ and $z \in C^1([a,b],\mathbb{R})$ that extremize (minimize or maximize)

$$z(b)$$

where the pair (x,z) satisfies the differential equation:

$$z'(t) = L\left(t, x(t), {}^C\mathrm{D}_{a^+}^{\phi(\alpha),\psi}x(t), {}^C\mathrm{D}_{b^-}^{\varphi(\alpha),\psi}x(t), z(t)\right), \quad t \in [a,b],$$

and

$$z(a) = \gamma \in \mathbb{R}.$$

It is assumed that ${}^C\mathrm{D}_{a^+}^{\phi(\alpha),\psi}x$ and ${}^C\mathrm{D}_{b^-}^{\varphi(\alpha),\psi}x$ are of class C^1, $L : [a,b] \times \mathbb{R}^4 \longrightarrow \mathbb{R}$ is of class C^1, and the maps exist and are continuous on $[a,b]$:

$$t \mapsto \mathrm{D}_{b^-}^{\phi(\alpha),\psi}\left(\frac{\lambda(t)\cdot\partial_3 L[x,z](t)}{\psi'(t)}\right) \quad \text{and} \quad t \mapsto \mathrm{D}_{a^+}^{\varphi(\alpha),\psi}\left(\frac{\lambda(t)\cdot\partial_4 L[x,z](t)}{\psi'(t)}\right),$$

for all admissible pairs (x,z), where

$$\lambda(t) := e^{-\int_a^t \partial_5 L[x,z](s)ds}, \quad t \in [a,b]. \tag{1}$$

The following result gives a necessary condition of the Euler–Lagrange type and natural boundary conditions, for an admissible pair (x, z) to be a solution of the problem $(\mathcal{P}_{\mathcal{H}1})$.

Theorem 2. *(Necessary optimality conditions for Problem $(\mathcal{P}_{\mathcal{H}1})$) If the pair (x, z) is a solution of Problem $(\mathcal{P}_{\mathcal{H}1})$, then (x, z) satisfies the generalized fractional Euler–Lagrange equation:*

$$\lambda(t)\partial_2 L[x,z](t) + \left(\mathrm{D}_{b^-}^{\phi(\alpha),\psi} \frac{\lambda(t)\partial_3 L[x,z](t)}{\psi'(t)}\right)\psi'(t) + \left(\mathrm{D}_{a^+}^{\varphi(\alpha),\psi} \frac{\lambda(t)\partial_4 L[x,z](t)}{\psi'(t)}\right)\psi'(t) = 0, \quad (2)$$

for all $t \in [a, b]$. Furthermore, if $x(a)$ is free, then (x, z) satisfies the following condition:

$$\mathrm{I}_{b^-}^{1-\phi(\alpha),\psi} \frac{\lambda(t)\partial_3 L[x,z](t)}{\psi'(t)} = \mathrm{I}_{a^+}^{1-\varphi(\alpha),\psi} \frac{\lambda(a)\partial_4 L[x,z](t)}{\psi'(t)}, \quad \text{at } t = a, \quad (3)$$

and if $x(b)$ is free, then (x, z) satisfies the following condition:

$$\mathrm{I}_{b^-}^{1-\phi(\alpha),\psi} \frac{\lambda(t)\partial_3 L[x,z](t)}{\psi'(t)} = \mathrm{I}_{a^+}^{1-\varphi(\alpha),\psi} \frac{\lambda(t)\partial_4 L[x,z](t)}{\psi'(t)}, \quad \text{at } t = b. \quad (4)$$

Proof. Suppose that the pair (x, z) is a solution of Problem $(\mathcal{P}_{\mathcal{H}1})$ and $h \in C^1([a, b], \mathbb{R})$ is an arbitrary function, such that its Caputo distributed-order fractional derivatives, ${}^C\mathrm{D}_{a^+}^{\phi(\alpha),\psi} h$ and ${}^C\mathrm{D}_{b^-}^{\varphi(\alpha),\psi} h$, are continuously differentiable. Define the function $\beta : [a, b] \to \mathbb{R}$ by

$$\beta(t) := \frac{d}{d\epsilon} z[x + \epsilon h](t) \mid_{\epsilon=0} .$$

Since $z(a)$ is fixed, we have that $\beta(a) = 0$. Now, we define $g :]-r, r[\to \mathbb{R}$, where $r > 0$, by

$$g(\epsilon) = z[x + \epsilon h](b).$$

We have that zero is a local extremizer of g, since $z(b)$ is a local extremum, and therefore,

$$\beta(b) := \frac{d}{d\epsilon} z[x + \epsilon h](b) \mid_{\epsilon=0} = g'(0) = 0.$$

Since

$$\begin{aligned}
\beta'(t) &= \frac{d}{dt}\frac{d}{d\epsilon} z[x + \epsilon h](t) \mid_{\epsilon=0} = \frac{d}{d\epsilon}\frac{d}{dt} z[x + \epsilon h](t) \mid_{\epsilon=0} = \frac{d}{d\epsilon} L[x + \epsilon h, z](t) \mid_{\epsilon=0} \\
&= \partial_2 L[x,z](t) \cdot h(t) + \partial_3 L[x,z](t) \cdot {}^C\mathrm{D}_{a^+}^{\phi(\alpha),\psi} h(t) + \partial_4 L[x,z](t) \cdot {}^C\mathrm{D}_{b^-}^{\varphi(\alpha),\psi} h(t) \\
&\quad + \partial_5 L[x,z](t) \cdot \frac{d}{d\epsilon} z[x + \epsilon h](t) \mid_{\epsilon=0} \\
&= \partial_2 L[x,z](t) \cdot h(t) + \partial_3 L[x,z](t) \cdot {}^C\mathrm{D}_{a^+}^{\phi(\alpha),\psi} h(t) + \partial_4 L[x,z](t) \cdot {}^C\mathrm{D}_{b^-}^{\varphi(\alpha),\psi} h(t) \\
&\quad + \partial_5 L[x,z](t) \cdot \beta(t),
\end{aligned}$$

then

$$\beta'(t) - \partial_5 L[x,z](t) \cdot \beta(t) = \partial_2 L[x,z](t) \cdot h(t) + \partial_3 L[x,z](t) \cdot {}^C\mathrm{D}_{a+}^{\phi(\alpha),\psi} h(t) + \partial_4 L[x,z](t) \cdot {}^C\mathrm{D}_{b-}^{\varphi(\alpha),\psi} h(t).$$

Solving this equation, we obtain

$$e^{-\int_a^t \partial_5 L[x,z](s)ds}\beta(t) - \beta(a) = \int_a^t e^{-\int_a^s \partial_5 L[x,z](p)dp} \cdot \Big(\partial_2 L[x,z](s) \cdot h(s) + \partial_3 L[x,z](s) \cdot {}^C\mathrm{D}_{a+}^{\phi(\alpha),\psi} h(s) + \partial_4 L[x,z](s) \cdot {}^C\mathrm{D}_{b-}^{\varphi(\alpha),\psi} h(s)\Big) ds.$$

Considering $t = b$, we obtain

$$\int_a^b \lambda(s) \cdot \Big(\partial_2 L[x,z](s) \cdot h(s) + \partial_3 L[x,z](s) \cdot {}^C\mathrm{D}_{a+}^{\phi(\alpha),\psi} h(s) + \partial_4 L[x,z](s) \cdot {}^C\mathrm{D}_{b-}^{\varphi(\alpha),\psi} h(s)\Big) ds = 0. \quad (5)$$

Using fractional integration by parts (Theorem 1) in Equation (5), we obtain

$$\int_a^b \left(\lambda(s)\partial_2 L[x,z](s) + \left(\mathrm{D}_{b^-}^{\phi(\alpha),\psi} \frac{\lambda(s)\partial_3 L[x,z](s)}{\psi'(s)}\right)\psi'(s) + \left(\mathrm{D}_{a^+}^{\varphi(\alpha),\psi} \frac{\lambda(s)\partial_4 L[x,z](s)}{\psi'(s)}\right)\psi'(s)\right) h(s) ds + \left[h(s)\left(\mathrm{I}_{b^-}^{1-\phi(\alpha),\psi} \frac{\lambda(s)\partial_3 L[x,z](s)}{\psi'(s)}\right) - h(s)\left(\mathrm{I}_{a^+}^{1-\varphi(\alpha),\psi} \frac{\lambda(s)\partial_4 L[x,z](s)}{\psi'(s)}\right)\right]_{s=a}^{s=b} = 0. \quad (6)$$

Considering $h(a) = h(b) = 0$ in Equation (6), we have

$$\int_a^b \left(\lambda(s)\partial_2 L[x,z](s) + \left(\mathrm{D}_{b^-}^{\phi(\alpha),\psi} \frac{\lambda(s)\partial_3 L[x,z](s)}{\psi'(s)}\right)\psi'(s) + \left(\mathrm{D}_{a^+}^{\varphi(\alpha),\psi} \frac{\lambda(s)\partial_4 L[x,z](s)}{\psi'(s)}\right)\psi'(s)\right) h(s) ds = 0.$$

From the fundamental lemma of the calculus of variations (see [24]), we obtain

$$\lambda(s)\partial_2 L[x,z](s) + \left(\mathrm{D}_{b^-}^{\phi(\alpha),\psi} \frac{\lambda(s)\partial_3 L[x,z](s)}{\psi'(s)}\right)\psi'(s) + \left(\mathrm{D}_{a^+}^{\varphi(\alpha),\psi} \frac{\lambda(s)\partial_4 L[x,z](s)}{\psi'(s)}\right)\psi'(s) = 0,$$

for all $s \in [a,b]$, proving the generalized fractional Euler–Lagrange Equation (2). Since $h(a)$ is arbitrary if $x(a)$ is free, using (2) and considering $h(a) \neq 0$ and $h(b) = 0$ in (6), we obtain

$$\mathrm{I}_{a^+}^{1-\varphi(\alpha),\psi} \frac{\lambda(s)\partial_4 L[x,z](s)}{\psi'(s)} = \mathrm{I}_{b^-}^{1-\phi(\alpha),\psi} \frac{\lambda(s)\partial_3 L[x,z](s)}{\psi'(s)}, \quad \text{at } s = a,$$

proving the natural boundary condition (3). Similarly, since $h(b)$ is arbitrary if $x(b)$ is free, considering $h(a) = 0$ and $h(b) \neq 0$ in (6) and using (2), we obtain the natural boundary condition (4). □

Remark 1. *We note that if the Lagrangian L does not depend on z, then we obtain as a corollary Theorem 3.2 of [9].*

3.2. Herglotz Fractional Variational Problem—Case 2

For this problem, let us consider the case where the Lagrangian depends on higher-order distributed-order fractional derivatives (see Definition 2).

Consider the distribution functions ϕ_i, φ_i with domains $[i-1, i]$, $i = 1, \ldots, n$, where $n \in \mathbb{N}$ is fixed, satisfying the following conditions

$$\int_{i-1}^{i} \phi_i(\alpha) d\alpha > 0 \quad \text{and} \quad \int_{i-1}^{i} \varphi_i(\alpha) d\alpha > 0, \text{ for all } i.$$

For the simplicity of notation, we consider the following:

$$[x, z]_n(t) := \left(t, x(t), {}^C\mathrm{D}_{a^+}^{\phi_1(\alpha),\psi} x(t), {}^C\mathrm{D}_{b^-}^{\varphi_1(\alpha),\psi} x(t), \ldots, {}^C\mathrm{D}_{a^+}^{\phi_n(\alpha),\psi} x(t), {}^C\mathrm{D}_{b^-}^{\varphi_n(\alpha),\psi} x(t), z(t)\right).$$

Problem ($\mathcal{P}_{\mathcal{H}n}$): Determine trajectories $x \in C^n([a, b], \mathbb{R})$ and $z \in C^1([a, b], \mathbb{R})$ that extremize

$$z(b)$$

where (x, z) satisfies the differential equation:

$$z'(t) = L[x, z]_n(t), \quad t \in [a, b],$$

subject to the boundary condition:

$$z(a) = \gamma \in \mathbb{R}.$$

We assume that, for each $i = 1, \ldots, n$, ${}^C\mathrm{D}_{a^+}^{\phi_i(\alpha),\psi} x$ and ${}^C\mathrm{D}_{b^-}^{\varphi_i(\alpha),\psi} x$ are all of class C^1, the Lagrangian function $L : [a, b] \times \mathbb{R}^{2n+2} \to \mathbb{R}$ is of class C^1, and the maps exist and are continuous on $[a, b]$:

$$t \mapsto \mathrm{D}_{b^-}^{\phi_i(\alpha),\psi} \left(\frac{\lambda(t) \cdot \partial_{2i+1} L[x, z]_n(t)}{\psi'(t)} \right) \quad \text{and} \quad t \mapsto \mathrm{D}_{a^+}^{\varphi_i(\alpha),\psi} \left(\frac{\lambda(t) \cdot \partial_{2i+2} L[x, z]_n(t)}{\psi'(t)} \right), \tag{7}$$

for all admissible pairs (x, z), where

$$\lambda(t) := e^{-\int_a^t \partial_{2n+3} L[x,z](s) ds}, \quad t \in [a, b]. \tag{8}$$

We are now in a position to present our second result.

Theorem 3. *(Necessary optimality conditions for Problem ($\mathcal{P}_{\mathcal{H}n}$)) If the pair (x, z) is a solution of Problem ($\mathcal{P}_{\mathcal{H}n}$), then (x, z) satisfies the generalized fractional Euler–Lagrange equation:*

$$\lambda(t)\partial_2 L[x, z]_n(t) + \sum_{i=1}^{n} \left[\left(\mathrm{D}_{b^-}^{\phi_i(\alpha),\psi} \frac{\lambda(t)\partial_{2i+1} L[x, z]_n(t)}{\psi'(t)} \right) \psi'(t) + \left(\mathrm{D}_{a^+}^{\varphi_i(\alpha),\psi} \frac{\lambda(t)\partial_{2i+2} L[x, z]_n(t)}{\psi'(t)} \right) \psi'(t) \right] = 0, \tag{9}$$

for all $t \in [a, b]$. Furthermore,

(i) *For a given $i = 0, \ldots, n-1$, if $x_\psi^{[i]}(a)$ is free, then*

$$\sum_{k=i+1}^{n} \left[\left(-\frac{1}{\psi'(t)} \frac{1}{dt} \right)^{k-i-1} \left(\mathrm{I}_{b^-}^{k-\phi_k(\alpha),\psi} \frac{\lambda(t)\partial_{2k+1} L[x, z]_n(t)}{\psi'(t)} \right) + (-1)^{i+1} \left(\frac{1}{\psi'(t)} \frac{1}{dt} \right)^{k-i-1} \left(\mathrm{I}_{a^+}^{k-\varphi_k(\alpha),\psi} \frac{\lambda(t)\partial_{2k+2} L[x, z]_n(t)}{\psi'(t)} \right) \right] = 0, \quad \text{at } t = a; \tag{10}$$

(ii) *For a given* $i = 0, \ldots, n-1$, *if* $x_\psi^{[i]}(b)$ *is free, then*

$$\sum_{k=i+1}^{n} \left[\left(-\frac{1}{\psi'(t)} \frac{1}{dt} \right)^{k-i-1} \left(\mathrm{I}_{b^-}^{k-\phi_k(\alpha),\psi} \frac{\lambda(t)\partial_{2k+1} L[x,z]_n(t)}{\psi'(t)} \right) + (-1)^{i+1} \left(\frac{1}{\psi'(t)} \frac{1}{dt} \right)^{k-i-1} \left(\mathrm{I}_{a^+}^{k-\varphi_k(\alpha),\psi} \frac{\lambda(t)\partial_{2k+2} L[x,z]_n(t)}{\psi'(t)} \right) \right] = 0, \quad at\ t = b. \quad (11)$$

Proof. Let $h \in C^n([a,b], \mathbb{R})$ be an arbitrary function such that $h_\psi^{[i]}(a) = 0$ or $h_\psi^{[i]}(b) = 0$, if $x_\psi^{[i]}(a)$ or $x_\psi^{[i]}(b)$ are fixed, respectively, for each $i = 0, \ldots, n-1$. Defining $\beta : [a,b] \to \mathbb{R}$ by

$$\beta(t) := \frac{d}{d\epsilon} z[x + \epsilon h](t) \mid_{\epsilon=0},$$

then $\beta(a) = 0$, $\beta(b) = 0$, and

$$\beta'(t) = \partial_2 L[x,z]_n(t) \cdot h(t) + \sum_{i=1}^{n} \left(\partial_{2i+1} L[x,z]_n(t) \cdot {}^C\mathrm{D}_{a^+}^{\phi_i(\alpha),\psi} h(t) + \partial_{2i+2} L[x,z]_n(t) \cdot {}^C\mathrm{D}_{b^-}^{\varphi_i(\alpha),\psi} h(t) \right) + \partial_{2n+3} L[x,z]_n(t) \cdot \beta(t). \quad (12)$$

The solution of Equation (12) is defined by

$$e^{-\int_a^t \partial_{2n+3} L[x,z](s)ds} \cdot \beta(t) - \beta(a) = \int_a^t e^{-\int_a^s \partial_{2n+3} L[x,z](p)dp} \cdot \left(\partial_2 L[x,z]_n(s) \cdot h(s) + \sum_{i=1}^{n} \left[\partial_{2i+1} L[x,z]_n(s) \cdot {}^C\mathrm{D}_{a^+}^{\phi_i(\alpha),\psi} h(s) + \partial_{2i+2} L[x,z]_n(s) \cdot {}^C\mathrm{D}_{b^-}^{\varphi_i(\alpha),\psi} h(s) \right] \right) ds. \quad (13)$$

Considering $\lambda(t) = e^{-\int_a^t \partial_{2n+3} L[x,z](s)ds}$ and taking $t = b$ in (13), Theorem 1 allows us to prove that

$$\int_a^b \left(\lambda(s)\partial_2 L[x,z]_n(s) + \sum_{i=1}^{n} \left[\left(\mathrm{D}_{b^-}^{\phi_i(\alpha),\psi} \frac{\lambda(s)\partial_{2i+1} L[x,z]_n(s)}{\psi'(s)} \right) \psi'(s) + \left(\mathrm{D}_{a^+}^{\varphi_i(\alpha),\psi} \frac{\lambda(s)\partial_{2i+2} L[x,z]_n(s)}{\psi'(s)} \right) \psi'(s) \right] \right) h(s)ds$$
$$+ \sum_{i=1}^{n} \sum_{k=0}^{i-1} \left[\left(\left(-\frac{1}{\psi'(s)} \frac{1}{ds} \right)^{k} \left(\mathrm{I}_{b^-}^{i-\phi_i(\alpha),\psi} \frac{\lambda(s)\partial_{2i+1} L[x,z]_n(s)}{\psi'(s)} \right) + (-1)^{i-k} \left(\frac{1}{\psi'(s)} \frac{1}{ds} \right)^{k} \left(\mathrm{I}_{a^+}^{i-\varphi_i(\alpha),\psi} \frac{\lambda(s)\partial_{2i+2} L[x,z]_n(s)}{\psi'(s)} \right) \right) h_\psi^{[i-k-1]}(s) \right]_{s=a}^{s=b} = 0.$$

Since

$$\sum_{i=1}^{n}\sum_{k=0}^{i-1}\Bigg[\Bigg(\bigg(-\frac{1}{\psi'(s)}\frac{1}{ds}\bigg)^{k}\bigg(\mathrm{I}_{b^-}^{i-\phi_i(\alpha),\psi}\frac{\lambda(s)\partial_{2i+1}L[x,z]_n(s)}{\psi'(s)}\bigg)$$
$$+(-1)^{i-k}\bigg(\frac{1}{\psi'(s)}\frac{1}{ds}\bigg)^{k}\bigg(\mathrm{I}_{a^+}^{i-\varphi_i(\alpha),\psi}\frac{\lambda(s)\partial_{2i+2}L[x,z]_n(s)}{\psi'(s)}\bigg)\Bigg)h_{\psi}^{[i-k-1]}(s)\Bigg]_{s=a}^{s=b}$$
$$=\sum_{i=0}^{n-1}h_{\psi}^{[i]}(s)\sum_{k=i+1}^{n}\Bigg[\Bigg(\bigg(-\frac{1}{\psi'(s)}\frac{1}{ds}\bigg)^{k-i-1}\bigg(\mathrm{I}_{b^-}^{k-\phi_k(\alpha),\psi}\frac{\lambda(s)\partial_{2k+1}L[x,z]_n(s)}{\psi'(s)}\bigg)$$
$$+(-1)^{i+1}\bigg(\frac{1}{\psi'(s)}\frac{1}{ds}\bigg)^{k-i-1}\bigg(\mathrm{I}_{a^+}^{k-\varphi_k(\alpha),\psi}\frac{\lambda(s)\partial_{2k+2}L[x,z]_n(s)}{\psi'(s)}\bigg)\Bigg)\Bigg]_{s=a}^{s=b};$$

from the arbitrariness of h and using the fundamental lemma of calculus of variations, we have proven the generalized fractional Euler–Lagrange Equation (9) and the necessary conditions (10) and (11). □

Remark 2. *It is easy to see that Theorem 9 of [25] is a corollary of Theorem 3.*

3.3. Herglotz Fractional Variational Problem—Case 3

It is well known that time delay is a common phenomenon that occurs in many engineering and nature problems. Therefore, it is extremely important to consider when formulating such problems the explicit dependence of a time delay, in order to better understand the evolution of the dynamical systems under observation. Motivated by the importance of considering a time delay in the formulation of variational problems, we now study the Herglotz variational problem with time delay. For the simplicity of presentation, we restrict ourselves to the case where $\alpha \in [0,1]$.

In what follows, τ is a fixed real number such that $0 \leq \tau < b-a$, and in order to simplify the notation, we write:

$$[x,z]_\tau(t) := \Big(t, x(t), x(t-\tau), {}^{C}\mathrm{D}_{a+}^{\phi(\alpha),\psi}x(t), {}^{C}\mathrm{D}_{b-}^{\varphi(\alpha),\psi}x(t), z(t)\Big).$$

Problem $(\mathcal{P}_{\mathcal{H}\tau})$: Determine $x \in C^1([a-\tau,b],\mathbb{R})$ and $z \in C^1([a,b],\mathbb{R})$ that extremize

$$z(b)$$

where

$$z'(t) = L[x,z]_\tau(t), \quad t \in [a,b];$$

$z(a) = \gamma \in \mathbb{R}$ and $x(t) = \mu(t)$ on $[a-\tau,a]$, where $\mu \in C^1([a-\tau,a],\mathbb{R})$ is a given initial function.

It is assumed that ${}^{C}\mathrm{D}_{a^+}^{\phi(\alpha),\psi}x$ and ${}^{C}\mathrm{D}_{b^-}^{\varphi(\alpha),\psi}x$ are of class C^1 and L satisfies the following conditions:

1. $L : [a,b]\times\mathbb{R}^5 \longrightarrow \mathbb{R}$ is of class C^1;
2. The functions exist and are continuous:

$$t \mapsto \mathrm{D}_{(b-\tau)^-}^{\phi(\alpha),\psi}\bigg(\frac{\lambda(t)\cdot\partial_4 L[x,z]_\tau(t)}{\psi'(t)}\bigg) \quad \text{and} \quad t \mapsto \mathrm{D}_{a^+}^{\varphi(\alpha),\psi}\bigg(\frac{\lambda(t)\cdot\partial_5 L[x,z]_\tau(t)}{\psi'(t)}\bigg)$$

on $[a, b-\tau]$, and

$$t \mapsto \mathrm{D}_{b^-}^{\phi(\alpha),\psi}\bigg(\frac{\lambda(t)\cdot\partial_4 L[x,z]_\tau(t)}{\psi'(t)}\bigg) \quad \text{and} \quad t \mapsto \mathrm{D}_{(b-\tau)^+}^{\varphi(\alpha),\psi}\bigg(\frac{\lambda(t)\cdot\partial_5 L[x,z]_\tau(t)}{\psi'(t)}\bigg)$$

on $[b-\tau, b]$, where

$$\lambda(t) := e^{-\int_a^t \partial_6 L[x,z]_\tau(s)ds}, \quad t \in [a,b].$$

The next result presents the necessary optimality conditions for the fractional variational problem of the Herglotz type with time delay.

Theorem 4 (Necessary optimality conditions for Problem $(\mathcal{P}_{\mathcal{H}\tau})$)**.** *If the pair (x,z) is a solution of Problem $(\mathcal{P}_{\mathcal{H}\tau})$, then (x,z) satisfies the generalized fractional Euler–Lagrange equations:*

$$\begin{aligned}
&\lambda(t)\partial_2 L[x,z]_\tau(t) + \lambda(t+\tau)\partial_3 L[x,z]_\tau(t+\tau) \\
&\quad + \left(\mathrm{D}^{\phi(\alpha),\psi}_{(b-\tau)^-}\frac{\lambda(t)\partial_4 L[x,z]_\tau(t)}{\psi'(t)}\right)\psi'(t) + \left(\mathrm{D}^{\varphi(\alpha),\psi}_{a^+}\frac{\lambda(t)\partial_5 L[x,z]_\tau(t)}{\psi'(t)}\right)\psi'(t) \\
&\quad - \int_0^1 \frac{\phi(\alpha)}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{b-\tau}^b (\psi(s)-\psi(t))^{-\alpha}\lambda(s)\partial_4 L[x,z]_\tau(s)dsd\alpha = 0, \ \forall\, t \in [a, b-\tau]
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
&\lambda(t)\partial_2 L[x,z]_\tau(t) + \left(\mathrm{D}^{\phi(\alpha),\psi}_{b^-}\frac{\lambda(t)\partial_4 L[x,z]_\tau(t)}{\psi'(t)}\right)\psi'(t) \\
&\quad + \left(\mathrm{D}^{\varphi(\alpha),\psi}_{(b-\tau)^+}\frac{\lambda(t)\partial_5 L[x,z]_\tau(t)}{\psi'(t)}\right)\psi'(t) \\
&\quad + \int_0^1 \frac{\varphi(\alpha)}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^{b-\tau} (\psi(t)-\psi(s))^{-\alpha}\lambda(s)\partial_5 L[x,z]_\tau(s)dsd\alpha = 0, \ \forall\, t \in [b-\tau, b].
\end{aligned} \tag{15}$$

Furthermore, if $x(b)$ is free, then (x,z) satisfies the natural boundary condition:

$$\mathrm{I}^{1-\phi(\alpha),\psi}_{b^-}\frac{\lambda(t)\partial_4 L[x,z]_\tau(t)}{\psi'(t)} = \mathrm{I}^{1-\varphi(\alpha),\psi}_{a^+}\frac{\lambda(t)\partial_5 L[x,z]_\tau(t)}{\psi'(t)}, \quad \text{at } t = b. \tag{16}$$

Proof. Let $h \in C^1([a-\tau, b], \mathbb{R})$ be an arbitrary function such that $h(t) = 0$, $a-\tau \le t \le a$, ${}^C\mathrm{D}^{\phi(\alpha),\psi}_{a^+}h$, and ${}^C\mathrm{D}^{\varphi(\alpha),\psi}_{b^-}h$ are of class C^1. Defining function $\beta : [a,b] \to \mathbb{R}$ by

$$\beta(t) := \frac{d}{d\epsilon}z[x+\epsilon h]_\tau(t)\ |_{\epsilon=0}$$

we have that $\beta(a) = \beta(b) = 0$. Hence, we obtain

$$\begin{aligned}
\beta'(t) &= \frac{d}{dt}\frac{d}{d\epsilon}z[x+\epsilon h]_\tau(t)\ |_{\epsilon=0} = \frac{d}{d\epsilon}\frac{d}{dt}z[x+\epsilon h]_\tau(t)\ |_{\epsilon=0} = \frac{d}{d\epsilon}L[x+\epsilon h, z]_\tau(t)\ |_{\epsilon=0} \\
&= \partial_2 L[x,z]_\tau(t)\cdot h(t) + \partial_3 L[x,z]_\tau(t)\cdot h(t-\tau) + \partial_4 L[x,z]_\tau(t)\cdot {}^C\mathrm{D}^{\phi(\alpha),\psi}_{a^+}h(t) \\
&\quad + \partial_5 L[x,z]_\tau(t)\cdot {}^C\mathrm{D}^{\varphi(\alpha),\psi}_{b^-}h(t) + \partial_6 L[x,z]_\tau(t)\cdot\beta(t).
\end{aligned} \tag{17}$$

Solving the differential Equation (17), we obtain

$$\begin{aligned}
\lambda(t)\beta(t) - \beta(a) = \int_a^t \lambda(s)\Big(&\partial_2 L[x,z]_\tau(s)\cdot h(s) + \partial_3 L[x,z]_\tau(s)\cdot h(s-\tau) \\
&+ \partial_4 L[x,z]_\tau(s)\cdot {}^C\mathrm{D}^{\phi(\alpha),\psi}_{a^+}h(s) + \partial_5 L[x,z]_\tau(s)\cdot {}^C\mathrm{D}^{\varphi(\alpha),\psi}_{b^-}h(s)\Big)ds.
\end{aligned} \tag{18}$$

Considering $t = b$ and replacing $\beta(a) = \beta(b) = 0$ in (18), we have

$$\int_a^b \Big(\lambda(s)\partial_2 L[x,z]_\tau(s)\cdot h(s) + \lambda(s)\partial_3 L[x,z]_\tau(s)\cdot h(s-\tau) + \lambda(s)\partial_4 L[x,z]_\tau(s)\cdot {}^C\mathrm{D}_{a^+}^{\phi(\alpha),\psi} h(s) + \lambda(s)\partial_5 L[x,z]_\tau(s)\cdot {}^C\mathrm{D}_{b^-}^{\varphi(\alpha),\psi} h(s)\Big) ds = 0. \quad (19)$$

Since

$$\int_a^b \lambda(s)\partial_3 L[x,z]_\tau(s)\cdot h(s-\tau)ds = \int_a^{b-\tau} \lambda(s+\tau)\partial_3 L[x,z]_\tau(s+\tau)\cdot h(s)ds,$$

then, from (19), we obtain

$$\int_a^{b-\tau} \Big(\lambda(s)\partial_2 L[x,z]_\tau(s) + \lambda(s+\tau)\partial_3 L[x,z]_\tau(s+\tau)\Big)\cdot h(s)ds + \int_{b-\tau}^b \lambda(s)\partial_2 L[x,z]_\tau(s)\cdot h(s)ds + \int_a^b \Big(\lambda(s)\partial_4 L[x,z]_\tau(s)\cdot {}^C\mathrm{D}_{a^+}^{\phi(\alpha),\psi} h(s) + \lambda(s)\partial_5 L[x,z]_\tau(s)\cdot {}^C\mathrm{D}_{b^-}^{\varphi(\alpha),\psi} h(s)\Big) ds = 0. \quad (20)$$

Note also that

$$\mathrm{D}_{b^-}^{\phi(\alpha),\psi} \frac{\lambda(s)\partial_4 L[x,z]_\tau(s)}{\psi'(s)} = \mathrm{D}_{(b-\tau)^-}^{\phi(\alpha),\psi} \frac{\lambda(s)\partial_4 L[x,z]_\tau(s)}{\psi'(s)} - \int_0^1 \frac{\phi(\alpha)}{\Gamma(1-\alpha)}\left(\frac{1}{\psi'(s)}\frac{d}{ds}\right)\int_{b-\tau}^b (\psi(p)-\psi(s))^{-\alpha}\lambda(p)\partial_4 L[x,z]_\tau(p)dpd\alpha, \quad (21)$$

for all $s \in [a, b-\tau]$, and

$$\mathrm{D}_{a^+}^{\varphi(\alpha),\psi} \frac{\lambda(s)\partial_5 L[x,z]_\tau(s)}{\psi'(s)} = \mathrm{D}_{(b-\tau)^+}^{\varphi(\alpha),\psi} \frac{\lambda(s)\partial_5 L[x,z]_\tau(s)}{\psi'(s)} + \int_0^1 \frac{\varphi(\alpha)}{\Gamma(1-\alpha)}\left(\frac{1}{\psi'(s)}\frac{d}{ds}\right)\int_a^{b-\tau} (\psi(s)-\psi(p))^{-\alpha}\lambda(p)\partial_5 L[x,z]_\tau(p)dpd\alpha = 0, \quad (22)$$

for all $s \in [b-\tau, b]$. Using Equation (21) and Theorem 1, we conclude that

$$\begin{aligned}\int_a^b \lambda(s)\partial_4 L[x,z]_\tau(s)\cdot {}^C\mathrm{D}_{a^+}^{\phi(\alpha),\psi} h(s)ds = \int_a^{b-\tau}\Bigg(\left(\mathrm{D}_{(b-\tau)^-}^{\phi(\alpha),\psi}\frac{\lambda(s)\partial_4 L[x,z]_\tau(s)}{\psi'(s)}\right)\psi'(s) \\ - \int_0^1 \frac{\phi(\alpha)}{\Gamma(1-\alpha)}\frac{d}{ds}\int_{b-\tau}^b (\psi(p)-\psi(s))^{-\alpha}\lambda(p)\partial_4 L[x,z]_\tau(p)dpd\alpha\Bigg)h(s)ds \\ + \int_{b-\tau}^b \left(\mathrm{D}_{b^-}^{\phi(\alpha),\psi}\frac{\lambda(s)\partial_4 L[x,z]_\tau(s)}{\psi'(s)}\right)\psi'(s)h(s)ds + \left[\left(\mathrm{I}_{b^-}^{1-\phi(\alpha),\psi}\frac{\lambda(s)\partial_4 L[x,z]_\tau(s)}{\psi'(s)}\right)h(s)\right]_{s=a}^{s=b}.\end{aligned} \quad (23)$$

Similarly, using (22) and Theorem 1, we obtain

$$\begin{aligned}\int_a^b \lambda(s)\partial_5 L[x,z]_\tau(s)\cdot {}^C\mathrm{D}_{b^-}^{\varphi(\alpha),\psi} h(s)ds = \int_{b-\tau}^b\Bigg(\left(\mathrm{D}_{(b-\tau)^+}^{\varphi(\alpha),\psi}\frac{\lambda(s)\partial_5 L[x,z]_\tau(s)}{\psi'(s)}\right)\psi'(s) \\ + \int_0^1 \frac{\varphi(\alpha)}{\Gamma(1-\alpha)}\frac{d}{ds}\int_a^{b-\tau} (\psi(s)-\psi(p))^{-\alpha}\lambda(p)\partial_5 L[x,z]_\tau(p)dpd\alpha\Bigg)h(s)ds \\ + \int_a^{b-\tau} \left(\mathrm{D}_{a^+}^{\varphi(\alpha),\psi}\frac{\lambda(s)\partial_5 L[x,z]_\tau(s)}{\psi'(s)}\right)\psi'(s)h(s)ds - \left[\left(\mathrm{I}_{a^+}^{1-\varphi(\alpha),\psi}\frac{\lambda(s)\partial_5 L[x,z]_\tau(s)}{\psi'(s)}\right)h(s)\right]_{s=a}^{s=b}.\end{aligned} \quad (24)$$

Therefore, we obtain

$$
\begin{aligned}
&\int_a^{b-\tau}\Bigg(\lambda(s)\partial_2 L[x,z]_\tau(s)+\lambda(s+\tau)\partial_3 L[x,z]_\tau(s+\tau)\\
&+\left(\mathrm{D}^{\phi(\alpha),\psi}_{(b-\tau)^-}\frac{\lambda(s)\partial_4 L[x,z]_\tau(s)}{\psi'(s)}\right)\psi'(s)\\
&-\int_0^1\frac{\phi(\alpha)}{\Gamma(1-\alpha)}\frac{d}{ds}\int_{b-\tau}^{b}(\psi(p)-\psi(s))^{-\alpha}\lambda(p)\partial_4 L[x,z]_\tau(p)\,dp\,d\alpha\\
&+\left(\mathrm{D}^{\varphi(\alpha),\psi}_{a^+}\frac{\lambda(s)\partial_5 L[x,z]_\tau(s)}{\psi'(s)}\right)\psi'(s)\Bigg)h(s)ds+\int_{b-\tau}^{b}\Bigg(\lambda(s)\partial_2 L[x,z]_\tau(s)\\
&+\left(\mathrm{D}^{\phi(\alpha),\psi}_{b^-}\frac{\lambda(s)\partial_4 L[x,z]_\tau(s)}{\psi'(s)}\right)\psi'(s)+\left(\mathrm{D}^{\varphi(\alpha),\psi}_{(b-\tau)^+}\frac{\lambda(s)\partial_5 L[x,z]_\tau(s)}{\psi'(s)}\right)\psi'(s)\\
&+\int_0^1\frac{\varphi(\alpha)}{\Gamma(1-\alpha)}\frac{d}{ds}\int_a^{b-\tau}(\psi(s)-\psi(p))^{-\alpha}\lambda(p)\partial_5 L[x,z]_\tau(p)\,dp\,d\alpha\Bigg)h(s)ds\\
&+\left[\left(\mathrm{I}^{1-\phi(\alpha),\psi}_{b^-}\frac{\lambda(s)\partial_4 L[x,z]_\tau(s)}{\psi'(s)}\right)h(s)-\left(\mathrm{I}^{1-\varphi(\alpha),\psi}_{a^+}\frac{\lambda(s)\partial_5 L[x,z]_\tau(s)}{\psi'(s)}\right)h(s)\right]_{s=a}^{s=b}=0,
\end{aligned}
$$

introducing (23) and (24) into (20). Therefore, choosing the appropriate h and using the fundamental lemma of calculus of variations, we obtain the generalized fractional Euler–Lagrange Equations (14) and (15) and the natural boundary condition (16). □

Remark 3. *It is clear that Theorem 2 of [25] can be obtained from Theorem 4 in the particular case where the Lagrangian is independent of z.*

3.4. Herglotz Fractional Variational Problem—Case 4

For this the last problem, we consider the case where the state function depends on several independent variables. Here, we have the case where the fractional orders belong to the interval $[0,1]$.

We consider $U=[a,b]\times\Lambda$ and $\Lambda=\prod_{i=1}^{n}[a_i,b_i]$. We denote by $t\in[a,b]$ the time variable and $s=(s_1,\ldots,s_n)\in\Lambda$ the spacial coordinates. We use the notation:

$$[x,z](t,s):=\left(t,s,x(t,s),{}^C\mathrm{D}^{\phi(\alpha),\psi}_{+}x(t,s),{}^C\mathrm{D}^{\varphi(\alpha),\psi}_{-}x(t,s),z(t)\right),$$

where

$${}^C\mathrm{D}^{\phi(\alpha),\psi}_{+}x(t,s)=\left({}^C\mathrm{D}^{\phi(\alpha),\psi}_{a^+}x(t,s),{}^C\mathrm{D}^{\phi(\alpha),\psi}_{a_1^+}x(t,s),\ldots,{}^C\mathrm{D}^{\phi(\alpha),\psi}_{a_n^+}x(t,s)\right)\in\mathbb{R}^{n+1}$$

and

$${}^C\mathrm{D}^{\varphi(\alpha),\psi}_{-}x(t,s)=\left({}^C\mathrm{D}^{\varphi(\alpha),\psi}_{b^-}x(t,s),{}^C\mathrm{D}^{\varphi(\alpha),\psi}_{b_1^-}x(t,s),\ldots,{}^C\mathrm{D}^{\varphi(\alpha),\psi}_{b_n^-}x(t,s)\right)\in\mathbb{R}^{n+1},$$

and ${}^C\mathrm{D}^{\phi(\alpha),\psi}_{a^+}x$ and ${}^C\mathrm{D}^{\varphi(\alpha),\psi}_{b^-}x$ denote the left and right partial distributed-order fractional derivatives of x with respect to the variable t; ${}^C\mathrm{D}^{\phi(\alpha),\psi}_{a_i^+}x$ and ${}^C\mathrm{D}^{\varphi(\alpha),\psi}_{b_i^-}x$, for any $i\in\{1,\ldots,n\}$, denote the left and right partial distributed-order fractional derivatives of x with respect to the variable s_i. Furthermore, it is assumed that the domain of function ψ contains the intervals $[a,b]$ and $[a_i,b_i]$, for $i=1,\ldots,n$.

Problem $(\mathcal{P}_{\mathcal{H}*})$: Determine trajectories $x\in C^1(U,\mathbb{R})$ and $z\in C^1([a,b],\mathbb{R})$ that extremize

$$z(b)$$

such that the pair (x,z) satisfies the differential equation:

$$z'(t) = \int_\Lambda L[x,z](t,s)d^n s, \quad t \in [a,b], \quad d^n s = ds_1 \dots ds_n,$$

and

$$z(a) = \gamma \in \mathbb{R}.$$

Furthermore, we assume that $x(t,s)$ is fixed when $s_i = a_i$ and $s_i = b_i$, for all $t \in [a,b]$. We suppose that ${}^C\mathrm{D}_{a^+}^{\phi(\alpha),\psi} x$, ${}^C\mathrm{D}_{b^-}^{\varphi(\alpha),\psi} x$, ${}^C\mathrm{D}_{a_i^+}^{\phi(\alpha),\psi} x$, and ${}^C\mathrm{D}_{b_i^-}^{\varphi(\alpha),\psi} x$ are of class C^1 for $i = 1,\dots,n$, $L : [a,b] \times \mathbb{R}^{3n+4} \longrightarrow \mathbb{R}$ is a continuously differentiable function, and the maps exist and are continuous on U:

$$(t,s) \mapsto \mathrm{D}_{b^-}^{\phi(\alpha),\psi}\left(\frac{\lambda(t)\cdot\partial_{n+3}L[x,z](t,s)}{\psi'(t)}\right), (t,s) \mapsto \mathrm{D}_{a^+}^{\varphi(\alpha),\psi}\left(\frac{\lambda(t)\cdot\partial_{2n+4}L[x,z](t,s)}{\psi'(t)}\right),$$

and

$$(t,s) \mapsto \mathrm{D}_{b_i^-}^{\phi(\alpha),\psi}\left(\frac{\lambda(t)\cdot\partial_{n+3+i}L[x,z](t,s)}{\psi'(s_i)}\right), (t,s) \mapsto \mathrm{D}_{a_i^+}^{\varphi(\alpha),\psi}\left(\frac{\lambda(t)\cdot\partial_{2n+4+i}L[x,z](t,s)}{\psi'(s_i)}\right),$$

for all admissible pairs (x,z) and for all $i = 1,\dots,n$, where

$$\lambda(t) := e^{-\int_a^t \int_\Lambda \partial_{3n+5}L[x,z](p,s)d^n s dp}, \quad t \in [a,b].$$

Under these assumptions, we can prove our last result.

Theorem 5 (Necessary optimality conditions for Problem $(\mathcal{P}_{\mathcal{H}*})$). *If the pair (x,z) is a solution of Problem $(\mathcal{P}_{\mathcal{H}*})$, then (x,z) satisfies the generalized fractional Euler–Lagrange equation:*

$$\begin{aligned}\lambda(t)\partial_{n+2}L[x,z](t,s) &+ \left(\mathrm{D}_{b^-}^{\phi(\alpha),\psi}\frac{\lambda(t)\partial_{n+3}L[x,z](t,s)}{\psi'(t)}\right)\psi'(t) \\ &+ \left(\mathrm{D}_{a^+}^{\varphi(\alpha),\psi}\frac{\lambda(t)\partial_{2n+4}L[x,z](t,s)}{\psi'(t)}\right)\psi'(t) + \sum_{i=1}^n\left[\left(\mathrm{D}_{b_i^-}^{\phi(\alpha),\psi}\frac{\lambda(t)\partial_{n+3+i}L[x,z](t,s)}{\psi'(s_i)}\right)\psi'(s_i)\right. \\ &\left. + \left(\mathrm{D}_{a_i^+}^{\varphi(\alpha),\psi}\frac{\lambda(t)\partial_{2n+4+i}L[x,z](t,s)}{\psi'(s_i)}\right)\psi'(s_i)\right] = 0, \quad (25)\end{aligned}$$

for all $(t,s) \in U$. Furthermore, if $x(a,\cdot)$ is free, then

$$\int_\Lambda\left(\mathrm{I}_{b^-}^{1-\phi(\alpha),\psi}\frac{\lambda(t)\partial_{n+3}L[x,z](t,s)}{\psi'(t)}\right)d^n s = \int_\Lambda\left(\mathrm{I}_{a^+}^{1-\varphi(\alpha),\psi}\frac{\lambda(t)\partial_{2n+4}L[x,z](t,s)}{\psi'(t)}\right)d^n s, \quad \text{at } t = a, \quad (26)$$

and if $x(b,\cdot)$ is free, then

$$\int_\Lambda\left(\mathrm{I}_{a^+}^{1-\varphi(\alpha),\psi}\frac{\lambda(t)\partial_{2n+4}L[x,z](t,s)}{\psi'(t)}\right)d^n s = \int_\Lambda\left(\mathrm{I}_{b^-}^{1-\phi(\alpha),\psi}\frac{\lambda(t)\partial_{n+3}L[x,z](t,s)}{\psi'(t)}\right)d^n s, \quad \text{at } t = b. \quad (27)$$

Proof. Let $h \in C^1(U,\mathbb{R})$ be an arbitrary function such that its Caputo distributed-order fractional derivatives are continuously differentiable. Because the state function is fixed when $s_i = a_i$ and $s_i = b_i$, we suppose that, for any $i \in \{1,\dots,n\}$, if $s_i = a_i$ or $s_i = b_i$, then $h(t,s) = 0$ for all $t \in [a,b]$. Defining function $\beta : [a,b] \to \mathbb{R}$ by

$$\beta(t) := \frac{d}{d\epsilon} z[x+\epsilon h](t) \mid_{\epsilon=0},$$

then $\beta(a) = \beta(b) = 0$ and

$$\begin{aligned}
\beta'(t) &= \frac{d}{dt}\frac{d}{d\epsilon} z[x+\epsilon h](t) \mid_{\epsilon=0} = \frac{d}{d\epsilon}\frac{d}{dt} z[x+\epsilon h](t) \mid_{\epsilon=0} \\
&= \frac{d}{d\epsilon}\int_\Lambda L[x+\epsilon h, z](t,s) d^n s \mid_{\epsilon=0} = \int_\Lambda \frac{d}{d\epsilon} L[x+\epsilon h, z](t,s) d^n s \mid_{\epsilon=0} \\
&= \int_\Lambda \Bigg(\partial_{n+2} L[x,z](t,s) \cdot h(t,s) + \partial_{n+3} L[x,z](t,s) \cdot {}^C \mathrm{D}_{a^+}^{\phi(\alpha),\psi} h(t,s) \\
&\quad + \partial_{2n+4} L[x,z](t,s) \cdot {}^C \mathrm{D}_{b^-}^{\varphi(\alpha),\psi} h(t,s) + \sum_{i=1}^{n} \Big[\partial_{n+3+i} L[x,z](t,s) \cdot {}^C \mathrm{D}_{a_i^+}^{\phi(\alpha),\psi} h(t,s) \\
&\quad + \partial_{2n+4+i} L[x,z](t,s) \cdot {}^C \mathrm{D}_{b_i^-}^{\varphi(\alpha),\psi} h(t,s) \Big] \Bigg) d^n s + \beta(t) \cdot \int_\Lambda \partial_{3n+5} L[x,z](t) d^n s;
\end{aligned}$$

we obtain

$$\begin{aligned}
\int_a^b \lambda(t) \cdot \int_\Lambda \Bigg(& \partial_{n+2} L[x,z](t,s) \cdot h(t,s) + \partial_{n+3} L[x,z](t,s) \cdot {}^C \mathrm{D}_{a^+}^{\phi(\alpha),\psi} h(t,s) \\
& + \partial_{2n+4} L[x,z](t,s) \cdot {}^C \mathrm{D}_{b^-}^{\varphi(\alpha),\psi} h(t,s) + \sum_{i=1}^{n} \Big[\partial_{n+3+i} L[x,z](t,s) \cdot {}^C \mathrm{D}_{a_i^+}^{\phi(\alpha),\psi} h(t,s) \\
& + \partial_{2n+4+i} L[x,z](t,s) \cdot {}^C \mathrm{D}_{b_i^-}^{\varphi(\alpha),\psi} h(t,s) \Big] \Bigg) d^n s dt = 0.
\end{aligned}$$

Using Theorem 1, we obtain

$$\begin{aligned}
\int_a^b \int_\Lambda \Bigg(& \lambda(t) \partial_{n+2} L[x,z](t,s) + \left(\mathrm{D}_{b^-}^{\phi(\alpha),\psi} \frac{\lambda(t) \partial_{n+3} L[x,z](t,s)}{\psi'(t)} \right) \psi'(t) \\
& + \left(\mathrm{D}_{a^+}^{\varphi(\alpha),\psi} \frac{\lambda(t) \partial_{2n+4} L[x,z](t,s)}{\psi'(t)} \right) \psi'(t) + \sum_{i=1}^{n} \Bigg[\left(\mathrm{D}_{b_i^-}^{\phi(\alpha),\psi} \frac{\lambda(t) \partial_{n+3+i} L[x,z](t,s)}{\psi'(s_i)} \right) \psi'(s_i) \\
& + \left(\mathrm{D}_{a_i^+}^{\varphi(\alpha),\psi} \frac{\lambda(t) \partial_{2n+4+i} L[x,z](t,s)}{\psi'(s_i)} \right) \psi'(s_i) \Bigg] \Bigg) h(t,s) d^n s dt \\
& + \Bigg[\int_\Lambda \left(\mathrm{I}_{b^-}^{1-\phi(\alpha),\psi} \frac{\lambda(t) \partial_{n+3} L[x,z](t,s)}{\psi'(t)} \right) h(t,s) d^n s \\
& - \int_\Lambda \left(\mathrm{I}_{a^+}^{1-\varphi(\alpha),\psi} \frac{\lambda(t) \partial_{2n+4} L[x,z](t,s)}{\psi'(t)} \right) h(t,s) d^n s \Bigg]_{t=a}^{t=b} = 0.
\end{aligned}$$

From the arbitrariness of h and using the fundamental lemma of the calculus of variations, we obtain the generalized fractional Euler–Lagrange Equation (25) and the natural boundary conditions (26) and (27). □

4. Illustrative Examples

In order to illustrate our results, we present three examples.

Example 1. *Consider the following fractional differential equation:*

$$z'(t) = \left({}^C\mathrm{D}_{0^+}^{\phi(\alpha),\psi} x(t) \cdot (\psi(t) - \psi(0))^{\alpha+1} - (\psi(t) - \psi(0))^{\alpha+3}\right)^4 + (t - \cos t)z(t),$$

for $t \in [0,1]$, *where* $z(0) = 4$, $x(0) = 0$ *and* $x(1) = (\psi(1) - \psi(0))^{\alpha+2}$. *Define* $\phi : [0,1] \to [0,1]$ *by*

$$\phi(\alpha) = \frac{2}{\Gamma(\alpha + 3)}.$$

If we consider $\overline{x}(t) = (\psi(t) - \psi(0))^{\alpha+2}$, $t \in [0,1]$, *then by Lemma 1 of [26], we obtain*

$${}^C\mathrm{D}_{0^+}^{\alpha,\psi} \overline{x}(t) = \frac{\Gamma(\alpha + 3)}{2}(\psi(t) - \psi(0))^2.$$

Thus,

$${}^C\mathrm{D}_{0^+}^{\phi(\alpha),\psi} \overline{x}(t) = \int_0^1 \phi(\alpha) {}^C\mathrm{D}_{0^+}^{\alpha,\psi} \overline{x}(t) d\alpha = (\psi(t) - \psi(0))^2.$$

Note that $\overline{x}$ *satisfies the necessary optimality condition (2):*

$$\left(\mathrm{D}_{1^-}^{\phi(\alpha),\psi} \frac{4(\psi(t) - \psi(0))^{\alpha+1} \lambda(t) \left({}^C\mathrm{D}_{0^+}^{\phi(\alpha),\psi} x(t) \cdot (\psi(t) - \psi(0))^{\alpha+1} - (\psi(t) - \psi(0))^{\alpha+3}\right)^3}{\psi'(t)}\right) \cdot \psi'(t) = 0, \quad \forall t \in [0,1],$$

where $\lambda(t) = e^{-\frac{t^2}{2} + \sin t}$. *Therefore,*

$$\overline{x}(t) = (\psi(t) - \psi(0))^{\alpha+2} \quad \text{and} \quad z(t) = 4e^{\frac{t^2}{2} - \sin t}$$

is a candidate to be a local extremizer of the value $z(1)$.

Example 2. *Let* $x \in C^4([1,5], \mathbb{R})$ *and* $z \in C^1([1,5], \mathbb{R})$ *such that the pair* (x, z) *is a solution of the fractional differential equation:*

$$z'(t) = 4t + \left({}^C\mathrm{D}_{1^+}^{\phi_4(\alpha),\psi} x(t) - \frac{(\psi(t) - \psi(1))^2 - \psi(t) + \psi(1)}{\ln(\psi(t) - \psi(1))}\right)^5 - \frac{2}{t} z(t),$$

and for $i = 0, 1, 2, 3$, *we have* $x_\psi^{[i]}(1) = 0$, $x_\psi^{[i]}(5)$ *is free, and* $z(1) = 2$, *where* $\phi_4 : [3,4] \to [0,1]$ *is defined by*

$$\phi_4(\alpha) = \frac{\Gamma(6 - \alpha)}{120}.$$

Note that, if $\overline{x}(t) = (\psi(t) - \psi(1))^5$, $t \in [1,5]$, *then, by Lemma 1 of [26], we obtain*

$${}^C\mathrm{D}_{1^+}^{\alpha,\psi} \overline{x}(t) = \frac{120}{\Gamma(6 - \alpha)}(\psi(t) - \psi(1))^{5-\alpha}.$$

Hence,

$${}^C\mathrm{D}_{1^+}^{\phi_4(\alpha),\psi} \overline{x}(t) = \int_3^4 \phi_4(\alpha) {}^C\mathrm{D}_{1^+}^{\alpha,\psi} \overline{x}(t) d\alpha = \frac{(\psi(t) - \psi(1))^2 - \psi(t) + \psi(1)}{\ln(\psi(t) - \psi(1))}.$$

Note that $\overline{x}$ satisfies the generalized fractional Euler–Lagrange Equation (9):

$$\left(\mathrm{D}_{5^-}^{\phi_4(\alpha),\psi} \frac{5(t^2-1)\left({}^C\mathrm{D}_{1^+}^{\phi_4(\alpha),\psi} x(t) - \frac{(\psi(t)-\psi(1))^2 - \psi(t) + \psi(1)}{\ln(\psi(t)-\psi(1))} \right)^4}{\psi'(t)} \right) \psi'(t) = 0, \quad \forall\, t \in [1,5],$$

and the natural boundary conditions (11), for each $i = 0,1,2,3$,

$$\sum_{k=i+1}^{4} \left(-\frac{1}{\psi(t)} \frac{d}{dt} \right)^{k-i-1} \left(\mathrm{I}_{5^-}^{k-\phi_4(\alpha),\psi} \frac{5(t^2-1)\left({}^C\mathrm{D}_{1^+}^{\phi_4(\alpha),\psi} x(t) - \frac{(\psi(t)-\psi(1))^2 - \psi(t) + \psi(1)}{\ln(\psi(t)-\psi(1))} \right)^4}{\psi'(t)} \right) = 0, \quad \text{at } t = 5.$$

Therefore, by Theorem 3, the pair $(\overline{x}, z)$ where $\overline{x}(t) = (\psi(t) - \psi(1))^5$ and $z(t) = t^2 + \frac{1}{t^2}$, $t \in [1,5]$, is a candidate to be a local extremizer of the value $z(5)$.

Example 3. *Consider $x \in C^1([-2,4],\mathbb{R})$ and $z \in C^1([0,4],\mathbb{R})$ such that the pair (x,z) is a solution of the following fractional differential equation:*

$$z'(t) = \left(x(t-2) - (\psi(4) - \psi(t-2))^3 \right)^2 + \left({}^C\mathrm{D}_{4^-}^{\varphi(\alpha),\psi} x(t) \cdot (\psi(4)-\psi(t))^\alpha - \frac{(\psi(4)-\psi(t))^{\alpha+2}}{2\ln(\psi(4)-\psi(t))} \right)^2 + \frac{1}{\cos(t/4)} z(t),$$

where $z(0) = 3$, $\mu(t) = (\psi(4) - \psi(t))^3$, $t \in [-2,0]$, and $x(4) = 0$. Let $\varphi : [0,1] \to [0,1]$ defined by

$$\varphi(\alpha) = \frac{\Gamma(4-\alpha)}{6}.$$

If $\overline{x}(t) = (\psi(4) - \psi(t))^3$, $t \in [-2,4]$, then, by Lemma 1 of [26], we have

$${}^C\mathrm{D}_{4^-}^{\alpha,\psi} \overline{x}(t) = \frac{6}{\Gamma(4-\alpha)} (\psi(4)-\psi(t))^{3-\alpha},$$

and the distributed-order fractional derivative of $\overline{x}$ is given by

$${}^C\mathrm{D}_{4^-}^{\varphi(\alpha),\psi} \overline{x}(t) = \frac{(\psi(4)-\psi(t))^3 - (\psi(4)-\psi(t))^2}{\ln(\psi(4)-\psi(t))}.$$

Note that $\overline{x}$ satisfies the generalized fractional Euler–Lagrange Equations (14) and (15):

$$2\lambda(t+2)\left(x(t+2) - (\psi(4)-\psi(t+2))^3 \right) + \left(\mathrm{D}_{0^+}^{\varphi(\alpha),\psi} \frac{2\lambda(t)\left({}^C\mathrm{D}_{4^-}^{\varphi(\alpha),\psi} x(t) - \frac{(\psi(4)-\psi(t))^3 - (\psi(4)-\psi(t))^2}{\ln(\psi(4)-\psi(t))} \right)(\psi(4)-\psi(t))^\alpha}{\psi'(t)} \right) \psi'(t) = 0,$$

for all $t \in [0,2]$*, and*

$$\left(\mathrm{D}_{2^+}^{\varphi(\alpha),\psi} \frac{2\lambda(t)\left({}^{C}\mathrm{D}_{4^-}^{\varphi(\alpha),\psi} x(t) - \frac{(\psi(4)-\psi(t))^3 - (\psi(4)-\psi(t))^2}{\ln(\psi(4)-\psi(t))}\right)(\psi(4)-\psi(t))^{\alpha}}{\psi'(t)} \right)\psi'(t)$$

$$+ \int_0^1 \frac{\varphi(\alpha)}{\Gamma(1-\alpha)} \left(\frac{d}{dt} \int_0^2 (\psi(t)-\psi(s))^{-\alpha} 2\lambda(s) \right.$$

$$\left. \left({}^{C}\mathrm{D}_{4^-}^{\varphi(\alpha),\psi} x(s) - \frac{(\psi(4)-\psi(s))^3 - (\psi(4)-\psi(s))^2}{\ln(\psi(4)-\psi(s))}\right)(\psi(4)-\psi(s))^{\alpha} ds \right) d\alpha = 0,$$

for all $t \in [2,4]$*, where* $\lambda(t) = e^{-\int_0^t \sec(4s)ds}$*. Therefore, by Theorem 4,* $(\overline{x}, z)$*, where*

$$\overline{x}(t) = (\psi(4)-\psi(t))^3 \quad \text{and} \quad z(t) = 3(\sec(t/4) + \tan(t/4))^4$$

is a candidate to be a local extremizer of the value $z(4)$*.*

5. Concluding Remarks

In this paper, we studied four cases of the fractional-Herglotz-variational-type problems, where the Lagrangian depends on distributed-order fractional derivatives with arbitrary smooth kernels. In the first case, the distributed order belongs to $[0,1]$, by considering the definitions introduced in [9]. In the second case, we considered the higher-order case, that is when $\alpha \in [n-1, n]$ for a given $n \in \mathbb{N}$, considering the definitions recently introduced in [25]. In the third case, we studied the Herglotz variational problem with time delay, and in the last case, we considered the Herglotz variational problem with several independent variables. We proved the necessary optimality conditions for all of these Herglotz-type problems, and three examples were presented to illustrate our results. To finalize this paper, we point out that our theoretical contributions generalize several results recently proven in the context of the fractional calculus of variations.

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