Contents of Volume 126, Number 3

STUDIA MATHEMATICA

The journal publishes original papers in English, French, German and Russian, mainly in functional analysis, abstract methods of mathematical analysis and probability theory. Usually 3 issues constitute a volume.

Detailed information for authors is given on the inside back cover. Manuscripts and correspondence concerning editorial work should be addressed to

STUDIA MATHEMATICA

Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-6293997
E-mail: studia@impan.gov.pl

Subscription information (1997): Vols. 122–126 (15 issues); $30 per issue.

Correspondence concerning subscription, exchange and back numbers should be addressed to

Institute of Mathematics, Polish Academy of Sciences
Publications Department

Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-6293997
E-mail: publ@impan.gov.pl

Published by the Institute of Mathematics, Polish Academy of Sciences
Typeset in TeX at the Institute
Printed and bound by
hermann & hermann
02-240 Warszawa, ul. Jakobinów 23, tel: 846-79-66, tel/fax: 49-89-95

PRINTED IN POLAND

ISSN 0039-3223

STUDIA MATHEMATICA 126 (3) (1997)

Hermite and Laguerre wave packet expansions

by

JAY EPPERSON (Albuquerque, N.Mex.)

Abstract. This paper describes expansions in terms of Hermite and Laguerre functions similar to the Frazier–Jawerth expansion in Fourier analysis. The wave packets occurring in these expansions are finite linear combinations of Hermite and Laguerre functions. The Shannon sampling formula played an important role in the derivation of the Frazier–Jawerth expansion. In this paper we use the Christoffel–Darboux formula for orthogonal polynomials instead. We obtain estimates on the decay of the Hermite and Laguerre wave packets by investigating two closely related oscillatory integrals.

1. The Hermite expansion. Let $H_k(x)$ denote the kth Hermite polynomial, defined by

$$H_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k} e^{-x^2}, \quad k = 0, 1, \ldots,$$

and let

$$h_k(x) = \pi^{-1/4}(2^k k!)^{-1/2} H_k(x) e^{-x^2/2}$$

denote the kth L^2-normalized Hermite function. Recall that the collection $\{h_k\}_{k=0}^{\infty}$ forms a complete orthonormal basis for $L^2(\mathbb{R})$. The kth Hermite function h_k is an eigenfunction of the Hermite operator $H = -d^2/dx^2 + x^2$ with corresponding eigenvalue $2k+1$. If $m : \mathbb{R} \to \mathbb{C}$ is a bounded function and $\lambda \in \mathbb{R}$, then we let $m(\lambda H)$ denote the bounded linear operator on $L^2(\mathbb{R})$ defined by the property $m(\lambda H) h_k = m(\lambda(2k+1)) h_k$.

Now suppose $\varphi, \psi : \mathbb{R} \to \mathbb{C}$ are C^∞ and satisfy

(i) $\operatorname{supp}\varphi, \operatorname{supp}\psi \subset [1/2, 2]$,
(ii) $|\varphi(x)| \ge c > 0$ if $x \in [3/4, 7/4]$,
(iii) $\sum_{\mu=0}^{\infty} \overline{\varphi}(2^{-\mu}x)\psi(2^{-\mu}x) = 1$ for all $x \ge 1$.

Then for every $f \in L^2(\mathbb{R})$ we have $f = \sum_{\mu=0}^{\infty} \overline{\varphi}(2^{-\mu}H)\psi(2^{-\mu}H) f$, with convergence in the L^2 sense. We will decompose $\overline{\varphi}(2^{-\mu}H) f$ in a particular way.

1991 *Mathematics Subject Classification*: Primary 42C15.

For $N \geq 0$ let l^2_{N+1} denote the Hilbert space of maps $s : \{0, \ldots, N\} \to \mathbb{C}$ with inner product $(s,t) = s(0)\overline{t(0)} + \ldots + s(N)\overline{t(N)}$. Also, for $N \geq 1$ let $z_{N,1} < \ldots < z_{N,N}$ denote the N distinct real zeros of $h_N(x)$. According to the Christoffel–Darboux formula for Hermite polynomials (see [5]),

$$\text{(1)} \qquad \sum_{k=0}^{N} h_k(x)h_k(y) = \left(\frac{N+1}{2}\right)^{1/2} \frac{h_{N+1}(x)h_N(y) - h_{N+1}(y)h_N(x)}{x-y}.$$

It follows that

$$\sum_{k=0}^{N} h_k(z_{N+1,i})h_k(z_{N+1,j}) = 0 \quad \text{if } i \neq j.$$

Therefore, the functions $a_{N+1,1}, \ldots, a_{N+1,N+1} \in l^2_{N+1}$ defined by

$$a_{N+1,j}(k) = \Big(\sum_{n=0}^{N} h_n^2(z_{N+1,j})\Big)^{-1/2} h_k(z_{N+1,j}) = c_{N+1,j}h_k(z_{N+1,j})$$

form an orthonormal basis for l^2_{N+1}.

Now we return to

$$\overline{\varphi}(2^{-\mu}H)f = \sum_{k=0}^{2^\mu-1} \overline{\varphi}(2^{-\mu}(2k+1))(f,h_k)_{L^2(\mathbb{R})}h_k.$$

Regard $f_\mu(k) = \overline{\varphi}(2^{-\mu}(2k+1))(f,h_k)$ as an element of $l^2_{2^\mu}$. Then

$$\overline{\varphi}(2^{-\mu}H)f = \sum_{k=0}^{2^\mu-1}\sum_{n=1}^{2^\mu} (f_\mu, a_{2^\mu,n})_{l^2_{2^\mu}} a_{2^\mu,n}(k)h_k.$$

Since $\overline{\varphi}(2^{-\mu}H)$ and $\psi(2^{-\mu}H)$ commute,

$$\overline{\varphi}(2^{-\mu}H)\psi(2^{-\mu}H)f = \sum_{k=0}^{2^\mu-1}\sum_{n=1}^{2^\mu} (f_\mu, a_{2^\mu,n})_{l^2_{2^\mu}} a_{2^\mu,n}(k)\psi(2^{-\mu}(2k+1))h_k.$$

For $n = 1, \ldots, 2^\mu$ define

$$\varphi_{\mu n}(x) = c_{2^\mu,n}\sum_{k=0}^{2^\mu-1} \varphi(2^{-\mu}(2k+1))h_k(z_{2^\mu,n})h_k(x),$$

$$\psi_{\mu n}(x) = c_{2^\mu,n}\sum_{k=0}^{2^\mu-1} \psi(2^{-\mu}(2k+1))h_k(z_{2^\mu,n})h_k(x).$$

Then we have

$$\overline{\varphi}(2^{-\mu}H)\psi(2^{-\mu}H)f = \sum_{n=1}^{2^\mu} (f,\varphi_{\mu n})_{L^2(\mathbb{R})}\psi_{\mu n}.$$

Therefore, if $f \in L^2(\mathbb{R})$,

$$\text{(2)} \qquad f = \sum_{\mu=0}^{\infty}\sum_{n=1}^{2^\mu} (f,\varphi_{\mu n})_{L^2(\mathbb{R})}\psi_{\mu n}$$

with the sum over μ converging to f in L^2. This identity is an analogue of the Frazier–Jawerth expansion [2] in Fourier analysis. It will be shown in Section 3 that the wave packets $\varphi_{\mu n}$ and $\psi_{\mu n}$ are essentially localized near the point $z_{2^\mu,n}$.

We conclude this section with the L^2 theory associated with (2). Note that condition (ii) on φ has not been used yet.

Proposition 1. *Let φ satisfy conditions* (i), (ii). *Then there exist constants $c_1, c_2 > 0$ such that*

$$c_1\|f\|^2_{L^2} \leq \sum_{\mu=0}^{\infty}\sum_{n=1}^{2^\mu} |(f,\varphi_{\mu n})|^2 \leq c_2\|f\|^2_{L^2}.$$

If φ also satisfies the condition

$$\sum_{\mu=0}^{\infty} |\varphi(2^{-\mu}x)|^2 = 1 \quad \textit{for all } x \geq 1,$$

then $\|f\|^2_{L^2} = \sum_{\mu=0}^{\infty}\sum_{n=1}^{2^\mu} |(f,\varphi_{\mu n})|^2$.

Proof. (The author thanks Michael Frazier for explaining this simple method of proof.) Conditions (i) and (ii) on φ imply that there exist $c_1, c_2 > 0$ such that $c_1 \leq \sum_{\mu=0}^{\infty} |\varphi(2^{-\mu}x)|^2 \leq c_2$ for all $x \geq 1$. It follows that

$$\|f\|^2_{L^2} = \sum_{k=0}^{\infty} |(f,h_k)|^2 \approx \sum_{k=0}^{\infty}\sum_{\mu=0}^{\infty} |\overline{\varphi}(2^{-\mu}(2k+1))(f,h_k)|^2 = \sum_{\mu=0}^{\infty} \|f_\mu\|^2_{l^2_{2^\mu}}$$

where we define $f_\mu \in l^2_{2^\mu}$ by $f_\mu(k) = \overline{\varphi}(2^{-\mu}(2k+1))(f,h_k)$. Since $\{a_{2^\mu,n}\}_{n=1}^{2^\mu}$ is an orthonormal basis for $l^2_{2^\mu}$, $\|f_\mu\|^2_{l^2_{2^\mu}} = \sum_{n=1}^{2^\mu} |(f_\mu, a_{2^\mu,n})_{l^2_{2^\mu}}|^2$. But $(f_\mu, a_{2^\mu,n})_{l^2_{2^\mu}} = (f,\varphi_{\mu n})_{L^2(\mathbb{R})}$, which proves the first part of the proposition. If $\sum_{\mu=0}^{\infty} |\varphi(2^{-\mu}x)|^2 = 1$ for all $x \geq 1$, then $\|f\|^2_{L^2} = \sum_{\mu=0}^{\infty} \|f_\mu\|^2_{l^2_{2^\mu}}$, which proves the second part of the proposition. ■

2. The Laguerre expansion. The Laguerre polynomials $L_k^\alpha(x)$ of order $\alpha > -1$ are defined by

$$L_k^\alpha(x) = e^x x^{-\alpha}\frac{1}{k!}\frac{d^k}{dx^k}(e^{-x}x^{k+\alpha}), \qquad k = 0, 1, \ldots$$

They have the orthogonality properties

$$\int_0^\infty L_j^\alpha(x)L_k^\alpha(x)e^{-x}x^\alpha\,dx = \frac{\Gamma(k+\alpha+1)}{k!}\delta_{jk}.$$

We define the standard Laguerre functions $\mathcal{L}_k^\alpha(x)$ of order $\alpha > -1$ by

$$\mathcal{L}_k^\alpha(x) = \left(\frac{k!}{\Gamma(k+\alpha+1)}\right)^{1/2} e^{-x/2}x^{\alpha/2}L_k^\alpha(x),$$

and we define the Laguerre functions $\mathcal{M}_k^\alpha(x)$ of order $\alpha > -1$ by

$$\mathcal{M}_k^\alpha(x) = (2x)^{1/2}\mathcal{L}_k^\alpha(x^2), \qquad x \ge 0.$$

The standard Laguerre functions $\{\mathcal{L}_k^\alpha\}_{k=0}^\infty$ and the Laguerre functions $\{\mathcal{M}_k^\alpha\}_{k=0}^\infty$ both form complete orthonormal bases for $L^2(\mathbb{R}_+, dx)$. The kth Laguerre function $\mathcal{M}_k^\alpha$ is an eigenfunction of the operator $L = -d^2/dx^2 + x^2 + (\alpha^2 - 1/4)/x^2$ with corresponding eigenvalue $4k + 2\alpha + 2$. If $m : \mathbb{R} \to \mathbb{C}$ is a bounded function and $\lambda \in \mathbb{R}$, then we let $m(\lambda L)$ denote the bounded linear operator on $L^2(\mathbb{R}_+)$ defined by the property $m(\lambda L)\mathcal{M}_k^\alpha = m(\lambda(4k+2\alpha+2))\mathcal{M}_k^\alpha$.

Now suppose $\varphi, \psi : \mathbb{R} \to \mathbb{C}$ are C^∞ and satisfy

(i) $\operatorname{supp}\varphi, \operatorname{supp}\psi \subset [\alpha+1, 4(\alpha+1)]$,
(ii) $|\varphi(x)| \ge c > 0$ if $x \in [5(\alpha+1)/4, 7(\alpha+1)/2]$,
(iii) $\sum_{\mu=0}^\infty \overline{\varphi}(2^{-\mu}x)\psi(2^{-\mu}x) = 1$ for all $x \ge 2\alpha+2$.

Then for every $f \in L^2(\mathbb{R})$ we have $f = \sum_{\mu=0}^\infty \overline{\varphi}(2^{-\mu}L)\psi(2^{-\mu}L)f$, with convergence in the L^2 sense. As in the Hermite case we further decompose the terms in this sum. Let $0 < z_{N,1} < \ldots < z_{N,N}$ denote the N distinct real zeros of $\mathcal{M}_N^\alpha(x)$. According to the Christoffel–Darboux formula for Laguerre polynomials (see [5]),

$$(3)\quad \sum_{k=0}^N \mathcal{M}_k^\alpha(x)\mathcal{M}_k^\alpha(y) = ((N+1)(N+\alpha+1))^{1/2}\frac{\mathcal{M}_N^\alpha(x)\mathcal{M}_{N+1}^\alpha(y) - \mathcal{M}_N^\alpha(y)\mathcal{M}_{N+1}^\alpha(x)}{x^2-y^2}.$$

It follows that

$$\sum_{k=0}^N \mathcal{M}_k^\alpha(z_{N+1,i})\mathcal{M}_k^\alpha(z_{N+1,j}) = 0 \quad \text{if } i \ne j.$$

Therefore, the functions $a_{N+1,1}, \ldots, a_{N+1,N+1} \in l_{N+1}^2$ defined by

$$a_{N+1,j}(k) = \Big(\sum_{n=0}^N (\mathcal{M}_n^\alpha(z_{N+1,j}))^2\Big)^{-1/2}\mathcal{M}_k^\alpha(z_{N+1,j}) = c_{N+1,j}\mathcal{M}_k^\alpha(z_{N+1,j})$$

form an orthonormal basis for l_{N+1}^2. Now for $n = 1, \ldots, 2^\mu[\alpha+2]$ define

$$\varphi_{\mu n}(x) = c_{2^\mu[\alpha+2],n}\sum_{k=0}^{2^\mu[\alpha+2]-1}\varphi(2^{-\mu}(4k+2\alpha+2))\mathcal{M}_k^\alpha(z_{2^\mu[\alpha+2],n})\mathcal{M}_k^\alpha(x),$$

$$\psi_{\mu n}(x) = c_{2^\mu[\alpha+2],n}\sum_{k=0}^{2^\mu[\alpha+2]-1}\psi(2^{-\mu}(4k+2\alpha+2))\mathcal{M}_k^\alpha(z_{2^\mu[\alpha+2],n})\mathcal{M}_k^\alpha(x).$$

Then, as in the Hermite case, if $f \in L^2(\mathbb{R}_+)$,

$$(4)\qquad f = \sum_{\mu=0}^\infty \sum_{n=1}^{2^\mu[\alpha+2]} (f, \varphi_{\mu n})_{L^2(\mathbb{R}_+)}\psi_{\mu n}$$

with the sum over μ converging to f in $L^2(\mathbb{R}_+)$.

It will be shown in Section 4 that $\varphi_{\mu n}$ and $\psi_{\mu n}$ are localized near the point $z_{2^\mu[\alpha+2],n}$. This involves the estimation of an oscillatory integral which is very similar to the integral estimated in Section 3 for the Hermite case. It has been recognized for some time now that expansions in terms of the $\mathcal{M}_k^\alpha$ functions behave in many ways like Hermite expansions. See for example the comments in [6], p. 156, and the earlier reference [3].

We conclude this section with a proposition concerning the L^2 theory associated with (4). Its proof is identical to that of Proposition 1.

PROPOSITION 2. *Let φ satisfy conditions* (i), (ii) *of this section. Then there exist constants $c_1, c_2 > 0$ such that*

$$c_1\|f\|_{L^2(\mathbb{R}_+)}^2 \le \sum_{\mu=0}^\infty \sum_{n=1}^{2^\mu[\alpha+2]} |(f, \varphi_{\mu n})_{L^2(\mathbb{R}_+)}|^2 \le c_2\|f\|_{L^2(\mathbb{R}_+)}^2.$$

If φ also satisfies the condition

$$\sum_{\mu=0}^\infty |\varphi(2^{-\mu}x)|^2 = 1 \quad \textit{for all } x \ge 2\alpha+2,$$

then $\|f\|_{L^2(\mathbb{R}_+)}^2 = \sum_{\mu=0}^\infty \sum_{n=1}^{2^\mu[\alpha+2]} |(f, \varphi_{\mu n})_{L^2(\mathbb{R}_+)}|^2$.

3. Decay estimates (Hermite case). Let $m : \mathbb{R} \to \mathbb{C}$ be C^∞ and compactly supported. Suppose $m(t)$ vanishes for all $t \ge \Lambda > 0$. The purpose of this section is to prove an estimate on the integral kernel of the operator $m(\lambda H)$ for $0 < \lambda < \Lambda$. (If $\lambda \ge \Lambda$, then $m(\lambda H) = 0$.) This immediately leads to decay estimates on the Hermite wave packets $\varphi_{\mu n}$, $\psi_{\mu n}$ since the integral kernel of $m(\lambda H)$, denoted by $m(\lambda H)(x, y)$, is as follows:

$$m(\lambda H)(x,y) = \sum_{k=0}^\infty m(\lambda(2k+1))h_k(x)h_k(y).$$

THEOREM 1. *For every $p > 0$ and $\kappa > 0$ there exists a constant $c < \infty$ independent of $0 < \lambda < \Lambda$ such that*

$$|m(\lambda H)(x,y)| \le \frac{c\lambda^{-1/2}}{(1+\lambda^{-1/2}|x-y|)^p} + \frac{c\lambda^{-1/2}}{(1+\lambda^{-\kappa}|x+y|)^p}.$$

Reference [1] contains a proof with $p = 4$. The method there involved an integral representation for $m(\lambda H)(x,y)$ which we review now for the sake of keeping this paper self-contained. The classical Mehler formula states that if $z \in \mathbb{C}$, $|z| < 1$, then

$$\sum_{k=0}^{\infty} z^k h_k(x)h_k(y) = \pi^{-1/2}(1-z^2)^{-1/2}\exp\left(-\frac{1}{2}\cdot\frac{1+z^2}{1-z^2}(x^2+y^2)+\frac{2z}{1-z^2}xy\right).$$

(Here $(1-z^2)^{-1/2}$ is defined by cutting $\mathbb{C}$ along the negative real axis.) We need an extension of this formula to the boundary of the disk $|z| < 1$.

Let L^2_{f} denote the dense subspace of $L^2(\mathbb{R})$ consisting of finite linear combinations of Hermite functions. Suppose $g \in L^2_{\mathrm{f}}$, $|z| \le 1$, $z \ne \pm 1$. Then

$$\begin{aligned}
\sum_{k=0}^{\infty} z^k (g, h_k)_{L^2(\mathbb{R})} h_k(x)
&= \lim_{r\to 1^-} \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} (rz)^k h_k(x)h_k(y)g(y)\,dy \\
&= \lim_{r\to 1^-} \int_{-\infty}^{\infty} \sum_{k=0}^{\infty} (rz)^k h_k(x)h_k(y)g(y)\,dy \\
&= \lim_{r\to 1^-} \pi^{-1/2}(1-(rz)^2)^{-1/2} \\
&\quad \times \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\cdot\frac{1+(rz)^2}{1-(rz)^2}(x^2+y^2)+\frac{2rz}{1-(rz)^2}xy\right)g(y)\,dy \\
&= \pi^{-1/2}(1-z^2)^{-1/2} \\
&\quad \times \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\cdot\frac{1+z^2}{1-z^2}(x^2+y^2)+\frac{2z}{1-z^2}xy\right)g(y)\,dy.
\end{aligned}$$

The first equality is valid because the sum is a finite sum. The second equality holds by the dominated convergence theorem, since the Hermite functions h_k are known to be uniformly bounded and g is integrable. The last equality holds by another application of the dominated convergence theorem.

Now let $\widehat{m}(\xi) = (2\pi)^{-1}\int_{-\infty}^{\infty} m(x)e^{-ix\xi}\,dx$ be our convention for the

Fourier transform, and continue to let $g \in L^2_{\mathrm{f}}$. Then

$$\begin{aligned}
\int_{-\infty}^{\infty} \sum_{k=0}^{\infty} m(\lambda(2k+1))h_k(x)h_k(y)g(y)\,dy
&= \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} \widehat{m}(\xi)e^{i\lambda(2k+1)\xi}(g,h_k)h_k(x)\,d\xi \\
&= \int_{-\infty}^{\infty} \widehat{m}(\xi)e^{i\lambda\xi} \sum_{k=0}^{\infty} (e^{i2\lambda\xi})^k (g,h_k)h_k(x)\,d\xi.
\end{aligned}$$

If we let $z = z(\xi) = e^{i2\lambda\xi}$, then according to the extended version of Mehler's formula, the last integral equals

$$\pi^{-1/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \widehat{m}(\xi)e^{i\lambda\xi}(1-z^2)^{-1/2} \times \exp\left(-\frac{1}{2}\cdot\frac{1+z^2}{1-z^2}(x^2+y^2)+\frac{2z}{1-z^2}xy\right)g(y)\,dy\,d\xi.$$

Finally, an application of Fubini's theorem shows that this equals

$$\int_{-\infty}^{\infty} G_\lambda(x,y)g(y)\,dy,$$

where

$$G_\lambda(x,y) = \pi^{-1/2}\int_{-\infty}^{\infty} \widehat{m}(\xi)e^{i\lambda\xi}(1-z^2)^{-1/2} \times \exp\left(-\frac{1}{2}\cdot\frac{1+z^2}{1-z^2}(x^2+y^2)+\frac{2z}{1-z^2}xy\right)d\xi.$$

It will be useful to write $G_\lambda(x,y)$ in the more concise form

$$\pi^{-1/2}(2\lambda)^{-1}\int_{-\infty}^{\infty} \widehat{m}(\xi/(2\lambda))e^{i\xi/2}(1-e^{i2\xi})^{-1/2} \times \exp\left(-\frac{i}{2}((x^2+y^2)\cot\xi - 2xy\csc\xi)\right)d\xi. \tag{5}$$

We claim that $G_\lambda(x,y) = m(\lambda H)(x,y)$ pointwise. So far we have

$$\int_{-\infty}^{\infty} G_\lambda(x,y)g(y)\,dy = \int_{-\infty}^{\infty} m(\lambda H)(x,y)g(y)\,dy$$

for every $g \in L^2_{\mathrm{f}}$. We will prove below that the statement of Theorem 1 holds with $G_\lambda(x,y)$ in place of $m(\lambda H)(x,y)$. This implies, in particular, that for every fixed x, $G_\lambda(x,\cdot) \in L^2$. Of course, it is also true that for every fixed x, $m(\lambda H)(x,\cdot)\in L^2$. Since $L^2_{\rm f}$ is dense in L^2, it follows that for every fixed x, $G_\lambda(x,\cdot)=m(\lambda H)(x,\cdot)$ in the L^2 sense. Hence, for every fixed x, $G_\lambda(x,y)=m(\lambda H)(x,y)$ for a.e. y. But $G_\lambda(x,y)$ and $m(\lambda H)(x,y)$ are both continuous functions of y, so $G_\lambda(x,y)=m(\lambda H)(x,y)$ everywhere.

In summary, to prove Theorem 1 *it suffices to prove that the statement of Theorem* 1 *holds with* $G_\lambda(x,y)$ *in place of* $m(\lambda H)(x,y)$.

The strategy is to integrate by parts repeatedly in (5) using

$$\exp\left(-\frac{i}{2}((x^2+y^2)\cot\xi-2xy\csc\xi)\right)$$
$$=-2i\sin^2\xi\,(x^2+y^2-2xy\cos\xi)^{-1}\frac{d}{d\xi}\exp\left(-\frac{i}{2}((x^2+y^2)\cot\xi-2xy\csc\xi)\right).$$

We assume for now that $x\neq\pm y$, so that $(x^2+y^2-2xy\cos\xi)^{-1}$ is well behaved as a function of ξ. Two integrations by parts were performed in [1] to get a decay exponent $p=4$. This involved the analysis of many separate integrals. Here an inductive scheme is set up to allow integration by parts arbitrarily many times. Define operators

$$A_0f(\xi)=f(\xi),$$
$$A_1f(\xi)=\frac{d}{d\xi}f(\xi),$$
$$B_0f(\xi)=\sin^2\xi\,f(\xi),$$
$$B_1f(\xi)=\frac{d}{d\xi}\sin^2\xi\,f(\xi),$$
$$C_0f(\xi)=(x^2+y^2-2xy\cos\xi)^{-1}f(\xi),$$
$$C_1f(\xi)=\frac{d}{d\xi}(x^2+y^2-2xy\cos\xi)^{-1}f(\xi).$$

Let E_k denote the set of maps $\sigma:\{1,\ldots,k\}\to\{0,1\}$. If $\sigma\in E_k$, let $\tau(\sigma)=\sum_{i=1}^k\sigma(i)$. After doing k integrations by parts we get a sum of integrals of the form

$$(6)\quad c\lambda^{-1}\int_{-\infty}^{\infty}(A_{\sigma_1(k)}\ldots A_{\sigma_1(1)}\widehat{m}(\xi/(2\lambda)))(B_{\sigma_2(k)}\ldots B_{\sigma_2(1)}e^{i\xi/2}(1-e^{i2\xi})^{-1/2})$$
$$\times(C_{\sigma_3(k)}\ldots C_{\sigma_3(1)}1)\exp\left(-\frac{i}{2}((x^2+y^2)\cot\xi-2xy\csc\xi)\right)d\xi.$$

Here $\sigma_1,\sigma_2,\sigma_3\in E_k$ and c is some unimportant constant depending only on k. The maps $\sigma_1,\sigma_2,\sigma_3$ in (6) are required to have the property that for each $i=1,\ldots,k$, exactly one of the numbers $\sigma_1(i),\sigma_2(i),\sigma_3(i)$ equals one. (It is easy to check that the points $\xi=n\pi$ present no obstacle to doing these integrations by parts.) Now we consider the groupings of terms in (6).

LEMMA 1. *For every $L>0$ there exists some c depending only on m and k such that*
$$|A_{\sigma(k)}\ldots A_{\sigma(1)}\widehat{m}(\xi/(2\lambda))|\le c\lambda^{-\tau(\sigma)}(1+\lambda^{-1}|\xi|)^{-L}.$$

Proof. This follows immediately from the fact that m is C^∞ and compactly supported. ∎

LEMMA 2. *There exists a constant c depending only on k such that*
$$|B_{\sigma(k)}\ldots B_{\sigma(1)}e^{i\xi/2}(1-e^{i2\xi})^{-1/2}|\le c|\sin\xi|^{2k-\tau(\sigma)-1/2}.$$

Proof. The proof is by induction on k. Note that
$$B_{\sigma(k)}\ldots B_{\sigma(1)}e^{i\xi/2}(1-e^{i2\xi})^{-1/2}$$
can be written in a natural way as a linear combination of functions of the form
$$(7)\qquad \sin^l\xi\cos^m\xi\,e^{in\xi/2}(1-e^{i2\xi})^{-p/2},$$
where $l,m,n,p\in\mathbb{N}_0=\{0,1,\ldots\}$. The induction hypothesis is that each such term in this linear combination is bounded by $c|\sin\xi|^{2k-\tau(\sigma)-1/2}$. It is easy to check the truth of this hypothesis when $k=1$. So let $\sigma\in E_{k+1}$, $k\ge1$. If $\sigma(k+1)=0$, then $B_{\sigma(k+1)}$ applied to (7) results in a function bounded by
$$c|\sin\xi|^{2k+2-(\sigma(1)+\ldots+\sigma(k))-1/2}=c|\sin\xi|^{2(k+1)-\tau(\sigma)-1/2}.$$
On the other hand, if $\sigma(k+1)=1$, then $B_{\sigma(k+1)}$ applied to (7) results in a linear combination of functions each of which is bounded by
$$c|\sin\xi|^{2k+1-(\sigma(1)+\ldots+\sigma(k))-1/2}=c|\sin\xi|^{2(k+1)-\tau(\sigma)-1/2}.\ ∎$$

LEMMA 3. *There exists a constant c depending only on k such that*
$$|C_{\sigma(k)}\ldots C_{\sigma(1)}1|\le c|\xi-2\pi n|^{-\tau(\sigma)}(x^2+y^2-2xy\cos\xi)^{-k}$$
for all $\xi\in[-\pi/3+2\pi n,\pi/3+2\pi n]$, $n\in\mathbb{Z}$.

Proof. By periodicity it is good enough to consider the $n=0$ case. The proof is by induction on k. Note that $C_{\sigma(k)}\ldots C_{\sigma(1)}1$ can be written in a natural way as a linear combination of functions of the form
$$(8)\qquad (xy)^l\sin^m\xi\cos^n\xi(x^2+y^2-2xy\cos\xi)^{-(k+l)}$$
where $l,m,n\in\mathbb{N}_0$. The induction hypothesis is that each such term in this linear combination is bounded by $c|\xi|^{-\tau(\sigma)}(x^2+y^2-2xy\cos\xi)^{-k}$ for $\xi\in[-\pi/3,\pi/3]$. We rely on the fact that
$$(9)\qquad |xy\sin\xi|(x^2+y^2-2xy\cos\xi)^{-1}\le c|\xi|^{-1}$$
for $\xi\in[-\pi/3,\pi/3]$. Using this, it is easy to check the truth of the hypothesis when $k=1$. So let $\sigma\in E_{k+1}$, $k\ge1$. If $\sigma(k+1)=0$, then $C_{\sigma(k+1)}$ applied

to (8) results in a function bounded by

$$c|\xi|^{-(\sigma(1)+\ldots+\sigma(k))}(x^2+y^2-2xy\cos\xi)^{-(k+1)} = c|\xi|^{-\tau(\sigma)}(x^2+y^2-2xy\cos\xi)^{-(k+1)}$$

for $\xi\in[-\pi/3,\pi/3]$. On the other hand, if $\sigma(k+1)=1$, then $C_{\sigma(k+1)}$ applied to (8) results in a sum of functions

$$(10)\quad \left(\frac{d}{d\xi}(x^2+y^2-2xy\cos\xi)^{-1}\right)\cdot(8)+(x^2+y^2-2xy\cos\xi)^{-1}\cdot\frac{d}{d\xi}(8).$$

By (9) and the induction hypothesis, the first term in (10) is bounded by

$$c|\xi|^{-1}(x^2+y^2-2xy\cos\xi)^{-1}|\xi|^{-(\sigma(1)+\ldots+\sigma(k))}(x^2+y^2-2xy\cos\xi)^{-k} = c|\xi|^{-\tau(\sigma)}(x^2+y^2-2xy\cos\xi)^{-(k+1)}.$$

The second term in (10) is bounded by a linear combination of functions each of which has the same upper bound $c|\xi|^{-\tau(\sigma)}(x^2+y^2-2xy\cos\xi)^{-(k+1)}$ for $\xi\in[-\pi/3,\pi/3]$. ■

LEMMA 4. *There exists a constant c depending only on k such that*

$$|C_{\sigma(k)}\ldots C_{\sigma(1)}1|\le c(x^2+y^2-2xy\cos\xi)^{-k}$$

for all $\xi\in[-2\pi/3+2\pi n,-\pi/3+2\pi n]\cup[\pi/3+2\pi n,2\pi/3+2\pi n]$, $n\in\mathbb{Z}$.

Proof. The proof is similar to that of Lemma 3, except that we use the fact that $|xy\sin\xi|(x^2+y^2-2xy\cos\xi)^{-1}\le c$ for ξ in the allowed region. ■

LEMMA 5. *There exists a constant c depending only on k such that*

$$|C_{\sigma(k)}\ldots C_{\sigma(1)}1|\le c|\xi-\pi-2\pi n|^{-\tau(\sigma)}(x^2+y^2-2xy\cos\xi)^{-k}$$

for all $\xi\in[2\pi/3+2\pi n,4\pi/3+2\pi n]$, $n\in\mathbb{Z}$.

Proof. The proof is similar to that of Lemma 3, except that we use the fact that $|xy\sin\xi|\,(x^2+y^2-2xy\cos\xi)^{-1}\le c|\xi-\pi-2\pi n|^{-1}$ for $\xi\in[2\pi/3+2\pi n,4\pi/3+2\pi n]$. ■

Proof of Theorem 1. First note that for every $L>0$ there exists some $c<\infty$ such that

$$|G_\lambda(x,y)|\le c\lambda^{-1}\int_{-\infty}^{\infty}(1+|\xi|/\lambda)^{-L}|\sin\xi|^{-1/2}\,d\xi.$$

It is not difficult to show that this has a bound of the form $c\lambda^{-1/2}$, for $0<\lambda<\Lambda$. This suffices to prove the desired estimate if $\lambda^{-1/2}|x-y|\le1$ or $\lambda^{-\kappa}|x+y|\le1$.

So assume $\lambda^{-1/2}|x-y|>1$ and $\lambda^{-\kappa}|x+y|>1$. In this case we estimate $G_\lambda(x,y)$ using (6). Define sets

$$S_1=\bigcup_{n\in\mathbb{Z}}[-\pi/3+2\pi n,\pi/3+2\pi n],$$
$$S_2=\bigcup_{n\in\mathbb{Z}}([-2\pi/3+2\pi n,-\pi/3+2\pi n]\cup[\pi/3+2\pi n,2\pi/3+2\pi n]),$$
$$S_3=\bigcup_{n\in\mathbb{Z}}[2\pi/3+2\pi n,4\pi/3+2\pi n].$$

Note that $\mathbb{R}$ is the essentially disjoint union of S_1,S_2,S_3. By Lemmas 1, 2, and 3, the part of the integral (6) over S_1 is bounded by

$$c\lambda^{-1}\sum_{n\in\mathbb{Z}}\int_{-\pi/3}^{\pi/3}\lambda^{-\tau(\sigma_1)}(1+\lambda^{-1}|\xi+2\pi n|)^{-L} \times|\xi|^{2k-\tau(\sigma_2)-1/2}|\xi|^{-\tau(\sigma_3)}(x^2+y^2-2xy\cos\xi)^{-k}\,d\xi.$$

Since $x^2+y^2-2xy\cos\xi\ge(x-y)^2/2$ for $\xi\in[-\pi/3,\pi/3]$ and $\tau(\sigma_1)+\tau(\sigma_2)+\tau(\sigma_3)=k$, the above quantity is bounded by

$$\frac{c\lambda^{-1/2}}{(\lambda^{-1/2}|x-y|)^{2k}}\sum_{n\in\mathbb{Z}}\int_{-\pi/3}^{\pi/3}(1+\lambda^{-1}|\xi+2\pi n|)^{-L}(\lambda^{-1}|\xi|)^{k+\tau(\sigma_1)-1/2}\,\lambda^{-1}d\xi.$$

By taking L sufficiently large we see (with a little work) that this is bounded by a quantity of the form $c\lambda^{-1/2}(\lambda^{-1/2}|x-y|)^{-2k}$, for $0<\lambda<\Lambda$. Next, by Lemmas 1, 2 and 4, the part of the integral (6) over S_2 is bounded by

$$c\lambda^{-1}\sum_{n\in\mathbb{Z}}\Big(\int_{-2\pi/3}^{-\pi/3}+\int_{\pi/3}^{2\pi/3}\Big)\lambda^{-\tau(\sigma_1)}(1+\lambda^{-1}|\xi+2\pi n|)^{-L} \times(x^2+y^2-2xy\cos\xi)^{-k}\,d\xi.$$

Since $x^2+y^2-2xy\cos\xi\ge(x^2+y^2)/2$ for $\xi\in[-2\pi/3,-\pi/3]\cup[\pi/3,2\pi/3]$, we find that this is bounded by a quantity of the form $c\lambda^K(x^2+y^2)^{-k}$, for $0<\lambda<\Lambda$. Here K can be taken as large as desired, by taking L sufficiently large. Finally, by Lemmas 1, 2 and 5, the part of the integral (6) over S_3 is bounded by

$$c\lambda^{-1}\sum_{n\in\mathbb{Z}}\int_{2\pi/3}^{4\pi/3}\lambda^{-\tau(\sigma_1)}(1+\lambda^{-1}|\xi+2\pi n|)^{-L} \times|\xi-\pi|^{2k-\tau(\sigma_2)-1/2}|\xi-\pi|^{-\tau(\sigma_3)}(x^2+y^2-2xy\cos\xi)^{-k}\,d\xi.$$

Since $x^2+y^2-2xy\cos\xi\ge(x+y)^2/2$ for $\xi\in[2\pi/3,4\pi/3]$ and $\tau(\sigma_1)+$

$\tau(\sigma_2)+\tau(\sigma_3)=k$, the above quantity is bounded by

$$\frac{c\,\lambda^{-1-\tau(\sigma_1)}}{|x+y|^{2k}}\sum_{n\in\mathbb{Z}}\int_{2\pi/3}^{4\pi/3}(1+\lambda^{-1}|\xi+2\pi n|)^{-L}|\xi-\pi|^{k+\tau(\sigma_1)-1/2}\,d\xi.$$

This is bounded by a quantity of the form $c\lambda^K|x+y|^{-2k}$, for $0<\lambda<\Lambda$. Here K can be taken as large as desired, by taking L sufficiently large. These estimates suffice to prove the theorem, since k can be taken arbitrarily large. ■

COROLLARY 1. *Let $\varphi_{\mu n}$, $\psi_{\mu n}$ be the Hermite wave packets from Section* 1. *Also, let $c_{2^\mu,n}$ and $z_{2^\mu,n}$ be as in Section* 1. *Then for every $p>0$ and $\kappa>0$ there exists a constant $c<\infty$ independent of μ, n such that*

$$|\varphi_{\mu n}(x)|,|\psi_{\mu n}(x)|\le c_{2^\mu,n}\left\{\frac{c\,2^{\mu/2}}{(1+2^{\mu/2}|x-z_{2^\mu,n}|)^p}+\frac{c\,2^{\mu/2}}{(1+2^{\mu\kappa}|x+z_{2^\mu,n}|)^p}\right\}.$$

The constants $c_{2^\mu,n}$ appearing in this estimate can be analyzed further. Recall that for $N\ge1$, $c_{N,n}=(\sum_{k=0}^{N-1}h_k^2(z_N,n))^{-1/2}$. Using the Christoffel–Darboux formula (1) and the relation $H_n'=2nH_{n-1}$ it can be shown that

$$\sum_{k=0}^{N-1}h_k^2(z_{N,n})=Nh_{N-1}^2(z_{N,n})\qquad\text{for } n=1,\dots,N.$$

Some information about $h_{N-1}(z_{N,n})$ may be obtained from asymptotic formulas for Hermite polynomials, such as those due to Plancherel and Rotach (see [5]).

Remark. Another consequence of Theorem 1 is that the Hermite–Triebel–Lizorkin spaces $H_p^{\alpha q}$ introduced in [1] are well-defined for the whole parameter range $0<p<\infty$, $0<q\le\infty$. That is, the restriction $p,q>1$ in Theorem 1.1 of [1] may be dropped. This leaves open the very interesting problem of characterizing the "Hermite–Hardy" spaces H_p^{02}, $0<p\le1$.

4. Decay estimates (Laguerre case). Recall the Laguerre operator $L=-d^2/dx^2+x^2+(\alpha^2-1/4)/x^2$ from Section 2. Let $m:\mathbb{R}\to\mathbb{C}$ be C^∞ and compactly supported, and suppose $m(t)$ vanishes for all $t\ge\Lambda(2\alpha+2)>0$. The purpose of this section is to prove an estimate on the integral kernel of the operator $m(\lambda L)$ for $0<\lambda<\Lambda$. (If $\lambda\ge\Lambda$, then $m(\lambda L)=0$.) Of course, this kernel is given by

$$m(\lambda L)(x,y)=\sum_{k=0}^{\infty}m(\lambda(4k+2\alpha+2))\mathcal{M}_k^\alpha(x)\mathcal{M}_k^\alpha(y).$$

THEOREM 2. *Let $\alpha\ge-1/2$. Then for every $p>0$ there exists a constant $c<\infty$ independent of $0<\lambda<\Lambda$ such that*

$$|m(\lambda L)(x,y)|\le\frac{c\,\lambda^{-1/2}}{(1+\lambda^{-1/2}|x-y|)^p},\qquad x,y>0.$$

As in the Hermite case, we begin by deriving an integral representation. There is an old formula for Laguerre polynomials (see [4], Theorem 69) which, when translated into the $\mathcal{M}_k^\alpha$ functions, states that for $x,y>0$, $z\in\mathbb{C}$, $|z|<1$,

$$\sum_{k=0}^{\infty}z^k\mathcal{M}_k^\alpha(x)\mathcal{M}_k^\alpha(y)=2(xy)^{\alpha+1/2}(1-z)^{-1-\alpha}e^{-(1+z)(x^2+y^2)/(2(1-z))}\times\sum_{n=0}^{\infty}\frac{(x^2y^2z)^n}{n!\Gamma(n+\alpha+1)(1-z)^{2n}}.\tag{11}$$

(Here $(1-z)^{-1-\alpha}$ is defined by cutting $\mathbb{C}$ along the negative real axis.) We need an extension of this formula to the boundary of the disk $|z|<1$. For the purpose of this section let $L_{\rm f}^2$ denote the dense subspace of $L^2(\mathbb{R}_+)$ consisting of finite linear combinations of $\mathcal{M}_k^\alpha$ functions.

LEMMA 6. *Let $\alpha\ge-1/2$. Suppose $z=e^{i\theta}$, $z\ne1$, and let $g\in L_{\rm f}^2$. Then*

$$\sum_{k=0}^{\infty}z^k(g,\mathcal{M}_k^\alpha)_{L^2(\mathbb{R}_+)}\mathcal{M}_k^\alpha(x)=2^{1+\alpha}(1-e^{i\theta})^{-1-\alpha}|\sin\theta/2|^\alpha\times\int_0^\infty(xy)^{1/2}\exp\left(-\frac{i}{2}(x^2+y^2)\cot\frac{\theta}{2}\right)J_\alpha(xy|\csc\theta/2|)g(y)\,dy.$$

Proof. Let $x>0$ be fixed. According to (11) we have

$$\begin{aligned}\sum_{k=0}^{\infty}z^k(g,\mathcal{M}_k^\alpha)_{L^2(\mathbb{R}_+)}\mathcal{M}_k^\alpha(x)&=\lim_{r\to1^-}\sum_{k=0}^{\infty}\int_0^\infty(rz)^k\mathcal{M}_k^\alpha(x)\mathcal{M}_k^\alpha(y)g(y)\,dy\\&=\lim_{r\to1^-}\int_0^\infty\sum_{k=0}^{\infty}(rz)^k\mathcal{M}_k^\alpha(x)\mathcal{M}_k^\alpha(y)g(y)\,dy\\&=\lim_{r\to1^-}\int_0^\infty2(xy)^{\alpha+1/2}(1-re^{i\theta})^{-1-\alpha}e^{-(1+re^{i\theta})(x^2+y^2)/(2(1-re^{i\theta}))}\\&\quad\times\sum_{n=0}^{\infty}\frac{(x^2y^2re^{i\theta})^n}{n!\Gamma(n+\alpha+1)(1-re^{i\theta})^{2n}}\,g(y)\,dy.\end{aligned}\tag{12}$$

The first equality is valid because the sum is a finite sum. The second equality holds by an application of the dominated convergence theorem. This is allowed because the Laguerre functions $\mathcal{M}_k^\alpha$ are uniformly bounded when $\alpha \ge -1/2$ (see [6], Section 1.5) and g is integrable. Now we need to evaluate the last limit in (12). Since $g \in L_{\mathrm{f}}^2$ and $\alpha \ge -1/2$, $|g(y)| \le c(1+y^a)e^{-y^2/2}$ for some $c < \infty$ and $a \ge 0$. Note that

$$e^{-y^2/2}|e^{-(1+re^{i\theta})y^2/(2(1-re^{i\theta}))}| \le \exp\left(-\frac{1}{4}\min\{1, 1-\cos\theta\}y^2\right).$$

Also note that for every fixed $x \ge 0$ and every $\varepsilon > 0$ there exists some $c < \infty$ such that

$$\left|\sum_{n=0}^{\infty} \frac{(x^2y^2re^{i\theta})^n}{n!\,\Gamma(n+\alpha+1)(1-re^{i\theta})^{2n}}\right| \le ce^{\varepsilon y^2}.$$

Therefore, by the dominated convergence theorem, the last limit in (12) equals the value of the integral obtained by substituting $r = 1$. The proof is finished by observing that

$$\begin{aligned}(xy)^\alpha e^{-(1+e^{i\theta})(x^2+y^2)/2(1-e^{i\theta})} &\sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+\alpha+1)}\left(\frac{xye^{i\theta/2}}{1-e^{i\theta}}\right)^{2n} \\ &= (xy)^\alpha \exp\left(-\frac{i}{2}(x^2+y^2)\cot\frac{\theta}{2}\right)\left(\frac{2}{xy|\csc\theta/2|}\right)^\alpha \\ &\quad \times \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\alpha+1)}\left(\frac{xy|\csc\theta/2|}{2}\right)^{2n+\alpha} \\ &= 2^\alpha|\sin\theta/2|^\alpha \exp\left(-\frac{i}{2}(x^2+y^2)\cot\frac{\theta}{2}\right)J_\alpha(xy|\csc\theta/2|). \ \blacksquare\end{aligned}$$

Now let $\widehat{m}(\xi) = (2\pi)^{-1}\int_{-\infty}^{\infty} m(x)e^{-ix\xi}\,dx$, and continue to let $\alpha \ge -1/2$, $g \in L_{\mathrm{f}}^2$. Then according to Lemma 6,

$$\begin{aligned}&\int_{-\infty}^{\infty}\sum_{k=0}^{\infty} m(\lambda(4k+2\alpha+2))\mathcal{M}_k^\alpha(x)\mathcal{M}_k^\alpha(y)g(y)\,dy \\ &\quad = \int_{-\infty}^{\infty} \widehat{m}(\xi)e^{i\lambda(2\alpha+2)\xi}\sum_{k=0}^{\infty}(e^{i4\lambda\xi})^k(g,\mathcal{M}_k^\alpha)_{L^2(\mathbb{R}_+)}\mathcal{M}_k^\alpha(x)\,d\xi \\ &\quad = \int_{-\infty}^{\infty}\int_0^{\infty} \widehat{m}(\xi)e^{i\lambda(2\alpha+2)\xi}2^{1+\alpha}(1-e^{i4\lambda\xi})^{-1-\alpha}|\sin 2\lambda\xi|^\alpha \\ &\qquad \times (xy)^{1/2}\exp\left(-\frac{i}{2}(x^2+y^2)\cot 2\lambda\xi\right)J_\alpha(xy|\csc 2\lambda\xi|)g(y)\,dy\,d\xi.\end{aligned}$$

Since $\alpha \ge -1/2$, there exists a constant $c < \infty$ such that $|J_\alpha(xy|\csc 2\lambda\xi|)| \le c(xy|\csc 2\lambda\xi|)^{-1/2}$. It follows by an application of Fubini's theorem that the last double integral equals $\int_0^\infty G_\lambda(x,y)g(y)\,dy$, where

$$\begin{aligned}G_\lambda(x,y) = 2^{1+\alpha}(xy)^{1/2}\int_{-\infty}^{\infty} &\widehat{m}(\xi)e^{i\lambda(2\alpha+2)\xi}(1-e^{i4\lambda\xi})^{-1-\alpha}|\sin 2\lambda\xi|^\alpha \\ &\times \exp\left(-\frac{i}{2}(x^2+y^2)\cot 2\lambda\xi\right)J_\alpha(xy|\csc 2\lambda\xi|)\,d\xi.\end{aligned}$$

It will be useful to write $G_\lambda(x,y)$ in the more concise form

$$\begin{aligned}(13)\quad 2^\alpha(xy)^{1/2}\lambda^{-1}\int_{-\infty}^{\infty} &\widehat{m}(\xi/(2\lambda))e^{i(\alpha+1)\xi}(1-e^{i2\xi})^{-1-\alpha} \\ &\times |\sin\xi|^\alpha\exp\left(-\frac{i}{2}(x^2+y^2)\cot\xi\right)J_\alpha(xy|\csc\xi|)\,d\xi.\end{aligned}$$

As in the Hermite case, in order to prove Theorem 2, it suffices to prove that the statement of Theorem 2 holds with $G_\lambda(x,y)$ in place of $m(\lambda L)(x,y)$.

The analysis of (13) will be similar to the analysis of the integral (5) in Section 3. Note that because we are assuming $\alpha \ge -1/2$, we can write the Bessel function in (13) in the form $J_\alpha(t) = f_1(t)e^{it} + f_2(t)e^{-it}$, where f_1, f_2 are complex-valued functions satisfying $|f_1^{(k)}(t)|, |f_2^{(k)}(t)| \le c_k t^{-(1+2k)/2}$ for $t > 0$. This splits (13) into two absolutely convergent integrals of the form

$$\begin{aligned}(14)\quad (xy)^{1/2}\lambda^{-1}\int_{-\infty}^{\infty} &\widehat{m}(\xi/(2\lambda))e^{i(\alpha+1)\xi}(1-e^{i2\xi})^{-1-\alpha}|\sin\xi|^\alpha \\ &\times f(xy|\csc\xi|)\exp\left(-\frac{i}{2}((x^2+y^2)\cot\xi \pm 2xy|\csc\xi|)\right)d\xi,\end{aligned}$$

where f satisfies $|f^{(k)}(t)| \le c_k t^{-(1+2k)/2}$ for $t > 0$. We consider the integral with $-2xy|\csc\xi|$ in the exponential. The other integral leads to similar estimates.

Define sets

$$T_1 = \bigcup_{n\in\mathbb{Z}}(2\pi n, 2\pi n+\pi), \qquad T_2 = \bigcup_{n\in\mathbb{Z}}(2\pi n+\pi, 2\pi n+2\pi).$$

The strategy is to split (14) into separate integrals over T_1, T_2 and in both cases to integrate by parts repeatedly using

$$\begin{aligned}&\exp\left(-\frac{i}{2}((x^2+y^2)\cot\xi - 2xy|\csc\xi|)\right) \\ &= -2i\sin^2\xi\,(x^2+y^2\mp 2xy\cos\xi)^{-1}\frac{d}{d\xi}\exp\left(-\frac{i}{2}((x^2+y^2)\cot\xi - 2xy|\csc\xi|)\right).\end{aligned}$$

(We take the $-$ sign if $\xi \in T_1$ and the $+$ sign if $\xi \in T_2$.) Recall the notation used for the integration by parts in Section 3. The only modification to be made is that we define new operators

$$C_0^j g(\xi) = (x^2+y^2+2(-1)^j xy\cos\xi)^{-1} g(\xi), \quad j=1,2,$$
$$C_1^j g(\xi) = \frac{d}{d\xi}(x^2+y^2+2(-1)^j xy\cos\xi)^{-1} g(\xi), \quad j=1,2,$$

in place of the old C_0, C_1 operators. After doing k integrations by parts on the portion of (14) over T_j we get a sum of integrals of the form

$$(15)\quad c(xy)^{1/2}\lambda^{-1}\int_{T_j}(A_{\sigma_1(k)}\ldots A_{\sigma_1(1)}\widehat{m}(\xi/(2\lambda))) \times (B_{\sigma_2(k)}\ldots B_{\sigma_2(1)}\, e^{i(\alpha+1)\xi}(1-e^{i2\xi})^{-1-\alpha}|\sin\xi|^\alpha f(xy|\csc\xi|)) \times (C^j_{\sigma_3(k)}\ldots C^j_{\sigma_3(1)}1)\exp\left(-\frac{i}{2}((x^2+y^2)\cot\xi - 2xy|\csc\xi|)\right)d\xi.$$

As before, the maps $\sigma_1,\sigma_2,\sigma_3$ are required to have the property that for each $i=1,\ldots,k$, exactly one of the numbers $\sigma_1(i),\sigma_2(i),\sigma_3(i)$ equals one. (Note that the integration by parts incurs no boundary terms.)

LEMMA 7. *Let f satisfy $|f^{(k)}(t)| \le c_k t^{-(1+2k)/2}$, $k=0,1,\ldots$ Then there exists a constant c depending only on k such that*

$$|B_{\sigma(k)}\ldots B_{\sigma(1)}\, e^{i(\alpha+1)\xi}(1-e^{i2\xi})^{-1-\alpha}|\sin\xi|^\alpha f(xy|\csc\xi|)| \le c(xy)^{-1/2}|\sin\xi|^{2k-\tau(\sigma)-1/2}$$

for all $\xi \in T_1\cup T_2$, $x,y>0$.

Proof. The proof is by induction on k. Note that the quantity to be bounded can be written in a natural way as a linear combination of functions of the form

$$(16)\qquad \sin^l\xi\cos^m\xi\, e^{ir\xi}(1-e^{i2\xi})^{-n-\alpha}|\sin\xi|^\alpha(xy)^p f^{(p)}(xy|\csc\xi|)$$

where $l,m,n,p\in\mathbb{N}_0$, $r\in\mathbb{R}$. The induction hypothesis is that each such term in this linear combination has the property $l-n+p+1/2 \ge 2k-\tau(\sigma)-1/2$. (Note that this implies that (16) is bounded by $c(xy)^{-1/2}|\sin\xi|^{2k-\tau(\sigma)-1/2}$.) It is easy to check the truth of the hypothesis when $k=1$. So let $\sigma\in E_{k+1}$, $k\ge1$. If $\sigma(k+1)=0$, then $B_{\sigma(k+1)}$ applied to (16) results in a function of the form

$$\sin^{l+2}\xi\cos^m\xi\, e^{ir\xi}(1-e^{i2\xi})^{-n-\alpha}|\sin\xi|^\alpha(xy)^p f^{(p)}(xy|\csc\xi|).$$

In this case we certainly have $(l+2)-n+p+1/2 \ge 2(k+1)-\tau(\sigma)-1/2$. We leave it to the reader to check the result of applying B_1 to (16). ∎

LEMMA 8. *There exists a constant c depending only on k such that*

$$|C^1_{\sigma(k)}\ldots C^1_{\sigma(1)}1| \le c|\xi-2\pi n|^{-\tau(\sigma)}(x^2+y^2-2xy\cos\xi)^{-k}$$

for all $\xi\in[-\pi/3+2\pi n, \pi/3+2\pi n]\cap T_1$, $n\in\mathbb{Z}$, $x,y>0$, and

$$|C^2_{\sigma(k)}\ldots C^2_{\sigma(1)}1| \le c(x^2+y^2+2xy\cos\xi)^{-k}$$

for all $\xi\in[-\pi/3+2\pi n, \pi/3+2\pi n]\cap T_2$, $n\in\mathbb{Z}$, $x,y>0$.

Proof. The first inequality is just a special case of Lemma 3. The proof of the second inequality is similar to that of Lemma 3, except that we use the fact that $|xy|(x^2+y^2+2xy\cos\xi)^{-1}\le c$ for $\xi\in[-\pi/3+2\pi n, \pi/3+2\pi n]$, $x,y>0$. ∎

LEMMA 9. *There exists a constant c depending only on k such that*

$$|C^j_{\sigma(k)}\ldots C^j_{\sigma(1)}1| \le c(x^2+y^2+2(-1)^j xy\cos\xi)^{-k}$$

for all $\xi\in[-2\pi/3+2\pi n, -\pi/3+2\pi n]\cup[\pi/3+2\pi n, 2\pi/3+2\pi n]$, $n\in\mathbb{Z}$.

Proof. The proof is similar to that of Lemma 3, except that we use the fact that $|xy|(x^2+y^2+2(-1)^j xy\cos\xi)^{-1}\le c$ for ξ in the allowed region. ∎

LEMMA 10. *There exists a constant c depending only on k such that*

$$|C^1_{\sigma(k)}\ldots C^1_{\sigma(1)}1| \le c(x^2+y^2-2xy\cos\xi)^{-k}$$

for all $\xi\in[2\pi/3+2\pi n, 4\pi/3+2\pi n]\cap T_1$, $n\in\mathbb{Z}$, $x,y>0$, and

$$|C^2_{\sigma(k)}\ldots C^2_{\sigma(1)}1| \le c|\xi-\pi-2\pi n|^{-\tau(\sigma)}(x^2+y^2+2xy\cos\xi)^{-k}$$

for all $\xi\in[2\pi/3+2\pi n, 4\pi/3+2\pi n]\cap T_2$, $n\in\mathbb{Z}$, $x,y>0$.

Proof. The proof is similar to that of Lemma 3. For the first inequality we use the fact that $|xy|(x^2+y^2-2xy\cos\xi)^{-1}\le c$ for $\xi\in[2\pi/3+2\pi n, 4\pi/3+2\pi n]$, $x,y>0$. For the second inequality we use the fact that $|xy\sin\xi|(x^2+y^2+2xy\cos\xi)^{-1}\le c|\xi-\pi-2\pi n|^{-1}$ for $\xi\in[2\pi/3+2\pi n, 4\pi/3+2\pi n]$, $x,y>0$. ∎

Proof of Theorem 2. First note that (14) is bounded by a quantity of the form

$$c\lambda^{-1}\int_{-\infty}^{\infty}(1+|\xi|/\lambda)^{-L}|\sin\xi|^{-1/2}\,d\xi.$$

Here L can be taken as large as desired. As in the proof of Theorem 1, this has a bound of the form $c\lambda^{-1/2}$, for $0<\lambda<\Lambda$. This suffices to prove the theorem if $\lambda^{-1/2}|x-y|\le1$.

So assume $\lambda^{-1/2}|x-y|>1$. Let S_1,S_2,S_3 be the sets defined in the proof of Theorem 1. By Lemmas 1, 7 and 8, the part of the integral (15) over $S_1 \cap T_1$ is bounded by

$$c(xy)^{1/2}\lambda^{-1}\sum_{n\in\mathbb{Z}}\int_0^{\pi/3}\lambda^{-\tau(\sigma_1)}(1+\lambda^{-1}|\xi+2\pi n|)^{-L} \times (xy)^{-1/2}|\xi|^{2k-\tau(\sigma_2)-1/2}|\xi|^{-\tau(\sigma_3)}(x^2+y^2-2xy\cos\xi)^{-k}\,d\xi.$$

As was observed in the proof of Theorem 1, this is bounded by a quantity of the form $c\lambda^{-1/2}(\lambda^{-1/2}|x-y|)^{-2k}$, for $0<\lambda<\Lambda$. The part of the integral (15) over $S_1 \cap T_2$ is bounded by

$$c(xy)^{1/2}\lambda^{-1}\sum_{n\in\mathbb{Z}}\int_{-\pi/3}^{0}\lambda^{-\tau(\sigma_1)}(1+\lambda^{-1}|\xi+2\pi n|)^{-L} \times (xy)^{-1/2}|\xi|^{2k-\tau(\sigma_2)-1/2}(x^2+y^2+2xy\cos\xi)^{-k}\,d\xi.$$

This is bounded by a quantity of the form $c\lambda^{-1/2}(\lambda^{-1/2}(x+y))^{-2k}$, for $0<\lambda<\Lambda$. Next, by Lemmas 1, 7 and 9, the part of the integral (15) over $S_2 \cap T_1$ is bounded by

$$c(xy)^{1/2}\lambda^{-1}\sum_{n\in\mathbb{Z}}\int_{\pi/3}^{2\pi/3}\lambda^{-\tau(\sigma_1)}(1+\lambda^{-1}|\xi+2\pi n|)^{-L} \times (xy)^{-1/2}(x^2+y^2-2xy\cos\xi)^{-k}\,d\xi.$$

This is bounded by a quantity of the form $c\lambda^K(x+y)^{-2k}$, for $0<\lambda<\Lambda$. Here K can be taken as large as desired, by taking L sufficiently large. The part of the integral (15) over $S_2 \cap T_2$ has the same bound. Finally, by Lemmas 1, 7 and 10, the part of the integral (15) over $S_3 \cap T_1$ is bounded by

$$c(xy)^{1/2}\lambda^{-1}\sum_{n\in\mathbb{Z}}\int_{2\pi/3}^{\pi}\lambda^{-\tau(\sigma_1)}(1+\lambda^{-1}|\xi+2\pi n|)^{-L} \times (xy)^{-1/2}|\xi-\pi|^{2k-\tau(\sigma_2)-1/2}(x^2+y^2-2xy\cos\xi)^{-k}\,d\xi.$$

This is bounded by a quantity of the form $c\lambda^K(x+y)^{-2k}$, for $0<\lambda<\Lambda$. Here K can be taken as large as desired, by taking L sufficiently large. The part of the integral (15) over $S_3 \cap T_2$ is bounded by

$$c(xy)^{1/2}\lambda^{-1}\sum_{n\in\mathbb{Z}}\int_{\pi}^{4\pi/3}\lambda^{-\tau(\sigma_1)}(1+\lambda^{-1}|\xi+2\pi n|)^{-L} \times (xy)^{-1/2}|\xi-\pi|^{2k-\tau(\sigma_2)-1/2}|\xi-\pi|^{-\tau(\sigma_3)}(x^2+y^2+2xy\cos\xi)^{-k}\,d\xi.$$

This is bounded by $c\lambda^K|x-y|^{-2k}$, for $0<\lambda<\Lambda$. Again, K can be taken as large as desired, by taking L sufficiently large. These estimates suffice to prove the theorem, since k can be taken arbitrarily large. ∎

COROLLARY 2. *Let $\alpha \ge -1/2$, and let $\varphi_{\mu n}$, $\psi_{\mu n}$ be the Laguerre wave packets from Section 2. Also, let $c_{2^\mu[\alpha+2],n}$ and $z_{2^\mu[\alpha+2],n}$ be as in Section 2. Then for every $p>0$ there exists a constant $c<\infty$ independent of μ, n such that*

$$|\varphi_{\mu n}(x)|, |\psi_{\mu n}(x)| \le \frac{c\,c_{2^\mu[\alpha+2],n}\,2^{\mu/2}}{(1+2^{\mu/2}|x-z_{2^\mu[\alpha+2],n}|)^p}.$$

As in the Hermite case, it is possible to analyze the $c_{2^\mu[\alpha+2],n}$ constants further.

References

[1] J. Epperson, *Triebel–Lizorkin spaces for Hermite expansions*, Studia Math. 114 (1995), 87–103.

[2] M. Frazier and B. Jawerth, *A discrete transform and decompositions of distribution spaces*, J. Funct. Anal. 93 (1990), 34–170.

[3] C. Markett, *Mean Cesàro summability of Laguerre expansions and norm estimates with shifted parameter*, Anal. Math. 8 (1982), 19–37.

[4] E. Rainville, *Special Functions*, Macmillan, New York, 1960.

[5] G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ. 23, Providence, R.I., 1975.

[6] S. Thangavelu, *Lectures on Hermite and Laguerre Expansions*, Math. Notes 42, Princeton Univ. Press, 1993.

Department of Mathematics and Statistics
University of New Mexico
Albuquerque, New Mexico 87131
U.S.A.
E-mail: jeppers@math.unm.edu

Received October 9, 1995
Revised version December 9, 1996 (3541)