# HERMITE-BIRKHOFF TRIGONOMETRIC INTERPOLATION <br> IN THE (0, 1, 2, M) CASE 

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## 1. Introduction

Following four important papers on Birkhoff interpolation by Turán and his associates ([2], [3], [4], [14]), Kis ([8], [9]) proved the following theorems.

Theorem 1. (Kis). Let $f(z)$ be analytic in $|z|<1$ and continuous on $|z| \leqq 1$. Let $R_{n}(z)$ be the unique interpolation polynomial of degree $\leqq 2 n-1$ in $z$ such that

$$
\begin{align*}
& R_{n}\left(z_{k n}\right)=f\left(z_{k n}\right), f\left(z_{k n}\right), R_{n}^{\prime \prime}\left(z_{k n}\right)=\beta_{k n}, k=1,2, \cdots, n  \tag{1.1}\\
& z_{k n}=\exp \frac{2 \pi k i}{n}, k=1,2, \cdots, n  \tag{1.2}\\
& \beta_{k n}=o\left(\frac{n^{2}}{\log n}\right), k=1,2, \cdots, n  \tag{1.3}\\
& \omega(\delta) \log \delta=0  \tag{1.4}\\
& \delta \rightarrow 0+
\end{align*}
$$

Then $R_{n}(z)$ converges uniformly to $f(z)$ in $|z| \leqq 1$.
Here $\omega(\delta)$ denotes the modulus of continuity of $f(z)$. Theorem 1 is best possible in the sense that the freedom of $\beta_{k n}$ cannot be improved. The above theorem is surprising, in view of the arbitrariness of the numbers $\beta_{k n}$ which satisfy only the order relation (1.3).

Theorem 2. (Kis). Let $f(x)$ be a $2 \pi$ periodic continuous function satisfying Zygmund's condition

$$
\begin{equation*}
f(x+h)-2 f(x)+f(x-h)=o(h) \tag{1.5}
\end{equation*}
$$

Let $R_{n}(x)$ be the unique trigonometric polynomial of order $n$ satisfying the conditions

$$
\begin{gather*}
R_{n}\left(x_{k n}\right)=f\left(x_{k n}\right), R_{n}^{\prime \prime}\left(x_{k n}\right)=\delta_{k n}, k=1,2, \cdots, n  \tag{1.6}\\
{\left[R_{n}(x)=a_{0}+\sum_{k=1}^{n-1}\left(a_{k} \cos k x+b_{k} \sin k x\right)+a_{n} \cos n x\right]} \\
x_{k n}=\frac{2 k \pi}{n}, k=0,1, \cdots, n-1 . \tag{1.7}
\end{gather*}
$$

Then $R_{n}(x)$ will converge uniformly to $f(x)$ on the real axis provided that

$$
\begin{equation*}
\delta_{k n}=o(n) \tag{1.8}
\end{equation*}
$$

Theorem 2 is best possible in the sense that Zygmund's condition cannot be replaced by

$$
\begin{equation*}
f(x+h)-2 f(x)+f(x-h)=O(h) \tag{1.9}
\end{equation*}
$$

For this result we refer to the work of Vertesi [20].
Theorem 2 of Kis has received the following generalization. Sharma and the author [11] have considered the problem of $(0, M)$ interpolation. Here the interpolation trigonometric polynomial $R_{n}(x)$ of order $n$ is given by

$$
\begin{equation*}
R_{n}\left(x_{k n}\right)=f\left(x_{k n}\right), R_{n}^{(M)}\left(x_{k n}\right)=\beta_{k n}, x_{k n}=\frac{2 k \pi}{n}, k=0,1, \cdots, n-1 \tag{1.10}
\end{equation*}
$$

( $M$ being a fixed positive integer $\geqq 1$ ). The trigonometric polynomial $R_{n}(x)$ given by (1.10) has the following form:

$$
\begin{align*}
R_{n}(x)=a_{0}+\sum_{k=1}^{n-1}\left(a_{k} \cos k x+b_{k} \sin k x\right) & +b_{n} \sin n x(M-\text { odd })  \tag{1.11}\\
& +a_{n} \cos n x(M \text {-even })
\end{align*}
$$

The main theorem of the above paper is as follows.
Theorem 3. (Sharma and Varma). Let $f(x)$ be a $2 \pi$ periodic continuous function. Let $M$ be any fixed odd positive integer. Suppose $\beta_{k n}$ as stated in (1.10) satisfy

$$
\begin{equation*}
\beta_{k n}=O\left(\frac{n^{M}}{\log n}\right), \quad k=0,1, \cdots, n-1 \tag{1.12}
\end{equation*}
$$

Then $R_{n}(x)$ as defined by (1.10) and (1.11) converges uniformly to $f(x)$ on the real line.

For the case M-even, let $f(x)$ satisfy the $Z y g m u n d$ condition (1.5) and

$$
\begin{equation*}
\beta_{k n}=o\left(n^{M-1}\right), \quad k=0,1, \cdots, n-1 \tag{1.13}
\end{equation*}
$$

Then $R_{n}(x)$ will converge uniformly to $f(x)$ on the real line,

Motivated by the above theorem on trigonometric interpolation, the author has considered the problem of $(0,1, M)$ interpolation on the nodes $x_{k n}=2 k \pi / n$, $k=0,1, \cdots, n-1$. By ( $0,1, M$ ) interpolation we mean the problem of finding interpolatory polynomials of suitable form for which the values, first derivative and $M$ th derivative, are prescribed at $n$ distinct points. It turns out that these interpolation polynomials exist uniquely only when $M$ is an even integer. In this case we proved that the interpolation polynomials converge uniformly to $f(x)$, provided $f(x) \in \operatorname{Lip} \alpha 0<\alpha \leqq 1$. For details we refer to Theorem 2.2 in [15]. Thus, by prescribing also the first derivative of the interpolation polynomials, one obtains a convergence theorem for a much wider class of functions than in ( $0, M$ ) interpolation (for $M$-even). But, by doing so, we have increased the order of the trigonometric polynomials. It is also interesting to compare the results of $(0,2,3)$ interpolation [12] with $(0,3)$ case as well. We know from Theorem 3 that $R_{n}(x)$, obtained from the consideration of $(0,3)$ interpolation, converges uniformly to $f(x)$ for just $2 \pi$ periodic continuous functions, whereas, in the case of $(0,2,3)$ interpolation we need at least $f(x) \in \operatorname{Lip} \alpha, 0<\alpha<1$. Thus, by prescribing also the second derivative of the interpolation polynomials one obtains convergence theorems for much narrower class than in $(0,3)$ interpolation.

## 2. Statement of results

The object of this paper is to consider the following problem: Let $M$ be a fixed odd positive integer $\geqq 3$. Let $R_{n}(x)$ be a trigonometric sum of order $2 n$ (of the form)

$$
\begin{equation*}
d_{0}+\sum_{j=1}^{2 n-1}\left(d_{j} \cos j x+e_{j} \sin j x\right)+e_{2 n} \sin 2 n x \tag{2.1}
\end{equation*}
$$

We ask the following question; Does there exist a unique trigonometric sum of order $2 n$ which satisfies (2.1) and

$$
\begin{align*}
& R_{n}\left(x_{i n}\right)=f\left(x_{i n}\right), R_{n}^{\prime}\left(x_{k n}\right)=\alpha_{i n}, \quad i=0, \cdots, n-1 ?  \tag{2.2}\\
& R_{n}^{\prime \prime}\left(x_{i n}\right)=\beta_{i n}, R_{n}^{(M)}\left(x_{i n}\right)=\delta_{i n},
\end{align*}
$$

Here $x_{i n}$ are given by (1.7). It turns out that the answer to the above question is in the affirmative. We call it ( $0,1,2, M$ ) trigonometric interpolation. We will show that under suitable restrictions on $\alpha_{i n}, \beta_{i n}, \delta_{i n}$ and $f(x) \in c_{2 \pi}, R_{n}(x)$ will converge uniformly to $f(x)$ on the real line. We will also prove some inequalities on trigonometric polynomials analogous to Fejer [7]. For the case when $M$ is even, the results are analogous to the case $(0,1,2,4)$ trigonometric interpolation, which has been dealt with already in my earlier work [17]. Now, we state the main theorem of this paper.

Theorem 4. Let $f(x) \in C_{2 \pi}$. Then $R_{n}(f)$ defined by (2.1) and (2.2) converges uniformly to $f(x)$ on the real line provided that

$$
\begin{equation*}
\alpha_{i n}=o\left(\frac{n}{\log n}\right), \beta_{i n}=o\left(n^{2}\right), \delta_{i n}=o\left(\frac{n^{M}}{\log n}\right) i=0,1, \cdots, n-1 \tag{2.3}
\end{equation*}
$$

The freedom of these numbers is best possible.
The main interest of the above theorem lies in the fact that as far as the freedom of $\beta_{k n}$ is concerned, we need only $\beta_{k n}=o\left(n^{2}\right)$. See also the remarks at the end of the paper.

Theorem 5. Let $\phi_{n}(x)$ be any trigonometric polynomial of an order $\leqq 2 n$ and satisfying (2.1). Let further

$$
\begin{equation*}
\left|\phi_{n}^{(j)}\left(x_{i n}\right)\right| \leqq a_{j}, j=0,1,2, M, i=0,1, \cdots, n-1 \tag{2.4}
\end{equation*}
$$

Then we have for $0 \leqq x \leqq 2 \pi$,

$$
\begin{equation*}
\left|\phi_{n}(x)\right| \leqq c_{0}\left(a_{0}+a_{1} \frac{\log n}{n}+\frac{a_{2}}{n^{2}}+\frac{a_{M}}{n^{M}} \log n\right) \tag{2.5}
\end{equation*}
$$

Here $c_{0}$ is a definite constant independent of $n$ and $x$. (2.5) is best possible in the sense that there exists a trigonometric polynomial $g_{n}(x)$ of the order $2 n$ satisfying (2.1) and $\left|g_{n}^{j}\left(x_{i n}\right)\right|=a_{j}, j=0,1,2, M, i=0,1, \cdots, n-1$, and for which

$$
\begin{equation*}
\left|g_{n}(\pi)\right|>c_{1}\left(a_{0}+a_{1} \frac{\log n}{n}+\frac{a_{2}}{n^{2}}+\frac{a_{M}}{n^{M}} \log n\right) \tag{2.6}
\end{equation*}
$$

Theorem 6. Let $f(x) \in c_{2 \pi}$ have $\omega(\delta)$ as its module of continuity. Then under the assumption $\alpha_{i n}=\beta_{\text {in }}=\delta_{i n}=0$,

$$
\begin{equation*}
\left|R_{n}(x)-f(x)\right| \leqq c_{2} \omega\left(\frac{1}{\sqrt{n}}\right) \tag{2.7}
\end{equation*}
$$

THEOREM 7. The explicit representation of $R_{n}(x)$ is given by

$$
\begin{align*}
R_{n}(x) & =\sum_{k \pm 0}^{n-1} f\left(x_{k n}\right) A\left(x-x_{k n}\right)+\sum_{k=0}^{n-1} \alpha_{k n} B\left(x-x_{k n}\right)  \tag{2.8}\\
& +\sum_{k=0}^{n-1} \beta_{k n} C\left(x-x_{k n}\right)+\sum_{k=0}^{n-1} \delta_{k n} D\left(x-x_{k n}\right)
\end{align*}
$$

where $A(x), B(x), C(x)$ and $D(x)$ are defined in (2.15), (2.13), (2.11) and (2.9) respectively.

Here $A(x), B(x), C(x)$ and $D(x)$ are given by:

$$
\begin{equation*}
D(x)=\frac{2(-1)^{(M+1) / 2}(1-\cos n x)}{n}\left[2 \sum_{j=1}^{n-1} \frac{\sin j x}{a_{j, M}}+\frac{\sin n x}{a_{n, M}}\right] \tag{2,9}
\end{equation*}
$$

where

$$
\begin{align*}
& a_{j, M}=(2 n-j)^{M}+(n+j)^{M}-3\left\{(n-j)^{M}+j^{M}\right\}  \tag{2.10}\\
& C(x)=\frac{(1-\cos n x)}{n^{3}}\left[1+2 \sum_{j=1}^{n-1} \frac{b_{j, M}}{a_{j, M}} \cos j x\right] \tag{2.11}
\end{align*}
$$

where

$$
\begin{gather*}
b_{j, M}=(2 n-j)^{M}-2(n-j)^{M}-j^{M}  \tag{2.12}\\
B(x)=G(x)+\frac{(1-\cos n x)}{n^{3}}\left[2 \sum_{j=1}^{n-1} \frac{c_{j, M}}{a_{j, M}} \sin j x+\frac{c_{n, M}}{a_{n, M}} \sin n x\right], \tag{2.13}
\end{gather*}
$$

where

$$
\begin{equation*}
c_{j, M}=(3 n-2 j) j^{M}+4(n-j)^{M+1}-(n-2 j)(2 n-j)^{M} \tag{2.14}
\end{equation*}
$$

where

$$
\begin{equation*}
A(x)=F(x)+\frac{2(1-\cos n x)}{n^{3}} \sum_{j=1}^{n-1} \frac{d_{j, M}}{a_{j, M}} \cos j x \tag{2.15}
\end{equation*}
$$

$$
\begin{equation*}
d_{j, M}=j(n-j)(2 n-j)\left\{(2 n-j)^{M-1}-2(n-j)^{M-1}+j^{M-1}\right\} \tag{2.16}
\end{equation*}
$$

Here $F(x)$ and $G(x)$ are fundamental polynomials of Hermite interpolation (see [11]) and they are given by

$$
\begin{align*}
& F(x)=\frac{1}{n}\left[1+\frac{2}{n} \sum_{j=1}^{n-1}(n-j) \cos j x\right]  \tag{2.17}\\
& G(x)=\frac{1}{n^{2}}\left[2 \sum_{j=1}^{n-1} \sin j x+\sin n x\right] \tag{2.18}
\end{align*}
$$

REMARK 1. It is interesting to mention that for $M=3$. the fundamental polynomials $A(x)$ and $C(x)$ are nonnegative, but for the cases when $M>3$, this property, in general, breaks down. Indeed for $M=3$ we have

$$
\begin{equation*}
D(x)=\frac{(1-\cos n x) G(x)}{3 n^{2}} \tag{2.19}
\end{equation*}
$$

$$
\begin{equation*}
C(x)=\frac{(1-\cos n x) F(x)}{n^{2}} \tag{2.20}
\end{equation*}
$$

and

$$
\begin{equation*}
A(x)=F^{2}(x)+\frac{\left(n^{2}-1\right)}{3} C(x) \tag{2.22}
\end{equation*}
$$

## 3. Preliminaries

Here we state those results which we shall require in the proof of theorems stated in Article 2.

Following identities are easy to obtain from (2.8):

$$
\begin{align*}
& \sum_{k=0}^{n-1} A\left(x-x_{k n}\right) \equiv 1  \tag{3.1}\\
& \sum_{k=0}^{n-1} C\left(x-x_{k n}\right)=\frac{1-\cos n x}{n^{2}} . \tag{3.2}
\end{align*}
$$

From (see Zygmund [21]) the known results due to Jackson we have:

$$
\begin{equation*}
\sum_{k=0}^{n-1} F\left(x-x_{k n}\right) \equiv 1, \sum_{k=0}^{n-1}\left|G\left(x-x_{k n}\right)\right| \leqq \frac{2}{n} \log n \tag{3.3}
\end{equation*}
$$

$$
\begin{equation*}
\left|G\left(x-x_{k n}\right)\right| \leqq \frac{2}{n}, k=0,1, \cdots, n-1 \tag{3.4}
\end{equation*}
$$

Following the arguments given in Jackson [5] we have for $x \neq x_{k n}$ :

$$
\begin{equation*}
\sum_{k=0}^{n-1} \max _{1 \leqq p \leqq n}\left|\sum_{j=1}^{p} \sin j\left(x-x_{k n}\right)\right| \leqq 4 n \log n \tag{3.5}
\end{equation*}
$$

Let $0 \leqq \alpha_{1} \leqq \alpha_{2} \leqq \cdots \leqq \alpha_{p}$ then we have

$$
\begin{equation*}
\left|\sum_{j=1}^{p} \alpha_{j} \sin j x\right| \leqq 2 \alpha_{p} \max _{1 \leqq v \leqq p}\left|\sum_{j=1}^{v} \sin j x\right| \tag{3.6}
\end{equation*}
$$

Similarly, if $\alpha_{1} \geqq \alpha_{2} \geqq \alpha_{3} \geqq \cdots \geqq \alpha_{p}$ then we have

$$
\begin{equation*}
\left|\sum_{j=1}^{p} \alpha_{j} \sin j x\right| \leqq 2 \alpha_{1} \max _{1 \leqq v \leqq p}\left|\sum_{j=1}^{v} \sin j x\right| \tag{3.7}
\end{equation*}
$$

Proof of (3.6 and (3.7) follows easily from Abel's Lemma.
We denote the Fejer-kernel by

$$
\begin{equation*}
\tau_{j, k}(x)=1+\frac{2}{j} \sum_{i=1}^{j-1}(j-i) \cos i\left(x-x_{k n}\right) \tag{3.8}
\end{equation*}
$$

It is easy to verify the following properties of Fejér-kernel:

$$
\begin{equation*}
\sum_{k=0}^{n-1} \tau_{j, k}(x)=n, \tau_{j, k}(x)=\frac{1}{j}\left[\frac{\sin \frac{j\left(x-x_{k n}\right)}{2}}{\sin \frac{x-x_{k n}}{2}}\right]^{2} \tag{3.9}
\end{equation*}
$$

and

$$
\begin{equation*}
(j+1) \tau_{j+1, k}(x)-2 j \tau_{j, k}(x)+(j-1) \tau_{j-1, k}^{(x)}=2 \cos j\left(x-x_{k n}\right) \tag{3.10}
\end{equation*}
$$

Let $a_{j, M}$ be defined as given in (2.10). Denote $a_{j, M}^{\prime \prime}$, the second derivative of $a_{j, M}$, with respect to $j$. By using

$$
\begin{equation*}
(2 n-j)^{M} \geqq 2^{M}(n-j)^{M}+j^{M},(n+j)^{M} \geqq(n-j)^{M}+2^{M} j^{M}, \tag{3.11}
\end{equation*}
$$

it follows that

$$
\begin{equation*}
a_{j, M}>0, a_{j, M}^{\prime \prime}>0 \text { for } M \geqq 3 \tag{3.12}
\end{equation*}
$$

By using (3.12), we conclude that $a_{j, M}^{\prime}$ is an increasing function of $j$, for $j=0,1, \cdots, n$. But $a_{m, M}^{\prime} \leqq 0(n=2 m+1)$, or $(n=2 m)$. On account of

$$
a_{j, M}=a_{n-j, M} \text { for } j=1,2, \cdots, m
$$

we finally obtain that $a_{j, M}$ is a decreasing function of $j$, for $j=0,1, \cdots, m$ and an increasing function of $j$ for $j=m+1, \cdots, n$. From these observations we easily conclude that:

$$
\begin{align*}
& n(3 m)^{M-1}<a_{j, M}<(2 n)^{M},  \tag{3.13}\\
& \left|a_{j, M}^{\prime}\right|<2 M(2 n)^{M-1}  \tag{3.14}\\
& \left|a_{j, M}^{\prime \prime}\right|<M(M-1)(2 n)^{M-2},  \tag{3.15}\\
& \frac{\left|a_{j+1, M}-a_{j: M}\right|}{\left|a_{j, M}\right|\left|a_{j+1, M}\right|} \leqq \frac{M}{n^{M+1}} \tag{3.16}
\end{align*}
$$

## 4. Upper estimates of the fundamental polynomials

Here, we shall prove the following result:
Lemma 4.1. The following estimates are valid:

$$
\begin{align*}
& \sum_{k=0}^{n-1}\left|D\left(x-x_{k n}\right)\right| \leqq \frac{25 \log n}{n^{M}}  \tag{4.1}\\
& \sum_{k=0}^{n-1}\left|C\left(x-x_{k n}\right)\right| \leqq \frac{f_{1}}{n^{2}}  \tag{4.2}\\
& \sum_{k=0}^{n-1}\left|B\left(x-x_{k n}\right)\right| \leqq \frac{f_{2} \log n}{n}  \tag{4.3}\\
& \sum_{k=0}^{n-1}\left|A\left(x-x_{k n}\right)\right| \leqq f_{3} \tag{4.4}
\end{align*}
$$

Here $f_{1}, f_{2}, f_{3}$ are positive constants independent of $n$ and $x$.
Proof. We note that for $M=3$, we have more precise constants. In this case, we have:

$$
\begin{align*}
& \sum_{k=0}^{n-1}\left|D\left(x-x_{k n}\right)\right| \leqq \frac{2 \log n}{n^{3}},  \tag{4.1a}\\
& \sum_{k=0}^{n-1}\left|C\left(x-x_{k n}\right)\right| \leqq \frac{2}{n^{2}},  \tag{4.2a}\\
& \sum_{k=0}^{n-1}\left|B\left(x-x_{k n}\right)\right| \leqq \frac{2 \log n}{n},  \tag{4.3a}\\
& \sum_{k=0}^{n-1}\left|A\left(x-x_{k n}\right)\right| \leqq 1 \tag{4.4a}
\end{align*}
$$

This follows immediately by using (2.19)-(2.22), (3.3) and (3.4).
First, we prove (4.1). We note that (4.1)-(4.4) are valid for $x=x_{i n}, i=0,1, \cdots$, $n-1$. Let $x \neq x_{i n}$ and let $n=2 m$ (Proof for $n=2 m+1$ is similar). From (2.9) we have

$$
\begin{aligned}
\left|D\left(x-x_{k n}\right)\right| & \leqq \frac{4}{n}\left[2\left|\sum_{j=1}^{m} \frac{\sin j\left(x-x_{k n}\right)}{a_{j, M}}\right|+\frac{1}{\left(2^{M}-2\right) n^{M}}\right. \\
& \left.+2\left|\sum_{j=m+1}^{n-1} \frac{\sin j\left(x-x_{k n}\right)}{a_{j, M}}\right|\right] .
\end{aligned}
$$

Since $a_{j, M}$ is a decreasing function of $j$ for $j=0,1, \cdots$, and increasing function of $j$ for $j=m+1, \cdots, n$ (see Art 3), by using (3.6) and (3.7) we obtain

$$
\left|D\left(x-x_{k n}\right)\right| \leqq \frac{4}{n}\left[\frac{4}{m^{M}\left(3^{M}-3\right)} \max _{1 \leqq v \leqq n-1}\left|\sum_{j=1}^{v} \sin j\left(x-x_{k n}\right)\right|+\frac{1}{\left(2^{M}-2\right) n^{M}}\right]
$$

Now, we note that for $M \geqq 3$ we have

$$
\frac{1}{3^{M}-3}<3.2^{-M-3} .
$$

Therefore, by using (3.5) and the above estimates we obtain

$$
\begin{aligned}
\sum_{k=0}^{n-1}\left|D\left(x-x_{k n}\right)\right| & \leqq \frac{4}{n}\left[\frac{3}{2 n^{M}} 4 n \log n+\frac{1}{4 n^{M}-1}\right] \\
& \leqq \frac{25 \log n}{n^{M}}
\end{aligned}
$$

which proves (4.1). To prove (4.2) we need some estimates of the coefficients involved in $C\left(x-x_{k n}\right)$. First we observe from (2.12) and (3.11) that

$$
b_{j, M}=(2 n-j)^{M}-2(n-j)^{M}-j^{M} \geqq 0 .
$$

Next, we note that

$$
b_{j, M}^{\prime \prime}=M(M-1) b_{j, M-2} \geqq 0 .
$$

From this it follows that $b_{j, M}^{\prime}$ is a monotonic increasing function of $j$. It is easy to check that $b_{j, M}^{\prime} \leqq 0$ for $j=0,1, \cdots, n$. Therefore, $b_{j, M}$ is a monotonic decreasing function of $j$ for $j=0,1, \cdots, n$. Thus, we obtain the following estimates:

$$
0 \leqq b_{j, M} \leqq 2 n^{M},\left|b_{j, M}^{\prime}\right|<M(2 n)^{M-1}
$$

By using the estimates of $a_{j, M}, a_{j, M}^{\prime}, a_{j, M}^{\prime \prime}$, as given in (3.13)-(3.15), and the above estimates of $b_{j M}$, we finally obtain

$$
\left|\left[\frac{b_{j, M}}{a_{j, M}}\right]^{\prime \prime}\right| \leqq \frac{f_{4}(M)}{n^{2}}
$$

With the help of (3.10) and (3.8), $C(x)$ (as stated in (2.11)) can be rewritten in the form

$$
\begin{aligned}
C\left(x-x_{k n}\right)=\frac{(1-\cos n x)}{n^{3}}[1 & +\sum_{j=1}^{n-1} \frac{b_{j, M}}{a_{j, M}}\left\{(j+1) \tau_{j+1, k}(x)\right. \\
& \left.-2 j \tau_{j, k}(x)+(j-1) \tau_{j-1, k}(x)\right\}
\end{aligned}
$$

Let us write:

$$
\begin{equation*}
p(j, M)=\frac{b_{j, M}}{a_{j, M}}, \quad g(j, M)=p(j+1, M)-2 p(j, M)+p(j-1, M) \tag{4.6}
\end{equation*}
$$

so that $p(o, M)=1$ and $p(n, M)=0$. Thus, $C\left(x-x_{k n}\right)$ can be expressed in the form
(4.7) $C\left(x-x_{k n}\right)=\frac{(1-\cos n x)}{n^{3}}\left[\sum_{j=1}^{n-1} g(j, M) j \tau_{j, k}(x)+n \tau_{n, k}(x) p(n-1, M)\right]$.

From (4.5) it follows that

$$
\begin{equation*}
|g(j, M)|=\left|\left(\xi_{2}-\xi_{1}\right) p^{\prime \prime}(\xi, M)\right|<\frac{2 f_{4}(M)}{n^{2}} \tag{4.8}
\end{equation*}
$$

where $j-1<\xi_{2}<j<\xi_{1}<j+1$. Further, it is easy to verify that

$$
\begin{equation*}
|p(n-1, M)| \leqq \frac{f_{5}(M)}{n^{2}} \tag{4.9}
\end{equation*}
$$

On using (3.9) and (4.7)-(4.9), we obtain

$$
\begin{aligned}
\sum_{k=0}^{n-1}\left|C\left(x-x_{k n}\right)\right| & \leqq \frac{2}{n^{3}}\left[(n-1) f_{4}(M)+f_{M}(M)\right] \\
& \leqq \frac{f_{1}(M)}{n^{2}}
\end{aligned}
$$

This proves (4.2). The proof for (4.3) is similar to (4.1) and the proof for (4.4) is similar to (4.2). We omit the proof.

## 5. Lower estimates of the fundamental polynomials

The inequalities of Lemma 4.1 are, in a sense, best possible as is shown by the following lemma.

Lemma 5.1. There exist positive constants $f_{7}(M)$ and $f_{8}(M)$ for which the following inequalities hold true for $n=2 m+1$ :

$$
\begin{align*}
& \sum_{k=0}^{n-1}\left|A\left(\pi-x_{k n}\right)\right| \geqq 1,  \tag{5.1}\\
& \sum_{k=0}^{n-1}\left|B\left(\pi-x_{k n}\right)\right| \geqq \frac{f_{7}(M) \log n}{n},  \tag{5.2}\\
& \sum^{n-1}\left|C\left(\pi-x_{k n}\right)\right| \geqq \frac{2}{n^{2}},  \tag{5.3}\\
& \sum_{k=0}^{n-1}\left|D\left(\pi-x_{k n}\right)\right| \geqq \frac{f_{8}(M) \log n}{n^{M}} . \tag{5.4}
\end{align*}
$$

Proof. We observe from (3.1) that

$$
\sum_{k=0}^{n-1}\left|A\left(\pi-x_{k n}\right)\right|>\sum_{k=0}^{n+1} A\left(\pi-x_{k n}\right)=1
$$

which proves (5.1). Similarly from (3.2) we have

$$
\sum_{k=0}^{n-1}\left|C\left(\pi-x_{k n}\right)\right| \geqq \sum_{k=0}^{n-1} C\left(\pi-x_{k n}\right)=\frac{1-\cos n \pi}{n^{2}}=\frac{2}{n^{2}}
$$

which proves (5.3). Proofs for (5.2) and (5.4) are similar. We will only prove (5.4). First, we note that

$$
D\left(\pi-x_{k n}\right)=\frac{8(-1)^{(M+1) / 2}}{n} \sum_{j=1}^{n-1} \frac{\sin j\left(\pi-x_{k n}\right)}{a_{j, M}} .
$$

Therefore, we have

$$
\begin{aligned}
& \sin \frac{\pi-x_{k n}}{2} D\left(\pi-x_{k n}\right)=\frac{4(-1)^{(M+1) / 2}}{n}\left\{\cos \frac{\left[\pi-x_{k n} / 2\right]}{a_{1, M}}\right. \\
&\left.+\sum_{j=1}^{n-1} \frac{\left(a_{j+1, M}-a_{j, M}\right)}{a_{j, M} a_{j+1, M}} \cos \left(j+\frac{1}{2}\right)\left(\pi-x_{k n}\right)\right\} .
\end{aligned}
$$

From (2.10) it follows that

$$
\begin{equation*}
a_{1, M}<2^{M} n^{M} \tag{5.6}
\end{equation*}
$$

It is well known that

$$
\begin{equation*}
\sum_{k=0}^{n-1}\left|\cot \frac{\pi-x_{k n}}{2}\right| \geqq f_{9} n \log n . \tag{5.7}
\end{equation*}
$$

From (5.5)-(5.7) and (3.16) we obtain

$$
\sum_{k=0}^{n-1}\left|D\left(\pi-x_{k n}\right)\right| \geqq \frac{4 f_{9} \log n}{(2 n)^{M}}-\frac{4^{M}}{n^{M}} \geqq \frac{f_{8}(m) \log n}{n^{M}}
$$

which proves (5.4) as well.

## 6. Proof of Theorems

The upper and lower estimates of fundamental polynomials obtained in Articles 4 and 5 lead to the proof of theorems very easily.

Proof of Theorem 4. From (3.1) and (2.8) we obtain

$$
\begin{gather*}
R_{n}(x)-f(x)=\sum_{i=0}^{n-1}\left[f\left(x_{i n}\right)-f(x)\right] A\left(x-x_{i n}\right) \\
+\sum_{i=0}^{n-1} \alpha_{i n} B\left(x-x_{i n}\right)+\sum_{i=0}^{n-1} \beta_{i n} C\left(x-x_{i n}\right)  \tag{6.1}\\
+\sum_{i=0}^{n-1} \delta_{i n} D\left(x-x_{i n}\right)
\end{gather*}
$$

Let us denote the expression on the right-hand side by $I_{1}, I_{2}, I_{3}$ and $I_{4}$ respectively.
From (2.3) and (4.1) we have

$$
\begin{equation*}
\left|I_{4}\right|=o\left(\frac{n^{M}}{\log n}\right) \sum_{i=0}^{n-1}\left|D\left(x-x_{i n}\right)\right|=o\left(\frac{n^{M}}{\log n}\right) \frac{25 \log n}{n^{M}}=o(1) \tag{6.2}
\end{equation*}
$$

From (2.3) and (4.2) we have

$$
\begin{equation*}
\left|I_{3}\right|=o\left(n^{2}\right) \sum_{i=0}^{n-1}\left|C\left(x-x_{i n}\right)\right|=\frac{o\left(n^{2}\right) f_{1}(M)}{n^{2}}=o(1) \tag{6.3}
\end{equation*}
$$

From (2.3) and (4.3) we obtain

$$
\begin{equation*}
\left|I_{2}\right|=o\left(\frac{n}{\log n}\right) \sum_{i=0}^{n-1}\left|B\left(x-x_{i n}\right)\right|=o\left(\frac{n}{\log n}\right) \frac{f_{2}(M) \log n}{n}=o(1) \tag{6.4}
\end{equation*}
$$

For the estimation of $I_{1}$ we use the fact that $f(x)$ is continuous $2 \pi$ periodic function. Given $\varepsilon>0 \exists \delta$ such that $\left|f(x)-f\left(x_{i n}\right)\right|<\varepsilon$ whenever $\left|x-x_{i n}\right| \leqq \delta$ $=\delta(\varepsilon)$. Put $\max _{0 \leqq x \leqq 2 \pi}|f(x)|=B$. From (3.9), it follows that

$$
0 \leqq \tau_{j, k}(x)<\frac{1}{j(\sin (\delta / 2))^{2}} \text { for }\left|x-x_{i n}\right|>\delta
$$

By expressing $A\left(x-x_{i n}\right)$ in terms of Fejér-kernel and using the above result, we obtain

$$
\begin{equation*}
\sum_{\left|x-x_{i n}\right|>\delta}\left|A\left(x-x_{i n}\right)\right| \leqq \frac{C}{n \sin ^{2}(\delta / 2)} \tag{6.5}
\end{equation*}
$$

Next, we express $I_{1}$ as

$$
\begin{aligned}
& I_{1}=\sum_{\left|x-x_{i n}\right| \leqq \delta}\left[f(x)-f\left(x_{i n}\right)\right] A\left(x-x_{i n}\right) \\
&+\sum_{\left|x-x_{i n}\right|>\delta}\left[f(x)-f\left(x_{i n}\right)\right] A\left(x-x_{i n}\right) .
\end{aligned}
$$

By using (6.5) and (5.4), we obtain

$$
\begin{aligned}
\left|I_{1}\right| & \left.\leqq \varepsilon \sum_{\left|x-x_{i n}\right| \leqq \delta}\left|A\left(x-x_{i n}\right)\right|+2 B \sum_{\left|x-x_{i n}\right|>\delta} \mid A\left(x-x_{i n}\right)\right] \\
& \leqq \varepsilon \sum_{i=1}^{n}\left|A\left(x-x_{i n}\right)\right|+\frac{2 B C}{n \sin (\delta / 2)} \leqq \varepsilon f_{3}(M)+\frac{2 B C}{n \sin ^{2}(\delta / 2)} .
\end{aligned}
$$

Since the second term on the right-hand side can be made as small as we please by choosing $n$ sufficiently large, we have

$$
\begin{equation*}
I_{1}=o(1) \tag{6.6}
\end{equation*}
$$

From (6.1)-(6.4) and (6.6) we have $R_{n}[x]-f(x)=o(1)$ which proves Theorem 4.

Proof of Theorem 5. From the uniqueness of $(0,1,2, M)$ trigonometric interpolation it follows that for an arbitrary trigonometric polynomial $\phi_{n}(x)$ of the order $2 n$ (satisfying (2.1.)) we have

$$
\begin{aligned}
\phi_{n}(x) & =\sum_{i=0}^{n-1} \phi_{n}\left(x_{i n}\right) A_{i n}(x)+\sum_{i=0}^{n-1} \phi_{n}^{\prime}\left(x_{i n}\right) B_{i n}(x) \\
& +\sum_{i=0}^{n-1} \phi_{n}^{\prime \prime}\left(x_{i n}\right) C_{i n}(x)+\sum_{i=0}^{n-1} \phi_{n}^{(M)}\left(x_{i n}\right) D_{i n}(x)
\end{aligned}
$$

Let $\phi_{n}(x)$ satisfy further the condition (2.4). On using (4.1)-(4.4) it follows that

$$
\begin{aligned}
\left|\phi_{n}(x)\right| \leqq f_{3}(M) a_{0}+f_{2}(M) & \frac{a_{1} \log n}{n}+\frac{f_{1} a_{2}(M)}{n^{2}} \\
& +\frac{25 \log n}{n^{M}} a_{M}
\end{aligned}
$$

Therefore for $0 \leqq x \leqq 2 \pi$, we have

$$
\left|\phi_{n}(x)\right| \leqq c_{0}\left(a_{0}+\frac{a_{1} \log n}{n}+\frac{a_{2}}{n^{2}}+\frac{a_{M} \log n}{n^{M}}\right)
$$

where $c_{0}=\max \left(f_{3}, f_{2}, f_{1}, 25\right)$.
To prove (2.6), we denote $q_{n}(x)$ to be the trigonometric polynomial

$$
\begin{aligned}
q_{n}(x) & =\sum_{i=0}^{n-1} a_{0} A\left(x-x_{i n}\right) \operatorname{sign} A\left(\pi-x_{i n}\right) \\
& +\sum_{i=0}^{n-1} a_{1} B\left(x-x_{i n}\right) \operatorname{sign} B\left(\pi-x_{i n}\right) \\
& +\sum_{i=0}^{n-1} a_{2} C\left(x-x_{i n}\right) \operatorname{sign} C\left(\pi-x_{i n}\right) \\
& +\sum_{i=0}^{n-1} a_{M} D\left(x-x_{i n}\right) \operatorname{sign} D\left(\pi-x_{i n}\right)
\end{aligned}
$$

Let $x=\pi$. By using (5.1)-(5.4) we can deduce (2.6) and this proves Theorem 5.
Proof of Theorem 6. Let $\alpha_{i n}=\beta_{i n}=\delta_{i n}=0$. Then (2.8) reduces to

$$
R_{n}(x)=\sum_{i=0}^{n-1} f\left(x_{i n}\right) A\left(x-x_{i n}\right)
$$

By using (3.1) we have

$$
\begin{equation*}
f(x)-R_{n}(x)=\sum_{i=0}^{n-1}\left[f(x)-f\left(x_{i n}\right)\right] A\left(x-x_{i n}\right) \tag{6.7}
\end{equation*}
$$

Let us denote $\omega(\delta)$ as modulus of continuity of $f(x)$. From the result of Shisha and Mond [12], we have for any $\delta>0$ and all $x, y$

$$
\begin{equation*}
|f(x)-f(y)| \leqq\left(1+\frac{\pi^{2}}{\delta^{2}} \sin ^{2} \frac{x-y}{2}\right) \omega(\delta) \tag{6.8}
\end{equation*}
$$

By using (6.7) and (6.8) we obtain

$$
\begin{equation*}
\left|f(x)-R_{n}(x)\right| \leqq \sum_{i=0}^{n-1} \omega(\delta)\left(1+\frac{\pi^{2}}{\delta^{2}} \sin ^{2} \frac{\left(x-x_{i n}\right)}{2}\right)\left|A\left(x-x_{i n}\right)\right| \tag{6.9}
\end{equation*}
$$

Following [17], it can be shown that

$$
\begin{equation*}
\sum_{i=0}^{n-1} \sin ^{2} \frac{x-x_{i n}}{2}\left|A\left(x-x_{i n}\right)\right| \leqq \frac{f_{10}(M)}{n} \tag{6.1}
\end{equation*}
$$

Let $\delta=1 / \sqrt{n}$ in (6.9) and use (6.10) and (4.4) to obtain

$$
\left|f(x)-R_{n}(x)\right|=O\left(\omega_{\delta}(1 / \sqrt{ } n)\right)
$$

This proves Theorem 6 as well.

Remark. Let $M$ be an even positive integer. Let $S_{n}(x)$ be the unique trigonometric polynomial determined by $(0, M)$ interpolation. From Theorem 3 we know that $S_{n}(x)$ converges uniformly to $f(x)$ (satisfying (1.5)) provided that the freedom of $S_{n}^{(M)}(x)$ at the points $x_{k}$ 's is given by

$$
S_{n}^{(M)}\left(x_{n}\right)=o\left(n^{M-1}\right), k=0,1, \cdots, n-1
$$

This is best possible. In [15] the author has considered the problem of $(0,1, M)$ ( $M$-even) interpolation. One of the main features of this result is that by prescribing the first derivative, the freedom of $S_{n}^{(M)}\left(x_{k}\right)$ has considerably increased.

$$
S_{n}^{(M)}\left(x_{k}\right)=o\left(\frac{n^{M}}{\log n}\right), k=0,1, \cdots, n-1
$$

It may be noted that for $M$-odd ( $0,1, M$ ) trigonometric interpolation does not exist uniquely. This is shown also in [15].

Let $S_{n}(x)$ be the unique trigonometric polynomial determined by $(0, M)$ interpolation ( $M$-odd). From Theorem 3 we know that $S_{n}(x)$ converges uniformly to $f(x)\left(f(x) \in c_{2 \pi}\right)$ provided the freedom of $S_{n}^{(M)}(x)$ at the points $x_{k}$ 's is given by

$$
S_{n}^{(M)}\left(x_{k}\right)=o\left(\frac{n^{M}}{\log n}\right)
$$

Further, this is best possible. Let $R_{n}(x)$ be the unique trigonometric polynomial determined by ( $0,1,2, M$ ) interpolation ( $M$-odd). Here we prescribed $R_{n}(x)$ and $R_{n}^{\prime \prime}(x)$ at $x=x_{k}$ as well. One of the main features of Theorem 4 is that even with these new restrictions, the freedom of $R_{n}^{(M)}(x)$ at $x=x_{k}$ 's can not be improved.

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