HERMITE GEOMETRIC INTERPOLATION BY RATIONAL BÉZIER SPATIAL CURVES

GAŠPER JAKLIČ*, JERNEJ KOZAK†, MARJETA KRAJNC‡, VITO VITRIH§, AND EMIL ŽAGAR¶

Abstract. Polynomial geometric interpolation by parametric curves became one of the standard techniques for interpolation of geometric data. An obvious generalization leads to rational geometric interpolation schemes, which are a much less investigated research topic. The aim of this paper is to present a general framework for Hermite geometric interpolation by rational Bézier spatial curves. In particular, cubic G^2 and quartic G^3 interpolations are analyzed in detail. Systems of nonlinear equations are derived in a simplified form and the existence of admissible solutions is studied. For the cubic case, geometric conditions implying solvability of the nonlinear system are also stated. The asymptotic analysis is done in both cases and optimal approximation orders are proved. Numerical examples are given, which confirm the theoretical results.

Key words. interpolation, rational, geometric, Hermite

AMS subject classifications. 65D05, 65D17, 65D10

1. Introduction.

Geometric interpolation by parametric curves is a modern research topic dealing with interpolation of given geometric data (points, tangent directions, etc.) and is based on geometric properties, independent of parameterizations. In comparison to classical interpolation schemes, this allows additional shape parameters and consequently a higher approximation order. The shape of the interpolant is more pleasant since the parameters that need to be prescribed in advance in classical schemes are here chosen automatically. The first rigorous analysis of a particular geometric interpolation scheme goes back to [2]. Later, Höllig and Koch ([8]) stated a general conjecture on polynomial geometric interpolation asserting that these interpolants could, in general, interpolate much more data than their classical counterparts of the same degree. However, the analysis of geometric interpolation schemes is a challenging task since it involves analysis of systems of nonlinear equations. Several results on existence, uniqueness, geometric conditions for solvability, and algorithms for construction are known (see [9], [15], [14], [6] and the references therein). While planar polynomial geometric interpolation is a well investigated topic, not much is known for interpolation in higher dimensional spaces $\mathbb{R}^d$. Obviously the rational geometric interpolation is even more challenging topic and even less results are known. A nice survey on known facts is given, e.g., in [20]. Some recent results are mainly dealing with the planar case ([7], [17], [19]) while the spatial case is still to be investigated. Some interesting results can be found in [12], [4], [3], [10] and [21].

In this paper, Hermite interpolation by rational Bézier spatial curves is tackled. A general system of $6\,r$ polynomial equations for G^r Hermite interpolation by a rational Bézier curve of degree $n = r+1$ is derived. This system can be further simplified to a system of $3\,r$ equations not involving control points and weights. Sufficient conditions for the optimal asymptotic approximation order $2\,n$ of a solution are given.

Quadratic rational Bézier curves are planar and thus they do not posses enough

*FMF and IMFM, University of Ljubljana (gasper.jaklic@fmf.uni-lj.si).
†FMF and IMFM, University of Ljubljana (jernej.kozak@fmf.uni-lj.si).
‡FMF and IMFM, University of Ljubljana (marjetka.krajnc@fmf.uni-lj.si).
§FAMNIT and PINT, University of Primorska (vito.vitrih@upr.si).
¶FMF and IMFM, University of Ljubljana (emil.zagar@fmf.uni-lj.si).

flexibility to interpolate spatial data. In this paper, the next two interesting cases, i.e., cubic G^2 and quartic G^3 rational Bézier interpolants will be considered. Quite surprisingly, the analysis of the cubic rational geometric Hermite interpolation is a much more straightforward than its polynomial counterpart. There is only one nontrivial solution of the interpolation problem, and the geometric conditions for its existence can be derived. The interpolant is given in a closed form and the asymptotic approximation order is 6. The quartic case is harder to tackle, but the solutions can still be derived explicitly by solving a quartic polynomial equation. There can be 0 to 4 admissible solutions. The asymptotic approximation order of solutions is 8.

Since only data points are usually given in practical applications, G^r data need to be obtained by a suitable approximation. The most natural way would be to use local approach - apply a polynomial interpolant through 5 successive data points, and compute its suitable higher order derivatives, which would be used as G^r data.

The outline of the paper is as follows. In the next section, the Hermite interpolation problem is analyzed, and the system of equations, that needs to be solved, is derived. It is further simplified, and sufficient conditions for the optimal approximation order of the solution are derived. In the third section, the solution for the cubic case is given, together with geometric conditions for its existence, and the optimal approximation order is confirmed. In the penultimate section, the quartic case is studied. The paper is concluded by some examples.

2. Hermite interpolation. Let us consider a Hermite geometric interpolation problem in $\mathbb{R}^3$. We are given two data points $\boldsymbol{P}^L, \boldsymbol{P}^R$, with the associated vectors $\boldsymbol{t}_i^L$ and $\boldsymbol{t}_i^R$, $i = 1, 2, \ldots, r$, respectively. The notation $(\cdot)^L$ and $(\cdot)^R$ refers to the data at the left and the right endpoint of the domain interval of a parametric interpolant. The vectors $\boldsymbol{t}_i^L$ and $\boldsymbol{t}_i^R$ represent i-th derivatives. Without loss of generality we can assume that the vectors $\boldsymbol{t}_1^L$ and $\boldsymbol{t}_1^R$, that represent tangent directions, are of the unit length.

We are looking for a particular interpolant of the form $\boldsymbol{r}(t) = \sum_{i=0}^n \boldsymbol{b}_i \Phi_i(t)$ on $[0, 1]$, where $\boldsymbol{b}_i$ are control points and Φ_i are chosen basis functions with $\sum_{i=0}^n \Phi_i(t) \equiv 1$. Our goal is to interpolate the given data in G^r sense, i.e., the data points, tangent directions, etc.,

$$\begin{aligned} \boldsymbol{r}(0) &= \boldsymbol{P}^L, \quad \boldsymbol{r}(1) = \boldsymbol{P}^R, \\ \boldsymbol{r}^{(i)}(0) &= \sum_{k=1}^{i} \alpha_{i,k}^L \, \boldsymbol{t}_k^L, \quad i = 1, 2, \ldots, r, \\ \boldsymbol{r}^{(i)}(1) &= \sum_{k=1}^{i} \alpha_{i,k}^R \, \boldsymbol{t}_k^R, \quad i = 1, 2, \ldots, r. \end{aligned} \tag{2.1}$$

The coefficients $\alpha_{i,j}^L$ and $\alpha_{i,j}^R$ are elements of a lower triangular *connection matrix* at parameter 0 and 1, respectively. They describe geometric continuity conditions (see [5], [18], e.g.). As an example, for $r \leq 4$, the connection matrix is given as the $r \times r$ leading principal submatrix of the matrix

$$\begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ \alpha_2 & \alpha_1^2 & 0 & 0 \\ \alpha_3 & 3\alpha_1\alpha_2 & \alpha_1^3 & 0 \\ \alpha_4 & 3\alpha_2^2 + 4\alpha_1\alpha_3 & 6\alpha_1^2\alpha_2 & \alpha_1^4 \end{pmatrix}. \tag{2.2}$$

The entries of (2.2) are obtained by Faa di Bruno's formula for differentiation of a reparameterized curve ([11]). Note that there are r unknown parameters α_i in a general G^r connection matrix. The elements $\alpha_{1,1}^L$ and $\alpha_{1,1}^R$ need to be positive to preserve the given tangent directions.

In this paper we will consider the rational Bézier basis, i.e.,

$$\Phi_i(t) = \frac{w_i B_i^n(t)}{w(t)},$$

where w_i denotes the weight and B_i^n the i-th Bernstein basis polynomial of degree n. The polynomial w is defined as $w(t) := \sum_{j=0}^n w_j B_j^n(t)$. Thus the interpolant can be written as

$$\boldsymbol{r}(t) = \frac{\boldsymbol{p}(t)}{w(t)},$$

with

$$\boldsymbol{p}(t) := \sum_{i=0}^n w_i \boldsymbol{b}_i B_i^n(t),$$

where $\boldsymbol{b}_i$ are control points of the rational Bézier curve and w_i are its weights. By using the normalized form of the curve [5], we can without loss of generality assume that $w_0 = w_n = 1$.

Let us consider the interpolation problem (2.1) more precisely now. Since the endpoint interpolation property of a rational Bézier curve automatically fulfills the first two conditions, there are $6r$ equations left. The unknowns are $w_1, w_2, \dots, w_{n-1}$, $\boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_{n-1}$ and r parameters of each connection matrix at both endpoints. Thus there are $4(n-1)+2r$ unknowns. In order for the number of unknowns to match the number of equations, this yields $n = r+1$.

The geometric nature of the unknown weights w_i and the elements of the connection matrices differs from the unknown control points $\boldsymbol{b}_i$ significantly. But the equation set can be split into two parts. The first, the tough one, involves the scalar unknowns only. Let $[t_\ell, t_{\ell+1}, \dots, t_{\ell+k}]$ denote the divided difference based upon the knots $t_\ell, t_{\ell+1}, \dots, t_{\ell+k}$. If we apply divided differences

$$[\underbrace{0,\dots,0}_{n-i}, \underbrace{1,\dots,1}_{i+2}], \quad i = 0, 1, \dots, n-2,$$

to the identity $\boldsymbol{p} = w\boldsymbol{r}$, the right-hand side must vanish since $\boldsymbol{p}$ is a polynomial curve of degree $\leq n$. The closed form of the equations follows from the following lemma. This, together with the following lemma, gives the closed form of the equations.

Lemma 2.1. *Suppose that $i, j, k, m \in \mathbb{N}_0$ are given integers, and*

$$g_{k,m}(x) := x^k (1-x)^m.$$

Then

$$[\underbrace{0,\dots,0}_{i}, \underbrace{1,\dots,1}_{j}] g_{k,m} = \begin{cases} (-1)^{j+i-k-1} \binom{m-j}{i-k-1}, & k \leq i-1,\ m \geq j, \\ (-1)^m \binom{k-i}{j-m-1}, & k \geq i,\ m \leq j-1, \\ 0, & \textit{otherwise.} \end{cases} \tag{2.3}$$

Proof. Suppose that $k \le i-1$, $m \ge j$. Since

$$g_{k,m}(x) = x\, g_{k-1,m}(x) = (1-x)\, g_{k,m-1}(x),$$

the Leibniz rule reveals

$$\begin{aligned}
[\underbrace{0,\ldots,0}_{i},\underbrace{1,\ldots,1}_{j}]g_{k,m} &= [\underbrace{0,\ldots,0}_{i},\underbrace{1,\ldots,1}_{j}]\,(\,(\cdot)\, g_{k-1,m}(\cdot)\,) = \\
&= [0]\,(\cdot)\;[\underbrace{0,\ldots,0}_{i},\underbrace{1,\ldots,1}_{j}]g_{k-1,m} + [0,0]\,(\cdot)\;[\underbrace{0,\ldots,0}_{i-1},\underbrace{1,\ldots,1}_{j}]g_{k-1,m} \\
&= [\underbrace{0,\ldots,0}_{i-1},\underbrace{1,\ldots,1}_{j}]g_{k-1,m} = \cdots = [\underbrace{0,\ldots,0}_{i-k},\underbrace{1,\ldots,1}_{j}]g_{0,m},
\end{aligned}$$

and

$$\begin{aligned}
[\underbrace{0,\ldots,0}_{i-k},\underbrace{1,\ldots,1}_{j}]g_{0,m} &= [\underbrace{0,\ldots,0}_{i-k},\underbrace{1,\ldots,1}_{j}]\,(\,(1-\cdot)\, g_{0,m-1}(\cdot)\,) = \\
&= -[\underbrace{0,\ldots,0}_{i-k},\underbrace{1,\ldots,1}_{j-1}]g_{0,m-1} = \cdots = (-1)^j[\underbrace{0,\ldots,0}_{i-k}]g_{0,m-j}.
\end{aligned}$$

Further,

$$\begin{aligned}
[\underbrace{0,\ldots,0}_{i-k}]g_{0,m-j} &= \frac{1}{(i-k-1)!}\frac{d^{i-k-1}}{dx^{i-k-1}}(1-x)^{m-j}\Big|_{x=0} \\
&= (-1)^{i-k-1}\binom{m-j}{i-k-1},
\end{aligned}$$

which proves the first possibility in (2.3). The second one follows similarly. If $k \le i-1$, $m \le j-1$, the degree of the polynomial $g_{k,m}$ is $\le i+j-2$, so any divided difference based upon $i+j$ knots maps it to zero. If $m \ge j, k \ge i$,

$$[\underbrace{0,\ldots,0}_{i},\underbrace{1,\ldots,1}_{j}]g_{k,m} = (-1)^{j-1}[0,1]g_{k-i+1,m-j+1} = 0,$$

which completes the proof. ◻

Note that the result of Lemma 2.1 could be derived from the Opitz's formula for divided differences of monomials [1], but a direct proof is quite straightforward.

THEOREM 2.2. *The equations that determine the scalar parameters of the rational interpolatory curve $\boldsymbol{r}$ are given by*

$$\begin{aligned}
&\sum_{\ell=0}^{n-2-i}(-1)^{n-\ell}\binom{n}{\ell}w_\ell \sum_{k=1}^{n-1-i-\ell}\boldsymbol{t}_k^L \sum_{s=k}^{n-1-i-\ell}(-1)^{s+1}\binom{n-2-i-\ell}{s-1}\frac{\alpha_{s,k}^L}{s!} \\
&\quad + (-1)^{i+1}\binom{n}{i+1}w_{n-1-i}\,\Delta\boldsymbol{P} \qquad\qquad (2.4)\\
&\quad + \sum_{\ell=n-i}^{n}(-1)^{n-\ell}\binom{n}{\ell}w_\ell \sum_{k=1}^{\ell+1-n+i}\boldsymbol{t}_k^R \sum_{s=k}^{\ell+1-n+i}\binom{\ell-n+i}{s-1}\frac{\alpha_{s,k}^R}{s!} = \boldsymbol{0}, \\
&\qquad\qquad i = 0,1,\ldots,n-2,
\end{aligned}$$

where $\alpha_{s,k}^L$ and $\alpha_{s,k}^R$ are the elements of the connection matrices involved in the interpolation conditions (2.1)*, and $\Delta\boldsymbol{P} := \boldsymbol{P}^R - \boldsymbol{P}^L$.*

Proof. Suppose that $0 \le \ell \le n-2-i$. The Leibniz rule gives

$$\begin{aligned}
[\underbrace{0,\dots,0}_{n-i},\underbrace{1,\dots,1}_{i+2}]\,(\boldsymbol{r}\,B_\ell^n) &= \sum_{s=1}^{n-i}[\underbrace{0,\dots,0}_{s}]\boldsymbol{r}\,[\underbrace{0,\dots,0}_{n-i+1-s},\underbrace{1,\dots,1}_{i+2}]B_\ell^n \\
&+ \sum_{s=n-i+1}^{n+2}[\underbrace{0,\dots,0}_{n-i},\underbrace{1,\dots,1}_{s-n+i}]\boldsymbol{r}\,[\underbrace{1,\dots,1}_{n+3-s}]B_\ell^n.
\end{aligned} \tag{2.5}$$

But the terms $[\underbrace{1,\dots,1}_{n+3-s}]B_\ell^n$ vanish by Lemma 2.1 since

$$\ell \ge 0, \quad n-\ell \ge i+2 \ge n+3-s, \qquad s = n-i+1, n-i+2, \dots, n+2,$$

and so does the term at $s=1$. Lemma 2.1 simplifies (2.5) further to

$$\sum_{s=2}^{n-i-\ell}(-1)^{n-\ell-s}\binom{n}{\ell}\binom{n-\ell-i-2}{s-2}[\underbrace{0,\dots,0}_{s}]\boldsymbol{r}. \tag{2.6}$$

If $\ell \ge n-i$, we obtain

$$\begin{aligned}
[\underbrace{0,\dots,0}_{n-i},\underbrace{1,\dots,1}_{i+2}]\,(\boldsymbol{r}\,B_\ell^n) &= \sum_{s=1}^{i+2}[\underbrace{1,\dots,1}_{s}]\boldsymbol{r}\,[\underbrace{0,\dots,0}_{n-i},\underbrace{1,\dots,1}_{i+3-s}]B_\ell^n \\
&+ \sum_{s=i+3}^{n+2}[\underbrace{0,\dots,0}_{s-i-2},\underbrace{1,\dots,1}_{i+2}]\boldsymbol{r}\,[\underbrace{0,\dots,0}_{n+3-s}]B_\ell^n \\
&= \sum_{s=2}^{\ell-n+i+2}(-1)^{n-\ell}\binom{n}{\ell}\binom{\ell-n+i}{s-2}[\underbrace{1,\dots,1}_{s}]\boldsymbol{r},
\end{aligned} \tag{2.7}$$

again by Lemma 2.1. Similarly, if $\ell = n-1-i$,

$$\begin{aligned}
[\underbrace{0,\dots,0}_{n-i},\underbrace{1,\dots,1}_{i+2}]\left(\boldsymbol{r}\,B_{n-1-i}^n\right) &= [0]\boldsymbol{r}\,[\underbrace{0,\dots,0}_{n-i},\underbrace{1,\dots,1}_{i+2}]B_{n-1-i}^n \\
+ [0,1]\boldsymbol{r}\,[\underbrace{0,\dots,0}_{n-i-1},\underbrace{1,\dots,1}_{i+2}]B_{n-1-i}^n &+ \cdots = (-1)^{i+1}\binom{n}{i+1}[0,1]\boldsymbol{r}.
\end{aligned} \tag{2.8}$$

Let us combine (2.6)-(2.8). We obtain the system of equations

$$\begin{aligned}
&\sum_{\ell=0}^{n-2-i}(-1)^{n-\ell}\binom{n}{\ell}w_\ell\sum_{s=0}^{n-2-i-\ell}(-1)^s\binom{n-i-2-\ell}{s}\frac{1}{(s+1)!}\boldsymbol{r}^{(s+1)}(0) \\
&\quad + (-1)^{i+1}\binom{n}{i+1}w_{n-1-i}\,[0,1]\boldsymbol{r} \\
&\quad + \sum_{\ell=n-i}^{n}(-1)^{n-\ell}\binom{n}{\ell}w_\ell\sum_{s=0}^{\ell-n+i}\binom{\ell-n+i}{s}\frac{1}{(s+1)!}\boldsymbol{r}^{(s+1)}(1) = \boldsymbol{0}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 0,1,\dots,n-2,
\end{aligned}$$

and the interpolation conditions (2.1) give the final form (2.4). □

There is a natural way to rewrite the system of vector equations (2.4) in a scalar form that allows to separate the equations that determine the weights w_ℓ and the connection parameters. Let $\lambda_j : \mathbb{R}^3 \to \mathbb{R},\ j = 1,2,3$, be linear functionals, defined as

$$\lambda_1 := \det\left(\,\cdot\,, \Delta\boldsymbol{P}, \boldsymbol{t}_1^R\right),\ \lambda_2 := \det\left(\boldsymbol{t}_1^L, \,\cdot\,, \boldsymbol{t}_1^R\right),\ \lambda_3 := \det\left(\boldsymbol{t}_1^L, \Delta\boldsymbol{P}, \,\cdot\,\right). \tag{2.9}$$

These functionals are clearly linearly independent iff the vectors $\boldsymbol{t}_1^L, \Delta\boldsymbol{P}, \boldsymbol{t}_1^R$ are linearly independent, i.e.,

$$\omega := \det\left(\boldsymbol{t}_1^L, \Delta\boldsymbol{P}, \boldsymbol{t}_1^R\right) \neq 0. \tag{2.10}$$

So if one applies λ_j, $j = 1,2,3$, to the system (2.4), the equivalent scalar system of $3(n-1)$ equations emerges. But, for a fixed i, $0 \le i \le n-2$, the part of (2.4), contributed by terms that involve w_{n-2-i}, w_{n-1-i} and w_{n-i}, reads

$$\begin{aligned} v_i :=&(-1)^i \binom{n}{i+2} w_{n-2-i}\, \alpha_{1,1}^L\, \boldsymbol{t}_1^L + (-1)^{i+1}\binom{n}{i+1} w_{n-1-i}\Delta\boldsymbol{P} \\ &+ (-1)^i \binom{n}{i} w_{n-i}\alpha_{1,1}^R \boldsymbol{t}_1^R, \end{aligned}$$

and

$$\begin{aligned} \lambda_1 v_i &= (-1)^i \binom{n}{i+2} \omega\, \alpha_{1,1}^L\, w_{n-2-i}, \\ \lambda_2 v_i &= (-1)^{i+1}\binom{n}{i+1} \omega\, w_{n-1-i}, \quad \lambda_3 v_i = (-1)^i \binom{n}{i} \omega\, \alpha_{1,1}^R\, w_{n-i}. \end{aligned}$$

So one may use $\lambda_2 v_i$, for $i = 0,1,\dots,n-2$, to obtain w_{n-1-i}, and eliminate it from all the other equations. This separates the system of equations into two parts: the first should determine the connection parameters, and the second one provides the weights w_ℓ in terms of them.

Once the weights w_i and the connection parameters $\alpha_{i,k}^L$, $\alpha_{i,k}^R$, $i = 1,2,\dots,n-1$, have been determined, it is straightforward to compute the control points $\boldsymbol{b}_i, i = 0,1,\dots,n$, from

$$\begin{aligned} \frac{n!}{(n-s)!}\Delta^s(w_0\boldsymbol{b}_0) &= \sum_{i=0}^{s} \binom{s}{i} \boldsymbol{r}^{(i)}(0) \frac{n!}{(n-s+i)!}\Delta^{s-i} w_0, \\ \frac{n!}{(n-s)!}\Delta^s(w_{n-s}\boldsymbol{b}_{n-s}) &= \sum_{i=0}^{s} \binom{s}{i} \boldsymbol{r}^{(i)}(1) \frac{n!}{(n-s+i)!}\Delta^{s-i} w_{n-s+i}, \end{aligned} \tag{2.11}$$

and (2.1), for $s = 0,1,\dots,r$, where Δ denotes the standard forward difference, i.e., $\Delta(\cdot)_i := (\cdot)_{i+1} - (\cdot)_i$. Of course all the control points could be obtained by either of equations in (2.11), but by using both, more elegant symmetric expressions can be obtained.

The system (2.4) may have none or several feasible solutions, depending on the data supplied. Let us consider the asymptotic approximation order of the feasible solutions. The asymptotic analysis could not be carried out in advance for a general n, but a significant preparation step is right at hand. First of all, we recall the parametric distance (see [16], e.g.) as a measure of distance between parametric curves $\boldsymbol{f} : [a,b] \to \mathbb{R}^d$ and $\boldsymbol{g} : [c,d] \to \mathbb{R}^d$, defined as

$$\operatorname{dist}_P(\boldsymbol{f}, \boldsymbol{g}) := \inf_{\varphi} \max_{a \le t \le b} \|\boldsymbol{f}(t) - \boldsymbol{g}(\varphi(t))\|, \tag{2.12}$$

where the infimum is taken among all diffeomorphisms $\varphi : [a,b] \to [c,d]$, and $\|\cdot\|$ is the usual Euclidean norm.

Theorem 2.3. *Suppose that the interpolation data*

$$\begin{aligned} \boldsymbol{P}^L &= \boldsymbol{f}\left(-\frac{h}{2}\right), \quad \boldsymbol{t}_s^L = \boldsymbol{f}^{(s)}\left(-\frac{h}{2}\right), \ s = 1,2,\ldots,n-1, \\ \boldsymbol{P}^R &= \boldsymbol{f}\left(\frac{h}{2}\right), \quad \boldsymbol{t}_s^R = \boldsymbol{f}^{(s)}\left(\frac{h}{2}\right), \ s = 1,2,\ldots,n-1, \end{aligned}$$

are sampled from an analytic curve $\boldsymbol{f} : [-\frac{h}{2}, \frac{h}{2}] \to \mathbb{R}^3$, parameterized by the arc-length. Suppose that a rational G^{n-1} interpolant $\boldsymbol{r}_h$ exists, and depends continuously on h for all $0 < h \le h_0$, for some constant $h_0 > 0$. Further, let the corresponding unknowns, determined as a solution of the system (2.4), additionally satisfy

$$w(t) = 1 + \mathcal{O}(h), \quad w^{(s)}(t) = \mathcal{O}(h^s), \quad s = 1,2,\ldots,n, \tag{2.13}$$

and

$$\varphi'(t) = h + \mathcal{O}\left(h^2\right), \quad \varphi^{(i)}(t) = \mathcal{O}\left(h^i\right), \quad i = 2,3,\ldots,2n-1, \tag{2.14}$$

where $\varphi : [0,1] \to [-h/2, h/2]$ is a polynomial of degree $\le\ 2n-1$, determined by the values

$$\varphi(0) = -\frac{h}{2}, \ \varphi(1) = \frac{h}{2}, \quad \varphi^{(i)}(0) = \alpha_i^L, \ \varphi^{(i)}(1) = \alpha_i^R, \ i = 1,2,\ldots,n-1. \tag{2.15}$$

Then

$$dist_P(\boldsymbol{f}, \boldsymbol{r}_h) = \mathcal{O}\left(h^{2n}\right).$$

Proof. Let $\boldsymbol{q}$ be the polynomial curve of degree $\le\ 2n-1$ that interpolates the data (2.1). Since

$$\operatorname{dist}_P(\boldsymbol{f}, \boldsymbol{r}_h) \le \operatorname{dist}_P(\boldsymbol{f}, \boldsymbol{q}) + \operatorname{dist}_P(\boldsymbol{q}, \boldsymbol{r}_h), \tag{2.16}$$

it is enough to estimate each of the right-hand side terms separately. Any particular reparameterization in (2.12) gives an upper bound on the parametric distance. Let h_0 be small enough so that φ as defined in the theorem is a diffeomorphism for all $0 < h \le h_0$. Such an h_0 should exist by (2.14). Then

$$\operatorname{dist}_P(\boldsymbol{f}, \boldsymbol{q}) \le \max_{0 \le t \le 1} \|\boldsymbol{f}(\varphi(t)) - \boldsymbol{q}(t)\|.$$

But the polynomial curve $\boldsymbol{q}$ of degree $\le\ 2n-1$ agrees with $\boldsymbol{f} \circ \varphi$ n-fold at 0 and at 1, respectively. Thus the interpolation error is

$$\boldsymbol{f}(\varphi(t)) - \boldsymbol{q}(t) = t^n (t-1)^n [\underbrace{0,\ldots,0}_{n}, \underbrace{1,\ldots,1}_{n}, t]\,(\boldsymbol{f} \circ \varphi) = \mathcal{O}\left(h^{2n}\right).$$

The last equality follows from the chain rule applied to $\frac{d^s}{dt^s}\boldsymbol{f}\left(\varphi(t)\right)$, and (2.14), which proves

$$\frac{d^s}{dt^s}\boldsymbol{f}\left(\varphi(t)\right) = \mathcal{O}\left(h^s\right), \quad s = 1, 2, \ldots, 2n. \tag{2.17}$$

The second term in (2.16) is bounded above by

$$\max_{0\le t\le 1} \|\boldsymbol{q}\left(t\right) - \boldsymbol{r}_h(t)\|,$$

and

$$\boldsymbol{q}\left(t\right) - \boldsymbol{r}_h(t) = t^n(t-1)^n[\underbrace{0,\ldots,0}_{n}, \underbrace{1,\ldots,1}_{n}, t\,]\boldsymbol{r}_h,$$

since $\boldsymbol{q}$ interpolates $\boldsymbol{r}_h$ too. But

$$\begin{aligned} 0 &= [\underbrace{0,\ldots,0}_{n}, \underbrace{1,\ldots,1}_{n}, t\,]\left(w\boldsymbol{r}_h\right) = \\ &= w(t)[\underbrace{0,\ldots,0}_{n}, \underbrace{1,\ldots,1}_{n}, t\,]\boldsymbol{r}_h + \sum_{i=1}^{n}[\underbrace{0,\ldots,0}_{i}, t]w[\underbrace{0,\ldots,0}_{n+1-i}, \underbrace{1,\ldots,1}_{n}]\boldsymbol{r}_h. \end{aligned}$$

Since $w(t) = 1 + \mathcal{O}\left(h\right)$, and $\boldsymbol{r}_h$ interpolates $\boldsymbol{f}\circ\varphi$ $(n-1)$-fold at 0 and at 1, (2.13) and (2.17) imply

$$\begin{aligned} &\left(1 + \mathcal{O}\left(h\right)\right)[\underbrace{0,\ldots,0}_{n}, \underbrace{1,\ldots,1}_{n}, t\,]\boldsymbol{r}_h = \\ &-\sum_{i=1}^{n}[\underbrace{0,\ldots,0}_{i}, t]w\,[\underbrace{0,\ldots,0}_{n+1-i}, \underbrace{1,\ldots,1}_{n}]\left(\boldsymbol{f}\circ\varphi\right) = \mathcal{O}\left(h^{2n}\right). \end{aligned}$$

This concludes the proof. □

3. Cubic rational G^2 interpolation. In this section, we will analyze the cubic rational G^2 Hermite interpolation in detail. We are given the following data: points $\boldsymbol{P}^L, \boldsymbol{P}^R \in \mathbb{R}^3$, the corresponding unit tangent directions $\boldsymbol{t}_1^L, \boldsymbol{t}_1^R \in \mathbb{R}^3$, and (second derivative) vectors $\boldsymbol{t}_2^L, \boldsymbol{t}_2^R \in \mathbb{R}^3$. The system (2.1) simplifies to

$$\begin{aligned} \boldsymbol{r}(0) &= \boldsymbol{P}^L, & \boldsymbol{r}(1) &= \boldsymbol{P}^R,\\ \boldsymbol{r}'(0) &= \alpha_1^L\,\boldsymbol{t}_1^L, & \boldsymbol{r}'(1) &= \alpha_1^R\,\boldsymbol{t}_1^R,\\ \boldsymbol{r}''(0) &= \alpha_2^L\,\boldsymbol{t}_1^L + (\alpha_1^L)^2\,\boldsymbol{t}_2^L, & \boldsymbol{r}''(1) &= \alpha_2^R\,\boldsymbol{t}_1^R + (\alpha_1^R)^2\,\boldsymbol{t}_2^R, \end{aligned} \tag{3.1}$$

where $\alpha_1^L > 0, \alpha_1^R > 0$ and α_2^L, α_2^R are unknowns, arising from the connection matrix (2.2) at both endpoints of the domain interval $[0, 1]$. The nonlinear part of the system of equations, established in Theorem 2.2, reads

$$\begin{aligned} \boldsymbol{e}_1 &:= \frac{w_0}{2}\left(\left(\alpha_2^L - 2\alpha_1^L\right)\boldsymbol{t}_1^L + \left(\alpha_1^L\right)^2\boldsymbol{t}_2^L\right) + 3w_1\alpha_1^L\boldsymbol{t}_1^L - 3w_2\Delta\boldsymbol{P} + \alpha_1^R\boldsymbol{t}_1^R = \boldsymbol{0},\\ \boldsymbol{e}_2 &:= -\alpha_1^L\boldsymbol{t}_1^L + 3w_1\Delta\boldsymbol{P} - 3w_2\alpha_1^R\boldsymbol{t}_1^R + \frac{w_3}{2}\left(\left(2\alpha_1^R + \alpha_2^R\right)\boldsymbol{t}_1^R + \left(\alpha_1^R\right)^2\boldsymbol{t}_2^R\right) = \boldsymbol{0}, \end{aligned}$$

where $w_0 = 1, w_3 = 1$. Let the linear functionals λ_ℓ be as introduced in (2.9), and let us define data constants

$$\nu_{\ell,j}^L := \lambda_\ell \boldsymbol{t}_j^L, \quad \nu_{\ell,j}^R := \lambda_\ell \boldsymbol{t}_j^R, \quad \ell = 1,2,3;\ j = 1,2,\dots,n-1. \tag{3.2}$$

Then, with $\omega \neq 0$, defined in (2.10), equations $\lambda_2 \boldsymbol{e}_j = 0, j = 1,2$, give relations that determine the weights w_i as functions of α_1^L and α_1^R,

$$\frac{1}{2}\nu_{2,2}^L \left(\alpha_1^L\right)^2 - 3\omega w_2 = 0, \quad 3\omega w_1 + \frac{1}{2}\nu_{2,2}^R \left(\alpha_1^R\right)^2 = 0. \tag{3.3}$$

Thus

$$w_1 = -\frac{\nu_{2,2}^R}{6\omega}\left(\alpha_1^R\right)^2, \quad w_2 = \frac{\nu_{2,2}^L}{6\omega}\left(\alpha_1^L\right)^2. \tag{3.4}$$

Similarly, equations $\lambda_1 \boldsymbol{e}_1 = 0$ and $\lambda_3 \boldsymbol{e}_2 = 0$, together with (3.4), yield

$$\alpha_2^L = \frac{\alpha_1^L\left(\nu_{2,2}^R\left(\alpha_1^R\right)^2 - \alpha_1^L \nu_{1,2}^L + 2\omega\right)}{\omega}, \ \alpha_2^R = \frac{\alpha_1^R\left(\nu_{2,2}^L\left(\alpha_1^L\right)^2 - \nu_{3,2}^R \alpha_1^R - 2\omega\right)}{\omega}. \tag{3.5}$$

Finally, $\lambda_1 \boldsymbol{e}_2 = 0$ and $\lambda_3 \boldsymbol{e}_1 = 0$, reveal a system of quadratic equations

$$\frac{1}{2}\nu_{1,2}^R\left(\alpha_1^R\right)^2 - \omega\alpha_1^L = 0, \quad \frac{1}{2}\nu_{3,2}^L\left(\alpha_1^L\right)^2 + \omega\alpha_1^R = 0, \tag{3.6}$$

that can be straightforwardly solved. The other unknowns are then obtained from (3.5) and (3.4) by a backward substitution, and the control points follow from (2.11) as

$$\boldsymbol{b}_0 = \boldsymbol{P}^L, \quad \boldsymbol{b}_1 = \boldsymbol{P}^L + \frac{\alpha_1^L}{3w_1}\boldsymbol{t}_1^L, \quad \boldsymbol{b}_2 = \boldsymbol{P}^R - \frac{\alpha_1^R}{3w_2}\boldsymbol{t}_1^R, \quad \boldsymbol{b}_3 = \boldsymbol{P}^R. \tag{3.7}$$

Note that the control points (3.7) and the weights (3.4) are independent of α_2^L and α_2^R. Let us summarize the discussion.

THEOREM 3.1. *There exists a unique cubic rational Bézier curve, satisfying the G^2 interpolation conditions* (3.1) *iff*

$$\nu_{2,2}^L\omega > 0,\ \nu_{2,2}^R\omega < 0,\ \nu_{3,2}^L\omega < 0,\ \nu_{1,2}^R\omega > 0.$$

In this case,

$$\alpha_1^L = \frac{2\omega}{\sqrt[3]{\left(\nu_{3,2}^L\right)^2\ \nu_{1,2}^R}}, \quad \alpha_1^R = -\frac{2\omega}{\sqrt[3]{\nu_{3,2}^L\ \left(\nu_{1,2}^R\right)^2}}. \tag{3.8}$$

The rest of the parameters are determined from (3.4) *and* (3.7). *If $\omega = 0$, the solution may exist only if the vectors $\boldsymbol{t}_1^L, \boldsymbol{t}_2^L, \Delta\boldsymbol{P}, \boldsymbol{t}_1^R$ and $\boldsymbol{t}_2^R$ are coplanar.*

Proof. If $\omega \neq 0$, the assertion follows from (3.4), (3.6), and the requirements $\alpha_1^L > 0, \alpha_1^R > 0, w_1 > 0, w_2 > 0$. So only the case $\omega = 0$ is left to be examined. If $\omega = 0$, the vectors $\boldsymbol{t}_1^L, \Delta\boldsymbol{P}$, and $\boldsymbol{t}_1^R$ are coplanar. Further, (3.3) implies that the vectors $\boldsymbol{t}_1^L, \boldsymbol{t}_2^L, \boldsymbol{t}_2^R$, and $\boldsymbol{t}_1^R$ should be coplanar too, and the proof is completed. ◻

The asymptotic approximation order of the cubic rational G^2 interpolation scheme is also not too hard to establish.

Theorem 3.2. *Suppose that the data are sampled from an analytic parametric curve $\boldsymbol{f} : [-h/2, h/2] \to \mathbb{R}^3$ with a nonvanishing curvature and torsion, parameterized by the arc-length,*

$$\boldsymbol{P}^L = \boldsymbol{f}\left(-\frac{h}{2}\right), \ \boldsymbol{P}^R = \boldsymbol{f}\left(\frac{h}{2}\right), \quad \boldsymbol{t}_s^L = \boldsymbol{f}^{(s)}\left(-\frac{h}{2}\right), \ \boldsymbol{t}_s^R = \boldsymbol{f}^{(s)}\left(\frac{h}{2}\right), \ s = 1, 2.$$

The asymptotic approximation order of the cubic rational interpolant $\boldsymbol{r}_h$ is 6.

Proof. Without loss of generality we may assume $\boldsymbol{f}(0) = \boldsymbol{0}$. Since the Frenet frame of the curve $\boldsymbol{f}$ exists by assumption, we may assume that at $s = 0$ it is equal to the columns of the identity matrix. This simplifies the Taylor series of the curve $\boldsymbol{f}$ at 0 to

$$\boldsymbol{f}(s) = \begin{pmatrix} s - \frac{1}{6}\kappa_0^2 s^3 - \frac{1}{8}\kappa_0\kappa_1 s^4 + \dots \\ \frac{1}{2}\kappa_0 s^2 + \frac{1}{6}\kappa_1 s^3 - \frac{1}{24}\left(\kappa_0^3 - \kappa_2 + \kappa_0\tau_0^2\right) s^4 + \dots \\ \frac{1}{6}\kappa_0\tau_0 s^3 + \frac{1}{24}\left(2\kappa_1\tau_0 + \kappa_0\tau_1\right) s^4 + \dots \end{pmatrix}, \tag{3.9}$$

where $\kappa(s) = \kappa_0 + \frac{1}{1!}\kappa_1\, s + \frac{1}{2!}\kappa_2\, s^2 + \dots$ and $\tau(s) = \tau_0 + \frac{1}{1!}\tau_1\, s + \frac{1}{2!}\tau_2\, s^2 + \dots$ denote Taylor series of the principal curvatures of $\boldsymbol{f}$. From (3.9) it is straightforward to obtain the expansions of the rest of the data, and the data constants $\nu_{\ell,j}^L, \nu_{\ell,j}^R, \dots$ involved. In particular,

$$\omega = \frac{1}{12}\kappa_0^2\tau_0 h^4 + \mathcal{O}\left(h^6\right),$$

so $\omega > 0$ for h small enough, and (3.8), (3.5), and (3.4) give expansions of the unknowns

$$\begin{aligned} \alpha_1^L &= h + \theta_2 h^2 + \theta_3 h^3 + \theta_4 h^4 + \mathcal{O}\left(h^5\right), & \alpha_1^R &= h - \theta_2 h^2 + \theta_3 h^3 - \theta_4 h^4 + \mathcal{O}\left(h^5\right), \\ \alpha_2^L &= -2\theta_2 h^2 - 6\theta_3 h^3 + \theta_5 h^4 + \mathcal{O}\left(h^5\right), & \alpha_2^R &= -2\theta_2 h^2 + 6\theta_3 h^3 + \theta_5 h^4 + \mathcal{O}\left(h^5\right), \\ w_1 &= 1 + \xi_2 h^2 + \mathcal{O}\left(h^3\right), & w_2 &= 1 + \xi_2 h^2 + \mathcal{O}\left(h^3\right), \end{aligned} \tag{3.10}$$

with

$$\begin{aligned} \theta_2 &:= \frac{1}{12}\left(\frac{2\kappa_1}{\kappa_0} + \frac{\tau_1}{\tau_0}\right), \\ \theta_3 &:= \frac{-12\kappa_0^4\tau_0^2 - 12\kappa_0^2\tau_0^4 + 25\kappa_0^2\tau_1^2 - 18\kappa_0^2\tau_0\tau_2 - 24\kappa_2\kappa_0\tau_0^2 + 16\kappa_1\kappa_0\tau_0\tau_1 + 40\kappa_1^2\tau_0^2}{720\kappa_0^2\tau_0^2}, \\ \xi_2 &:= \frac{-36\kappa_0^4\tau_0^2 - 36\kappa_0^2\tau_0^4 + 35\kappa_0^2\tau_1^2 - 24\kappa_0^2\tau_0\tau_2 - 12\kappa_2\kappa_0\tau_0^2 + 8\kappa_1\kappa_0\tau_0\tau_1 + 20\kappa_1^2\tau_0^2}{720\kappa_0^2\tau_0^2}, \end{aligned}$$

and θ_4, θ_5 are similar, but rather lengthy expressions depending on the curvature expansions only. Thus the denominator w by (3.10) satisfies

$$\begin{gathered} w(0) = 1, \quad [0,0]w = 3\xi_2 h^2 + \mathcal{O}\left(h^3\right), \\ [0,0,0]w = -3\xi_2 h^2 + \mathcal{O}\left(h^3\right), \quad [0,0,0,0]w = \mathcal{O}\left(h^3\right), \end{gathered}$$

and the assumption (2.13) of Theorem 2.3 is confirmed. For the polynomial reparameterization φ of degree $\leq 2n - 1 = 5$, determined by the conditions (2.15), we observe from (3.10) that the first four columns of the corresponding divided difference table, that determines a particular Newton form of φ, read

$$\begin{array}{cccc}
0 & -\frac{h}{2} & & \\
 & & h+\theta_2h^2+\theta_3h^3+\theta_4h^4+\mathcal{O}\left(h^5\right) & \\
0 & -\frac{h}{2} & & -\theta_2h^2-3\theta_3h^3+\frac{\theta_5}{2}h^4+\mathcal{O}\left(h^5\right) \\
 & & h+\theta_2h^2+\theta_3h^3+\theta_4h^4+\mathcal{O}\left(h^5\right) & \\
0 & -\frac{h}{2} & & -\theta_2h^2-\theta_3h^3-\theta_4h^4+\mathcal{O}\left(h^5\right) \\
 & & h & \\
1 & \frac{h}{2} & & -\theta_2h^2+\theta_3h^3-\theta_4h^4+\mathcal{O}\left(h^5\right) \\
 & & h-\theta_2h^2+\theta_3h^3-\theta_4h^4+\mathcal{O}\left(h^5\right) & \\
1 & \frac{h}{2} & & -\theta_2h^2+3\theta_3h^3+\frac{\theta_5}{2}h^4+\mathcal{O}\left(h^5\right) \\
 & & h-\theta_2h^2+\theta_3h^3-\theta_4h^4+\mathcal{O}\left(h^5\right) & \\
1 & \frac{h}{2} & &
\end{array}$$

and the last three ones are

$$\begin{array}{ccc}
2\theta_3h^3-\left(\theta_4+\frac{\theta_5}{2}\right)h^4+\mathcal{O}\left(h^5\right) & & \\
 & \left(\theta_4+\frac{\theta_5}{2}\right)h^4+\mathcal{O}\left(h^5\right) & \\
2\theta_3h^3+\mathcal{O}\left(h^5\right) & & \mathcal{O}\left(h^5\right) \\
 & \left(\theta_4+\frac{\theta_5}{2}\right)h^4+\mathcal{O}\left(h^5\right) & \\
2\theta_3h^3+\left(\theta_4+\frac{\theta_5}{2}\right)h^4+\mathcal{O}\left(h^5\right) & &
\end{array}$$

Thus the reparameterization φ satisfies (2.14), and one may use Theorem 2.3 to complete the proof. □

4. Quartic rational G^3 interpolation. For the quartic G^3 rational Hermite interpolation, the nonlinear system of equations that follows from (2.4), with $w_0 = w_4 = 1$, is given as

$$\begin{aligned}
\boldsymbol{e}_1 &:= \frac{1}{6}\left(\left(6\alpha_1^L-6\alpha_2^L+\alpha_3^L\right)\boldsymbol{t}_1^L+\alpha_1^L\left(\left(3\alpha_2^L-6\alpha_1^L\right)\boldsymbol{t}_2^L+\left(\alpha_1^L\right)^2\boldsymbol{t}_3^L\right)\right)\\
&\quad + w_1\left(\left(2\alpha_2^L-4\alpha_1^L\right)\boldsymbol{t}_1^L+2\left(\alpha_1^L\right)^2\boldsymbol{t}_2^L\right)+6w_2\alpha_1^L\boldsymbol{t}_1^L-4w_3\Delta\boldsymbol{P}+\alpha_1^R\boldsymbol{t}_1^R=\boldsymbol{0},\\
\boldsymbol{e}_2 &:= \left(\alpha_1^L-\frac{\alpha_2^L}{2}\right)\boldsymbol{t}_1^L-\frac{1}{2}\left(\alpha_1^L\right)^2\boldsymbol{t}_2^L-4w_1\alpha_1^L\boldsymbol{t}_1^L+6w_2\Delta\boldsymbol{P}-4w_3\alpha_1^R\boldsymbol{t}_1^R\\
&\quad +\frac{1}{2}\left(\left(2\alpha_1^R+\alpha_2^R\right)\boldsymbol{t}_1^R+\left(\alpha_1^R\right)^2\boldsymbol{t}_2^R\right)=\boldsymbol{0}, \qquad (4.1)\\
\boldsymbol{e}_3 &:= \alpha_1^L\boldsymbol{t}_1^L-4w_1\Delta\boldsymbol{P}+6w_2\alpha_1^R\boldsymbol{t}_1^R-2w_3\left(\left(2\alpha_1^R+\alpha_2^R\right)\boldsymbol{t}_1^R+\left(\alpha_1^R\right)^2\boldsymbol{t}_2^R\right)\\
&\quad +\frac{1}{6}\left(\left(6\alpha_1^R+6\alpha_2^R+\alpha_3^R\right)\boldsymbol{t}_1^R+\alpha_1^R\left(3\left(2\alpha_1^R+\alpha_2^R\right)\boldsymbol{t}_2^R+\left(\alpha_1^R\right)^2\boldsymbol{t}_3^R\right)\right)=\boldsymbol{0}.
\end{aligned}$$

This system of three vector equations is obviously harder to tackle than its cubic counterpart. But similar elimination steps can be carried out, which will result in the final quartic equation.

Recall the linear functionals λ_ℓ, defined in (2.9), and the data constants, introduced in (3.2). Let us simplify the discussion by assuming $\omega = \det\left(\boldsymbol{t}_1^L, \Delta\boldsymbol{P}, \boldsymbol{t}_1^R\right) \neq 0$. If we apply $\lambda_\ell, \ell = 1, 2, 3$, to the system (4.1), we obtain nine scalar equations. From $\lambda_1\boldsymbol{e}_1 = 0$ we may express

$$\begin{aligned}
\alpha_3^L =& \frac{1}{\omega}\Big(\left(6\left(1-2w_1\right)\nu_{1,2}^L\alpha_1^L-\nu_{1,3}^L\left(\alpha_1^L\right)^2+6\left(4w_1-6w_2-1\right)\omega\right)\alpha_1^L\\
&-3\left(\nu_{1,2}^L\alpha_1^L+4w_1\omega-2\omega\right)\alpha_2^L\Big). \qquad (4.2)
\end{aligned}$$

Similarly, $\lambda_3 \boldsymbol{e}_3 = 0$ gives

$$\begin{aligned}\alpha_3^R =&\frac{1}{\omega}\Big(- 3\left(\nu_{3,2}^R \alpha_2^R + 2\left(6w_2 - 4w_3 + 1\right)\omega\right)\alpha_1^R \\ &+ 6\left(2w_3 - 1\right)\nu_{3,2}^R\left(\alpha_1^R\right)^2 - \nu_{3,3}^R\left(\alpha_1^R\right)^3 + 6\left(2w_3 - 1\right)\omega\alpha_2^R\Big).\end{aligned} \tag{4.3}$$

The unknown weights appear linearly in (4.1). Since $\omega \neq 0$, we may use the equations

$$\lambda_2 \boldsymbol{e}_i = 0, \quad i = 1, 2, 3,$$

and express them as

$$\begin{aligned}w_1 &= \frac{\nu_{2,2}^R\left(\left(\left(12\omega - \nu_{2,3}^L\left(\alpha_1^L\right)^3 + 3\nu_{2,2}^L\left(2\alpha_1^L - \alpha_2^L\right)\alpha_1^L\right)\alpha_1^R + 6\omega\alpha_2^R\right)\alpha_1^R + 2\omega\nu_{2,3}^R\left(\alpha_1^R\right)^3}{12\,\pi_1\left(\alpha_1^L, \alpha_1^R\right)}, \\ w_2 &= \frac{\nu_{2,2}^L\left(\alpha_1^L\right)^2 - \nu_{2,2}^R\left(\alpha_1^R\right)^2}{12\omega}, \\ w_3 &= \frac{\nu_{2,2}^L\left(\left(\left(\nu_{2,3}^R\left(\alpha_1^R\right)^3 + 3\left(2\alpha_1^R + \alpha_2^R\right)\alpha_1^R\nu_{2,2}^R - 12\omega\right)\alpha_1^L + 6\omega\alpha_2^L\right)\alpha_1^L + 2\omega\nu_{2,3}^L\left(\alpha_1^L\right)^3}{12\,\pi_1\left(\alpha_1^L, \alpha_1^R\right)},\end{aligned} \tag{4.4}$$

with

$$\pi_1(x, y) := 4\omega^2 + \nu_{2,2}^L\nu_{2,2}^R x^2 y^2. \tag{4.5}$$

Note that w_i in (4.4) depend on $\alpha_1^L, \alpha_2^L, \alpha_1^R$ and α_2^R only, and the unknowns α_2^L, α_2^R appear linearly. If we insert the weights (4.4) in

$$\begin{aligned}\lambda_1 \boldsymbol{e}_2 &= \frac{1}{2}\left(-\nu_{1,2}^L\left(\alpha_1^L\right)^2 + \nu_{1,2}^R\left(\alpha_1^R\right)^2 + 2\omega\alpha_1^L - \omega\alpha_2^L\right) - 4\omega w_1\alpha_1^L = 0, \\ \lambda_3 \boldsymbol{e}_2 &= \frac{1}{2}\left(-\nu_{3,2}^L\left(\alpha_1^L\right)^2 + \nu_{3,2}^R\left(\alpha_1^R\right)^2 + 2\omega\alpha_1^R + \omega\alpha_2^R\right) - 4\omega w_3\alpha_1^R = 0,\end{aligned}$$

we derive rather long expressions for α_2^L, α_2^R, depending on data constants and α_1^L, α_1^R only,

$$\alpha_2^L = \frac{\pi_2\left(\alpha_1^L, \alpha_1^R\right)}{3\omega\,\pi_1\left(\alpha_1^L, \alpha_1^R\right)}, \quad \alpha_2^R = \frac{\pi_3\left(\alpha_1^L, \alpha_1^R\right)}{3\omega\,\pi_1\left(\alpha_1^L, \alpha_1^R\right)}, \tag{4.6}$$

with

$$\begin{aligned}\pi_2(x, y) :=& \,\nu_{2,2}^R\left(3\nu_{1,2}^L\nu_{2,2}^L - 2\omega\nu_{2,3}^L\right)x^4y^2 - 3\nu_{1,2}^R\nu_{2,2}^L\nu_{2,2}^R x^2y^4 + 6\omega\nu_{2,2}^L\nu_{2,2}^R x^3y^2 \\ &- 12\omega\nu_{2,2}^R\nu_{3,2}^L x^3y - 4\omega\left(\omega\nu_{2,3}^R - 3\nu_{2,2}^R\nu_{3,2}^R\right)xy^3 - 12\omega^2\nu_{1,2}^L x^2 \\ &+ 12\omega^2\nu_{1,2}^R y^2 + 24\omega^3 x, \\ \pi_3(x, y) :=& \,\nu_{2,2}^L\left(3\nu_{2,2}^R\nu_{3,2}^R - 2\omega\nu_{2,3}^R\right)x^2y^4 - 3\nu_{2,2}^L\nu_{2,2}^R\nu_{3,2}^L x^4y^2 - 6\omega\nu_{2,2}^L\nu_{2,2}^R x^2y^3 \\ &+ 4\omega\left(\omega\nu_{2,3}^L - 3\nu_{1,2}^L\nu_{2,2}^L\right)x^3y + 12\omega\nu_{1,2}^R\nu_{2,2}^L xy^3 + 12\omega^2\nu_{3,2}^L x^2 \\ &- 12\omega^2\nu_{3,2}^R y^2 - 24\omega^3 y.\end{aligned}$$

There are only two equations left,

$$\begin{aligned}\lambda_1 \boldsymbol{e}_3 =& \,\omega\alpha_1^L + \frac{1}{6}\nu_{1,3}^R\left(\alpha_1^R\right)^3 + \left(\nu_{1,2}^R - 2w_3\nu_{1,2}^R\right)\left(\alpha_1^R\right)^2 + \frac{1}{2}\nu_{1,2}^R\alpha_1^R\alpha_2^R = 0, \\ \lambda_3 \boldsymbol{e}_1 =& \,\omega\alpha_1^R + \frac{1}{6}\nu_{3,3}^L\left(\alpha_1^L\right)^3 + \left(2w_1\nu_{3,2}^L - \nu_{3,2}^L\right)\left(\alpha_1^L\right)^2 + \frac{1}{2}\nu_{3,2}^L\alpha_1^L\alpha_2^L = 0.\end{aligned} \tag{4.7}$$

Let us insert in (4.7) the weights w_1 and w_3 from (4.4) and α_2^L, α_2^R from (4.6). The equations (4.7) transform to

$$6\lambda_1 \boldsymbol{e}_3 = \pi_4(\alpha_1^L, \alpha_1^R) + \frac{\pi_6(\alpha_1^L, \alpha_1^R)}{\pi_1(\alpha_1^L, \alpha_1^R)} = 0, \quad 6\lambda_3 \boldsymbol{e}_1 = \pi_5(\alpha_1^L, \alpha_1^R) + \frac{\pi_7(\alpha_1^L, \alpha_1^R)}{\pi_1(\alpha_1^L, \alpha_1^R)} = 0, \tag{4.8}$$

where π_1 is defined in (4.5),

$$\pi_4(x,y) := 2\left(3\omega + \frac{\omega\nu_{2,3}^L - 3\nu_{1,2}^L\nu_{2,2}^L}{\nu_{2,2}^L\nu_{2,2}^R}\nu_{1,2}^R\right)x + \left(\nu_{1,3}^R - \frac{\nu_{1,2}^R\nu_{2,3}^R}{\nu_{2,2}^R}\right)y^3,$$
$$\pi_5(x,y) := 2\left(3\omega - \frac{\omega\nu_{2,3}^R - 3\nu_{3,2}^R\nu_{2,2}^R}{\nu_{2,2}^L\nu_{2,2}^R}\nu_{3,2}^L\right)y + \left(\nu_{3,3}^L - \frac{\nu_{2,3}^L\nu_{3,2}^L}{\nu_{2,2}^L}\right)x^3,$$

and

$$\begin{aligned}\pi_6(x,y) &:= 6\nu_{1,2}^R\left(2\omega\nu_{3,2}^L x + \nu_{2,2}^L\nu_{1,2}^R y^3\right)xy \\ &\quad - 8\omega^2\nu_{1,2}^R\frac{\omega\nu_{2,3}^L - 3\nu_{1,2}^L\nu_{2,2}^L}{\nu_{2,2}^L\nu_{2,2}^R}x + 4\omega\nu_{1,2}^R\left(\frac{\omega\nu_{2,3}^R}{\nu_{2,2}^R} - 3\nu_{3,2}^R\right)y^3, \\ \pi_7(x,y) &:= 6\nu_{3,2}^L\left(2\omega\nu_{1,2}^R y - \nu_{3,2}^L\nu_{2,2}^R x^3\right)xy \\ &\quad + 8\omega^2\nu_{3,2}^L\frac{\omega\nu_{2,3}^R - 3\nu_{2,2}^R\nu_{3,2}^R}{\nu_{2,2}^L\nu_{2,2}^R}y + 4\omega\nu_{3,2}^L\left(\frac{\omega\nu_{2,3}^L}{\nu_{2,2}^L} - 3\nu_{1,2}^L\right)x^3.\end{aligned}$$

Let us introduce new variables z_1 and z_2 with

$$z_1 := \frac{\alpha_1^L}{\left(\alpha_1^R\right)^3}, \quad z_2 := \alpha_1^L\alpha_1^R. \tag{4.9}$$

Thus

$$\alpha_1^L = \sqrt[4]{z_1 z_2^3}, \quad \alpha_1^R = \sqrt[4]{\frac{z_2}{z_1}}. \tag{4.10}$$

Note that only positive values z_1 and z_2 are admissible. The first equation in (4.8), divided by $\left(\alpha_1^R\right)^3$, is linear in z_1, which yields

$$z_1 = \frac{4\omega\left(3\nu_{1,2}^R\nu_{3,2}^R - \omega\nu_{1,3}^R\right) - 6\nu_{2,2}^L\left(\nu_{1,2}^R\right)^2 z_2 + \nu_{2,2}^L\left(\nu_{1,2}^R\nu_{2,3}^R - \nu_{1,3}^R\nu_{2,2}^R\right)z_2^2}{24\omega^3 + 12\omega\nu_{3,2}^L\nu_{1,2}^R z_2 + 2\left(\omega\nu_{2,3}^L\nu_{1,2}^R + 3\omega\nu_{2,2}^L\nu_{2,2}^R - 3\nu_{1,2}^L\nu_{2,2}^L\nu_{1,2}^R\right)z_2^2}. \tag{4.11}$$

Let us insert the new variables (4.9) in the second equation in (4.8), divided by α_1^R, and let us apply the substitution (4.11). The numerator of the expression obtained gives the last equation as

$$\begin{aligned}\pi_8(z_2) :=& 144\omega^4 + 144\omega^2\nu_{3,2}^L\nu_{1,2}^R z_2 + 4\Big(3\omega^2\nu_{2,3}^L\nu_{1,2}^R - \omega^2\nu_{3,3}^L\nu_{1,3}^R + 9\omega^2\nu_{2,2}^L\nu_{2,2}^R \\ & - 3\omega^2\nu_{3,2}^L\nu_{2,3}^R + 3\omega\nu_{3,2}^R\left(\nu_{3,3}^L\nu_{1,2}^R + 3\nu_{3,2}^L\nu_{2,2}^R\right) \\ & + \nu_{1,2}^L\left(3\nu_{3,2}^L\left(\omega\nu_{1,3}^R - 3\nu_{1,2}^R\nu_{3,2}^R\right) - 9\omega\nu_{2,2}^L\nu_{1,2}^R\right) + 9\left(\nu_{3,2}^L\right)^2\left(\nu_{1,2}^R\right)^2\Big)z_2^2 \\ & + 6\left(\left(\nu_{3,2}^L\right)^2\left(\nu_{1,3}^R\nu_{2,2}^R - \nu_{1,2}^R\nu_{2,3}^R\right) + \left(\nu_{2,3}^L\nu_{3,2}^L - \nu_{2,2}^L\nu_{3,3}^L\right)\left(\nu_{1,2}^R\right)^2\right)z_2^3 \\ & + \left(\nu_{2,3}^L\nu_{3,2}^L - \nu_{2,2}^L\nu_{3,3}^L\right)\left(\nu_{1,3}^R\nu_{2,2}^R - \nu_{1,2}^R\nu_{2,3}^R\right)z_2^4 = 0.\end{aligned} \tag{4.12}$$

Thus the main part of the quartic rational G^3 interpolation problem is to solve the quartic equation (4.12). All the other unknowns are obtained by the backward substitution. Any positive root of π_8 is an admissible z_2. If the corresponding z_1, computed from (4.11), is positive too, the equations (4.10) determine an admissible pair α_1^L, α_1^R. With the help of (4.6), we then obtain the corresponding unknowns α_2^L, α_2^R. The weights w_1, w_2, and w_3 naturally follow from (4.4). If all weights are positive, the solution set is altogether admissible. In case α_3^L, α_3^R are needed too, they can be computed from (4.2) and (4.3), respectively. Finally, the unknown control points are determined from (2.11) as

$$\boldsymbol{b}_0 = \boldsymbol{P}^L, \quad \boldsymbol{b}_4 = \boldsymbol{P}^R, \quad \boldsymbol{b}_1 = \boldsymbol{P}^L + \frac{\alpha_1^L}{4w_1}\boldsymbol{t}_1^L, \quad \boldsymbol{b}_3 = \boldsymbol{P}^R - \frac{\alpha_1^R}{4w_3}\boldsymbol{t}_1^R,$$
$$\boldsymbol{b}_2 = \boldsymbol{P}^L + \frac{(8w_1-2)\,\alpha_1^L + \alpha_2^L}{12w_2}\boldsymbol{t}_1^L + \frac{\left(\alpha_1^L\right)^2}{12w_2}\boldsymbol{t}_2^L.$$

There is no way to state existence conditions of the quartic case solution so precisely as in Theorem 3.1 for the cubic one. Quite clearly we observe from (4.12) that the number of admissible solutions is at most 4, but the actual number depends on the particular set of the independent data constants

$$\boldsymbol{\nu} := \left(\omega, \nu_{1,2}^L, \nu_{2,2}^L, \nu_{3,2}^L, \nu_{1,3}^L, \nu_{2,3}^L, \nu_{3,3}^L, \nu_{1,2}^R, \nu_{2,2}^R, \nu_{3,2}^R, \nu_{1,3}^R, \nu_{2,3}^R, \nu_{3,3}^R\right) \in \mathbb{R}^{13}.$$

Since a data change $\boldsymbol{\nu} \to \text{const}\ \boldsymbol{\nu}, \text{const} \neq 0$, does not change the equations involved, it is enough to consider the unit sphere $\mathcal{S}$ in $\mathbb{R}^{13}$ as parameter space only. Its surface could be split in open subsets and their boundaries such that the number of admissible solutions on each subset is constant. However, there is no point to list all the varieties that define these boundaries. Instead, as an example, let us consider data from a particular curve

$$f(t) := \left(\frac{\cos t}{\sqrt{1+t^2}}, \sin t, \sqrt{1+t^2}\right)^T, \quad t \in [a,b], \tag{4.13}$$

and let us examine the number of admissible solutions depending on the parameter interval, $-1 \leq a < b \leq 1$. Figure 4.1 clearly suggests that any number of admissible solutions between 0 and 4 is possible. Even in the asymptotic case, no entirely precise answer can be found. Let us start with two lemmas.

LEMMA 4.1. *Suppose that the weights w_j for h small enough expand as*

$$w_j = \sum_{\ell=0}^{3} w_{j,\ell} h^\ell + \mathcal{O}\left(h^4\right), \quad j = 1,2,3. \tag{4.14}$$

The polynomial $w = \sum_{j=0}^4 w_j B_j^4$ satisfies the assumptions (2.13) *of Theorem 2.3 iff*

$$\begin{gathered} w_{1,0} = 1,\ w_{2,0} = 1,\ w_{3,0} = 1, \quad w_{1,1} = 0,\ w_{2,1} = 0,\ w_{3,1} = 0, \\ 3w_{1,2} - 3w_{2,2} + w_{3,2} = 0,\ w_{1,2} - 3w_{2,2} + 3w_{3,2} = 0, \\ -2w_{1,3} + 3w_{2,3} - 2w_{3,3} = 0. \end{gathered} \tag{4.15}$$

Proof. Recall $w_0 = w_4 = 1$. Since w is a Bézier polynomial, the assumptions (2.13) are by the convex hull property equivalent to

$$\begin{gathered} w_j = 1 + \mathcal{O}\left(h\right), \quad j = 0,1,\ldots,4, \\ \Delta^\ell w_j = \mathcal{O}\left(h^\ell\right), \quad j = 0,1,\ldots,4-\ell;\ \ell = 2,3,4. \end{gathered} \tag{4.16}$$

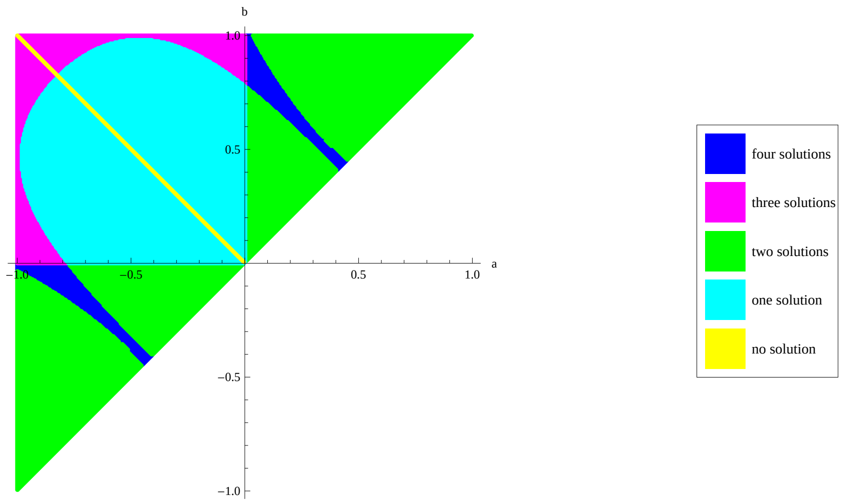

FIG. 4.1. *The number of admissible solutions for the curve f, defined in* (4.13)*, that depends on a particular parameter interval $[a, b]$ in the range $-1 \leq a < b \leq 1$, with a legend on the right.*

Let us insert the expansions (4.14) in the finite difference table $\Delta^{\ell} w_j$. The relations (4.16) are fulfilled if and only if the constants $w_{j,\ell}$ satisfy the relations (4.15). Since the relations $w_{j,0} = 1, j = 1, 2, 3$, are obvious, let us verify (4.16) for $\ell = 2$. Since $\mathcal{O}(h)$ terms should vanish, we obtain

$$w_{2,1} - 2w_{1,1} = 0, \quad w_{1,1} - 2w_{2,1} + w_{3,1} = 0, \quad w_{2,1} - 2w_{3,1} = 0,$$

which implies $w_{1,1} = 0,\ w_{2,1} = 0,\ w_{3,1} = 0$. The other relations follow similarly. ◻

LEMMA 4.2. *Suppose that φ is a polynomial of degree $\leq 2n - 1 = 7$ determined by the interpolation conditions* (2.15)*, and let α_j^L, α_j^R expand as*

$$\alpha_j^L = \sum_{\ell=j}^{6} \alpha_{j,\ell}^L h^{\ell} + \mathcal{O}\left(h^7\right), \ \alpha_j^R = \sum_{\ell=j}^{6} \alpha_{j,\ell}^R h^{\ell} + \mathcal{O}\left(h^7\right), \quad j = 1, 2, 3. \tag{4.17}$$

Then φ satisfies (2.14) *iff the following relations hold,*

$$\alpha_{1,1}^L - 1 = 0, \ \alpha_{1,1}^R - 1 = 0, \tag{4.18}$$

$$2\alpha_{1,2}^L + \alpha_{2,2}^L = 0, \ \alpha_{1,2}^L + \alpha_{1,2}^R = 0, \ \alpha_{2,2}^R - 2\alpha_{1,2}^R = 0, \tag{4.19}$$

$$\begin{gathered} 6\alpha_{1,3}^L + 3\alpha_{2,3}^L + \alpha_{3,3}^L = 0, \ 4\alpha_{1,3}^L + \alpha_{2,3}^L + 2\alpha_{1,3}^R = 0, \\ -2\alpha_{1,3}^L - 4\alpha_{1,3}^R + \alpha_{2,3}^R = 0, \ 6\alpha_{1,3}^R - 3\alpha_{2,3}^R + \alpha_{3,3}^R = 0, \end{gathered} \tag{4.20}$$

$$\begin{gathered} 18\alpha_{1,4}^L + 6\alpha_{2,4}^L + \alpha_{3,4}^L + 6\alpha_{1,4}^R = 0, \ -6\alpha_{1,4}^L - \alpha_{2,4}^L - 6\alpha_{1,4}^R + \alpha_{2,4}^R = 0, \\ 6\alpha_{1,4}^L + 18\alpha_{1,4}^R - 6\alpha_{2,4}^R + \alpha_{3,4}^R = 0, \end{gathered} \tag{4.21}$$

$$\begin{gathered} -36\alpha_{1,5}^L - 9\alpha_{2,5}^L - \alpha_{3,5}^L - 24\alpha_{1,5}^R + 3\alpha_{2,5}^R = 0, \\ 24\alpha_{1,5}^L + 3\alpha_{2,5}^L + 36\alpha_{1,5}^R - 9\alpha_{2,5}^R + \alpha_{3,5}^R = 0, \end{gathered} \tag{4.22}$$

$$60\alpha_{1,6}^L + 12\alpha_{2,6}^L + \alpha_{3,6}^L + 60\alpha_{1,6}^R - 12\alpha_{2,6}^R + \alpha_{3,6}^R = 0. \tag{4.23}$$

Proof. Let $(\xi_i)_{i=0}^7 := (0,0,0,0,1,1,1,1)$ be the knot sequence at which φ interpolates the data $\left(-\frac{h}{2}, \alpha_1^L, \alpha_2^L, \alpha_3^L, \frac{h}{2}, \alpha_1^R, \alpha_2^R, \alpha_3^R\right)$. Quite clearly, (2.14) is equivalent to

$$[\xi_i, \xi_{i+1}]\varphi = h + \mathcal{O}\left(h^2\right), \quad i = 0, 1, \ldots, 6, \tag{4.24}$$

and

$$[\xi_i, \xi_{i+1}, \ldots, \xi_{i+\ell}]\varphi = \mathcal{O}\left(h^\ell\right), \quad i = 0, 1, \ldots, 7-\ell;\ \ell = 2, 3, \ldots, 7. \tag{4.25}$$

If we insert the expansions (4.17) in the divided difference table (4.25), we observe that the relation (4.25) for $\ell = 2$ is equivalent to (4.18), for $\ell = 3$ to (4.19), etc. Finally, (4.18) implies (4.24) too, which concludes the proof. ☐

Theorem 4.3. *Suppose that the data* (2.1) *are determined by an analytic curve* $\boldsymbol{f} : [-\frac{h}{2}, \frac{h}{2}] \to \mathbb{R}^3$ *with nonvanishing curvature and torsion as in Theorem 3.2. Then there exists* $h_0 > 0$ *such that for all* h, $0 < h \le h_0$, *the number of the corresponding solutions is constant. This number depends on the curvatures expansion coefficients*

$$\kappa_0, \kappa_1, \ldots, \kappa_5, \tau_0, \tau_1, \ldots, \tau_4,$$

and it is equal to $0, 2$ *or* 4. *The asymptotic approximation order of any regular solution (i.e., with a nonvanishing derivative) is* 8.

Proof. Let us recall the expansion (3.9) that gives, after quite a bit of symbolic computer work, the expansions of the data constants $\omega, \nu_{i,j}^L, \nu_{i,j}^R$. If we insert the expansions obtained in (4.12) and rearrange the quartic equation $\pi_8(z_2) = 0$ by introducing a new variable

$$Z := \frac{h^2 - z_2}{h^4},$$

we obtain

$$\pi_8(z_2) = \pi_8(h^2 - h^4 Z) = h^{24}\left(\pi_9(Z) + o_1(Z)h^2 + o_2(Z)h^4 + \ldots\right) = 0. \tag{4.26}$$

Here π_9 and o_i are quite lengthy polynomials of degree ≤ 4 with coefficients that depend on curvatures expansions only, but not on h. In particular, the leading polynomial reads

$$\pi_9(Z) = \frac{1}{144}\kappa_0^8 \tau_0^4 Z^4 + \ldots .$$

So it is precisely of the degree 4 since $\kappa_0, \tau_0 \neq 0$. So the equation $\pi_9(Z) = 0$ has an even number of real solutions. Suppose that $\widetilde{Z}$ is a simple real root of π_9. Then (4.26) gives an expansion of Z for h small enough as

$$Z = \widetilde{Z} - \frac{o_1(\widetilde{Z})}{\pi_9'(\widetilde{Z})}h^2 - \left(\frac{o_1(\widetilde{Z})^2 \pi_9''(\widetilde{Z})}{2\pi_9'(\widetilde{Z})^3} - \frac{o_1(\widetilde{Z}) o_1'(\widetilde{Z})}{\pi_9'(\widetilde{Z})^2} + \frac{o_2(\widetilde{Z})}{\pi_9'(\widetilde{Z})}\right) h^4 + \mathcal{O}\left(h^6\right).$$

Consequently,

$$\alpha_1^L \alpha_1^R = z_2 = h^2 - \widetilde{Z} h^4 + \frac{o_1(\widetilde{Z})}{\pi_9'(\widetilde{Z})} h^6 + \mathcal{O}\left(h^8\right). \tag{4.27}$$

The identity (4.11) gives z_1 as a rational function of the z_2 expansion (4.27). This gives the expansions

$$\alpha_1^R = \sqrt[4]{\frac{z_2}{z_1}} = h + \alpha_{1,2}^R h^2 + \alpha_{1,3}^R h^3 + \alpha_{1,4}^R h^4 + \alpha_{1,5}^R h^5 + \alpha_{1,6}^R h^6 + \mathcal{O}\left(h^7\right) \tag{4.28}$$

and

$$\alpha_1^L = \frac{z_2}{\alpha_1^R} = h + \alpha_{1,2}^L h^2 + \alpha_{1,3}^L h^3 + \alpha_{1,4}^L h^4 + \alpha_{1,5}^L h^5 + \alpha_{1,6}^L h^6 + \mathcal{O}\left(h^7\right). \tag{4.29}$$

Note that (4.27) implies

$$\begin{gathered} \alpha_{1,2}^L = -\alpha_{1,2}^R,\ \alpha_{1,4}^L + \alpha_{1,4}^R + \alpha_{1,3}^L\alpha_{1,2}^R + \alpha_{1,2}^L\alpha_{1,3}^R = 0, \\ \alpha_{1,6}^L + \alpha_{1,6}^R + \alpha_{1,5}^L\alpha_{1,2}^R + \alpha_{1,4}^L\alpha_{1,3}^R + \alpha_{1,3}^L\alpha_{1,4}^R + \alpha_{1,2}^L\alpha_{1,5}^R = 0, \end{gathered} \tag{4.30}$$

and

$$\alpha_{1,3}^L + \alpha_{1,3}^R + \alpha_{1,2}^L\alpha_{1,2}^R = -\widetilde{Z}.$$

Also, the coefficients $\alpha_{1,\ell}^L, \alpha_{1,\ell}^R$ depend on curvature expansions as well on $\widetilde{Z}$, and they are quite lengthy. A straightforward substitution of (4.29) and (4.28) in (4.6), (4.4), (4.3), and (4.2) confirms that the unknowns have asymptotic expansions of the form (4.14) and (4.17). But, due to the computer power at will, we have not been able to produce enough expansion constants to confirm the asserted approximation order by a straightforward backward substitution approach. Instead, let us consider the expansion of the original system (4.1), where we assume also that the unknowns have expansions of the form (4.14) and (4.17). Additionally, suppose that coefficients in the expansions of α_1^L and α_1^R satisfy (4.30) too, but for a moment we neglect their actual values that were determined in (4.28) and (4.29). Note that (4.18) is satisfied already. The left-hand sides of the equations expand as

$$\boldsymbol{e}_i = \left(e_{i,j}\right)_{j=1}^3 = \left(h^j\left(e_{i,j,0} + e_{i,j,1}h + e_{i,j,2}h^2 + \dots\right)\right)_{j=1}^3, \quad i = 1,2,3.$$

The leading terms simplify to

$$\begin{gathered} e_{1,1,0} = -4w_{1,0} + 6w_{2,0} - 4w_{3,0} + 2,\ e_{1,2,0} = \kappa_0\left(4w_{1,0} - 3w_{2,0} - 1\right), \\ e_{1,3,0} = \frac{1}{12}\kappa_0\tau_0\left(-18w_{1,0} + 9w_{2,0} - 2w_{3,0} + 11\right), \\ e_{2,1,0} = e_{1,1,0},\ e_{2,2,0} = 2\kappa_0\left(w_{1,0} - w_{3,0}\right),\ e_{2,3,0} = \frac{1}{4}\kappa_0\tau_0\left(-2w_{1,0} + w_{2,0} - 2w_{3,0} + 3\right), \\ e_{3,1,0} = e_{1,1,0},\ e_{3,2,0} = \kappa_0\left(3w_{2,0} - 4w_{3,0} + 1\right), \\ e_{3,3,0} = \frac{1}{12}\kappa_0\tau_0\left(-2w_{1,0} + 9w_{2,0} - 18w_{3,0} + 11\right). \end{gathered}$$

It is obvious that they can vanish iff

$$w_{1,0} = w_{2,0} = w_{3,0} = 1,$$

and the first three relations in (4.15) are satisfied. We now continue in this way by determining the expansion coefficients from the equations

$$e_{i,j,\ell} = 0,\ j = 1,2,3;\ i = 1,2,3;\ \ell = 1,2,\dots,5.$$

At $\ell = 1$ we obtain

$$w_{1,1} = w_{2,1} = w_{3,1} = 0,\ \alpha^L_{2,2} = -2\alpha^L_{2,1}, \alpha^R_{2,2} = -2\alpha^L_{2,1},$$

and the equations at $\ell = 2$ determine

$$w_{1,2} = \frac{1}{80}c,\ w_{2,2} = \frac{1}{60}c,\ w_{3,2} = \frac{1}{80}c,$$
$$\alpha^L_{2,3} = -2\left(2\alpha^L_{1,3} + \alpha^R_{1,3}\right),\ \alpha^R_{2,3} = 2\left(\alpha^L_{1,3} + 2\alpha^R_{1,3}\right),\ \alpha^L_{3,3} = \alpha^R_{3,3} = 6\left(\alpha^L_{1,3} + \alpha^R_{1,3}\right),$$

with

$$c := \frac{1}{\kappa_0\tau_0}\Big(- \tau_0\left(\kappa_0\left(-60\alpha^L_{1,3} - 60\alpha^R_{1,3} + \kappa_0^2 + \tau_0^2\right) - 3\kappa_2\right) \\ + 60(\alpha^L_{1,2})^2\kappa_0\tau_0 - 20\alpha^L_{1,2}\left(2\kappa_1\tau_0 + \kappa_0\tau_1\right) + 3\kappa_1\tau_1 + \kappa_0\tau_2\Big).$$

In this way we determine the coefficients

$$w_{1,\ell},\ w_{2,\ell},\ w_{3,\ell},\ \alpha^L_{2,\ell+1},\ \alpha^R_{2,\ell+1},\ \alpha^L_{3,\ell+1},\ \alpha^L_{3,\ell+1},\quad \ell = 0, 1, \ldots, 5,$$

as functions of the coefficients in the expansions (4.28) and (4.29). It is tedious but straightforward to verify that all the relations required, i.e., (4.15), (4.18), (4.19), (4.20), (4.21), (4.22) are fulfilled but (4.23). If we substitute now the actual values of the coefficients that were obtained in (4.28) and (4.29), the left-hand side of (4.23) simplifies to

$$60\alpha^L_{1,6} + 12\alpha^L_{2,6} + \alpha^L_{3,6} + 60\alpha^R_{1,6} - 12\alpha^R_{2,6} + \alpha^R_{3,6} =$$
$$-\frac{10450944000\ \pi_9(\widetilde{Z})}{\kappa_0^4\left(\kappa_0\left(\tau_1\left(2\tau_0\left(2\kappa_0^2\tau_0 + 6\tau_0^3 - 9\tau_2\right) + 15\tau_1^2\right) + 4\tau_3\tau_0^2\right) - 2\kappa_1\tau_0\left(4\tau_0^4 + 2\tau_2\tau_0 - 3\tau_1^2\right)\right)}.$$

But $\widetilde{Z}$ is a root of π_9, and the proof of the theorem is completed. □

The Mathematica programs, used in computations, are available at [13].

5. Examples. Let us conclude the paper with some numerical examples. In Figure 5.1, a cubic G^2 rational spline is shown. The data were sampled from the curve

$$\left((\ln t + 3\ \ln 10)\cos t, \sqrt{1 + \ln t}\ \sin t, 3\ \ln t\right)^T \tag{5.1}$$

at $2\,\pi/3 + i\,\pi/9,\ i = 0, 1, \ldots, 39$. Each spline segment was constructed on the basis provided by Theorem 3.1. Of course, the interpolant is almost indistinguishable from the original curve. In Fig. 5.2, the corresponding curvature and torsion plots are shown. Fig. 5.3 shows jumps (differences of values at the breakpoints) in the torsion that are inevitable in general since the G^2 conditions at the breakpoints are satisfied only. For the jumps to be more clearly visible, the logarithmic scale is used.

In Fig. 5.1, right, a comparison of our cubic rational interpolant and a cubic C^2 polynomial spline is presented. Some differences in the shape can be observed. Of course, the C^2 spline has only approximation order 4. Note that for less sparse data the curve shapes become indistinguishable.

Let us consider now a quartic G^3 case, with the interpolated data sampled from

$$\left(\cos t\ \ln(1 + t), \frac{\sqrt{1 + t^2} + 2\ln(1 + t)\sin t}{\sqrt{5}}, \frac{2\sqrt{1 + t^2} - \ln(1 + t)\sin t}{\sqrt{5}}\right)^T, \quad t \in [0, h]. \tag{5.2}$$

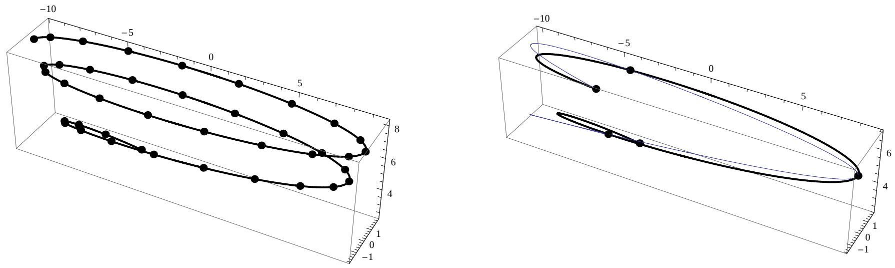

FIG. 5.1. *A cubic G^2 rational spline, interpolating the data, sampled from the curve* (5.1) *(left). A comparison of a cubic rational (black, thick) and a cubic C^2 polynomial interpolant (blue, thin) for more sparse data (right).*

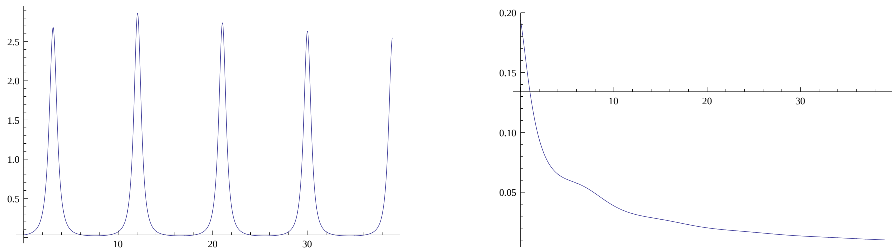

FIG. 5.2. *Curvature (left) and torsion (right) of the rational spline in Fig. 5.1.*

There are two quartic G^3 rational curves interpolating these data for h small enough, both with the approximation order 8. This is to be expected from Theorem 4.3. One computes the coefficients of the curvature expansion of (5.2) as $\kappa_0 = 2.2361, \kappa_1 = 4.0249, \kappa_2 = -4.0696, \kappa_3 = -20.9982, \kappa_4 = -8.2243, \kappa_5 = 75.6374$. Further, the leading part of the torsion is determined by $\tau_0 = 0.6, \tau_1 = -0.56, \tau_2 = -1.444, \tau_3 = 3.2744, \tau_4 = 3.8329$. This gives the quartic polynomial π_9 as

$$\pi_9(Z) = 0.5625\, Z^4 + 0.8088\, Z^3 + 0.6211\, Z^2 + 0.2425\, Z - 0.042,$$

with precisely two real solutions,

$$Z_1 = -0.85547, \quad Z_2 = 0.12565, \quad Z_{3,4} = -0.35398 \pm 0.7548\, i.$$

Table 5.1 shows a parametric upper bound and the approximation order of a G^2 rational curve for the data taken from (5.1) and for two G^3 rational curves for the data given by (5.2).

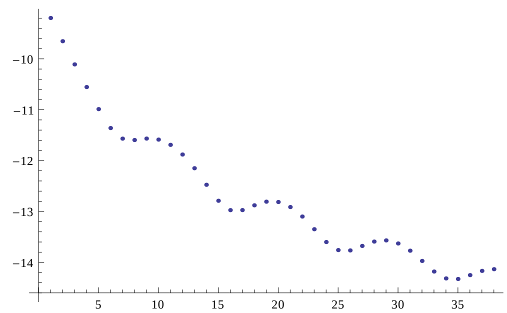

FIG. 5.3. *Jumps (differences of values at the breakpoints) in the torsion plot in Fig. 5.2, right. The logarithmic scale is used.*

TABLE 5.1
Parametric upper bound of the asymptotic approximation order based upon the data curve reparameterization φ *as defined in* (2.15)*, for the cubic* G^2 *rational curve and both quartic* G^3 *ones, interpolating data, sampled from* (5.2) *on* $[0,h]$.

	G^2 case		G^3 case, solution 1		G^3 case, solution 2	
h	Error	Order	Error	Order	Error	Order
$\frac{1}{100}$	$3.992\cdot10^{-15}$	–	$2.558\cdot10^{-12}$	–	$5.338\cdot10^{-13}$	–
$\frac{1}{200}$	$6.268\cdot10^{-17}$	5.99	$2.398\cdot10^{-15}$	10.06	$2.161\cdot10^{-17}$	14.59
$\frac{1}{400}$	$9.816\cdot10^{-19}$	6.00	$5.263\cdot10^{-18}$	8.83	$4.309\cdot10^{-19}$	5.65
$\frac{1}{800}$	$1.535\cdot10^{-20}$	6.00	$1.580\cdot10^{-20}$	8.38	$1.898\cdot10^{-21}$	7.83
$\frac{1}{1600}$	$2.400\cdot10^{-22}$	6.00	$5.438\cdot10^{-23}$	8.18	$7.531\cdot10^{-24}$	7.98
$\frac{1}{3200}$	$3.752\cdot10^{-24}$	6.00	$1.997\cdot10^{-25}$	8.09	$2.945\cdot10^{-26}$	8.00
$\frac{1}{6400}$	$5.863\cdot10^{-26}$	6.00	$7.565\cdot10^{-28}$	8.04	$1.149\cdot10^{-28}$	8.00

REFERENCES

[1] Carl de Boor, *Divided differences*, Surv. Approx. Theory, 1 (2005), pp. 46–69 (electronic).

[2] Carl de Boor, Klaus Höllig, and Malcolm Sabin, *High accuracy geometric Hermite interpolation*, Comput. Aided Geom. Design, 4 (1987), pp. 269–278.

[3] W. L. F. Degen, *High accurate rational approximation of parametric curves*, Comput. Aided Geom. Design, 10 (1993), pp. 293–313. Free-form curves and free-form surfaces (Oberwolfach, 1992).

[4] ———, *High accuracy approximation of parametric curves*, in Mathematical methods for curves and surfaces (Ulvik, 1994), Vanderbilt Univ. Press, Nashville, TN, 1995, pp. 83–98.

[5] Gerald Farin, *Curves and surfaces for computer-aided geometric design*, Computer Science and Scientific Computing, Academic Press Inc., San Diego, CA, fourth ed., 1997.

[6] Yu Yu Feng and Jernej Kozak, *On spline interpolation of space data*, in Mathematical methods for curves and surfaces, II (Lillehammer, 1997), Innov. Appl. Math., Vanderbilt Univ. Press, Nashville, TN, 1998, pp. 167–174.

[7] Zulfiqar Habib and Sakai Manabu, *Admissible regions for rational cubic spirals matching G^2 Hermite data*, Comput. Aided Design, 42 (2010), pp. 1117–1124.

[8] Klaus Höllig and Jürgen Koch, *Geometric Hermite interpolation with maximal order and smoothness*, Comput. Aided Geom. Design, 13 (1996), pp. 681–695.

[9] Gašper Jaklič, Jernej Kozak, Marjeta Krajnc, and Emil Žagar, *On geometric interpolation by planar parametric polynomial curves*, Math. Comp., 76 (2007), pp. 1981–1993.

[10] Gašper Jaklič, Jernej Kozak, Vito Vitrih, and Emil Žagar, *Lagrange geometric interpolation by rational spatial cubic Bézier curves*, Comput. Aided Geom. Design, (to appear).

[11] Donald E. Knuth, *The art of computer programming*, Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, second ed., 1975. Volume 1: Fundamental algorithms, Addison-Wesley Series in Computer Science and Information Processing.

[12] V. P. Kong and B. H. Ong, *Constrained space curve interpolation with constraint planes*, Comput. Aided Geom. Design, 22 (2005), pp. 531–550.

[13] Jernej Kozak, *Notebook of computations the paper relies upon*, 2012. http://www.fmf.uni-lj.si/~kozak/wikiang/index.php/Some_publications.

[14] Jernej Kozak and Marjeta Krajnc, *Geometric interpolation by planar cubic polynomial curves*, Comput. Aided Geom. Design, 24 (2007), pp. 67–78.

[15] Jernej Kozak and Emil Žagar, *On geometric interpolation by polynomial curves*, SIAM J. Numer. Anal., 42 (2004), pp. 953–967.

[16] Tom Lyche and Knut Mørken, *A metric for parametric approximation*, in Curves and surfaces in geometric design (Chamonix-Mont-Blanc, 1993), A K Peters, Wellesley, MA, 1994, pp. 311–318.

[17] D. S. Meek, B. H. Ong, and D. J. Walton, *Constrained interpolation with rational cubics*, Comput. Aided Geom. Design, 20 (2003), pp. 253–275.

[18] Knut Mørken and Karl Scherer, *A general framework for high-accuracy parametric interpolation*, Math. Comp., 66 (1997), pp. 237–260.

[19] Xing Liang Ou and Kui Fang, *G^2 rational cubic geometric Hermite interpolation algorithm*, Math. Theory Appl. (Changsha), 21 (2001), pp. 39–42.

[20] Robert Schaback, *Rational geometric curve interpolation*, in Mathematical methods in computer aided geometric design, II (Biri, 1991), Academic Press, Boston, MA, 1992, pp. 517–535.

[21] ———, *Optimal geometric Hermite interpolation of curves*, in Mathematical methods for curves and surfaces, II (Lillehammer, 1997), Innov. Appl. Math., Vanderbilt Univ. Press, Nashville, TN, 1998, pp. 417–428.