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Research Article

Hermite-Hadamard-Fejér Inequalities for Conformable Fractional Integrals via Preinvex Functions

Yousaf Khurshid,[1] Muhammad Adil Khan,[1] Yu-Ming Chu ,[2] and Zareen Abdulhameed Khan [3]

[1]*Department of Mathematics, University of Peshawar, Peshawar 25000, Pakistan*
[2]*Department of Mathematics, Huzhou University, Huzhou 313000, China*
[3]*Department of Mathematics, Princess Nora Bint Abdulrahman University, Riyadh 11538, Saudi Arabia*

Correspondence should be addressed to Yu-Ming Chu; chuyuming@zjhu.edu.cn

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In this paper, we present a Hermite-Hadamard-Fejér inequality for conformable fractional integrals by using symmetric preinvex functions. We also establish an identity associated with the right hand side of Hermite-Hadamard inequality for preinvex functions; then by using this identity and preinvexity of functions and some well-known inequalities, we find several new Hermite-Hadamard type inequalities for conformal fractional integrals.

1. Introduction

Let $I \in \mathbb{R}$ be an interval and $h : I \longrightarrow \mathbb{R}$ be a convex function defined on I such that $\kappa_1, \kappa_2 \in I$ with $\kappa_1 < \kappa_2$. Then the well-known $\mathscr{H}\mathscr{H}$ (Hermite-Hadamard) inequality [1] states that

$$h\left(\frac{\kappa_1 + \kappa_2}{2}\right) \leq \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} h(x)\,dx \leq \frac{h(\kappa_1) + h(\kappa_2)}{2} \quad (1)$$

holds. If the function h is concave on I, then both inequalities in (1) hold in the reverse direction.

In the last few years, many researchers have shown their extensive attention on the generalizations, extensions, variations, refinements, and applications of the $\mathscr{H}\mathscr{H}$ inequality (see [2–15]). The most well-known generalization of the $\mathscr{H}\mathscr{H}$ inequality is the Hermite-Hadamard-Fejér inequality [16]. In 1906, Fejér [16] established the following weighted generalization of the Hermite-Hadamard inequality for symmetric functions:

$$\begin{aligned} h\left(\frac{\kappa_1 + \kappa_2}{2}\right) \int_{\kappa_1}^{\kappa_2} g(x)\,dx &\leq \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} g(x)\,h(x)\,dx \\ &\leq \frac{h(\kappa_1) + h(\kappa_2)}{2} \int_{\kappa_1}^{\kappa_2} g(x)\,dx \end{aligned} \quad (2)$$

for all convex functions $h : I \longrightarrow \mathbb{R}$, $\kappa_1, \kappa_2 \in I$ with $\kappa_1 < \kappa_2$ and $g : [\kappa_1, \kappa_2] \longrightarrow \mathbb{R}^+$ is symmetric with respect to $(\kappa_1 + \kappa_2)/2$.

It is well known that the convex sets and convex functions play important roles in the nonlinear programming and optimization theory. Many generalizations and extensions have been considered for the classical convexity in the last few decades. A significant generalization of convex functions is that of invex functions introduced by Hanson in [17]. The basic properties of the preinvex functions and their roles in optimization theory can be found in [18]. The $\mathscr{H}\mathscr{H}$ inequalities for preinvex and log-preinvex functions were established by Noor [19, 20].

Now, we recall some notions and definitions in invexity analysis, which will be used throughout the paper (see [21, 22] and references therein).

Let $\mathfrak{A} \in \mathbb{R}$ be a nonempty set and the functions $h : \mathfrak{A} \longrightarrow \mathbb{R}$ and $\Psi : \mathfrak{A} \times \mathfrak{A} \longrightarrow \mathbb{R}$ be continuous.

Definition 1. The set $\mathfrak{A} \subseteq \mathbb{R}^n$ is said to be invex with respect to $\Psi(.,.)$ if

$$\mu_1 + s\Psi(\mu_2, \mu_1) \in \mathfrak{A} \quad (3)$$

for all $\mu_1, \mu_2 \in \mathfrak{A}$ and $s \in [0, 1]$.

The invex set $\mathfrak{A}$ is also called a Ψ-connected set. If $\Psi(\mu_2, \mu_1) = \mu_2 - \mu_1$, then the invex set is also a convex set, but some of the invex sets are not convex [21].

Definition 2. The function h is said to be preinvex with respect to Ψ on the invex set $\mathfrak{A}$ if

$$h(\mu_1 + s\Psi(\mu_2, \mu_1)) \le (1-s)h(\mu_1) + sh(\mu_2) \tag{4}$$

for all $\mu_1, \mu_2 \in \mathfrak{A}$ and $s \in [0,1]$. The function h is called preconcave if $-h$ is preinvex.

The following Condition C was introduced by Mohan and Neogy [23].

Condition C. Suppose $\mathfrak{A}$ is an open invex subset of $\mathbb{R}^n$ with respect to $\Psi : \mathfrak{A} \times \mathfrak{A} \longrightarrow \mathbb{R}$ and Ψ satisfies

$$\begin{aligned} \Psi(\mu_2, \mu_2 + s\Psi(\mu_1, \mu_2)) &= -s\Psi(\mu_1, \mu_2), \\ \Psi(\mu_1, \mu_2 + s\Psi(\mu_1, \mu_2)) &= (1-s)\Psi(\mu_1, \mu_2) \end{aligned} \tag{5}$$

for any $\mu_1, \mu_2 \in \mathfrak{A}$ and $s \in [0,1]$.

From Condition C, we clearly see that

$$\begin{aligned} &\Psi(\mu_2 + s_2\Psi(\mu_1, \mu_2), \mu_2 + s_1\Psi(\mu_1, \mu_2)) \\ &\quad = (s_2 - s_1)\Psi(\mu_1, \mu_2) \end{aligned} \tag{6}$$

for any $\mu_1, \mu_2 \in \mathfrak{A}$ and $s \in [0,1]$.

The following $\mathscr{H}\mathscr{H}$ inequality for the preinvex functions was proved by Noor [20].

Theorem 3. *Let $h : K = [\kappa_1, \kappa_1 + \Psi(\kappa_2, \kappa_1)] \longrightarrow (0, \infty)$ be a preinvex function on the interval K° (the interior of K) and κ_1, $\kappa_2 \in K^\circ$ with $\kappa_1 < \kappa_1 + \Psi(\kappa_2, \kappa_1)$. Then the following inequality holds:*

$$\begin{aligned} &h\left(\frac{2\kappa_1 + \Psi(\kappa_2, \kappa_1)}{2}\right) \\ &\quad \le \frac{1}{\Psi(\kappa_2, \kappa_1)} \int_{\kappa_1}^{\kappa_1 + \Psi(\kappa_2, \kappa_1)} h(x)\, dx \le \frac{h(\kappa_1) + h(\kappa_2)}{2}. \end{aligned} \tag{7}$$

Several important variants of $\mathscr{H}\mathscr{H}$ inequality for preinvex functions have been provided in the literature [24]. Recently, the authors in [25] defined a new well-behaved simple fractional derivative called the "conformable fractional derivative". Namely, the conformable fractional derivative of order $0 < \alpha \le 1$ at $s > 0$ for the function $h : [0, \infty) \longrightarrow \mathbb{R}$ is defined by

$$D_\alpha(h)(s) = \lim_{\epsilon \longrightarrow 0} \frac{h(s + \epsilon s^{1-\alpha}) - h(s)}{\epsilon}. \tag{8}$$

If the conformable fractional derivative of h of order α exists, then we say that h is α-fractional differentiable. The fractional derivative at 0 is defined as $h^\alpha(0) = \lim_{s \longrightarrow 0^+} h^\alpha(s)$.

Next, we present some basic results related to conformable fractional derivative in the following theorem.

Theorem 4 (see [25]). *Let $\alpha \in (0,1]$ and h_1, h_2 be α-differentiable at a point $s > 0$. Then*

(i) $(d_\alpha/d_\alpha s)(s^n) = ns^{n-\alpha}$ *for all* $n \in \mathbb{R}$.

(ii) $(d_\alpha/d_\alpha s)(c) = 0$ *for any constant* $c \in \mathbb{R}$.

(iii) $(d_\alpha/d_\alpha s)(\kappa_1 h_1(s) + \kappa_2 h_2(s)) = \kappa_1(d_\alpha/d_\alpha s)(h_1(s)) + \kappa_2(d_\alpha/d_\alpha s)(h_2(s))$ *for all* $\kappa_1, \kappa_2 \in \mathbb{R}$.

(iv) $(d_\alpha/d_\alpha s)(h_1(s)h_2(s)) = h_1(s)(d_\alpha/d_\alpha s)(h_2(s)) + h_2(s)(d_\alpha/d_\alpha s)(h_1(s))$.

(v) $(d_\alpha/d_\alpha s)(h_1(s)/h_2(s)) = (h_2(s)(d_\alpha/d_\alpha s)(h_1(s)) - h_1(s)(d_\alpha/d_\alpha s)(h_2(s)))/(h_2(s))^2$.

(vi) $(d_\alpha/d_\alpha s)((h_1 \circ h_2)(s)) = h_1'(h_2(s))(d_\alpha/d_\alpha s)(h_2(s))$ *if* h_1 *differentiable at* $h_2(s)$.

If in addition h_1 is differentiable, then

$$\frac{d_\alpha}{d_\alpha s}(h_1(s)) = s^{1-\alpha}\frac{d}{ds}(h_1(s)). \tag{9}$$

Definition 5 (see [25] conformable fractional integral). Let $\alpha \in (0,1]$ and $0 \le \kappa_1 < \kappa_2$. A function $h_1 : [\kappa_1, \kappa_2] \longrightarrow \mathbb{R}$ is α-fractional integrable on $[\kappa_1, \kappa_2]$ if the integral

$$\int_{\kappa_1}^{\kappa_2} h_1(x)\, d_\alpha x := \int_{\kappa_1}^{\kappa_2} h_1(x)\, x^{\alpha-1} dx \tag{10}$$

exists and is finite. All α-fractional integrable functions on $[\kappa_1, \kappa_2]$ are indicated by $L_\alpha([\kappa_1, \kappa_2])$.

Remark 6.

$$I_\alpha^{\kappa_1}(h_1)(s) = I_1^{\kappa_1}(s^{\alpha-1}h_1) = \int_{\kappa_1}^{s} \frac{h_1(x)}{x^{1-\alpha}} dx, \tag{11}$$

where the integral is the usual Riemann improper integral and $\alpha \in (0,1]$.

Recently, the conformable integrals and derivatives have been the subject of intensive research, and many remarkable properties and inequalities involving the conformable integrals and derivatives can be found in the literature [26–38].

In [39], Anderson provided the conformable integral version of $\mathscr{H}\mathscr{H}$ inequality as follows.

Theorem 7 (see [39]). *If $\alpha \in (0,1]$ and $h_1 : [\kappa_1, \kappa_2] \longrightarrow \mathbb{R}$ is an α-fractional differentiable function such that $D_\alpha h$ is increasing, then we have the following inequality:*

$$\frac{\alpha}{\kappa_2^\alpha - \kappa_1^\alpha} \int_{\kappa_1}^{\kappa_2} h(x)\, d_\alpha x \le \frac{h(\kappa_1) + h(\kappa_2)}{2}. \tag{12}$$

Moreover if the function h is decreasing on $[\kappa_1, \kappa_2]$, then we have

$$h\left(\frac{\kappa_1 + \kappa_2}{2}\right) \le \frac{\alpha}{\kappa_2^\alpha - \kappa_1^\alpha} \int_{\kappa_1}^{\kappa_2} h(x)\, d_\alpha x. \tag{13}$$

If $\alpha = 1$, then this reduces to the classical $\mathscr{H}\mathscr{H}$ inequality.

In this paper, we first establish the Hermite-Hadamard-Fejér inequality for conformable fractional integrals by using symmetric preinvex functions; then we present $\mathcal{HH}$ inequalities as their special cases (see Corollary 9). Secondly, we give an identity associated with the right side of $\mathcal{HH}$ inequality for preinvex functions using the conformable fractional integrals; then we establish $\mathcal{HH}$ inequalities for preinvex functions by use of Hölder inequality, power mean inequality, and preinvexity of functions.

2. Hermite-Hadamard-Fejér Inequalities for Conformable Fractional Integrals

The preinvex version of Fejer-Hermite-Hadamard inequality can be represented in conformable fractional integrals forms as follows.

Theorem 8. *Suppose that $\kappa_1, \kappa_2 \in K$ such that $\Psi(\kappa_2,\kappa_1) > 0$, $h : K = [\kappa_1, \kappa_1 + \Psi(\kappa_2,\kappa_1)] \longrightarrow (0,\infty)$ is a preinvex function and symmetric with respect to $(2\kappa_1 + \Psi(\kappa_2,\kappa_1))/2$, and $g : [\kappa_1,\kappa_2] \longrightarrow \mathbb{R}$ is a nonnegative integrable function. Also assume that Ψ satisfies Condition C; then the inequality*

$$h\left(\frac{2\kappa_1 + \Psi(\kappa_2,\kappa_1)}{2}\right)\int_{\kappa_1}^{\kappa_1+\Psi(\kappa_2,\kappa_1)} g(x)\, d_\alpha x \le \int_{\kappa_1}^{\kappa_1+\Psi(\kappa_2,\kappa_1)} h(x)\, g(x)\, d_\alpha x \le \frac{h(\kappa_1) + h(\kappa_1 + \Psi(\kappa_2,\kappa_1))}{2}\int_{\kappa_1}^{\kappa_1+\Psi(\kappa_2,\kappa_1)} g(x)\, d_\alpha x \quad (14)$$

holds for any $\alpha \in (0,1]$.

Proof. Since $h : K \longrightarrow \mathbb{R}$ is preinvex function and is symmetric with respect to $(2\kappa_1 + \Psi(\kappa_2,\kappa_1))/2$, then for any $x, y \in K$ and $t = 1/2$, we have

$$h\left(x + \frac{\Psi(y,x)}{2}\right) \le \frac{h(x) + h(y)}{2} \quad (15)$$

i.e., with $x = \kappa_1 + (1-s)\Psi(\kappa_2,\kappa_1)$ and $y = \kappa_1 + s\Psi(\kappa_2,\kappa_1)$, inequality (15) becomes

$$\begin{aligned} & h\left(\kappa_1 + (1-s)\Psi(\kappa_2,\kappa_1) + \frac{\Psi(\kappa_1 + s\Psi(\kappa_2,\kappa_1), \kappa_1 + (1-s)\Psi(\kappa_2,\kappa_1))}{2}\right) \\ &= h\left(\kappa_1 + (1-s)\Psi(\kappa_2,\kappa_1) + \frac{(s-1+s)\Psi(\kappa_2,\kappa_1)}{2}\right) \text{ (using Condition C)} \\ &= h\left(\kappa_1 + (1-s)\Psi(\kappa_2,\kappa_1) + \frac{(2s-1)\Psi(\kappa_2,\kappa_1)}{2}\right) \\ &= h\left(\frac{2\kappa_1 + \Psi(\kappa_2,\kappa_1)}{2}\right) \\ &\le \frac{h(\kappa_1 + (1-s)\Psi(\kappa_2,\kappa_1)) + h(\kappa_1 + s\Psi(\kappa_2,\kappa_1))}{2} \\ &= h(\kappa_1 + s\Psi(\kappa_2,\kappa_1)). \ (h \text{ is symmetric}) \end{aligned} \quad (16)$$

By change of variables, we have

$$\begin{aligned} & h\left(\frac{2\kappa_1 + \Psi(\kappa_2,\kappa_1)}{2}\right)\int_{\kappa_1}^{\kappa_1+\Psi(\kappa_2,\kappa_1)} g(x)\, d_\alpha x \\ &= \Psi(\kappa_2,\kappa_1)\, h\left(\frac{2\kappa_1 + \Psi(\kappa_2,\kappa_1)}{2}\right) \cdot \int_0^1 g(\kappa_1 + s\Psi(\kappa_2,\kappa_1))(\kappa_1 + s\Psi(\kappa_2,\kappa_1))^{\alpha-1} ds \\ &\le \Psi(\kappa_2,\kappa_1)\int_0^1 h(\kappa_1 + s\Psi(\kappa_2,\kappa_1)) \cdot g(\kappa_1 + s\Psi(\kappa_2,\kappa_1))(\kappa_1 + s\Psi(\kappa_2,\kappa_1))^{\alpha-1} ds \\ &= \int_{\kappa_1}^{\kappa_1+\Psi(\kappa_2,\kappa_1)} h(x)\, g(x)\, d_\alpha x. \end{aligned} \quad (17)$$

So we can write

$$h\left(\frac{2\kappa_1 + \Psi(\kappa_2,\kappa_1)}{2}\right)\int_{\kappa_1}^{\kappa_1+\Psi(\kappa_2,\kappa_1)} g(x)\, d_\alpha x \le \int_{\kappa_1}^{\kappa_1+\Psi(\kappa_2,\kappa_1)} h(x)\, g(x)\, d_\alpha x. \quad (18)$$

To prove the second inequality in (14), we know that h is preinvex and Ψ satisfies Condition C, so we have

$$\begin{aligned} & h(\kappa_1 + s\Psi(\kappa_2,\kappa_1)) \\ &= h(\kappa_1 + \Psi(\kappa_2,\kappa_1) + (1-s)\Psi(\kappa_1, \kappa_1 + \Psi(\kappa_2,\kappa_1))) \\ &\le sh(\kappa_1 + \Psi(\kappa_2,\kappa_1)) + (1-s)h(\kappa_1), \end{aligned} \quad (19)$$

and similarly

$$\begin{aligned} & h(\kappa_1 + (1-s)\Psi(\kappa_2,\kappa_1)) \\ &= h(\kappa_1 + \Psi(\kappa_2,\kappa_1) + s\Psi(\kappa_1, \kappa_1 + \Psi(\kappa_2,\kappa_1))) \\ &\le sh(\kappa_1) + (1-s)h(\kappa_1 + \Psi(\kappa_2,\kappa_1)). \end{aligned} \quad (20)$$

Now with $x = \kappa_1 + s\Psi(\kappa_2,\kappa_1)$, we have

$$\begin{aligned} & \int_{\kappa_1}^{\kappa_1+\Psi(\kappa_2,\kappa_1)} h(x)\, g(x)\, x^{\alpha-1} dx \\ &= \int_0^1 h(\kappa_1 + s\Psi(\kappa_2,\kappa_1))\, g(\kappa_1 + s\Psi(\kappa_2,\kappa_1)) \cdot (\kappa_1 + s\Psi(\kappa_2,\kappa_1))^{\alpha-1} ds \le \Psi(\kappa_2,\kappa_1) \\ &\cdot \int_0^1 [sh(\kappa_1 + \Psi(\kappa_2,\kappa_1)) + (1-s)h(\kappa_1)] \\ &\times g(\kappa_1 + s\Psi(\kappa_2,\kappa_1))(\kappa_1 + s\Psi(\kappa_2,\kappa_1))^{\alpha-1} ds \text{ (using (19))}. \end{aligned} \quad (21)$$

Also

$$\begin{aligned}
&\int_{\kappa_1}^{\kappa_1+\Psi(\kappa_2,\kappa_1)} h(x)\, g(x)\, x^{\alpha-1} dx \\
&= \int_0^1 h(\kappa_1 + s\Psi(\kappa_2,\kappa_1))\, g(\kappa_1 + s\Psi(\kappa_2,\kappa_1)) \\
&\cdot (\kappa_1 + s\Psi(\kappa_2,\kappa_1))^{\alpha-1} ds \\
&= \int_0^1 h(\kappa_1 + (1-s)\Psi(\kappa_2,\kappa_1))\, g(\kappa_1 + s\Psi(\kappa_2,\kappa_1)) \\
&\cdot (\kappa_1 + s\Psi(\kappa_2,\kappa_1))^{\alpha-1} ds \quad (h \text{ is symmetric}) \\
&\le \Psi(\kappa_2,\kappa_1) \\
&\cdot \int_0^1 [s h(\kappa_1) + (1-s) h(\kappa_1 + \Psi(\kappa_2,\kappa_1))] \\
&\times g(\kappa_1 + s\Psi(\kappa_2,\kappa_1))\, (\kappa_1 + s\Psi(\kappa_2,\kappa_1))^{\alpha-1} ds \quad (\text{using } (20)).
\end{aligned} \tag{22}$$

If we add (21) and (22), we obtain

$$\begin{aligned}
&2\int_{\kappa_1}^{\kappa_1+\Psi(\kappa_2,\kappa_1)} h(x)\, g(x)\, d_\alpha x \le \Psi(\kappa_2,\kappa_1) \\
&\cdot (h(\kappa_1) + h(\kappa_1 + \Psi(\kappa_2,\kappa_1))) \\
&\cdot \int_0^1 g(\kappa_1 + s\Psi(\kappa_2,\kappa_1))\, (\kappa_1 + s\Psi(\kappa_2,\kappa_1))^{\alpha-1} ds \\
&= (h(\kappa_1) + h(\kappa_1 + \Psi(\kappa_2,\kappa_1))) \\
&\cdot \int_{\kappa_1}^{\kappa_1+\Psi(\kappa_2,\kappa_1)} g(x)\, x^{\alpha-1} dx.
\end{aligned} \tag{23}$$

So we can write

$$\begin{aligned}
&\int_{\kappa_1}^{\kappa_1+\Psi(\kappa_2,\kappa_1)} h(x)\, g(x)\, d_\alpha x \\
&\le \frac{h(\kappa_1) + h(\kappa_1 + \Psi(\kappa_2,\kappa_1))}{2} \int_{\kappa_1}^{\kappa_1+\Psi(\kappa_2,\kappa_1)} g(x)\, d_\alpha x.
\end{aligned} \tag{24}$$

From inequalities (18) and (24), we obtain over required result. □

Corollary 9. *If we put $g(x) = 1$ in (14), then we get*

$$\begin{aligned}
&h\left(\frac{2\kappa_1 + \Psi(\kappa_2,\kappa_1)}{2}\right) \\
&\le \frac{\alpha}{(\kappa_1 + \Psi(\kappa_2,\kappa_1))^\alpha - \kappa_1^\alpha} \int_{\kappa_1}^{\kappa_1+\Psi(\kappa_2,\kappa_1)} h(x)\, d_\alpha x \\
&\le \frac{h(\kappa_1) + h(\kappa_1 + \Psi(\kappa_2,\kappa_1))}{2}.
\end{aligned} \tag{25}$$

3. $\mathscr{H}\mathscr{H}$ Type Inequalities for Conformable Fractional Integrals

Lemma 10. *Let $\kappa_1, \kappa_2 \in K$ with $\Psi(\kappa_2,\kappa_1) > 0$ and $h : K = [\kappa_1, \kappa_1 + \Psi(\kappa_2,\kappa_1)] \longrightarrow (0,\infty)$ be an α-fractional differentiable function on $(\kappa_1, \kappa_1 + \Psi(\kappa_2,\kappa_1))$ for $\alpha \in (0,1]$. If $D_\alpha(h) \in L_\alpha([\kappa_1, \kappa_1 + \Psi(\kappa_2,\kappa_1)])$, then the following identity holds:*

$$\begin{aligned}
&\frac{h(\kappa_1) + h(\kappa_1 + \Psi(\kappa_2,\kappa_1))}{2} - \frac{\alpha}{(\kappa_1 + \Psi(\kappa_2,\kappa_1))^\alpha - \kappa_1^\alpha} \\
&\cdot \int_{\kappa_1}^{\kappa_1+\Psi(\kappa_2,\kappa_1)} h(x)\, d_\alpha x \\
&= \frac{\Psi(\kappa_2,\kappa_1)}{2\left((\kappa_1 + \Psi(\kappa_2,\kappa_1))^\alpha - \kappa_1^\alpha\right)} \left[\int_0^1 \left((\kappa_1 + s\Psi(\kappa_2,\kappa_1))^{2\alpha-1}\right.\right. \\
&\left. - \kappa_1^\alpha (\kappa_1 + s\Psi(\kappa_2,\kappa_1))^{\alpha-1}\right) \times D_\alpha(h)(\kappa_1 + s\Psi(\kappa_2,\kappa_1)) \\
&\cdot s^{1-\alpha} d_\alpha s + \int_0^1 \left((\kappa_1 + s\Psi(\kappa_2,\kappa_1))^{2\alpha-1}\right. \\
&\left. - (\kappa_1 + \Psi(\kappa_2,\kappa_1))^\alpha (\kappa_1 + s\Psi(\kappa_2,\kappa_1))^{\alpha-1}\right) \times D_\alpha(h)(\kappa_1 \\
&\left. + s\Psi(\kappa_2,\kappa_1))\, s^{1-\alpha} d_\alpha s\right].
\end{aligned} \tag{26}$$

Proof. Integrating by parts, we have

$$\begin{aligned}
I &= \int_0^1 \left((\kappa_1 + s\Psi(\kappa_2,\kappa_1))^{2\alpha-1}\right. \\
&\left. - \kappa_1^\alpha (\kappa_1 + s\Psi(\kappa_2,\kappa_1))^{\alpha-1}\right) D_\alpha(h)(\kappa_1 \\
&+ s\Psi(\kappa_2,\kappa_1))\, ds + \int_0^1 \left((\kappa_1 + s\Psi(\kappa_2,\kappa_1))^{2\alpha-1}\right. \\
&\left. - (\kappa_1 + \Psi(\kappa_2,\kappa_1))^\alpha (\kappa_1 + s\Psi(\kappa_2,\kappa_1))^{\alpha-1}\right) \\
&\times D_\alpha(h)(\kappa_1 + s\Psi(\kappa_2,\kappa_1))\, ds \\
&= \int_0^1 \left((\kappa_1 + s\Psi(\kappa_2,\kappa_1))^\alpha - \kappa_1^\alpha\right) h'(\kappa_1 \\
&+ s\Psi(\kappa_2,\kappa_1))\, ds + \int_0^1 \left((\kappa_1 + s\Psi(\kappa_2,\kappa_1))^\alpha\right. \\
&\left. - (\kappa_1 + \Psi(\kappa_2,\kappa_1))^\alpha\right) h'(\kappa_1 + s\Psi(\kappa_2,\kappa_1))\, ds \\
&= \left((\kappa_1 + s\Psi(\kappa_2,\kappa_1))^\alpha - \kappa_1^\alpha\right) \frac{h(\kappa_1 + s\Psi(\kappa_2,\kappa_1))}{\Psi(\kappa_2,\kappa_1)}\Bigg|_0^1 \\
&- \int_0^1 \alpha(\kappa_1 + s\Psi(\kappa_2,\kappa_1))^{\alpha-1}\, \Psi(\kappa_2,\kappa_1) \\
&\cdot \frac{h(\kappa_1 + s\Psi(\kappa_2,\kappa_1))}{\Psi(\kappa_2,\kappa_1)} ds + \left((\kappa_1 + s\Psi(\kappa_2,\kappa_1))^\alpha\right. \\
&\left. - (\kappa_1 + \Psi(\kappa_2,\kappa_1))^\alpha\right) \frac{h(\kappa_1 + s\Psi(\kappa_2,\kappa_1))}{\Psi(\kappa_2,\kappa_1)}\Bigg|_0^1 \\
&- \int_0^1 \alpha(\kappa_1 + s\Psi(\kappa_2,\kappa_1))^{\alpha-1}\, \Psi(\kappa_2,\kappa_1) \\
&\cdot \frac{h(\kappa_1 + s\Psi(\kappa_2,\kappa_1))}{\Psi(\kappa_2,\kappa_1)} ds
\end{aligned}$$

$$= \frac{1}{\Psi(\kappa_2,\kappa_1)} \left[\left((\kappa_1 + \Psi(\kappa_2,\kappa_1))^{\alpha} - \kappa_1^{\alpha} \right) h(\kappa_1 + \Psi(\kappa_2,\kappa_1)) - \alpha \int_{\kappa_1}^{\kappa_1+\Psi(\kappa_2,\kappa_1)} h(x)\, d_{\alpha}x \right] + \frac{1}{\Psi(\kappa_2,\kappa_1)} \left[\left((\kappa_1 + \Psi(\kappa_2,\kappa_1))^{\alpha} - \kappa_1^{\alpha} \right) h(\kappa_1) - \alpha \int_{\kappa_1}^{\kappa_1+\Psi(\kappa_2,\kappa_1)} h(x)\, d_{\alpha}x \right] = \frac{\left((\kappa_1 + \Psi(\kappa_2,\kappa_1))^{\alpha} - \kappa_1^{\alpha} \right)}{\Psi(\kappa_2,\kappa_1)} \left(h(\kappa_1) + h(\kappa_1 + \Psi(\kappa_2,\kappa_1)) \right) - \frac{2\alpha}{\Psi(\kappa_2,\kappa_1)} \int_{\kappa_1}^{\kappa_1+\Psi(\kappa_2,\kappa_1)} h(x)\, d_{\alpha}x, \tag{27}$$

where we have used the change of variable $x = \kappa_1 + \Psi(\kappa_2,\kappa_1)$ and then multiplied both sides by $\Psi(\kappa_2,\kappa_1)/(2((\kappa_1 + \Psi(\kappa_2,\kappa_1))^{\alpha} - \kappa_1^{\alpha}))$ to get the desired result in (26). □

Remark 11. If we set $\alpha = 1$ in (26), then we obtain the result which is proved by Barani et al. in [40]

$$-\frac{h(\kappa_1) + h(\kappa_1 + \Psi(\kappa_2,\kappa_1))}{2} + \frac{1}{\Psi(\kappa_2,\kappa_1)} \int_{\kappa_1}^{\kappa_1+\Psi(\kappa_2,\kappa_1)} h(x)\, dx = \frac{\Psi(\kappa_2,\kappa_1)}{2} \int_0^1 (1-2s)\, h'(\kappa_1 + s\Psi(\kappa_2,\kappa_1))\, ds. \tag{28}$$

Theorem 12. *Let $\kappa_1, \kappa_2 \in K$ such that $\Psi(\kappa_2,\kappa_1) > 0$ and $h : K = [\kappa_1, \kappa_1 + \Psi(\kappa_2,\kappa_1)] \longrightarrow (0,\infty)$ be an α-differentiable function on $(\kappa_1, \kappa_1 + \Psi(\kappa_2,\kappa_1))$ for $\alpha \in (0,1]$ such that $D_{\alpha}(h) \in L_{\alpha}([\kappa_1, \kappa_1 + \Psi(\kappa_2,\kappa_1)])$. If $|h'|$ is preinvex, then we have the following inequality:*

$$\left| \frac{h(\kappa_1) + h(\kappa_1 + \Psi(\kappa_2,\kappa_1))}{2} - \frac{\alpha}{(\kappa_1 + \Psi(\kappa_2,\kappa_1))^{\alpha} - \kappa_1^{\alpha}} \int_{\kappa_1}^{\kappa_1+\Psi(\kappa_2,\kappa_1)} h(x)\, d_{\alpha}x \right| \le \frac{\Psi(\kappa_2,\kappa_1)}{4} \left[|h'(\kappa_1)| + |h'(\kappa_2)| \right]. \tag{29}$$

Proof. From Lemma 10, using the property of the modulus and preinvexity of $|h'|$, we have

$$\left| \frac{h(\kappa_1) + h(\kappa_1 + \Psi(\kappa_2,\kappa_1))}{2} - \frac{\alpha}{(\kappa_1 + \Psi(\kappa_2,\kappa_1))^{\alpha} - \kappa_1^{\alpha}} \cdot \int_{\kappa_1}^{\kappa_1+\Psi(\kappa_2,\kappa_1)} h(x)\, d_{\alpha}x \right| = \left| \frac{\Psi(\kappa_2,\kappa_1)}{2\left((\kappa_1 + \Psi(\kappa_2,\kappa_1))^{\alpha} - \kappa_1^{\alpha} \right)} \cdot \int_0^1 \left((\kappa_1 + s\Psi(\kappa_2,\kappa_1))^{2\alpha-1} - \kappa_1^{\alpha} (\kappa_1 + s\Psi(\kappa_2,\kappa_1))^{\alpha-1} \right) \times D_{\alpha}(h)(\kappa_1 + s\Psi(\kappa_2,\kappa_1))\, ds + \int_0^1 \left((\kappa_1 + s\Psi(\kappa_2,\kappa_1))^{2\alpha-1} - (\kappa_1 + \Psi(\kappa_2,\kappa_1))^{\alpha} (\kappa_1 + s\Psi(\kappa_2,\kappa_1))^{\alpha-1} \right) \times D_{\alpha}(h) \cdot (\kappa_1 + s\Psi(\kappa_2,\kappa_1))\, ds \right|$$
$$\le \frac{\Psi(\kappa_2,\kappa_1)}{2\left((\kappa_1 + \Psi(\kappa_2,\kappa_1))^{\alpha} - \kappa_1^{\alpha} \right)} \left[\int_0^1 \left((\kappa_1 + s\Psi(\kappa_2,\kappa_1))^{\alpha} - \kappa_1^{\alpha} \right) |h'(\kappa_1 + s\Psi(\kappa_2,\kappa_1))|\, ds + \int_0^1 \left((\kappa_1 + \Psi(\kappa_2,\kappa_1))^{\alpha} - (\kappa_1 + s\Psi(\kappa_2,\kappa_1))^{\alpha} \right) \cdot |h'(\kappa_1 + s\Psi(\kappa_2,\kappa_1))|\, ds \right]$$
$$\le \frac{\Psi(\kappa_2,\kappa_1)}{2\left((\kappa_1 + \Psi(\kappa_2,\kappa_1))^{\alpha} - \kappa_1^{\alpha} \right)} \left[\int_0^1 \left((\kappa_1 + s\Psi(\kappa_2,\kappa_1))^{\alpha} - \kappa_1^{\alpha} \right) \left[(1-s) |h'(\kappa_1)| + s |h'(\kappa_2)| \right] dt + \int_0^1 \left((\kappa_1 + \Psi(\kappa_2,\kappa_1))^{\alpha} - (\kappa_1 + s\Psi(\kappa_2,\kappa_1))^{\alpha} \right) \cdot \left[(1-s) |h'(\kappa_1)| + s |h'(\kappa_2)| \right] ds \right] = \frac{\Psi(\kappa_2,\kappa_1)}{4} \left[|h'(\kappa_1)| + |h'(\kappa_2)| \right]. \tag{30}$$

□

Theorem 13. *Let $\kappa_1, \kappa_2 \in K$ such that $\Psi(\kappa_2,\kappa_1) > 0$ and $h : K = [\kappa_1, \kappa_1 + \Psi(\kappa_2,\kappa_1)] \longrightarrow (0,\infty)$ be an α-differentiable function on $(\kappa_1, \kappa_1 + \Psi(\kappa_2,\kappa_1))$ for $\alpha \in (0,1]$ such that $D_{\alpha}(h) \in L_{\alpha}([\kappa_1, \kappa_1 + \Psi(\kappa_2,\kappa_1)])$. If $|h'|^q$ is preinvex for $q > 1$ and $q^{-1} + p^{-1} = 1$, then we have the following inequality:*

$$\left| \frac{h(\kappa_1) + h(\kappa_1 + \Psi(\kappa_2,\kappa_1))}{2} - \frac{\alpha}{(\kappa_1 + \Psi(\kappa_2,\kappa_1))^{\alpha} - \kappa_1^{\alpha}} \int_{\kappa_1}^{\kappa_1+\Psi(\kappa_2,\kappa_1)} h(x)\, d_{\alpha}x \right| \le \frac{\Psi(\kappa_2,\kappa_1)}{2\left((\kappa_1 + \Psi(\kappa_2,\kappa_1))^{\alpha} - \kappa_1^{\alpha} \right)} \left[(\mathscr{A}_1(\alpha,p))^{1/p} \cdot \left(\frac{|h'(\kappa_1)|^q + |h'(\kappa_2)|^q}{2} \right)^{1/q} + (\mathscr{A}_2(\alpha,p))^{1/p} \cdot \left(\frac{|h'(\kappa_1)|^q + |h'(\kappa_2)|^q}{2} \right)^{1/q} \right], \tag{31}$$

where

$$\mathscr{A}_1(\alpha,p)=\int_0^1\left((\kappa_1+s\Psi(\kappa_2,\kappa_1))^{\alpha}-\kappa_1^{\alpha}\right)^p ds,$$
$$\mathscr{A}_2(\alpha,p)=\int_0^1\left((\kappa_1+\Psi(\kappa_2,\kappa_1))^{\alpha}-(\kappa_1+s\Psi(\kappa_2,\kappa_1))^{\alpha}\right)^p ds. \quad (32)$$

Proof. From Lemma 10, using the property of the modulus and preinvexity of $|h'|^q$, we have

$$\begin{aligned}&\left|\frac{h(\kappa_1)+h(\kappa_1+\Psi(\kappa_2,\kappa_1))}{2}-\frac{\alpha}{(\kappa_1+\Psi(\kappa_2,\kappa_1))^{\alpha}-\kappa_1^{\alpha}}\cdot\int_{\kappa_1}^{\kappa_1+\Psi(\kappa_2,\kappa_1)}h(x)\,d_{\alpha}x\right|\\&\le\frac{\Psi(\kappa_2,\kappa_1)}{2\left((\kappa_1+\Psi(\kappa_2,\kappa_1))^{\alpha}-\kappa_1^{\alpha}\right)}\left[\int_0^1\left((\kappa_1+s\Psi(\kappa_2,\kappa_1))^{\alpha}-\kappa_1^{\alpha}\right)\left|h'(\kappa_1+s\Psi(\kappa_2,\kappa_1))\right|ds\right.\\&\left.+\int_0^1\left((\kappa_1+\Psi(\kappa_2,\kappa_1))^{\alpha}-(\kappa_1+s\Psi(\kappa_2,\kappa_1))^{\alpha}\right)\cdot\left|h'(\kappa_1+s\Psi(\kappa_2,\kappa_1))\right|ds\right].\end{aligned} \quad (33)$$

Now by Hölder's inequality

$$\begin{aligned}&\int_0^1\left((\kappa_1+s\Psi(\kappa_2,\kappa_1))^{\alpha}-\kappa_1^{\alpha}\right)\cdot\left|h'(\kappa_1+s\Psi(\kappa_2,\kappa_1))\right|ds\\&\le\left(\int_0^1\left((\kappa_1+s\Psi(\kappa_2,\kappa_1))^{\alpha}-\kappa_1^{\alpha}\right)^p ds\right)^{1/p}\cdot\left(\int_0^1\left|h'(\kappa_1+s\Psi(\kappa_2,\kappa_1))\right|^q ds\right)^{1/q}\\&\le\left(\int_0^1\left((\kappa_1+s\Psi(\kappa_2,\kappa_1))^{\alpha}-\kappa_1^{\alpha}\right)^p ds\right)^{1/p}\cdot\left(\int_0^1(1-s)\left|h'(\kappa_1)\right|^q+s\left|h'(\kappa_2)\right|^q ds\right)^{1/q}\\&=(\mathscr{A}_1(\alpha,p))^{1/p}\left(\frac{\left|h'(\kappa_1)\right|^q+\left|h'(\kappa_2)\right|^q}{2}\right)^{1/q}.\end{aligned} \quad (34)$$

Similarly, we have

$$\begin{aligned}&\int_0^1\left((\kappa_1+\Psi(\kappa_2,\kappa_1))^{\alpha}-(\kappa_1+s\Psi(\kappa_2,\kappa_1))^{\alpha}\right)\left|h'(\kappa_1+s\Psi(\kappa_2,\kappa_1))\right|ds\le\left(\int_0^1\left((\kappa_1+\Psi(\kappa_2,\kappa_1))^{\alpha}-(\kappa_1+s\Psi(\kappa_2,\kappa_1))^{\alpha}\right)^p ds\right)^{1/p}\\&\cdot\left(\int_0^1\left|h'(\kappa_1+s\Psi(\kappa_2,\kappa_1))\right|^q ds\right)^{1/q}\\&\le\left(\int_0^1\left((\kappa_1+\Psi(\kappa_2,\kappa_1))^{\alpha}-(\kappa_1+s\Psi(\kappa_2,\kappa_1))^{\alpha}\right)^p ds\right)^{1/p}\left(\int_0^1(1-s)\cdot\left|h'(\kappa_1)\right|^q+s\left|h'(\kappa_2)\right|^q ds\right)^{1/q}=(\mathscr{A}_2(\alpha,p))^{1/p}\\&\cdot\left(\frac{\left|h'(\kappa_1)\right|^q+\left|h'(\kappa_2)\right|^q}{2}\right)^{1/q}.\end{aligned} \quad (35)$$

Hence, we have the result in (31). □

Remark 14. If we set $\alpha=1$ in (31), then we have the following inequality:

$$\left|\frac{h(\kappa_1)+h(\kappa_1+\Psi(\kappa_2,\kappa_1))}{2}-\frac{1}{\Psi(\kappa_2,\kappa_1)}\int_{\kappa_1}^{\kappa_1+\Psi(\kappa_2,\kappa_1)}h(x)\,dx\right|\le\frac{\Psi(\kappa_2,\kappa_1)}{(1+p)^{1/p}}\left[\frac{\left|h'(\kappa_1)\right|^q+\left|h'(\kappa_2)\right|^q}{2}\right]^{1/q}. \quad (36)$$

Theorem 15. *Let $\kappa_1,\kappa_2\in K$ such that $\Psi(\kappa_2,\kappa_1)>0$ and $h:K=[\kappa_1,\kappa_1+\Psi(\kappa_2,\kappa_1)]\longrightarrow(0,\infty)$ be an α-differentiable function on $(\kappa_1,\kappa_1+\Psi(\kappa_2,\kappa_1))$ for $\alpha\in(0,1]$ such that $D_{\alpha}(h)\in L_{\alpha}([\kappa_1,\kappa_1+\Psi(\kappa_2,\kappa_1)])$. If $|h'|^q$ is preinvex for $q>1$ and $q^{-1}+p^{-1}=1$, then we have the following inequality:*

$$\begin{aligned}&\left|\frac{h(\kappa_1)+h(\kappa_1+\Psi(\kappa_2,\kappa_1))}{2}-\frac{\alpha}{(\kappa_1+\Psi(\kappa_2,\kappa_1))^{\alpha}-\kappa_1^{\alpha}}\int_{\kappa_1}^{\kappa_1+\Psi(\kappa_2,\kappa_1)}h(x)\,d_{\alpha}x\right|\\&\le\frac{\Psi(\kappa_2,\kappa_1)}{2\left((\kappa_1+\Psi(\kappa_2,\kappa_1))^{\alpha}-\kappa_1^{\alpha}\right)}\left[\ (\mathscr{A}_1(\alpha))^{1-1/q}\cdot\left(\mathscr{A}_2(\alpha)\left|h'(\kappa_1)\right|^q+\mathscr{A}_3(\alpha)\left|h'(\kappa_2)\right|^q\right)^{1/q}\right.\\&+(\mathscr{B}_1(\alpha))^{1-1/q}\left(\mathscr{B}_2(\alpha)\left|h'(\kappa_1)\right|^q+\mathscr{B}_3(\alpha)\left|h'(\kappa_2)\right|^q\right)^{1/q},\end{aligned} \quad (37)$$

where

$$\mathscr{A}_1(\alpha) = \frac{(\kappa_1 + \Psi(\kappa_2,\kappa_1))^{\alpha+1} - \kappa_1^{\alpha+1}}{(\alpha+1)\Psi(\kappa_2,\kappa_1)} - \kappa_1^{\alpha},$$

$$\mathscr{B}_2(\alpha) = (\kappa_1 + \Psi(\kappa_2,\kappa_1))^{\alpha} - \left[\frac{(\kappa_1 + \Psi(\kappa_2,\kappa_1))^{\alpha+1} - \kappa_1^{\alpha+1}}{(\alpha+1)\Psi(\kappa_2,\kappa_1)}\right],$$

$$\begin{aligned}
\mathscr{A}_2(\alpha) &= -\frac{\kappa_1^{\alpha+1}}{(\alpha+1)\Psi(\kappa_2,\kappa_1)}\left[\frac{(\alpha+2)\Psi(\kappa_2,\kappa_1)+\kappa_1}{(\alpha+2)\Psi(\kappa_2,\kappa_1)}\right] + \frac{(\kappa_1 + \Psi(\kappa_2,\kappa_1))^{\alpha+2}}{(\alpha+1)(\Psi(\kappa_2,\kappa_1))^2(\alpha+2)} - \frac{\kappa_1^{\alpha}}{2},\\
\mathscr{A}_3(\alpha) &= \frac{(\kappa_1 + \Psi(\kappa_2,\kappa_1))^{\alpha}}{2} - \frac{(\kappa_1 + \Psi(\kappa_2,\kappa_1))^{\alpha+1}}{(\alpha+1)\Psi(\kappa_2,\kappa_1)}\left[\frac{(\alpha+2)\Psi(\kappa_2,\kappa_1)-(\kappa_1+\Psi(\kappa_2,\kappa_1))}{(\alpha+2)\Psi(\kappa_2,\kappa_1)}\right] - \frac{\kappa_1^{\alpha+2}}{(\alpha+1)(\Psi(\kappa_2,\kappa_1))^2(\alpha+2)},\\
\mathscr{B}_2(\alpha) &= \frac{(\kappa_1 + \Psi(\kappa_2,\kappa_1))^{\alpha}}{2} + \frac{\kappa_1^{\alpha+1}}{(\alpha+1)\Psi(\kappa_2,\kappa_1)}\left[\frac{(\alpha+2)\Psi(\kappa_2,\kappa_1)+\kappa_1}{(\alpha+2)\Psi(\kappa_2,\kappa_1)}\right], - \frac{(\kappa_1 + \Psi(\kappa_2,\kappa_1))^{\alpha+2}}{(\alpha+1)(\Psi(\kappa_2,\kappa_1))^2(\alpha+2)}\\
\mathscr{B}_3(\alpha) &= \frac{(\kappa_1 + \Psi(\kappa_2,\kappa_1))^{\alpha}}{2} - \frac{(\kappa_1 + \Psi(\kappa_2,\kappa_1))^{\alpha+1}}{(\alpha+1)\Psi(\kappa_2,\kappa_1)}\left[\frac{(\alpha+2)\Psi(\kappa_2,\kappa_1)-(\kappa_1+\Psi(\kappa_2,\kappa_1))}{(\alpha+2)\Psi(\kappa_2,\kappa_1)}\right] - \frac{\kappa_1^{\alpha+2}}{(\alpha+1)(\Psi(\kappa_2,\kappa_1))^2(\alpha+2)}.
\end{aligned} \tag{38}$$

Proof. From Lemma 10, using the property of the modulus and preinvexity of $|h'|^q$, we have

$$\begin{aligned}
&\left|\frac{h(\kappa_1) + h(\kappa_1 + \Psi(\kappa_2,\kappa_1))}{2} - \frac{\alpha}{(\kappa_1 + \Psi(\kappa_2,\kappa_1))^{\alpha} - \kappa_1^{\alpha}} \cdot \int_{\kappa_1}^{\kappa_1+\Psi(\kappa_2,\kappa_1)} h(x)\, d_\alpha x\right|\\
&\le \frac{\Psi(\kappa_2,\kappa_1)}{2\left((\kappa_1 + \Psi(\kappa_2,\kappa_1))^{\alpha} - \kappa_1^{\alpha}\right)}\left[\int_0^1 \left((\kappa_1 + s\Psi(\kappa_2,\kappa_1))^{\alpha} - \kappa_1^{\alpha}\right)\left|h'(\kappa_1 + s\Psi(\kappa_2,\kappa_1))\right| ds\right.\\
&\quad \left. + \int_0^1 \left((\kappa_1 + \Psi(\kappa_2,\kappa_1))^{\alpha} - (\kappa_1 + s\Psi(\kappa_2,\kappa_1))^{\alpha}\right) \cdot \left|h'(\kappa_1 + s\Psi(\kappa_2,\kappa_1))\right| ds\right].
\end{aligned} \tag{39}$$

Now by the power-mean inequality

$$\begin{aligned}
&\int_0^1 \left((\kappa_1 + s\Psi(\kappa_2,\kappa_1))^{\alpha} - \kappa_1^{\alpha}\right)\left|h'(\kappa_1 + s\Psi(\kappa_2,\kappa_1))\right| ds\\
&\le \left(\int_0^1 \left((\kappa_1 + s\Psi(\kappa_2,\kappa_1))^{\alpha} - \kappa_1^{\alpha}\right) ds\right)^{1-1/q} \times \left(\int_0^1 \left((\kappa_1 + s\Psi(\kappa_2,\kappa_1))^{\alpha} - \kappa_1^{\alpha}\right) \cdot \left|h'(\kappa_1 + s\Psi(\kappa_2,\kappa_1))\right|^q ds\right)^{1/q},
\end{aligned} \tag{40}$$

and similarly, we have

$$\begin{aligned}
&\int_0^1 \left((\kappa_1 + \Psi(\kappa_2,\kappa_1))^{\alpha} - (\kappa_1 + s\Psi(\kappa_2,\kappa_1))^{\alpha}\right)\left|h'(\kappa_1 + s\Psi(\kappa_2,\kappa_1))\right| ds \le \left(\int_0^1 \left((\kappa_1 + \Psi(\kappa_2,\kappa_1))^{\alpha} - (\kappa_1 + s\Psi(\kappa_2,\kappa_1))^{\alpha}\right) ds\right)^{1-1/q}\\
&\quad \cdot \left(\int_0^1 \left((\kappa_1 + \Psi(\kappa_2,\kappa_1))^{\alpha} - (\kappa_1 + s\Psi(\kappa_2,\kappa_1))^{\alpha}\right) \cdot \left|h'(\kappa_1 + s\Psi(\kappa_2,\kappa_1))\right|^q ds\right)^{1/q}.
\end{aligned} \tag{41}$$

Now by the preinvexity of $|h'|^q$ from above, we have

$$\begin{aligned}
&\int_0^1 \left((\kappa_1 + s\Psi(\kappa_2,\kappa_1))^{\alpha} - \kappa_1^{\alpha}\right)\left[(1-s)\left|h'(\kappa_1)\right|^q + s\left|h'(\kappa_2)\right|^q\right] ds = \left|h'(\kappa_1)\right|^q \int_0^1 \left((\kappa_1 + s\Psi(\kappa_2,\kappa_1))^{\alpha} - \kappa_1^{\alpha}\right)(1-s)\, ds\\
&\quad + \left|h'(\kappa_2)\right|^q \int_0^1 \left((\kappa_1 + s\Psi(\kappa_2,\kappa_1))^{\alpha} - \kappa_1^{\alpha}\right) s ds = \left|h'(\kappa_1)\right|^q \left(-\frac{\kappa_1^{\alpha+1}}{(\alpha+1)\Psi(\kappa_2,\kappa_1)}\left[\frac{(\alpha+2)\Psi(\kappa_2,\kappa_1)+\kappa_1}{(\alpha+2)\Psi(\kappa_2,\kappa_1)}\right]\right.\\
&\quad \left. + \frac{(\kappa_1 + \Psi(\kappa_2,\kappa_1))^{\alpha+2}}{(\alpha+1)(\Psi(\kappa_2,\kappa_1))^2(\alpha+2)} - \frac{\kappa_1^{\alpha}}{2}\right) + \left|h'(\kappa_2)\right|^q \left(\frac{(\kappa_1 + \Psi(\kappa_2,\kappa_1))^{\alpha}}{2}\right.\\
&\quad \left. - \frac{(\kappa_1 + \Psi(\kappa_2,\kappa_1))^{\alpha+1}}{(\alpha+1)\Psi(\kappa_2,\kappa_1)}\left[\frac{(\alpha+2)\Psi(\kappa_2,\kappa_1)-(\kappa_1+\Psi(\kappa_2,\kappa_1))}{(\alpha+2)\Psi(\kappa_2,\kappa_1)}\right] - \frac{\kappa_1^{\alpha+2}}{(\alpha+1)(\Psi(\kappa_2,\kappa_1))^2(\alpha+2)}\right)
\end{aligned} \tag{42}$$

and

$$\begin{aligned}&\int_0^1 \left((\kappa_1+\Psi(\kappa_2,\kappa_1))^\alpha-(\kappa_1+s\Psi(\kappa_2,\kappa_1))^\alpha\right)\\&\cdot\left[(1-s)\left|h'(\kappa_1)\right|^q+s\left|h'(\kappa_2)\right|^q\right]ds=\left|h'(\kappa_1)\right|^q\\&\cdot\int_0^1\left((\kappa_1+\Psi(\kappa_2,\kappa_1))^\alpha-(\kappa_1+s\Psi(\kappa_2,\kappa_1))^\alpha\right)(1-s)\,ds\\&+\left|h'(\kappa_2)\right|^q\int_0^1\left((\kappa_1+\Psi(\kappa_2,\kappa_1))^\alpha-(\kappa_1+s\Psi(\kappa_2,\kappa_1))^\alpha\right)s\,ds\\&=\left|h'(\kappa_1)\right|^q\left(\frac{(\kappa_1+\Psi(\kappa_2,\kappa_1))^\alpha}{2}\right.\\&+\frac{\kappa_1^{\alpha+1}}{(\alpha+1)\Psi(\kappa_2,\kappa_1)}\left[\frac{(\alpha+2)\Psi(\kappa_2,\kappa_1)+\kappa_1}{(\alpha+2)\Psi(\kappa_2,\kappa_1)}\right]\\&\left.-\frac{(\kappa_1+\Psi(\kappa_2,\kappa_1))^{\alpha+2}}{(\alpha+1)(\Psi(\kappa_2,\kappa_1))^2(\alpha+2)}\right)+\left|h'(\kappa_2)\right|^q\\&\cdot\left(\frac{(\kappa_1+\Psi(\kappa_2,\kappa_1))^\alpha}{2}\right.\\&-\frac{(\kappa_1+\Psi(\kappa_2,\kappa_1))^{\alpha+1}}{(\alpha+1)\Psi(\kappa_2,\kappa_1)}\left[\frac{(\alpha+2)\Psi(\kappa_2,\kappa_1)-(\kappa_1+\Psi(\kappa_2,\kappa_1))}{(\alpha+2)\Psi(\kappa_2,\kappa_1)}\right]\\&\left.-\frac{\kappa_1^{\alpha+2}}{(\alpha+1)(\Psi(\kappa_2,\kappa_1))^2(\alpha+2)}\right),\end{aligned}\tag{43}$$

where we have the following:

$$\begin{aligned}\mathscr{A}_1(\alpha)&=\int_0^1\left((\kappa_1+s\Psi(\kappa_2,\kappa_1))^\alpha-\kappa_1^\alpha\right)ds\\&=\frac{(\kappa_1+\Psi(\kappa_2,\kappa_1))^{\alpha+1}-\kappa_1^{\alpha+1}}{(\alpha+1)\Psi(\kappa_2,\kappa_1)}-\kappa_1^\alpha,\\\mathscr{B}_1(\alpha)&\\&=\int_0^1\left((\kappa_1+\Psi(\kappa_2,\kappa_1))^\alpha-(\kappa_1+s\Psi(\kappa_2,\kappa_1))^\alpha\right)ds\\&=(\kappa_1+\Psi(\kappa_2,\kappa_1))^\alpha\\&\quad-\left[\frac{(\kappa_1+\Psi(\kappa_2,\kappa_1))^{\alpha+1}-\kappa_1^{\alpha+1}}{(\alpha+1)\Psi(\kappa_2,\kappa_1)}\right].\end{aligned}\tag{44}$$

Hence, we have the result in (37). □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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