

## Hermite-Hadamard-Fejér Type Inequalities for Harmonically Quasi-convex Functions via Fractional Integrals

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ABSTRACT. In this paper, some Hermite-Hadamard-Fejér type integral inequalities for harmonically quasi-convex functions in fractional integral forms have been obtained.

### 1. Introduction

Let  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a convex function defined on the interval  $I$  of real numbers and  $a, b \in I$  with  $a < b$ . The inequality

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

is well known in the literature as Hermite-Hadamard's inequality [5].

The most well-known inequalities related to the integral mean of a convex function  $f$  are the Hermite Hadamard inequalities or its weighted versions, the so-called Hermite-Hadamard-Fejér inequalities.

In [4], Fejér established the following Fejér inequality which is the weighted generalization of Hermite-Hadamard inequality (1.1):

**Theorem 1.1.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a convex function. Then the inequality*

$$(1.2) \quad f\left(\frac{a+b}{2}\right) \int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2} \int_a^b g(x)dx$$

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holds, where  $g : [a, b] \rightarrow \mathbb{R}$  is nonnegative, integrable and symmetric to  $a + b/2$ .

For some results which generalize, improve and extend the inequalities (1.1) and (1.2) see [1, 6, 7, 16, 18].

We recall the following inequality and special functions which are known as Beta and hypergeometric function respectively:

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1} dt, \quad x, y > 0,$$

$${}_2F_1(a, b; c; z) = \frac{1}{\beta(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt,$$

$$c > b > 0, |z| < 1 \text{ (see [13])}.$$

**Lemma 1.2.** ([15, 20]) For  $0 < \alpha \leq 1$  and  $0 \leq a < b$  we have  $|a^\alpha - b^\alpha| \leq (b-a)^\alpha$ .

The following definitions and mathematical preliminaries of fractional calculus theory are used further in this paper.

**Definition 1.3.** ([13]) Let  $f \in L[a, b]$ . The Riemann-Liouville integrals  $J_{a+}^\alpha f$  and  $J_{b-}^\alpha f$  of order  $\alpha > 0$  with  $a \geq 0$  are defined by

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b$$

respectively, where  $\Gamma(\alpha)$  is the Gamma function defined by  $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$  and  $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$ .

Because of the wide application of Hermite-Hadamard type inequalities and fractional integrals, many researchers extend their studies to Hermite-Hadamard type inequalities involving fractional integrals not limited to integer integrals. Recently, more and more Hermite-Hadamard inequalities involving fractional integrals have been obtained for different classes of functions; see [3, 8, 9, 17, 19, 20].

**Definition 1.4.** ([21]) A function  $f : I \subseteq (0, \infty) \rightarrow [0, \infty)$  is said to be harmonically quasi-convex, if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq \sup\{f(x), f(y)\}$$

for all  $x, y \in I$  and  $t \in [0, 1]$ .

In [11], İşcan defined the so-called harmonically convex functions and established following Hermite-Hadamard type inequality for them as follows:

**Definition 1.5.** Let  $I \subset \mathbb{R} \setminus \{0\}$  be a real interval. A function  $f : I \rightarrow \mathbb{R}$  is said to be harmonically convex, if

$$(1.3) \quad f\left(\frac{xy}{tx + (1-t)y}\right) \leq tf(y) + (1-t)f(x)$$

for all  $x, y \in I$  and  $t \in [0, 1]$ . If the inequality in (1.3) is reversed, then  $f$  is said to be harmonically concave.

**Theorem 1.6.**([11]) Let  $f : I \subset \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$  be a harmonically convex function and  $a, b \in I$  with  $a < b$ . If  $f \in L[a, b]$  then the following inequalities holds:

$$(1.4) \quad f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \leq \frac{f(a) + f(b)}{2}.$$

In [10], İşcan and Wu presented a Hermite-Hadamard type inequality for harmonically convex functions in fractional integral forms as follows:

**Theorem 1.7.** Let  $f : I \subset (0, \infty) \rightarrow \mathbb{R}$  be a function such that  $f \in L[a, b]$ , where  $a, b \in I$  with  $a < b$ . If  $f$  is a harmonically convex function on  $[a, b]$ , then the following inequalities for fractional integrals holds:

$$(1.5) \quad f\left(\frac{2ab}{a+b}\right) \leq \frac{\Gamma(\alpha+1)}{2} \left(\frac{ab}{b-a}\right)^\alpha \left\{ \begin{array}{l} J_{1/a-}^\alpha (f \circ h)(1/b) \\ + J_{1/b+}^\alpha (f \circ h)(1/a) \end{array} \right\} \\ \leq \frac{f(a) + f(b)}{2}$$

with  $\alpha > 0$  and  $h(x) = 1/x$ .

In [14] Latif et al. gave the following definition:

**Definition 1.8.** A function  $g : [a, b] \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$  is said to be harmonically symmetric with respect to  $2ab/a + b$ , if

$$g(x) = g\left(\frac{1}{\frac{1}{a} + \frac{1}{b} - \frac{1}{x}}\right)$$

holds for all  $x \in [a, b]$ .

In [2] Chan and Wu presented a Hermite-Hadamard-Fejér inequality for harmonically convex functions as follows:

**Theorem 1.9.** Let  $f : I \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$  be a harmonically convex function and  $a, b \in I$  with  $a < b$ . If  $f \in L[a, b]$  and  $g : [a, b] \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$  is nonnegative, integrable and harmonically symmetric with respect to  $2ab/a + b$ , then

$$(1.6) \quad f\left(\frac{2ab}{a+b}\right) \int_a^b \frac{g(x)}{x^2} dx \leq \int_a^b \frac{f(x)g(x)}{x^2} dx \\ \leq \frac{f(a) + f(b)}{2} \int_a^b \frac{g(x)}{x^2} dx.$$

In [12] İşcan and Kunt presented a Hermite–Hadamard–Fejér type inequality for harmonically convex functions in fractional integral forms and established following identity as follows:

**Theorem 1.10.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a harmonically convex function with  $a < b$  and  $f \in L[a, b]$ . If  $g : [a, b] \rightarrow \mathbb{R}$  is nonnegative, integrable and harmonically symmetric with respect to  $2ab/a+b$ , then the following inequalities for fractional integrals holds:*

$$\begin{aligned}
 (1.7) \quad & f\left(\frac{2ab}{a+b}\right) \left[ J_{1/b+}^{\alpha}(g \circ h)(1/a) + J_{1/a-}^{\alpha}(g \circ h)(1/b) \right] \\
 & \leq \left[ J_{1/b+}^{\alpha}(fg \circ h)(1/a) + J_{1/a-}^{\alpha}(fg \circ h)(1/b) \right] \\
 & \leq \frac{f(a) + f(b)}{2} \left[ J_{1/b+}^{\alpha}(g \circ h)(1/a) + J_{1/a-}^{\alpha}(g \circ h)(1/b) \right]
 \end{aligned}$$

with  $\alpha > 0$  and  $h(x) = 1/x$ ,  $x \in [\frac{1}{b}, \frac{1}{a}]$ .

**Lemma 1.11.**([12]) *Let  $f : I \subset (0, \infty) \rightarrow \mathbb{R}$  be a differentiable function on  $I^{\circ}$  such that  $f' \in L[a, b]$ , where  $a, b \in I$  and  $a < b$ . If  $g : [a, b] \rightarrow \mathbb{R}$  is integrable and harmonically symmetric with respect to  $2ab/a + b$ , then the following equality for fractional integrals holds:*

$$\begin{aligned}
 (1.8) \quad & \frac{f(a) + f(b)}{2} \left[ J_{1/b+}^{\alpha}(g \circ h)(1/a) + J_{1/a-}^{\alpha}(g \circ h)(1/b) \right] \\
 & - \left[ J_{1/b+}^{\alpha}(fg \circ h)(1/a) + J_{1/a-}^{\alpha}(fg \circ h)(1/b) \right] \\
 & = \frac{1}{\Gamma(\alpha)} \int_{\frac{1}{b}}^{\frac{1}{a}} \left[ \int_{\frac{1}{b}}^t \left(\frac{1}{a} - s\right)^{\alpha-1} (g \circ h)(s) ds \right. \\
 & \quad \left. - \int_t^{\frac{1}{a}} \left(s - \frac{1}{b}\right)^{\alpha-1} (g \circ h)(s) ds \right] (f \circ h)'(t) dt
 \end{aligned}$$

with  $\alpha > 0$  and  $h(x) = 1/x$ ,  $x \in [\frac{1}{b}, \frac{1}{a}]$ .

In this paper, we give some new inequalities connected with the right-hand side of Hermite–Hadamard–Fejér type integral inequality for harmonically quasi-convex function in fractional integrals.

## 2. Main Results

Throughout this section, we write  $\|g\|_{\infty} = \sup_{t \in [a, b]} |g(t)|$ , for the continuous function  $g : [a, b] \rightarrow \mathbb{R}$ .

**Theorem 2.1.** *Let  $f : I \subset (0, \infty) \rightarrow \mathbb{R}$  be a differentiable function on  $I^{\circ}$  such that  $f' \in L[a, b]$ , where  $a, b \in I$  and  $a < b$ . If  $|f'|$  is harmonically quasi-convex on  $[a, b]$ ,  $g : [a, b] \rightarrow \mathbb{R}$  is continuous and harmonically symmetric with respect to  $2ab/a + b$ ,*

then the following inequality for fractional integrals holds:

$$(2.1) \quad \left| \frac{f(a)+f(b)}{2} \left[ J_{1/b+}^\alpha (g \circ h) (1/a) + J_{1/a-}^\alpha (g \circ h) (1/b) \right] - \left[ J_{1/b+}^\alpha (fg \circ h) (1/a) + J_{1/a-}^\alpha (fg \circ h) (1/b) \right] \right| \leq \frac{\|g\|_\infty ab(b-a)}{\Gamma(\alpha+1)} \left( \frac{b-a}{ab} \right)^\alpha C_1(\alpha) \sup \{|f'(a)|, |f'(b)|\}$$

where

$$C_1(\alpha) = \left[ \begin{array}{c} \frac{b^{-2}}{\alpha+1} {}_2F_1(2, 1; \alpha+2; 1 - \frac{a}{b}) \\ -\frac{b^{-2}}{\alpha+1} {}_2F_1(2, \alpha+1; \alpha+2; 1 - \frac{a}{b}) \\ +\frac{4(a+b)^{-2}}{\alpha+1} {}_2F_1(2, \alpha+1; \alpha+2; \frac{b-a}{b+a}) \end{array} \right]$$

with  $0 < \alpha \leq 1$  and  $h(x) = 1/x, x \in [\frac{1}{b}, \frac{1}{a}]$ .

*Proof.* From Lemma 1.11 we have

$$(2.2) \quad \left| \frac{f(a)+f(b)}{2} \left[ J_{1/b+}^\alpha (g \circ h) (1/a) + J_{1/a-}^\alpha (g \circ h) (1/b) \right] - \left[ J_{1/b+}^\alpha (fg \circ h) (1/a) + J_{1/a-}^\alpha (fg \circ h) (1/b) \right] \right| \leq \frac{1}{\Gamma(\alpha)} \int_{\frac{1}{b}}^{\frac{1}{a}} \left| \int_{\frac{1}{b}}^t \left( \frac{1}{a} - s \right)^{\alpha-1} (g \circ h) (s) ds - \int_t^{\frac{1}{a}} \left( s - \frac{1}{b} \right)^{\alpha-1} (g \circ h) (s) ds \right| |(f \circ h)'(t)| dt.$$

Since  $g$  is harmonically symmetric with respect to  $2ab/a + b$ , using Definition 1.8 we have  $g(\frac{1}{x}) = g(\frac{1}{(\frac{1}{a})+(\frac{1}{b})-x})$  for all  $x \in [\frac{1}{b}, \frac{1}{a}]$ .

$$(2.3) \quad \left| \int_{\frac{1}{b}}^t \left( \frac{1}{a} - s \right)^{\alpha-1} (g \circ h) (s) ds - \int_t^{\frac{1}{a}} \left( s - \frac{1}{b} \right)^{\alpha-1} (g \circ h) (s) ds \right| = \left| \int_{\frac{1}{a}+\frac{1}{b}-t}^{\frac{1}{a}} \left( s - \frac{1}{b} \right)^{\alpha-1} (g \circ h) (s) ds + \int_{\frac{1}{a}}^t \left( s - \frac{1}{b} \right)^{\alpha-1} (g \circ h) (s) ds \right| = \left| \int_{\frac{1}{a}+\frac{1}{b}-t}^t \left( s - \frac{1}{b} \right)^{\alpha-1} (g \circ h) (s) ds \right| \leq \begin{cases} \int_t^{\frac{1}{a}+\frac{1}{b}-t} \left| \left( s - \frac{1}{b} \right)^{\alpha-1} (g \circ h) (s) \right| ds & , t \in [\frac{1}{b}, \frac{a+b}{2ab}] \\ \int_{\frac{1}{a}+\frac{1}{b}-t}^t \left| \left( s - \frac{1}{b} \right)^{\alpha-1} (g \circ h) (s) \right| ds & , t \in [\frac{a+b}{2ab}, \frac{1}{a}] \end{cases}.$$

If we use (2.3) in (2.2), we have

$$\begin{aligned}
& \left| \frac{f(a)+f(b)}{2} \left[ J_{1/b+}^{\alpha} (g \circ h) (1/a) + J_{1/a-}^{\alpha} (g \circ h) (1/b) \right] \right. \\
& \quad \left. - \left[ J_{1/b+}^{\alpha} (fg \circ h) (1/a) + J_{1/a-}^{\alpha} (fg \circ h) (1/b) \right] \right| \\
& \leq \frac{1}{\Gamma(\alpha)} \left[ \int_{\frac{1}{b}}^{\frac{a+b}{2ab}} \left( \int_t^{\frac{1}{a}+\frac{1}{b}-t} \left| \left( s - \frac{1}{b} \right)^{\alpha-1} (g \circ h) (s) \right| ds \right) |(f \circ h)'(t)| dt \right. \\
& \quad \left. + \int_{\frac{a+b}{2ab}}^{\frac{1}{a}} \left( \int_{\frac{1}{a}+\frac{1}{b}-t}^t \left| \left( s - \frac{1}{b} \right)^{\alpha-1} (g \circ h) (s) \right| ds \right) |(f \circ h)'(t)| dt \right] \\
& \leq \frac{\|g\|_{\infty}}{\Gamma(\alpha)} \left[ \int_{\frac{1}{b}}^{\frac{a+b}{2ab}} \left( \int_t^{\frac{1}{a}+\frac{1}{b}-t} \left( s - \frac{1}{b} \right)^{\alpha-1} ds \right) |(f \circ h)'(t)| dt \right. \\
& \quad \left. + \int_{\frac{a+b}{2ab}}^{\frac{1}{a}} \left( \int_{\frac{1}{a}+\frac{1}{b}-t}^t \left( s - \frac{1}{b} \right)^{\alpha-1} ds \right) |(f \circ h)'(t)| dt \right] \\
& \leq \frac{\|g\|_{\infty}}{\Gamma(\alpha)} \left[ \int_{\frac{1}{b}}^{\frac{a+b}{2ab}} \left( \int_t^{\frac{1}{a}+\frac{1}{b}-t} \left( s - \frac{1}{b} \right)^{\alpha-1} ds \right) \frac{1}{t^2} |f'(\frac{1}{t})| dt \right. \\
& \quad \left. + \int_{\frac{a+b}{2ab}}^{\frac{1}{a}} \left( \int_{\frac{1}{a}+\frac{1}{b}-t}^t \left( s - \frac{1}{b} \right)^{\alpha-1} ds \right) \frac{1}{t^2} |f'(\frac{1}{t})| dt \right].
\end{aligned}$$

Setting  $t = \frac{ub+(1-u)a}{ab}$  and  $dt = \left(\frac{b-a}{ab}\right) du$  gives

$$\begin{aligned}
(2.4) \quad & \left| \frac{f(a)+f(b)}{2} \left[ J_{1/b+}^{\alpha} (g \circ h) (1/a) + J_{1/a-}^{\alpha} (g \circ h) (1/b) \right] \right. \\
& \quad \left. - \left[ J_{1/b+}^{\alpha} (fg \circ h) (1/a) + J_{1/a-}^{\alpha} (fg \circ h) (1/b) \right] \right| \\
& \leq \frac{\|g\|_{\infty} ab(b-a)}{\Gamma(\alpha+1)} \left( \frac{b-a}{ab} \right)^{\alpha} \left[ \int_0^{\frac{1}{2}} \frac{(1-u)^{\alpha} - u^{\alpha}}{(ub+(1-u)a)^2} \left| f' \left( \frac{ab}{ub+(1-u)a} \right) \right| du \right. \\
& \quad \left. + \int_{\frac{1}{2}}^1 \frac{u^{\alpha} - (1-u)^{\alpha}}{(ub+(1-u)a)^2} \left| f' \left( \frac{ab}{ub+(1-u)a} \right) \right| du \right].
\end{aligned}$$

Since  $|f'|$  is harmonically quasi-convex on  $[a, b]$ , we have

$$(2.5) \quad \left| f' \left( \frac{ab}{ub+(1-u)a} \right) \right| \leq \sup \{ |f'(a)|, |f'(b)| \}.$$

If we use (2.5) in (2.4), we have

$$\begin{aligned}
(2.6) \quad & \left| \frac{f(a)+f(b)}{2} \left[ J_{1/b+}^{\alpha} (g \circ h) (1/a) + J_{1/a-}^{\alpha} (g \circ h) (1/b) \right] \right. \\
& \quad \left. - \left[ J_{1/b+}^{\alpha} (fg \circ h) (1/a) + J_{1/a-}^{\alpha} (fg \circ h) (1/b) \right] \right| \\
& \leq \frac{\|g\|_{\infty} ab(b-a)}{\Gamma(\alpha+1)} \left( \frac{b-a}{ab} \right)^{\alpha} \sup \{ |f'(a)|, |f'(b)| \} \\
& \quad \times \left[ \int_0^{\frac{1}{2}} \frac{(1-u)^{\alpha} - u^{\alpha}}{(ub+(1-u)a)^2} du + \int_{\frac{1}{2}}^1 \frac{u^{\alpha} - (1-u)^{\alpha}}{(ub+(1-u)a)^2} du \right].
\end{aligned}$$

Using Lemma 1.2, we have

$$\begin{aligned}
 (2.7) \quad & \int_0^{\frac{1}{2}} \frac{(1-u)^\alpha - u^\alpha}{(ub + (1-u)a)^2} du + \int_{\frac{1}{2}}^1 \frac{u^\alpha - (1-u)^\alpha}{(ub + (1-u)a)^2} du \\
 &= \int_0^1 \frac{u^\alpha - (1-u)^\alpha}{(ub + (1-u)a)^2} du + 2 \int_0^{\frac{1}{2}} \frac{(1-u)^\alpha - u^\alpha}{(ub + (1-u)a)^2} du \\
 &= \int_0^1 \frac{u^\alpha}{(ub + (1-u)a)^2} du - \int_0^1 \frac{(1-u)^\alpha}{(ub + (1-u)a)^2} du + 2 \int_0^{\frac{1}{2}} \frac{(1-u)^\alpha - u^\alpha}{(ub + (1-u)a)^2} du \\
 &\leq \int_0^1 \frac{u^\alpha}{(ub + (1-u)a)^2} du - \int_0^1 \frac{(1-u)^\alpha}{(ub + (1-u)a)^2} du + 2 \int_0^{\frac{1}{2}} \frac{(1-2u)^\alpha}{(ub + (1-u)a)^2} du.
 \end{aligned}$$

Calculating the following integrals, we have

$$\begin{aligned}
 (2.8) \quad & \int_0^1 \frac{u^\alpha}{(ub + (1-u)a)^2} du - \int_0^1 \frac{(1-u)^\alpha}{(ub + (1-u)a)^2} du + 2 \int_0^{\frac{1}{2}} \frac{(1-2u)^\alpha}{(ub + (1-u)a)^2} du \\
 &= \int_0^1 \frac{(1-u)^\alpha}{(ua + (1-u)b)^2} du - \int_0^1 \frac{u^\alpha}{(ua + (1-u)b)^2} du + \int_0^1 \frac{(1-u)^\alpha}{(\frac{u}{2}b + (1-\frac{u}{2})a)^2} du \\
 &= \int_0^1 (1-u)^\alpha b^{-2} \left(1 - u \left(1 - \frac{a}{b}\right)\right)^{-2} du - \int_0^1 u^\alpha b^{-2} \left(1 - u \left(1 - \frac{a}{b}\right)\right)^{-2} du \\
 &\quad + \int_0^1 v^\alpha \left(\frac{a+b}{2}\right)^{-2} \left(1 - v \left(\frac{b-a}{b+a}\right)\right)^{-2} dv \\
 &= \left[ \begin{array}{l} \frac{b^{-2}}{\alpha+1} {}_2F_1\left(2, 1; \alpha+2; 1 - \frac{a}{b}\right) \\ -\frac{b^{-2}}{\alpha+1} {}_2F_1\left(2, \alpha+1; \alpha+2; 1 - \frac{a}{b}\right) \\ +\frac{4(a+b)^{-2}}{\alpha+1} {}_2F_1\left(2, \alpha+1; \alpha+2; \frac{b-a}{b+a}\right) \end{array} \right] \\
 &= C_1(\alpha).
 \end{aligned}$$

If we use (2.7) and (2.8) in (2.6), we have (2.1). This completes the proof.  $\square$

**Corollary 2.2.** *In Theorem 2.1:*

(1) *If we take  $\alpha = 1$  we have the following Hermite-Hadamard-Fejér inequality for harmonically quasi-convex functions which is related to the right-hand side of (1.6):*

$$\begin{aligned}
 & \left| \frac{f(a) + f(b)}{2} \int_a^b \frac{g(x)}{x^2} dx - \int_a^b \frac{f(x)g(x)}{x^2} dx \right| \\
 & \leq \frac{\|g\|_\infty (b-a)^2}{2} C_1(1) \sup\{|f'(a)|, |f'(b)|\},
 \end{aligned}$$

(2) *If we take  $g(x) = 1$  we have following Hermite-Hadamard type inequality for harmonically quasi-convex functions in fractional integral forms which is related*

to the right-hand side of (1.5):

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2} \left( \frac{ab}{b-a} \right)^\alpha \left\{ \begin{array}{l} J_{1/a-}^\alpha (f \circ h)(1/b) \\ + J_{1/b+}^\alpha (f \circ h)(1/a) \end{array} \right\} \right| \\ & \leq \frac{ab(b-a)}{2} C_1(\alpha) \sup \{|f'(a)|, |f'(b)|\}, \end{aligned}$$

(3) If we take  $\alpha = 1$  and  $g(x) = 1$  we have the following Hermite-Hadamard type inequality for harmonically quasi-convex functions which is related to the right-hand side of (1.4):

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \\ & \leq \frac{ab(b-a)}{2} C_1(1) \sup \{|f'(a)|, |f'(b)|\}. \end{aligned}$$

**Theorem 2.3.** Let  $f : I \subset (0, \infty) \rightarrow \mathbb{R}$  be a differentiable function on  $I^\circ$  such that  $f' \in L[a, b]$ , where  $a, b \in I$  and  $a < b$ . If  $|f'|^q, q \geq 1$ , is harmonically quasi-convex on  $[a, b]$ ,  $g : [a, b] \rightarrow \mathbb{R}$  is continuous and harmonically symmetric with respect to  $2ab/a + b$ , then the following inequality for fractional integrals holds:

$$\begin{aligned} (2.9) \quad & \left| \begin{array}{l} \frac{f(a)+f(b)}{2} \left[ J_{1/b+}^\alpha (g \circ h)(1/a) + J_{1/a-}^\alpha (g \circ h)(1/b) \right] \\ - \left[ J_{1/b+}^\alpha (fg \circ h)(1/a) + J_{1/a-}^\alpha (fg \circ h)(1/b) \right] \end{array} \right| \\ & \leq \frac{\|g\|_\infty ab(b-a)}{\Gamma(\alpha + 1)} \left( \frac{b-a}{ab} \right)^\alpha C_2(\alpha) [\sup \{|f'(a)|^q, |f'(b)|^q\}]^{\frac{1}{q}} \end{aligned}$$

where

$$\begin{aligned} C_2(\alpha) &= \frac{b^{-2}}{\alpha + 1} {}_2F_1 \left( 2, 1; \alpha + 2; \frac{b-a}{b+a} \right) - \frac{b^{-2}}{\alpha + 1} {}_2F_1 \left( 2, \alpha + 1; \alpha + 2; \frac{b-a}{b+a} \right) \\ &+ \frac{4(a+b)^{-2}}{(\alpha + 1)} {}_2F_1 \left( 2, \alpha + 1; \alpha + 2; \frac{b-a}{b+a} \right), \end{aligned}$$

with  $0 < \alpha \leq 1$  and  $h(x) = 1/x, x \in [\frac{1}{b}, \frac{1}{a}]$ .

*Proof.* Using (2.4), power mean inequality and the harmonically quasi-convexity of



$|f'|^q$ , it follows that

$$\begin{aligned}
 (2.10) \quad & \left| \frac{f(a)+f(b)}{2} \left[ J_{1/b+}^\alpha (g \circ h) (1/a) + J_{1/a-}^\alpha (g \circ h) (1/b) \right] \right. \\
 & \left. - \left[ J_{1/b+}^\alpha (fg \circ h) (1/a) + J_{1/a-}^\alpha (fg \circ h) (1/b) \right] \right| \\
 & \leq \frac{\|g\|_\infty ab (b-a)}{\Gamma(\alpha+1)} \left( \frac{b-a}{ab} \right)^\alpha \left[ \int_0^{\frac{1}{2}} \frac{(1-u)^\alpha - u^\alpha}{(ub+(1-u)a)^2} \left| f' \left( \frac{ab}{ub+(1-u)a} \right) \right| du \right. \\
 & \quad \left. + \int_{\frac{1}{2}}^1 \frac{u^\alpha - (1-u)^\alpha}{(ub+(1-u)a)^2} \left| f' \left( \frac{ab}{ub+(1-u)a} \right) \right| du \right] \\
 & \leq \frac{\|g\|_\infty ab (b-a)}{\Gamma(\alpha+1)} \left( \frac{b-a}{ab} \right)^\alpha \left[ \left( \int_0^{\frac{1}{2}} \frac{(1-u)^\alpha - u^\alpha}{(ub+(1-u)a)^2} du \right)^{1-\frac{1}{q}} \right. \\
 & \quad \left. \times \left( \int_0^{\frac{1}{2}} \frac{(1-u)^\alpha - u^\alpha}{(ub+(1-u)a)^2} \left| f' \left( \frac{ab}{ub+(1-u)a} \right) \right|^q du \right)^{\frac{1}{q}} \right. \\
 & \quad \left. + \left( \int_{\frac{1}{2}}^1 \frac{u^\alpha - (1-u)^\alpha}{(ub+(1-u)a)^2} du \right)^{1-\frac{1}{q}} \right. \\
 & \quad \left. \times \left( \int_{\frac{1}{2}}^1 \frac{u^\alpha - (1-u)^\alpha}{(ub+(1-u)a)^2} \left| f' \left( \frac{ab}{ub+(1-u)a} \right) \right|^q du \right)^{\frac{1}{q}} \right] \\
 & \leq \frac{\|g\|_\infty ab (b-a)}{\Gamma(\alpha+1)} \left( \frac{b-a}{ab} \right)^\alpha \\
 & \quad \times \left[ \left( \int_0^{\frac{1}{2}} \frac{(1-u)^\alpha - u^\alpha}{(ub+(1-u)a)^2} du \right)^{1-\frac{1}{q}} \right. \\
 & \quad \left. \times \left( \int_0^{\frac{1}{2}} \frac{(1-u)^\alpha - u^\alpha}{(ub+(1-u)a)^2} \sup \{ |f'(a)|^q, |f'(b)|^q \} du \right)^{\frac{1}{q}} \right. \\
 & \quad \left. + \left( \int_{\frac{1}{2}}^1 \frac{u^\alpha - (1-u)^\alpha}{(ub+(1-u)a)^2} du \right)^{1-\frac{1}{q}} \right. \\
 & \quad \left. \times \left( \int_{\frac{1}{2}}^1 \frac{u^\alpha - (1-u)^\alpha}{(ub+(1-u)a)^2} \sup \{ |f'(a)|^q, |f'(b)|^q \} du \right)^{\frac{1}{q}} \right] \\
 & \leq \frac{\|g\|_\infty ab (b-a)}{\Gamma(\alpha+1)} \left( \frac{b-a}{ab} \right)^\alpha \left[ \sup \{ |f'(a)|^q, |f'(b)|^q \} \right]^{\frac{1}{q}} \\
 & \quad \times \left[ \int_0^{\frac{1}{2}} \frac{(1-u)^\alpha - u^\alpha}{(ub+(1-u)a)^2} du + \int_{\frac{1}{2}}^1 \frac{u^\alpha - (1-u)^\alpha}{(ub+(1-u)a)^2} du \right].
 \end{aligned}$$

Using Lemma 1.2, we have

$$\begin{aligned}
 (2.11) \quad & \int_0^{\frac{1}{2}} \frac{(1-u)^\alpha - u^\alpha}{(ub+(1-u)a)^2} du + \int_{\frac{1}{2}}^1 \frac{u^\alpha - (1-u)^\alpha}{(ub+(1-u)a)^2} du \\
 & = \int_0^1 \frac{u^\alpha - (1-u)^\alpha}{(ub+(1-u)a)^2} du + 2 \int_0^{\frac{1}{2}} \frac{(1-u)^\alpha - u^\alpha}{(ub+(1-u)a)^2} du \\
 & = \int_0^1 \frac{u^\alpha}{(ub+(1-u)a)^2} du - \int_0^1 \frac{(1-u)^\alpha}{(ub+(1-u)a)^2} du + 2 \int_0^{\frac{1}{2}} \frac{(1-u)^\alpha - u^\alpha}{(ub+(1-u)a)^2} du \\
 & \leq \int_0^1 \frac{u^\alpha}{(ub+(1-u)a)^2} du - \int_0^1 \frac{(1-u)^\alpha}{(ub+(1-u)a)^2} du + 2 \int_0^{\frac{1}{2}} \frac{(1-2u)^\alpha}{(ub+(1-u)a)^2} du.
 \end{aligned}$$

For the appearing integrals, we have

$$\begin{aligned}
 (2.12) \quad & \int_0^1 \frac{u^\alpha}{(ub + (1-u)a)^2} du - \int_0^1 \frac{(1-u)^\alpha}{(ub + (1-u)a)^2} du + 2 \int_0^{\frac{1}{2}} \frac{(1-2u)^\alpha}{(ub + (1-u)a)^2} du \\
 &= \int_0^1 (1-u)^\alpha b^{-2} \left(1 - u \left(1 - \frac{a}{b}\right)\right)^{-2} du - \int_0^1 u^\alpha b^{-2} \left(1 - u \left(1 - \frac{a}{b}\right)\right)^{-2} du \\
 &\quad + \int_0^1 \frac{(1-u)^\alpha}{\left(\frac{u}{2}b + \left(1 - \frac{u}{2}\right)a\right)^2} du \\
 &= \int_0^1 (1-u)^\alpha b^{-2} \left(1 - u \left(1 - \frac{a}{b}\right)\right)^{-2} du - \int_0^1 u^\alpha b^{-2} \left(1 - u \left(1 - \frac{a}{b}\right)\right)^{-2} du \\
 &\quad + \int_0^1 v^\alpha \left(\frac{a+b}{2}\right)^{-2} \left(1 - v \left(\frac{b-a}{b+a}\right)\right)^{-2} dv \\
 &= \frac{b^{-2}}{\alpha+1} {}_2F_1\left(2, 1; \alpha+2; \frac{b-a}{b+a}\right) - \frac{b^{-2}}{\alpha+1} {}_2F_1\left(2, \alpha+1; \alpha+2; \frac{b-a}{b+a}\right) \\
 &\quad + \frac{4(a+b)^{-2}}{(\alpha+1)} {}_2F_1\left(2, \alpha+1; \alpha+2; \frac{b-a}{b+a}\right) \\
 &= C_2(\alpha).
 \end{aligned}$$

If we use (2.11) and (2.12) in (2.10), we have (2.9). This completes the proof.  $\square$

**Corollary 2.4.** *In Theorem 2.3:*

(1) *If we take  $\alpha = 1$  we have the following Hermite-Hadamard-Fejér inequality for harmonically quasi-convex functions which is related to the right-hand side of (1.6):*

$$\begin{aligned}
 & \left| \frac{f(a) + f(b)}{2} \int_a^b \frac{g(x)}{x^2} dx - \int_a^b \frac{f(x)g(x)}{x^2} dx \right| \\
 & \leq \frac{\|g\|_\infty (b-a)^2}{2} C_2(1) [\sup\{|f'(a)|^q, |f'(b)|^q\}]^{\frac{1}{q}},
 \end{aligned}$$

(2) *If we take  $g(x) = 1$  we have following Hermite-Hadamard type inequality for harmonically quasi-convex functions in fractional integral forms which is related to the right-hand side of (1.5):*

$$\begin{aligned}
 & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2} \left(\frac{ab}{b-a}\right)^\alpha \left\{ \begin{array}{l} J_{1/a-}^\alpha (f \circ h)(1/b) \\ + J_{1/b+}^\alpha (f \circ h)(1/a) \end{array} \right\} \right| \\
 & \leq \frac{ab(b-a)}{2} C_2(\alpha) [\sup\{|f'(a)|^q, |f'(b)|^q\}]^{\frac{1}{q}},
 \end{aligned}$$

(3) If we take  $\alpha = 1$  and  $g(x) = 1$  we have the following Hermite-Hadamard type inequality for harmonically quasi-convex functions which is related to the right-hand side of (1.4):

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \\ & \leq \frac{ab(b-a)}{2} C_2(1) [\sup \{|f'(a)|^q, |f'(b)|^q\}]^{\frac{1}{q}}. \end{aligned}$$

We can state another inequality for  $q > 1$  as follows:

**Theorem 2.5.** Let  $f : I \subset (0, \infty) \rightarrow \mathbb{R}$  be a differentiable function on  $I^\circ$  such that  $f' \in L[a, b]$ , where  $a, b \in I$  and  $a < b$ . If  $|f'|^q, q > 1$ , is harmonically quasi-convex on  $[a, b]$ ,  $g : [a, b] \rightarrow \mathbb{R}$  is continuous and harmonically symmetric with respect to  $2ab/a + b$ , then the following inequality for fractional integrals holds:

$$\begin{aligned} (2.13) \quad & \left| \frac{f(a)+f(b)}{2} \left[ J_{1/b+}^\alpha (g \circ h)(1/a) + J_{1/a-}^\alpha (g \circ h)(1/b) \right] \right. \\ & \left. - \left[ J_{1/b+}^\alpha (fg \circ h)(1/a) + J_{1/a-}^\alpha (fg \circ h)(1/b) \right] \right| \\ & \leq \frac{\|g\|_\infty ab(b-a)}{2^{\frac{1}{q}} \Gamma(\alpha+1)} \left( \frac{b-a}{ab} \right)^\alpha \\ & \quad \times (\sup \{|f'(a)|^q, |f'(b)|^q\})^{\frac{1}{q}} \left[ C_3^{\frac{1}{p}}(\alpha) + C_4^{\frac{1}{p}}(\alpha) \right] \end{aligned}$$

where

$$\begin{aligned} C_3(\alpha) &= \left( \frac{a+b}{2} \right)^{-2p} \frac{1}{2(\alpha p + 1)} {}_2F_1 \left( 2p, \alpha p + 1; \alpha p + 2; \frac{b-a}{b+a} \right), \\ C_4(\alpha) &= b^{-2p} \frac{1}{2(\alpha p + 1)} {}_2F_1 \left( 2p, 1; \alpha p + 2; \frac{1}{2} \left( 1 - \frac{a}{b} \right) \right), \end{aligned}$$

with  $0 < \alpha \leq 1, h(x) = 1/x, x \in [\frac{1}{b}, \frac{1}{a}]$  and  $1/p + 1/q = 1$ .

*Proof.* Using (2.4), Hölder’s inequality and the harmonically quasi-convexity of  $|f'|^q$ , it follows that

$$\begin{aligned}
(2.14) \quad & \left| \frac{f(a)+f(b)}{2} \left[ J_{1/b+}^{\alpha} (g \circ h) (1/a) + J_{1/a-}^{\alpha} (g \circ h) (1/b) \right] \right. \\
& \left. - \left[ J_{1/b+}^{\alpha} (fg \circ h) (1/a) + J_{1/a-}^{\alpha} (fg \circ h) (1/b) \right] \right| \\
\leq & \frac{\|g\|_{\infty} ab(b-a)}{\Gamma(\alpha+1)} \left( \frac{b-a}{ab} \right)^{\alpha} \left[ \int_0^{\frac{1}{2}} \frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2} \left| f' \left( \frac{ab}{ub+(1-u)a} \right) \right| du \right. \\
& \left. + \int_{\frac{1}{2}}^1 \frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2} \left| f' \left( \frac{ab}{ub+(1-u)a} \right) \right| du \right] \\
\leq & \frac{\|g\|_{\infty} ab(b-a)}{\Gamma(\alpha+1)} \left( \frac{b-a}{ab} \right)^{\alpha} \left[ \left( \int_0^{\frac{1}{2}} \frac{[(1-u)^{\alpha}-u^{\alpha}]^p}{(ub+(1-u)a)^{2p}} du \right)^{\frac{1}{p}} \right. \\
& \left. \times \left( \int_0^{\frac{1}{2}} \left| f' \left( \frac{ab}{ub+(1-u)a} \right) \right|^q du \right)^{\frac{1}{q}} \right. \\
& \left. + \left( \int_{\frac{1}{2}}^1 \frac{[u^{\alpha}-(1-u)^{\alpha}]^p}{(ub+(1-u)a)^{2p}} du \right)^{\frac{1}{p}} \right. \\
& \left. \times \left( \int_{\frac{1}{2}}^1 \left| f' \left( \frac{ab}{ub+(1-u)a} \right) \right|^q du \right)^{\frac{1}{q}} \right] \\
\leq & \frac{\|g\|_{\infty} ab(b-a)}{\Gamma(\alpha+1)} \left( \frac{b-a}{ab} \right)^{\alpha} \\
& \times \left[ \left( \int_0^{\frac{1}{2}} \frac{[(1-u)^{\alpha}-u^{\alpha}]^p}{(ub+(1-u)a)^{2p}} du \right)^{\frac{1}{p}} \left( \int_0^{\frac{1}{2}} \sup \{ |f'(a)|^q, |f'(b)|^q \} du \right)^{\frac{1}{q}} \right. \\
& \left. + \left( \int_{\frac{1}{2}}^1 \frac{[u^{\alpha}-(1-u)^{\alpha}]^p}{(ub+(1-u)a)^{2p}} du \right)^{\frac{1}{p}} \left( \int_{\frac{1}{2}}^1 \sup \{ |f'(a)|^q, |f'(b)|^q \} du \right)^{\frac{1}{q}} \right] \\
\leq & \frac{\|g\|_{\infty} ab(b-a)}{2^{\frac{1}{q}} \Gamma(\alpha+1)} \left( \frac{b-a}{ab} \right)^{\alpha} (\sup \{ |f'(a)|^q, |f'(b)|^q \})^{\frac{1}{q}} \\
& \times \left[ \left( \int_0^{\frac{1}{2}} \frac{[(1-u)^{\alpha}-u^{\alpha}]^p}{(ub+(1-u)a)^{2p}} du \right)^{\frac{1}{p}} + \left( \int_{\frac{1}{2}}^1 \frac{[u^{\alpha}-(1-u)^{\alpha}]^p}{(ub+(1-u)a)^{2p}} du \right)^{\frac{1}{p}} \right].
\end{aligned}$$

Using Lemma 1.2, we have

$$(2.15) \quad \int_0^{\frac{1}{2}} \frac{[(1-u)^{\alpha}-u^{\alpha}]^p}{(ub+(1-u)a)^{2p}} du \leq \int_0^{\frac{1}{2}} \frac{(1-2u)^{\alpha p}}{(ub+(1-u)a)^{2p}} du$$

and

$$(2.16) \quad \int_{\frac{1}{2}}^1 \frac{[u^{\alpha}-(1-u)^{\alpha}]^p}{(ub+(1-u)a)^{2p}} du \leq \int_{\frac{1}{2}}^1 \frac{(2u-1)^{\alpha p}}{(ub+(1-u)a)^{2p}} du.$$

For the appearing integrals, we have

$$\begin{aligned}
 (2.17) \quad & \int_0^{\frac{1}{2}} \frac{(1-2u)^{\alpha p}}{(ub+(1-u)a)^{2p}} du \\
 &= \frac{1}{2} \int_0^1 \frac{(1-u)^{\alpha p}}{(\frac{u}{2}b+(1-\frac{u}{2})a)^{2p}} du \\
 &= \frac{1}{2} \int_0^1 v^{\alpha p} \left(\frac{a+b}{2}\right)^{-2p} \left[1-v\left(\frac{b-a}{b+a}\right)\right]^{-2p} dv \\
 &= \left(\frac{a+b}{2}\right)^{-2p} \frac{1}{2(\alpha p+1)} {}_2F_1\left(2p, \alpha p+1; \alpha p+2; \frac{b-a}{b+a}\right) \\
 &= C_3(\alpha)
 \end{aligned}$$

and

$$\begin{aligned}
 (2.18) \quad & \int_{\frac{1}{2}}^1 \frac{(2u-1)^{\alpha p}}{(ub+(1-u)a)^{2p}} du \\
 &= \int_0^{\frac{1}{2}} \frac{(1-2u)^{\alpha p}}{(ua+(1-u)b)^{2p}} du \\
 &= \frac{1}{2} \int_0^1 \frac{(1-v)^{\alpha p}}{(\frac{v}{2}a+(1-\frac{v}{2})b)^{2p}} dv \\
 &= \frac{1}{2} \int_0^1 (1-v)^{\alpha p} b^{-2p} \left(1-\frac{v}{2}\left(1-\frac{a}{b}\right)\right)^{-2p} dv \\
 &= b^{-2p} \frac{1}{2(\alpha p+1)} {}_2F_1\left(2p, 1; \alpha p+2; \frac{1}{2}\left(1-\frac{a}{b}\right)\right) \\
 &= C_4(\alpha).
 \end{aligned}$$

If we use (2.15), (2.16), (2.17) and (2.18) in (2.14), we have (2.13). This completes the proof. □

**Corollary 2.6.** *In Theorem 2.5:*

(1) *If we take  $\alpha = 1$  we have the following Hermite-Hadamard-Fejér inequality for harmonically quasi-convex functions which is related to the right-hand side of (1.6):*

$$\begin{aligned}
 & \left| \frac{f(a)+f(b)}{2} \int_a^b \frac{g(x)}{x^2} dx - \int_a^b \frac{f(x)g(x)}{x^2} dx \right| \\
 & \leq \frac{\|g\|_{\infty} (b-a)^2}{2^{\frac{1}{q}+1}} (\sup\{|f'(a)|^q, |f'(b)|^q\})^{\frac{1}{q}} \left[ C_3^{\frac{1}{p}}(1) + C_4^{\frac{1}{p}}(1) \right],
 \end{aligned}$$

(2) If we take  $g(x) = 1$  we have following Hermite-Hadamard type inequality for harmonically quasi-convex functions in fractional integral forms which is related to the right-hand side of (1.5):

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2} \left( \frac{ab}{b-a} \right)^\alpha \left\{ \begin{array}{l} J_{1/a-}^\alpha (f \circ h)(1/b) \\ + J_{1/b+}^\alpha (f \circ h)(1/a) \end{array} \right\} \right| \\ & \leq \frac{ab(b-a)}{2^{\frac{1}{q}+1}} (\sup \{|f'(a)|^q, |f'(b)|^q\})^{\frac{1}{q}} \left[ C_3^{\frac{1}{p}}(\alpha) + C_4^{\frac{1}{p}}(\alpha) \right], \end{aligned}$$

(3) If we take  $\alpha = 1$  and  $g(x) = 1$  we have the following Hermite-Hadamard type inequality for harmonically quasi-convex functions which is related to the right-hand side of (1.4):

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \\ & \leq \frac{ab(b-a)}{2^{\frac{1}{q}+1}} (\sup \{|f'(a)|^q, |f'(b)|^q\})^{\frac{1}{q}} \left[ C_3^{\frac{1}{p}}(1) + C_4^{\frac{1}{p}}(1) \right]. \end{aligned}$$

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## References

- [1] M. Bombardelli and S. Varošanec, *Properties of h-convex functions related to the Hermite Hadamard Fejér inequalities*, Comp. Math. with Appl., **58**(2009), 1869–1877.
- [2] F. Chen and S. Wu, *Fejér and Hermite-Hadamard type inequalities for harmonically convex functions*, J. Appl. Math., volume 2014, article id:386806.
- [3] Z. Dahmani, *On Minkowski and Hermite-Hadamard integral inequalities via fractional integration*, Ann. Funct. Anal., **1**(1)(2010), 51-58.
- [4] L. Fejér, *Über die Fourierreihen*, II, Math. Naturwiss. Anz Ungar. Akad., Wiss, **24**(1906), 369-390, (in Hungarian).
- [5] J. Hadamard, *Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann*, J. Math. Pures Appl., **58**(1893), 171-215.
- [6] İ. İşcan, *New estimates on generalization of some integral inequalities for s-convex functions and their applications*, Int. J. Pure Appl. Math., **86**(4)(2013), 727-746.
- [7] İ. İşcan, *Some new general integral inequalities for h-convex and h-concave functions*, Adv. Pure Appl. Math., **5**(1)(2014), 21-29 . doi: 10.1515/apam-2013-0029.
- [8] İ. İşcan, *Generalization of different type integral inequalities for s-convex functions via fractional integrals*, Appl. Anal., 2013. doi: 10.1080/00036811.2013.851785.

- [9] İ. İşcan, *On generalization of different type integral inequalities for  $s$ -convex functions via fractional integrals*, Math. Sci. Appl. E-Not., **2(1)**(2014), 55-67.
- [10] İ. İşcan, S. Wu, *Hermite-Hadamard type inequalities for harmonically convex functions via fractional integrals*, Appl. Math. Comput., **238**(2014) 237-244.
- [11] İ. İşcan, *Hermite-Hadamard type inequalities for harmonically convex functions*, Hacet. J. Math. Stat., **43(6)**(2014), 935-942.
- [12] İ. İşcan, M. Kunt, *Hermite-Hadamard-Fejér type inequalities for harmonically convex functions via fractional integrals*, RGMIA, **18**(2015), Article 107, 16 pp.
- [13] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*. Elsevier, Amsterdam (2006).
- [14] M. A. Latif, S. S. Dragomir and E. Momoniat, *Some Fejér type inequalities for harmonically-convex functions with applications to special means*, RGMIA, 18(2015), Article 24, 17 pp.
- [15] A. P. Prudnikov, Y. A. Brychkov, O. J. Marichev, *Integral and series, Elementary Functions*, vol. 1, Nauka, Moscow, 1981.
- [16] M. Z. Sarıkaya, *On new Hermite Hadamard Fejér type integral inequalities*, Stud. Univ. Babeş-Bolyai Math., **57(3)**(2012), 377–386.
- [17] M. Z. Sarıkaya, E. Set, H. Yıldız and N. Başak, *Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities*, Math. Comput. Mod., **57(9)**(2013), 2403-2407.
- [18] K. -L. Tseng, G. -S. Yang and K. -C. Hsu, *Some inequalities for differentiable mappings and applications to Fejér inequality and weighted trapezoidal formula*, Taiwanese J. Math., **15(4)**(2011), 1737-1747.
- [19] J. Wang, X. Li, M. Fečkan and Y. Zhou, *Hermite-Hadamard-type inequalities for Riemann-Liouville fractional integrals via two kinds of convexity*, Appl. Anal., **92(11)**(2012), 2241-2253. doi:10.1080/00036811.2012.727986
- [20] J. Wang, C. Zhu and Y. Zhou, *New generalized Hermite-Hadamard type inequalities and applications to special means*, J. Inequal. Appl., **2013(325)**(2013), 15 pages.
- [21] T. Y. Zhang, A. P. Ji, F. Qi, *Integral inequalities of Hermite-Hadamard type for harmonically quasi-convex functions*, Proc. Jangjeon Math. Soc., **16(3)**(2013), 399-407.