Mehreen and Anwar *Journal of Inequalities and Applications* (2019) 2019:92
https://doi.org/10.1186/s13660-019-2047-1

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Hermite–Hadamard type inequalities for exponentially p-convex functions and exponentially s-convex functions in the second sense with applications

Naila Mehreen[1*] and Matloob Anwar[1]

*Correspondence:
nailamehreen@gmail.com;
naila.mehreen@sns.nust.edu.pk
[1]School of Natural Sciences, National University of Sciences and Technology, Islamabad, Pakistan

Abstract

In this paper, we introduce the notion of exponentially p-convex function and exponentially s-convex function in the second sense. We establish several Hermite–Hadamard type inequalities for exponentially p-convex functions and exponentially s-convex functions in second sense. The present investigation is an extension of several well known results.

MSC: 26A51; 26D15

Keywords: Hermite–Hadamard inequalities; Convex functions; Exponentially convex functions; Exponentially p-convex functions; Exponentially s-convex functions in the second sense

1 Introduction

Recently, the study of convex functions has become more important due to variety of their nature. Many generalizations of this notion have been established. For more details see [1–6, 13, 16–19].

Convex functions satisfy many integral inequalities. Among these, the Hermite–Hadamard inequality is well known. The Hermite–Hadamard inequality [14, 15] for a convex function $\psi:\mathcal{K}\to\mathbb{R}$ on an interval $\mathcal{K}$ is

$$\psi\left(\frac{u_1+u_2}{2}\right)\le\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\psi(w)\,dw\le\frac{\psi(u_1)+\psi(u_2)}{2}, \tag{1.1}$$

for all $u_1,u_2\in\mathcal{K}$ with $u_1<u_2$. Many authors have made generalizations to inequality (1.1). For more results and details, see [3, 4, 6, 7, 17–19, 22, 24–26].

Definition 1.1 ([19, 20]) Consider an interval $\mathcal{K}\subset(0,\infty)=\mathbb{R}_+$, and $p\in\mathbb{R}\backslash\{0\}$. A function $\psi:\mathcal{K}\to\mathbb{R}$ is called p-convex, if

$$\psi\left(\left[ru_1^p+(1-r)u_2^p\right]^{\frac{1}{p}}\right)\le r\psi(u_1)+(1-r)\psi(u_2), \tag{1.2}$$

for all $u_1, u_2 \in \mathcal{K}$ and $r \in [0,1]$. If the inequality in (1.2) is reversed, then ψ is called p-concave.

Example 1.1 A function $\psi : (0,\infty) \to \mathbb{R}$, defined by $\psi(u) = u^p$ for $p \in \mathbb{R}\backslash\{0\}$, is p-convex as well as p-concave.

Iscan [19] gave the following results.

Theorem 1.2 ([19]) *Consider an interval $\mathcal{K} \subset (0,\infty)$, and $p \in \mathbb{R}\backslash\{0\}$. Let $\psi : \mathcal{K} \to \mathbb{R}$ be p-convex and $u_1, u_2 \in \mathcal{K}$, $u_1 < u_2$. If $\psi \in L_1[u_1,u_2]$, then we have*

$$\psi\left(\left[\frac{u_1^p + u_2^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{u_2^p - u_1^p}\int_{u_1}^{u_2} \frac{\psi(w)}{w^{1-p}}\, dw \leq \frac{\psi(u_1)+\psi(u_2)}{2}. \tag{1.3}$$

Lemma 1.1 ([19]) *Let $\psi : \mathcal{K} \to \mathbb{R}$ be a differentiable function on $\mathcal{K}^\circ$, i.e., the interior of $\mathcal{K}$, and $u_1, u_2 \in \mathcal{K}$, $u_1 < u_2$, and $p \in \mathbb{R}\backslash\{0\}$. If $\psi' \in L_1[u_1,u_2]$, then*

$$\begin{aligned}&\frac{\psi(u_1)+\psi(u_2)}{2} - \frac{p}{u_2^p - u_1^p}\int_{u_1}^{u_2} \frac{\psi(w)}{w^{1-p}}\, dw \\ &\quad = \frac{u_2^p - u_1^p}{2p}\int_0^1 \frac{1-2r}{[ru_1^p + (1-r)u_2^p]^{1-\frac{1}{p}}}\psi'\left(\left[ru_1^p + (1-r)u_2^p\right]^{\frac{1}{p}}\right) dr.\end{aligned} \tag{1.4}$$

Definition 1.2 ([16]) Let $s \in (0,1]$. A function $\psi : \mathcal{K} \subset \mathbb{R}_0 \to \mathbb{R}_0$, where $\mathbb{R}_0 = [0,\infty)$, is called s-convex in the second sense, if

$$\psi\left(ru_1 + (1-r)u_2\right) \leq r^s\psi(u_1) + (1-r)^s\psi(u_2), \tag{1.5}$$

for all $u_1, u_2 \in \mathcal{K}$ and $r \in [0,1]$.

Example 1.3 A function $\psi : (0,\infty) \to (0,\infty)$, defined by $\psi(u) = u^s$ for $s \in (0,1)$, is s-convex in the second sense.

Dragomir et al. [8, 9] gave the following important results.

Theorem 1.4 ([9]) *Let $s \in (0,1)$ and $\psi : \mathbb{R}_0 \to \mathbb{R}_0$ be s-convex in the second sense. Let $u_1, u_2 \in [0,\infty)$, $u_1 \leq u_2$. If $\psi \in L_1[u_1,u_2]$, then*

$$2^{s-1}\psi\left(\frac{u_1+u_2}{2}\right) \leq \frac{1}{u_2-u_1}\int_{u_1}^{u_2} \psi(w)\, dw \leq \frac{\psi(u_1)+\psi(u_2)}{s+1}. \tag{1.6}$$

Lemma 1.2 ([8]) *Let $\psi : \mathcal{K} \to \mathbb{R}$ be a differentiable mapping on $\mathcal{K}^\circ$, the interior of $\mathcal{K}$, and $u_1, u_2 \in \mathcal{K}$ be two distinct points. If $\psi' \in L_1[u_1,u_2]$, then*

$$\begin{aligned}&\frac{\psi(u_1)+\psi(u_2)}{2} - \frac{1}{u_2-u_1}\int_{u_1}^{u_2} \psi(w)\, dw \\ &\quad = \frac{u_2-u_1}{2}\int_0^1 (1-2r)\psi'\left(ru_1 + (1-r)u_2\right) dr.\end{aligned} \tag{1.7}$$

Awan et al. [4] introduced the following new class of convex functions.

Definition 1.3 ([4]) A function $\psi : \mathcal{K} \subseteq \mathbb{R} \to \mathbb{R}$ is called exponentially convex, if

$$\psi\bigl(ru_1 + (1-r)u_2\bigr) \leq r\frac{\psi(u_1)}{e^{\alpha u_1}} + (1-r)\frac{\psi(u_2)}{e^{\alpha u_2}}, \tag{1.8}$$

for all $u_1, u_2 \in \mathcal{K}$, $r \in [0,1]$ and $\alpha \in \mathbb{R}$. If the inequality in (1.8) is reversed, then ψ is called exponentially concave.

Example 1.5 A function $\psi : \mathbb{R} \to \mathbb{R}$, defined by $\psi(u) = -u^2$, is an exponentially convex for all $\alpha > 0$.

The Beta and Hypergeometric functions are defined as:

$$\beta(u_1, u_2) = \int_0^1 w^{u_1-1}(1-w)^{u_2-1}\,dw, \quad u_1, u_2 > 0,$$

and

$${}_2F_1(u_1, u_2; t; z) = \frac{1}{\beta(u_2, t-u_2)}\int_0^1 w^{u_2-1}(1-w)^{t-u_2-1}(1-zw)^{-u_1}\,dw, \quad t > u_2 > 0, |z| < 1,$$

respectively, see [21].

2 Exponentially p-convex functions

Now we introduce exponentially p-convex functions.

Definition 2.1 Consider an interval $\mathcal{K} \subset (0,\infty) = \mathbb{R}_+$ and $p \in \mathbb{R}\backslash\{0\}$. A function $\psi : \mathcal{K} \to \mathbb{R}$ is called exponentially p-convex, if

$$\psi\bigl(\bigl[ru_1^p + (1-r)u_2^p\bigr]^{\frac{1}{p}}\bigr) \leq r\frac{\psi(u_1)}{e^{\alpha u_1}} + (1-r)\frac{\psi(u_2)}{e^{\alpha u_2}}, \tag{2.1}$$

for all $u_1, u_2 \in \mathcal{K}$, $r \in [0,1]$ and $\alpha \in \mathbb{R}$. If the inequality in (2.1) is reversed, then ψ is called exponentially p-concave.

It is easy to note that, by taking $\alpha = 0$, an exponentially p-convex function becomes p-convex.

Example 2.1 Consider a function $\psi : (\sqrt{2}, \infty) \to \mathbb{R}$, defined by $\psi(u) = (\ln(u))^p$ for $p \geq 2$. Then ψ is exponentially p-convex for all $\alpha < 0$, and not p-convex.

Note that ψ satisfies inequality (2.1) for all $\alpha < 0$. But for $u_1 = 2$, $u_2 = 3$ and $p = 5$, inequality (1.2) does not hold.

2.1 Integral inequalities

Throughout this section, we denote by $\mathcal{K} \subset (0,\infty) = \mathbb{R}_+$ an interval with interior $\mathcal{K}^\circ$ and $p \in \mathbb{R}\backslash\{0\}$. We start with our results for exponentially p-convex functions.

Theorem 2.2 *Let $\psi : \mathcal{K} \to \mathbb{R}$ be an integrable exponentially p-convex function. Let $u_1, u_2 \in \mathcal{K}$ with $u_1 < u_2$. Then for $\alpha \in \mathbb{R}$, we have*

$$\psi\left(\left[\frac{u_1^p + u_2^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{u_2^p - u_1^p}\int_{u_1}^{u_2} \frac{\psi(w)}{w^{1-p}e^{\alpha w}}\,dw \leq A_1(r)\frac{\psi(u_1)}{e^{\alpha u_1}} + A_2(r)\frac{\psi(u_2)}{e^{\alpha u_2}}, \tag{2.2}$$

where

$$A_1(r) = \int_0^1 \frac{r\,dr}{e^{\alpha(ru_1^p + (1-r)u_2^p)^{\frac{1}{p}}}} \quad \text{and} \quad A_2(r) = \int_0^1 \frac{(1-r)\,dr}{e^{\alpha(ru_1^p + (1-r)u_2^p)^{\frac{1}{p}}}}.$$

Proof By using the exponential p-convexity of ψ, we have

$$2\psi\left(\left[\frac{w^p + z^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{\psi(w)}{e^{\alpha w}} + \frac{\psi(z)}{e^{\alpha z}}. \tag{2.3}$$

Letting $w^p = ru_1^p + (1-r)u_2^p$ and $z^p = (1-r)u_1^p + ru_2^p$, we get

$$2\psi\left(\left[\frac{u_1^p + u_2^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{\psi([ru_1^p + (1-r)u_2^p]^{\frac{1}{p}})}{e^{\alpha(ru_1^p + (1-r)u_2^p)^{\frac{1}{p}}}} + \frac{\psi([(1-r)u_1^p + ru_2^p]^{\frac{1}{p}})}{e^{\alpha((1-r)u_1^p + ru_2^p)^{\frac{1}{p}}}}. \tag{2.4}$$

Integrating with respect to $r \in [0, 1]$ and applying a change of variable, we find

$$\psi\left(\left[\frac{u_1^p + u_2^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{u_2^p - u_1^p}\int_{u_1}^{u_2} \frac{\psi(w)}{w^{1-p}e^{\alpha w}}\,dw. \tag{2.5}$$

Hence the first inequality of (2.2) has been established. For the next inequality, again using the exponential p-convexity of ψ, we have

$$\frac{\psi([ru_1^p + (1-r)u_2^p]^{\frac{1}{p}})}{e^{\alpha(ru_1^p + (1-r)u_2^p)^{\frac{1}{p}}}} \leq \frac{r\frac{\psi(u_1)}{e^{\alpha u_1}} + (1-r)\frac{\psi(u_2)}{e^{\alpha u_2}}}{e^{\alpha(ru_1^p + (1-r)u_2^p)^{\frac{1}{p}}}}. \tag{2.6}$$

Integrating with respect to $r \in [0, 1]$, we get

$$\begin{aligned} &\frac{p}{u_2^p - u_1^p}\int_{u_1}^{u_2} \frac{\psi(w)}{w^{1-p}e^{\alpha w}}\,dw \\ &\quad \leq \frac{\psi(u_1)}{e^{\alpha u_1}}\int_0^1 \frac{r\,dr}{e^{\alpha(ru_1^p + (1-r)u_2^p)^{\frac{1}{p}}}} + \frac{\psi(u_2)}{e^{\alpha u_2}}\int_0^1 \frac{(1-r)\,dr}{e^{\alpha(ru_1^p + (1-r)u_2^p)^{\frac{1}{p}}}}. \end{aligned} \tag{2.7}$$

By combining (2.5) and (2.7), we get (2.2). □

Remark 2.1 In Theorem 2.2, by taking $\alpha = 0$, we attain inequality (1.3) in Theorem 1.2.

Theorem 2.3 *Let $\psi : \mathcal{K} \to \mathbb{R}$ be a differentiable function on $\mathcal{K}^\circ$ and $u_1, u_2 \in \mathcal{K}$ with $u_1 < u_2$ and $\psi' \in L_1[u_1, u_2]$. If $|\psi'|^q$ is exponentially p-convex on $[u_1, u_2]$ for $q \geq 1$ and $\alpha \in \mathbb{R}$, then*

$$\begin{aligned} &\left|\frac{\psi(u_1)+\psi(u_2)}{2} - \frac{p}{u_2^p - u_1^p}\int_{u_1}^{u_2}\frac{\psi(w)}{w^{1-p}}\,dw\right| \\ &\quad\leq \frac{u_2^p - u_1^p}{2p} B_1^{1-\frac{1}{q}}\left[B_2\left|\frac{\psi'(u_1)}{e^{\alpha u_1}}\right|^q + B_3\left|\frac{\psi'(u_2)}{e^{\alpha u_2}}\right|^q\right]^{\frac{1}{q}}, \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} B_1 &= B_1(u_1, u_2; p) = \frac{1}{4}\left(\frac{u_1^p + u_2^p}{2}\right)^{\frac{1}{p}-1} \\ &\quad\times\left[{}_2F_1\left(1-\frac{1}{p}, 2; 3; \frac{u_1^p - u_2^p}{u_1^p + u_2^p}\right) + {}_2F_1\left(1-\frac{1}{p}, 2; 3; \frac{u_2^p - u_1^p}{u_1^p + u_2^p}\right)\right], \\ B_2 &= B_2(u_1, u_2; p) = \frac{1}{24}\left(\frac{u_1^p + u_2^p}{2}\right)^{\frac{1}{p}-1}\left[{}_2F_1\left(1-\frac{1}{p}, 2; 4; \frac{u_1^p - u_2^p}{u_1^p + u_2^p}\right)\right. \\ &\quad\left. + 6\,{}_2F_1\left(1-\frac{1}{p}, 2; 3; \frac{u_2^p - u_1^p}{u_1^p + u_2^p}\right) + {}_2F_1\left(1-\frac{1}{p}, 2; 4; \frac{u_2^p - u_1^p}{u_1^p + u_2^p}\right)\right], \\ B_3 &= B_3(u_1, u_2; p) = B_1 - B_2. \end{aligned}$$

Proof Applying the power mean inequality to (1.4) of Lemma 1.1, we get

$$\begin{aligned} &\left|\frac{\psi(u_1)+\psi(u_2)}{2} - \frac{p}{u_2^p - u_1^p}\int_{u_1}^{u_2}\frac{\psi(w)}{w^{1-p}}\,dw\right| \\ &\quad\leq \frac{u_2^p - u_1^p}{2p}\int_0^1\left|\frac{1-2r}{[ru_1^p + (1-r)u_2^p]^{1-\frac{1}{p}}}\right|\left|\psi'\left(\left[ru_1^p + (1-r)u_2^p\right]^{\frac{1}{p}}\right)\right|dr \\ &\quad\leq \frac{u_2^p - u_1^p}{2p}\left(\int_0^1\frac{|1-2r|}{[ru_1^p + (1-r)u_2^p]^{1-\frac{1}{p}}}\,dr\right)^{1-\frac{1}{q}} \\ &\qquad\times\left(\int_0^1\frac{|1-2r|}{[ru_1^p + (1-r)u_2^p]^{1-\frac{1}{p}}}\left|\psi'\left(\left[ru_1^p + (1-r)u_2^p\right]^{\frac{1}{p}}\right)\right|^q dr\right)^{\frac{1}{q}}. \end{aligned} \tag{2.9}$$

Since $|\psi'|^q$ is exponentially p-convex on $[u_1, u_2]$, we have

$$\begin{aligned} &\left|\frac{\psi(u_1)+\psi(u_2)}{2} - \frac{p}{u_2^p - u_1^p}\int_{u_1}^{u_2}\frac{\psi(w)}{w^{1-p}}\,dw\right| \\ &\quad\leq \frac{u_2^p - u_1^p}{2p}\left(\int_0^1\frac{|1-2r|}{[ru_1^p + (1-r)u_2^p]^{1-\frac{1}{p}}}\,dr\right)^{1-\frac{1}{q}} \\ &\qquad\times\left(\int_0^1\frac{||1-2r||[r|\frac{\psi'(u_1)}{e^{\alpha u_1}}|^q + (1-r)|\frac{\psi'(u_2)}{e^{\alpha u_2}}|^q]}{[ru_1^p + (1-r)u_2^p]^{1-\frac{1}{p}}}\,dr\right)^{\frac{1}{q}} \\ &\quad\leq \frac{u_2^p - u_1^p}{2p} B_1^{1-\frac{1}{q}}\left[B_2\left|\frac{\psi'(u_1)}{e^{\alpha u_1}}\right|^q + B_3\left|\frac{\psi'(u_2)}{e^{\alpha u_2}}\right|^q\right]^{\frac{1}{q}}. \end{aligned} \tag{2.10}$$

It is easy to note that

$$\int_0^1 \frac{|1-2r|}{[ru_1^p+(1-r)u_2^p]^{1-\frac{1}{p}}}\,dr = B_1(u_1,u_2;p),$$
$$\int_0^1 \frac{|1-2r|r}{[ru_1^p+(1-r)u_2^p]^{1-\frac{1}{p}}}\,dr = B_2(u_1,u_2;p),$$
$$\int_0^1 \frac{|1-2r|(1-r)}{[ru_1^p+(1-r)u_2^p]^{1-\frac{1}{p}}}\,dr = B_1(u_1,u_2;p)-B_2(u_1,u_2;p).$$

Hence the proof is completed. □

Remark 2.2 In Theorem 2.3,

(a) by taking $\alpha = 0$, we attain Theorem 7 in [19];

(b) by taking $p = 1$, we attain Theorem 5 in [4].

Corollary 2.4 *Let $\psi : \mathcal{K} \to \mathbb{R}$ be a differentiable function on $\mathcal{K}^\circ$ and $u_1, u_2 \in \mathcal{K}$, $u_1 < u_2$, and $\psi' \in L_1[u_1,u_2]$. If $|\psi'|$ is exponentially p-convex on $[u_1,u_2]$, then*

$$\left|\frac{\psi(u_1)+\psi(u_2)}{2} - \frac{p}{u_2^p-u_1^p}\int_{u_1}^{u_2}\frac{\psi(w)}{w^{1-p}}\,dw\right| \leq \frac{u_2^p-u_1^p}{2p}\left[B_2\left|\frac{\psi'(u_1)}{e^{\alpha u_1}}\right| + B_3\left|\frac{\psi'(u_2)}{e^{\alpha u_2}}\right|\right], \tag{2.11}$$

where B_2 and B_3 are given in Theorem 2.3.

Remark 2.3 In Corollary 2.4,

(a) by taking $\alpha = 0$, we attain Corollary 1 in [19];

(b) by taking $p = 1$, we attain Theorem 3 in [4].

Theorem 2.5 *Let $\psi : \mathcal{K} \to \mathbb{R}$ be a differentiable function on $\mathcal{K}^\circ$. Let $u_1, u_2 \in \mathcal{K}$, $u_1 < u_2$, and $\psi' \in L_1[u_1,u_2]$. If $|\psi'|^q$ is exponentially p-convex on $[u_1,u_2]$, and $q, l > 1$, $1/q + 1/l = 1$, and $\alpha \in \mathbb{R}$, then*

$$\left|\frac{\psi(u_1)+\psi(u_2)}{2} - \frac{p}{u_2^p-u_1^p}\int_{u_1}^{u_2}\frac{\psi(w)}{w^{1-p}}\,dw\right| \leq \frac{u_2^p-u_1^p}{2p}\left(\frac{1}{l+1}\right)^{\frac{1}{l}}\left[B_4\left|\frac{\psi'(u_1)}{e^{\alpha u_1}}\right|^q + B_5\left|\frac{\psi'(u_2)}{e^{\alpha u_2}}\right|^q\right]^{\frac{1}{q}}, \tag{2.12}$$

where

$$B_4 = B_4(u_1,u_2;p;q) = \begin{cases} \frac{1}{2u_1^{qp-q}}\,{}_2F_1(q-\frac{q}{p},1;3;1-(\frac{u_2}{u_1})^p), & p<0,\\ \frac{1}{2u_2^{qp-q}}\,{}_2F_1(q-\frac{q}{p},2;3;1-(\frac{u_1}{u_2})^p), & p>0,\end{cases}$$

$$\begin{aligned} B_5 &= B_5(u_1,u_2;p;q) \\ &= \begin{cases} \frac{1}{2u_1^{qp-q}}\,{}_2F_1(q-\frac{q}{p},2;3;1-(\frac{u_2}{u_1})^p), & p<0, \\ \frac{1}{2u_2^{qp-q}}\,{}_2F_1(q-\frac{q}{p},1;3;1-(\frac{u_1}{u_2})^p), & p>0. \end{cases} \end{aligned}$$

Proof Using Hölder's inequality on (1.4) of Lemma 1.1 and then applying the exponential p-convexity of $|\psi'|^q$ on $[u_1,u_2]$, we get

$$\begin{aligned} &\left|\frac{\psi(u_1)+\psi(u_2)}{2} - \frac{p}{u_2^p-u_1^p}\int_{u_1}^{u_2}\frac{\psi(w)}{w^{1-p}}\,dw\right| \\ &\le \frac{u_2^p-u_1^p}{2p}\left(\int_0^1 |1-2r|^l\,dr\right)^{\frac{1}{l}} \\ &\quad\times\left(\int_0^1 \frac{1}{[ru_1^p+(1-r)u_2^p]^{q(1-\frac{1}{p})}}\left|\psi'\left(\left[ru_1^p+(1-r)u_2^p\right]^{\frac{1}{p}}\right)\right|^q\,dr\right)^{\frac{1}{q}} \\ &\le \frac{u_2^p-u_1^p}{2p}\left(\frac{1}{l+1}\right)^{\frac{1}{l}}\left(\int_0^1 \frac{r|\frac{\psi'(u_1)}{e^{\alpha u_1}}|^q+(1-r)|\frac{\psi'(u_2)}{e^{\alpha u_2}}|^q}{[ru_1^p+(1-r)u_2^p]^{q-\frac{q}{p}}}\,dr\right)^{\frac{1}{q}} \\ &\le \frac{u_2^p-u_1^p}{2p}\left(\frac{1}{l+1}\right)^{\frac{1}{l}}\left[B_4\left|\frac{\psi'(u_1)}{e^{\alpha u_1}}\right|^q + B_5\left|\frac{\psi'(u_2)}{e^{\alpha u_2}}\right|^q\right]^{\frac{1}{q}}, \end{aligned} \tag{2.13}$$

where after calculations, we have

$$\begin{aligned} B_4 &= \int_0^1 \frac{r}{[ru_1^p+(1-r)u_2^p]^{q-\frac{q}{p}}}\,dr \\ &= \begin{cases} \frac{1}{2u_1^{qp-q}}\,{}_2F_1(q-\frac{q}{p},1;3;1-(\frac{u_2}{u_1})^p), & p<0, \\ \frac{1}{2u_2^{qp-q}}\,{}_2F_1(q-\frac{q}{p},2;3;1-(\frac{u_1}{u_2})^p), & p>0, \end{cases} \\ B_5 &= \int_0^1 \frac{1-r}{[ru_1^p+(1-r)u_2^p]^{q-\frac{q}{p}}}\,dr \\ &= \begin{cases} \frac{1}{2u_1^{qp-q}}\,{}_2F_1(q-\frac{q}{p},2;3;1-(\frac{u_2}{u_1})^p), & p<0, \\ \frac{1}{2u_2^{qp-q}}\,{}_2F_1(q-\frac{q}{p},1;3;1-(\frac{u_1}{u_2})^p), & p>0. \end{cases} \end{aligned}$$

□

Remark 2.4 In Theorem 2.5,

(a) by letting $\alpha = 0$, we attain Theorem 8 in [19];

(b) by letting $p = 1$, we attain Theorem 4 in [4].

Theorem 2.6 *Let $\psi : \mathcal{K} \to \mathbb{R}$ be a differentiable function on $\mathcal{K}^\circ$ and $u_1,u_2 \in \mathcal{K}$, $u_1 < u_2$, and $\psi' \in L_1[u_1,u_2]$. If $|\psi'|^q$ is exponentially p-convex on $[u_1,u_2]$, and $q,l > 1$, $1/q+1/l = 1$, and $\alpha \in \mathbb{R}$, then*

$$\begin{aligned} &\left|\frac{\psi(u_1)+\psi(u_2)}{2} - \frac{p}{u_2^p-u_1^p}\int_{u_1}^{u_2}\frac{\psi(w)}{w^{1-p}}\,dw\right| \\ &\le \frac{u_2^p-u_1^p}{2p}B_6^{\frac{1}{l}}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left(\frac{|\frac{\psi'(u_1)}{e^{\alpha u_1}}|^q+|\frac{\psi'(u_2)}{e^{\alpha u_2}}|^q}{2}\right)^{\frac{1}{q}}, \end{aligned} \tag{2.14}$$

where

$$B_6 = B_6(u_1, u_2; p; l) = \begin{cases} \frac{1}{2u_1^{pl-l}} {}_2F_1(l - \frac{l}{p}, 1; 2; 1 - (\frac{u_2}{u_1})^p), & p < 0, \\ \frac{1}{2u_2^{pl-l}} {}_2F_1(l - \frac{l}{p}, 1; 2; 1 - (\frac{u_1}{u_2})^p), & p > 0. \end{cases}$$

Proof Using Hölder's inequality on (1.4) of Lemma 1.1 and then applying the exponential p-convexity of $|\psi'|^q$ on $[u_1, u_2]$, we get

$$\begin{aligned} &\left| \frac{\psi(u_1) + \psi(u_2)}{2} - \frac{p}{u_2^p - u_1^p} \int_{u_1}^{u_2} \frac{\psi(w)}{w^{1-p}} \, dw \right| \\ &\quad \le \frac{u_2^p - u_1^p}{2p} \left(\int_0^1 \frac{1}{[ru_1^p + (1-r)u_2^p]^{l-\frac{l}{p}}} \, dr \right)^{\frac{1}{l}} \\ &\qquad \times \left(\int_0^1 |1 - 2r|^q \left| \psi'\left(\left[ru_1^p + (1-r)u_2^p \right]^{\frac{1}{p}} \right) \right|^q dr \right)^{\frac{1}{q}} \\ &\quad \le \frac{u_2^p - u_1^p}{2p} B_6^{\frac{1}{l}} \left(\frac{1}{q+1} \right)^{\frac{1}{q}} \left(\frac{|\frac{\psi'(u_1)}{e^{\alpha u_1}}|^q + |\frac{\psi'(u_2)}{e^{\alpha u_2}}|^q}{2} \right)^{\frac{1}{q}}, \end{aligned} \tag{2.15}$$

where a simple calculation implies

$$\begin{aligned} B_6(u_1, u_2; p; l) &= \int_0^1 \frac{1}{[ru_1^p + (1-r)u_1^p]^{l-\frac{l}{p}}} \, dr \\ &= \begin{cases} \frac{1}{2u_1^{pl-l}} {}_2F_1(l - \frac{l}{p}, 1; 2; 1 - (\frac{u_2}{u_1})^p), & p < 0, \\ \frac{1}{2u_2^{pl-l}} {}_2F_1(l - \frac{l}{p}, 1; 2; 1 - (\frac{u_1}{u_2})^p), & p > 0, \end{cases} \end{aligned} \tag{2.16}$$

and

$$\int_0^1 r|1 - 2r|^q \, dr = \int_0^1 (1-r)|1 - 2r|^q \, dr = \frac{1}{2(q+1)}. \tag{2.17}$$

By substituting (2.16) and (2.17) into (2.15), we get (2.14). □

Remark 2.5 In Theorem 2.6, by letting $\alpha = 0$, we attain Theorem 9 in [19].

2.2 Applications

Consider some special means of two positive numbers u_1, u_2, $u_1 < u_2$:

(1) The arithmetic mean

$$A = A(u_1, u_2) = \frac{u_1 + u_2}{2};$$

(2) The harmonic mean

$$H = H(u_1, u_2) = \frac{2u_1 u_2}{u_1 + u_2};$$

(3) The p-logarithmic mean

$$L_p = L_p(u_1, u_2) = \left(\frac{u_2^{p+1} - u_1^{p+1}}{(p+1)(u_2 - u_1)} \right)^{\frac{1}{p}}, \quad p \in \mathbb{R} \setminus \{-1, 0\}.$$

In the next three propositions we consider $0 < u_1 < u_2$ and $q > 1$.

Proposition 2.1 *Let $\alpha \in \mathbb{R}$ and $p < 1$. Then we have*

$$\left|L_{p-1}^{p-1} - HL_{p-2}^{p-2}\right| \le \frac{u_2^p - u_1^p}{2p} B_6^{\frac{1}{l}} \left(\frac{1}{q+1} \right)^{\frac{1}{q}} A^{\frac{1}{q}} \left(\left| \frac{1}{u_1^2 e^{\alpha u_1}} \right|^q, \left| \frac{1}{u_2^2 e^{\alpha u_2}} \right|^q \right) HL_{p-1}^{p-1},$$

where B_6 is defined as in Theorem 2.6.

Proof The proof ensues from Theorem 2.6, for a function $\psi : (0, \infty) \to \mathbb{R}$, $\psi(w) = \frac{1}{w}$. Here note that $|\psi'(w)|^q = |\frac{1}{w^2}|^q$ is exponentially p-convex for all $p < 1$ and $\alpha \in \mathbb{R}$. □

Proposition 2.2 *Let $\alpha \le 0$ and $p > 1$. Then we have*

$$\left|L_{p-1}^{p-1} A\left(u_1^p, u_2^p\right) - L_{2p-1}^{2p-1}\right| \le \frac{u_2^p - u_1^p}{2} B_6^{\frac{1}{l}} \left(\frac{1}{q+1} \right)^{\frac{1}{q}} A^{\frac{1}{q}} \left(\left| \frac{1}{u_1^{p-1} e^{\alpha u_1}} \right|^q, \left| \frac{1}{u_2^{p-1} e^{\alpha u_2}} \right|^q \right) L_{p-1}^{p-1},$$

where B_6 is defined as in Theorem 2.6.

Proof The proof ensues from Theorem 2.6, for $\psi : (0, \infty) \to \mathbb{R}$, $\psi(w) = w^p$. Here note that $|\psi'(w)|^q = |pw^{p-1}|^q$ is exponentially p-convex for all $p > 1$ and $\alpha \le 0$. □

Proposition 2.3 *Let $\alpha \le 0$ and $p > 1$. Then we have*

$$\left|L_{p-1}^{p-1} A - L_p^p\right| \le \frac{u_2^p - u_1^p}{2p} B_6^{\frac{1}{l}} \left(\frac{1}{q+1} \right)^{\frac{1}{q}} A^{\frac{1}{q}} \left(\left| \frac{1}{e^{\alpha u_1}} \right|^q, \left| \frac{1}{e^{\alpha u_2}} \right|^q \right) L_{p-1}^{p-1},$$

where B_6 is given as in Theorem 2.6.

Proof The proof ensues from Theorem 2.6, for $\psi : (0, \infty) \to \mathbb{R}$, $\psi(w) = w$. Here note that $|\psi'(w)|^q = 1$ is exponentially p-convex for all $p > 1$ and $\alpha \le 0$. □

3 Exponentially s-convex functions in the second sense

We first generalize Definition 1.2.

Definition 3.1 Let $s \in (0, 1]$ and $\mathcal{K} \subset \mathbb{R}_0$ be an interval. A function $\psi : \mathcal{K} \to \mathbb{R}$ is called exponentially s-convex in the second sense, if

$$\psi\left(ru_1 + (1-r)u_2\right) \le r^s \frac{\psi(u_1)}{e^{\alpha u_1}} + (1-r)^s \frac{\psi(u_2)}{e^{\alpha u_2}}, \tag{3.1}$$

for all $u_1, u_2 \in \mathcal{K}$, $r \in [0, 1]$ and $\alpha \in \mathbb{R}$. If the inequality in (3.1) is reversed then ψ is called exponentially s-concave.

Observe that, by taking $\alpha = 0$, an exponentially s-convex function becomes s-convex.

Example 3.1 Consider a function $\psi : [0,\infty) \to \mathbb{R}$, defined by $\psi(u) = \ln(u)$ for $s \in (0,1)$. Then ψ is exponentially s-convex, for all $\alpha \leq -1$, but not s-convex in the second sense.

3.1 Integral inequalities

Throughout this section, we denote by $\mathcal{K} \subset \mathbb{R}_0$ an interval with nonempty interior $\mathcal{K}^\circ$ and $s \in (0,1]$. We start our new results with the following theorem.

Theorem 3.2 *Let $\psi : \mathcal{K} \subset \mathbb{R}_0 \to \mathbb{R}$ be an integrable exponentially s-convex function in the second sense on $\mathcal{K}^\circ$. Then for $u_1, u_2 \in \mathcal{K}$ with $u_1 < u_2$ and $\alpha \in \mathbb{R}$, we have*

$$2^{s-1}\psi\left(\frac{u_1+u_2}{2}\right) \leq \frac{1}{u_2-u_1}\int_{u_1}^{u_2}\frac{\psi(w)}{e^{\alpha w}}\,dw \leq A_3(r)\frac{\psi(u_1)}{e^{\alpha u_1}} + A_4(r)\frac{\psi(u_2)}{e^{\alpha u_2}}, \tag{3.2}$$

where

$$A_3(r) = \int_0^1 \frac{r^s\,dr}{e^{\alpha(ru_1+(1-r)u_2)}} \quad \text{and} \quad A_4(r) = \int_0^1 \frac{(1-r)^s\,dr}{e^{\alpha(ru_1+(1-r)u_2)}}.$$

Proof Applying exponential s-convexity of ψ, we have

$$2^s\psi\left(\frac{w+z}{2}\right) \leq \frac{\psi(w)}{e^{\alpha w}} + \frac{\psi(z)}{e^{\alpha z}}. \tag{3.3}$$

Letting $w = ru_1 + (1-r)u_2$ and $z = (1-r)u_1 + ru_2$, we get

$$2^s\psi\left(\frac{u_1+u_2}{2}\right) \leq \frac{\psi(ru_1+(1-r)u_2)}{e^{\alpha(ru_1+(1-r)u_2)}} + \frac{\psi((1-r)u_1+ru_2)}{e^{\alpha((1-r)u_1+ru_2)}}. \tag{3.4}$$

Integrating with respect to $r \in [0,1]$ and applying a change of variable, we find

$$2^{s-1}\psi\left(\frac{u_1+u_2}{2}\right) \leq \frac{1}{u_2-u_1}\int_{u_1}^{u_2}\frac{\psi(w)}{e^{\alpha w}}\,dw. \tag{3.5}$$

Hence the proof of the first inequality of (3.2) has been completed. For the next inequality, again using the exponential s-convexity of ψ, we have

$$\frac{\psi(ru_1+(1-r)u_2)}{e^{\alpha(ru_1+(1-r)u_2)}} \leq \frac{r^s\frac{\psi(u_1)}{e^{\alpha u_1}} + (1-r)^s\frac{\psi(u_2)}{e^{\alpha u_2}}}{e^{\alpha(ru_1+(1-r)u_2)}}. \tag{3.6}$$

Integrating with respect to $r \in [0,1]$, we get

$$\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\frac{\psi(w)}{e^{\alpha w}}\,dw \leq \frac{\psi(u_1)}{e^{\alpha u_1}}\int_0^1\frac{r^s\,dr}{e^{\alpha(ru_1+(1-r)u_2)}} + \frac{\psi(u_2)}{e^{\alpha u_2}}\int_0^1\frac{(1-r)^s\,dr}{e^{\alpha(ru_1+(1-r)u_2)}}. \tag{3.7}$$

By combining (3.5) and (3.7), we get (3.2). □

Remark 3.1 In Theorem 3.2, by letting $\alpha = 0$, we get inequality (1.6) in Theorem 1.4.

Theorem 3.3 *Let $\psi:\mathcal{K}\to\mathbb{R}$ be a differentiable function on $\mathcal{K}^\circ$ and $u_1,u_2\in\mathcal{K}$ with $u_1<u_2$ and $\psi'\in L_1[u_1,u_2]$. If $|\psi'|$ is exponentially s-convex in the second sense on $[u_1,u_2]$, then*

$$\begin{aligned}&\left|\frac{\psi(u_1)+\psi(u_2)}{2}-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\psi(w)\,dw\right|\\&\quad\le\frac{u_2-u_1}{2(s+1)(s+2)}\left[(3s+4)\left|\frac{\psi'(u_1)}{e^{\alpha u_1}}\right|+(s+4)\left|\frac{\psi'(u_2)}{e^{\alpha u_2}}\right|\right].\end{aligned}\tag{3.8}$$

Proof From Lemma 1.2, we have

$$\begin{aligned}&\left|\frac{\psi(u_1)+\psi(u_2)}{2}-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\psi(w)\,dw\right|\\&\quad=\frac{u_2-u_1}{2}\left|\int_0^1(1-2r)\psi'\bigl(ru_1+(1-r)u_2\bigr)\,dr\right|\\&\quad\le\frac{u_2-u_1}{2}\int_0^1|1-2r|\bigl|\psi'\bigl(ru_1+(1-r)u_2\bigr)\bigr|\,dr.\end{aligned}\tag{3.9}$$

Using the exponential s-convexity of ψ', we get

$$\begin{aligned}&\left|\frac{\psi(u_1)+\psi(u_2)}{2}-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\psi(w)\,dw\right|\\&\quad\le\frac{u_2-u_1}{2}\int_0^1|1-2r|\left[r^s\left|\frac{\psi'(u_1)}{e^{\alpha u_1}}\right|+(1-r)^s\left|\frac{\psi'(u_2)}{e^{\alpha u_2}}\right|\right]dr\\&\quad\le\frac{u_2-u_1}{2}\int_0^1(1+2r)\left[r^s\left|\frac{\psi'(u_1)}{e^{\alpha u_1}}\right|+(1-r)^s\left|\frac{\psi'(u_2)}{e^{\alpha u_2}}\right|\right]dr\\&\quad=\frac{u_2-u_1}{2}\int_0^1\left[(1+2r)r^s\left|\frac{\psi'(u_1)}{e^{\alpha u_1}}\right|+(1+2r)(1-r)^s\left|\frac{\psi'(u_2)}{e^{\alpha u_2}}\right|\right]dr.\end{aligned}\tag{3.10}$$

Since

$$\int_0^1(1+2r)r^s\,dr=\frac{3s+4}{(s+1)(s+2)},\tag{3.11}$$

$$\int_0^1(1+2r)(1-r)^s\,dr=\frac{s+4}{(s+1)(s+2)},\tag{3.12}$$

by substituting equalities (3.11) and (3.12) into (3.10), we get inequality (3.8). □

Corollary 3.4 *Under the assumptions of Theorem 3.3, we have the following:*

(a) *If $s=1$, then*

$$\begin{aligned}&\left|\frac{\psi(u_1)+\psi(u_2)}{2}-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\psi(w)\,dw\right|\\&\quad\le\frac{u_2-u_1}{12}\left[7\left|\frac{\psi'(u_1)}{e^{\alpha u_1}}\right|+5\left|\frac{\psi'(u_2)}{e^{\alpha u_2}}\right|\right].\end{aligned}\tag{3.13}$$

(b) *If* $\alpha = 0$, *then*

$$\left|\frac{\psi(u_1)+\psi(u_2)}{2} - \frac{1}{u_2-u_1}\int_{u_1}^{u_2}\psi(w)\,dw\right| \le \frac{u_2-u_1}{2(s+1)(s+2)}\left[(3s+4)\left|\psi'(u_1)\right| + (s+4)\left|\psi'(u_2)\right|\right]. \tag{3.14}$$

Theorem 3.5 *Let* $\psi : \mathcal{K} \to \mathbb{R}$ *be a differentiable function on* $\mathcal{K}^\circ$ *and* $u_1, u_2 \in \mathcal{K}$, $u_1 < u_2$, *and* $\psi' \in L_1[u_1, u_2]$. *If* $|\psi'|$ *is exponentially s-convex in the second sense on* $[u_1, u_2]$, *then*

$$\left|\frac{\psi(u_1)+\psi(u_2)}{2} - \frac{1}{u_2-u_1}\int_{u_1}^{u_2}\psi(w)\,dw\right| \le \frac{u_2-u_1}{2}\frac{1}{(s+1)(s+2)}\left(s+\frac{1}{2^s}\right)\left[\left|\frac{\psi'(u_1)}{e^{\alpha u_1}}\right| + \left|\frac{\psi'(u_2)}{e^{\alpha u_2}}\right|\right]. \tag{3.15}$$

Proof From Lemma 1.2, we have

$$\begin{aligned}\left|\frac{\psi(u_1)+\psi(u_2)}{2} - \frac{1}{u_2-u_1}\int_{u_1}^{u_2}\psi(w)\,dw\right| &= \frac{u_2-u_1}{2}\left|\int_0^1 (1-2r)\psi'\big(ru_1+(1-r)u_2\big)\,dr\right| \\ &\le \frac{u_2-u_1}{2}\int_0^1 |1-2r|\left|\psi'\big(ru_1+(1-r)u_2\big)\right|dr.\end{aligned} \tag{3.16}$$

Using the exponential s-convexity of ψ', we get

$$\begin{aligned}\left|\frac{\psi(u_1)+\psi(u_2)}{2} - \frac{1}{u_2-u_1}\int_{u_1}^{u_2}\psi(w)\,dw\right| &\le \frac{u_2-u_1}{2}\int_0^1 |1-2r|\left[r^s\left|\frac{\psi(u_1)}{e^{\alpha u_1}}\right| + (1-r)^s\left|\frac{\psi(u_2)}{e^{\alpha u_2}}\right|\right]dr \\ &= \frac{u_2-u_1}{2}\int_0^1\left[|1-2r|r^s\left|\frac{\psi(u_1)}{e^{\alpha u_1}}\right| + |1+2r|(1-r)^s\left|\frac{\psi(u_2)}{e^{\alpha u_2}}\right|\right]dr \\ &= \frac{u_2-u_1}{2}\left[C_1(s)\left|\frac{\psi'(u_1)}{e^{\alpha u_1}}\right| + C_2(s)\left|\frac{\psi'(u_2)}{e^{\alpha u_2}}\right|\right].\end{aligned} \tag{3.17}$$

It is easily seen that

$$C_1(s) = \int_0^1 |1-2r|r^s\,dr = \frac{s}{(s+1)(s+2)} + \frac{1}{2^s(s+1)(s+2)}, \tag{3.18}$$

$$C_2(s) = \int_0^1 |1-2r|(1-r)^s\,dr = \frac{s}{(s+1)(s+2)} + \frac{1}{2^s(s+1)(s+2)}. \tag{3.19}$$

Thus by substituting equalities (3.18) and (3.19) into (3.17), we achieve inequality (3.15). □

Remark 3.2 In Theorem 3.5,

(a) by taking $\alpha = 0$, we obtain Theorem 1, for $q = 1$, in [23];

(b) by taking $s = 1$, we obtain Theorem 3 in [4].

Theorem 3.6 *Let $\psi : \mathcal{K} \to \mathbb{R}$ be a differentiable function on $\mathcal{K}^\circ$ and $u_1, u_2 \in \mathcal{K}$ with $u_1 < u_2$ and $\psi' \in L_1[u_1, u_2]$. If $|\psi'|^q$ is exponentially s-convex in the second sense on $[u_1, u_2]$ with $q > 1$, then we have*

$$\left|\frac{\psi(u_1)+\psi(u_2)}{2} - \frac{1}{u_2-u_1}\int_{u_1}^{u_2}\psi(w)\,dw\right| \leq \frac{u_2-u_1}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{s+\frac{1}{2^s}}{(s+1)(s+2)}\right)^{\frac{1}{q}}\left[\left|\frac{\psi'(u_1)}{e^{\alpha u_1}}\right|^q + \left|\frac{\psi'(u_2)}{e^{\alpha u_2}}\right|^q\right]^{\frac{1}{q}}. \tag{3.20}$$

Proof From Lemma 1.2, we have

$$\begin{aligned}\left|\frac{\psi(u_1)+\psi(u_2)}{2} - \frac{1}{u_2-u_1}\int_{u_1}^{u_2}\psi(w)\,dw\right| &= \frac{u_2-u_1}{2}\left|\int_0^1 (1-2r)\psi'\big(ru_1+(1-r)u_2\big)\,dr\right| \\ &\leq \frac{u_2-u_1}{2}\int_0^1 |1-2r|\left|\psi'\big(ru_1+(1-r)u_2\big)\right|\,dr. \end{aligned}\tag{3.21}$$

Applying the power-mean inequality, we find

$$\begin{aligned}&\frac{u_2-u_1}{2}\int_0^1 |1-2r|\left|\psi'\big(ru_1+(1-r)u_2\big)\right|\,dr \\ &\quad\leq \frac{u_2-u_1}{2}\left(\int_0^1 |1-2r|\,dr\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2r|\left|\psi'\big(ru_1+(1-r)u_2\big)\right|^q\,dr\right)^{\frac{1}{q}}.\end{aligned}\tag{3.22}$$

Since $|\psi'|^q$ is exponentially s-convex, we get

$$\begin{aligned}&\int_0^1 |1-2r|\left|\psi'\big(ru_1+(1-r)u_2\big)\right|^q\,dr \\ &\quad\leq \int_0^1 |1-2r|\left[r^s\left|\frac{\psi'(u_1)}{e^{\alpha u_1}}\right|^q + (1-r)^s\left|\frac{\psi'(u_2)}{e^{\alpha u_2}}\right|^q\right]dr \\ &\quad= \left[C_1(s)\left|\frac{\psi'(u_1)}{e^{\alpha u_1}}\right|^q + C_2(s)\left|\frac{\psi'(u_2)}{e^{\alpha u_2}}\right|^q\right],\end{aligned}\tag{3.23}$$

where

$$\int_0^1 |1-2r|\,dr = \frac{1}{2}. \tag{3.24}$$

Using (3.22)–(3.24) in (3.21), we get (3.20). □

Remark 3.3 In Theorem 3.6,

(a) by putting $\alpha = 0$, we get Theorem 1, for $q > 1$, in [23];
(b) by putting $s = 1$, we get Theorem 5 in [4].

Theorem 3.7 *Let $\psi : \mathcal{K} \to \mathbb{R}$ be a differentiable function on $\mathcal{K}^\circ$ and $u_1, u_2 \in \mathcal{K}$ with $u_1 < u_2$ and $\psi' \in L_1[u_1, u_2]$. If $|\psi'|^q$ is exponentially s-convex in the second sense on $[u_1, u_2]$ and*

$q, l > 1$, $\frac{1}{l} + \frac{1}{q} = 1$, then we have

$$\left|\frac{\psi(u_1) + \psi(u_2)}{2} - \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} \psi(w)\,dw\right| \leq \frac{u_2 - u_1}{2(l+1)^{\frac{1}{l}}} \left[\frac{|\frac{\psi'(u_1)}{e^{\alpha u_1}}|^q + |\frac{\psi'(u_2)}{e^{\alpha u_2}}|^q}{s+1}\right]^{\frac{1}{q}}. \tag{3.25}$$

Proof From Lemma 1.2 and using Hölder's inequality, we have

$$\left|\frac{\psi(u_1) + \psi(u_2)}{2} - \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} \psi(w)\,dw\right| \leq \frac{u_2 - u_1}{2} \left(\int_0^1 |1 - 2r|^l \,dr\right)^{\frac{1}{l}} \left(\int_0^1 \left|\psi'(ru_1 + (1-r)u_2)\right|^q \,dr\right)^{\frac{1}{q}}. \tag{3.26}$$

Since $|\psi'|^q$ is exponentially s-convex, we get

$$\begin{aligned}
&\left|\frac{\psi(u_1) + \psi(u_2)}{2} - \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} \psi(w)\,dw\right| \\
&\quad \leq \frac{u_2 - u_1}{2} \left(\int_0^1 |1 - 2r|^l \,dr\right)^{\frac{1}{l}} \left(\int_0^1 \left[r^s \left|\frac{\psi'(u_1)}{e^{\alpha u_1}}\right|^q + (1-r)^s \left|\frac{\psi'(u_2)}{e^{\alpha u_2}}\right|^q\right]\right)^{\frac{1}{q}} \\
&\quad = \frac{u_2 - u_1}{2(l+1)^{\frac{1}{l}}} \left[\frac{|\frac{\psi'(u_1)}{e^{\alpha u_1}}|^q + |\frac{\psi'(u_2)}{e^{\alpha u_2}}|^q}{s+1}\right]^{\frac{1}{q}}.
\end{aligned} \tag{3.27}$$

Hence the proof is completed. □

Remark 3.4 In Theorem 3.7,

(a) by letting $\alpha = 0$, we get

$$\left|\frac{\psi(u_1) + \psi(u_2)}{2} - \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} \psi(w)\,dw\right| \leq \frac{u_2 - u_1}{2(l+1)^{\frac{1}{l}}} \left[\frac{|\psi'(u_1)|^q + |\psi'(u_2)|^q}{s+1}\right]^{\frac{1}{q}}; \tag{3.28}$$

(b) by letting $s = 1$, we get Theorem 4 in [4].

3.2 Applications

Suppose d is a partition of the interval $[u_1, u_2]$, that is, $d : u_1 = w_0 < w_1 < \cdots < w_{m-1} < w_m = u_2$, then the trapezoidal formula is given as

$$T(\psi, d) = \sum_{n=0}^{m-1} \frac{\psi(w_n) + \psi(w_{n+1})}{2}(w_{n+1} - w_n).$$

We known that if $\psi : [u_1, u_2] \to \mathbb{R}$ is twice differentiable on (u_1, u_2) and $\mathcal{M} = \max_{w \in (u_1, u_2)} |\psi''(w)| < \infty$, then

$$\int_{u_1}^{u_2} \psi(w)\,dw = T(\psi, d) + R(\psi, d), \tag{3.29}$$

where the remainder term is given as

$$\bigl|R(\psi,d)\bigr| \leq \frac{\mathcal{M}}{12}\sum_{n=0}^{m-1}(w_{n+1}-w_n)^3. \tag{3.30}$$

It is noticed that if ψ'' does not exist or ψ'' is unbounded, then (3.29) is invalid. However, Dragomir and Wang [10–12] have shown that the term $R(\psi,d)$ can be obtained by using the first derivative only. These estimates surely have several applications. In this section, we estimate the remainder term $R(\psi,d)$ in a new sense.

Proposition 3.1 *Let $\psi:\mathcal{K}\subseteq\mathbb{R}_0\to\mathbb{R}$ be a differentiable function on $\mathcal{K}^\circ$. Let $u_1,u_2\in\mathcal{K}$, $u_1<u_2$. If $|\psi'|$ is exponentially s-convex in the second sense on $[u_1,u_2]$ and $s\in(0,1]$, then in (3.29), for every partition d of $[u_1,u_2]$, we have*

$$\begin{aligned}\bigl|R(\psi,d)\bigr| &\leq \frac{1}{2}\frac{1}{(s+1)(s+2)}\left(s+\frac{1}{2^s}\right)\sum_{n=0}^{m-1}(w_{n+1}-w_n)^2\left[\left|\frac{\psi'(w_n)}{e^{\alpha w_n}}\right|+\left|\frac{\psi'(w_{n+1})}{e^{\alpha w_{n+1}}}\right|\right]\\ &\leq \max\left\{\left|\frac{\psi'(u_1)}{e^{\alpha u_1}}\right|,\left|\frac{\psi'(u_2)}{e^{\alpha u_2}}\right|\right\}\\ &\quad\times\frac{1}{(s+1)(s+2)}\left(s+\frac{1}{2^s}\right)\sum_{n=0}^{m-1}(w_{n+1}-w_n)^2.\end{aligned} \tag{3.31}$$

Proof Applying Theorem 3.5 on the subinterval $[w_n,w_{n+1}]$ $(n=0,1,\dots,m-1)$ of the partition d, we obtain

$$\begin{aligned}&\left|\frac{\psi(w_n)+\psi(w_{n+1})}{2}(w_{n+1}-w_n)-\int_{w_n}^{w_{n+1}}\psi(w)\,dw\right|\\ &\quad\leq\frac{(w_{n+1}-w_n)^2}{2}\frac{1}{(s+1)(s+2)}\left(s+\frac{1}{2^s}\right)\left[\left|\frac{\psi'(w_n)}{e^{\alpha w_n}}\right|+\left|\frac{\psi'(w_{n+1})}{e^{\alpha w_{n+1}}}\right|\right].\end{aligned} \tag{3.32}$$

Summing over n from 0 to $m-1$, we get

$$\begin{aligned}&\left|T(\psi,d)-\int_{u_1}^{u_2}\psi(w)\,dw\right|\\ &\quad\leq\frac{1}{2}\sum_{n=0}^{m-1}(w_{n+1}-w_n)^2\frac{1}{(s+1)(s+2)}\left(s+\frac{1}{2^s}\right)\left[\left|\frac{\psi'(w_n)}{e^{\alpha w_n}}\right|+\left|\frac{\psi'(w_{n+1})}{e^{\alpha w_{n+1}}}\right|\right]\\ &\quad\leq\max\left\{\left|\frac{\psi'(u_1)}{e^{\alpha u_1}}\right|,\left|\frac{\psi'(u_2)}{e^{\alpha u_2}}\right|\right\}\frac{1}{(s+1)(s+2)}\left(s+\frac{1}{2^s}\right)\sum_{n=0}^{m-1}(w_{n+1}-w_n)^2.\end{aligned} \tag{3.33}$$

□

Proposition 3.2 *Let $\psi:\mathcal{K}\subseteq\mathbb{R}_0\to\mathbb{R}$ be a differentiable function on $\mathcal{K}^\circ$ and $u_1,u_2\in\mathcal{K}$ with $u_1<u_2$. If $|\psi'|^q$ is exponentially s-convex in the second sense on $[u_1,u_2]$ and $s\in(0,1]$*

and $q,l>1$ such that $\frac{1}{l}+\frac{1}{q}=1$, then in (3.29), *for every partition d of $[u_1,u_2]$, we have*

$$\begin{aligned}\left|R(\psi,d)\right| &\leq \frac{1}{2(l+1)^{\frac{1}{l}}}\sum_{n=0}^{m-1}(w_{n+1}-w_n)^2\left[\frac{|\frac{\psi'(w_n)}{e^{\alpha w_n}}|^q+|\frac{\psi'(w_{n+1})}{e^{\alpha w_{n+1}}}|^q}{s+1}\right]^{\frac{1}{q}} \\ &\leq \frac{\max\{\frac{2|\frac{\psi'(u_1)}{e^{\alpha u_1}}|}{s+1},\frac{2|\frac{\psi'(u_2)}{e^{\alpha u_2}}|}{s+1}\}}{2(l+1)^{\frac{1}{l}}}\sum_{n=0}^{m-1}(w_{n+1}-w_n)^2. \end{aligned} \tag{3.34}$$

Proof Using Theorem 3.7 and similar arguments as in Proposition 3.1, we get the required result. □

Funding
The present investigation is supported by National University of Science and Technology (NUST), Islamabad, Pakistan.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally to this work. Both authors read and approved the final manuscript.

Publisher's Note

Received: 27 September 2018 Accepted: 28 March 2019 Published online: 03 April 2019

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