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# HERMITE-HADAMARD TYPE INEQUALITIES FOR GA-*s*-CONVEX FUNCTIONS

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In this paper, the author introduces the concepts of the GA-*s*-convex functions in the first sense and second sense and establishes some integral inequalities of Hermite-Hadamard type related to the GA-*s*-convex functions. Some applications to special means of real numbers are also given.

## 1. Introduction

In this section, we firstly list several definitions and some known results.

The following concept was introduced by Orlicz in [11]:

**Definition 1.1.** Let $0 < s \leq 1$. A function $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ where $\mathbb{R}_0 = [0,\infty)$, is said to be *s*-convex in the first sense if

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y)$$

for all $x, y \in I$ and $\alpha, \beta \geq 0$ with $\alpha^s + \beta^s = 1$. We denote this class of real functions by $K_s^1$.

In [4], Hudzik and Maligranda considered the following class of functions:

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**Definition 1.2.** A function $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ where $\mathbb{R}_0 = [0,\infty)$, is said to be $s$-convex in the second sense if

$$f(\alpha x+\beta y) \leq \alpha^s f(x)+\beta^s f(y)$$

for all $x,y \in I$ and $\alpha,\beta \geq 0$ with $\alpha+\beta = 1$ and $s$ fixed in $(0,1]$. These authors denoted the set of such functions by $K_s^2$.

It can be easily seen that for $s = 1$, $s$-convexity reduces to ordinary convexity of functions defined on $[0,\infty)$.

In [2], Dragomir and Fitzpatrick proved a variant of Hermite-Hadamard inequality which holds for the $s$-convex functions.

**Theorem 1.3.** *Suppose that $f : \mathbb{R}_0 \to \mathbb{R}_0$ is an s-convex function in the second sense, where $s \in (0,1]$ and let $a,b \in [0,\infty)$, $a < b$. If $f \in L[a,b]$, then the following inequalities hold*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{s+1}. \tag{1}$$

*the constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1).*

The above inequalities are sharp. For recent results and generalizations concerning $s$-convex functions see [1, 2, 5, 6, 8].

**Definition 1.4** ([9, 10])**.** A function $f : I \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ is said to be a GA-convex function on $I$ if

$$f(x^t y^{1-t}) \leq t f(x) + (1-t) f(y)$$

holds for all $x,y \in I$ and $t \in [0,1]$, where $x^t y^{1-t}$ and $t f(x) + (1-t) f(y)$ are respectively called the weighted geometric mean of two positive numbers $x$ and $y$ and the weighted arithmetic mean of $f(x)$ and $f(y)$.

For $b > a > 0$, let $G(a,b) = \sqrt{ab}$, $L(a,b) = (b-a)/(\ln b - \ln a)$, $I(a,b) = (1/e)\left(b^b/a^a\right)^{1/(b-a)}$, $A(a,b) = \frac{a+b}{2}$ and $L_p(a,b) = \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}$, $p \in \mathbb{R}\backslash\{-1,0\}$, be the geometric, logarithmic, identric, arithmetic and $p$-logarithmic means of $a$ and $b$, respectively. Then

$$\min\{a,b\} < G(a,b) < L(a,b) < I(a,b) < A(a,b) < \max\{a,b\}.$$

In [14], Zhang et al. established some Hermite-Hadamard type integral inequalities for GA-convex functions and applied these inequalities to construct several inequalities for special means and they used the following lemma to prove their results:

**Lemma 1.5.** *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable mapping on $I^\circ$, and $a,b \in I^\circ$, with $a < b$. If $f' \in L[a,b]$, then*

$$bf(b) - af(a) - \int_a^b f(x)dx = (\ln b - \ln a)\int_0^1 b^{2t}a^{2(1-t)}f'\left(b^t a^{1-t}\right)dt.$$

Also, the main inequalities in [14] are pointed out as follows:

**Theorem 1.6.** *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be differentiable on $I^\circ$, and $a,b \in I$ with $a < b$ and $f' \in L[a,b]$. If $|f'|^q$ is GA-convex on $[a,b]$ for $q \geq 1$, then*

$$\left|bf(b) - af(a) - \int_a^b f(x)dx\right| \leq \frac{[(b-a)A(a,b)]^{1-1/q}}{2^{1/q}}$$
$$\times\left\{\left[L(a^2,b^2) - a^2\right]\left|f'(a)\right|^q + \left[b^2 - L(a^2,b^2)\right]\left|f'(b)\right|^q\right\}^{1/q}.$$

**Theorem 1.7.** *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be differentiable on $I^\circ$, and $a,b \in I$ with $a < b$ and $f' \in L[a,b]$. If $|f'|^q$ is GA-convex on $[a,b]$ for $q > 1$, then*

$$\left|bf(b) - af(a) - \int_a^b f(x)dx\right| \leq (\ln b - \ln a)$$
$$\times\left[L(a^{2q/(q-1)},b^{2q/(q-1)}) - a^{2q/(q-1)}\right]^{1-1/q}\left[A\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)\right]^{1/q}.$$

**Theorem 1.8.** *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be differentiable on $I^\circ$, and $a,b \in I$ with $a < b$ and $f' \in L[a,b]$. If $|f'|^q$ is GA-convex on $[a,b]$ for $q > 1$ and $2q > p > 0$, then*

$$\left|bf(b) - af(a) - \int_a^b f(x)dx\right|$$
$$\leq \frac{(\ln b - \ln a)^{1-1/q}}{p^{1/q}} \times \left[L(a^{(2q-p)/(q-1)},b^{(2q-p)/(q-1)})\right]^{1-1/q}$$
$$\times\left\{[L(a^p,b^p) - a^p]\left|f'(a)\right|^q + [b^p - L(a^p,b^p)]\left|f'(b)\right|^q\right\}^{1/q}.$$

In [13], Zhang et al. established the following Hermite-Hadamard type inequality for GA-convex (concave) functions:

**Theorem 1.9.** *If $b > a > 0$ and $f : [a,b] \to \mathbb{R}$ is a differentiable GA-convex (concave) function then*

$$f(I(a,b)) \leq (\geq)\frac{1}{b-a}\int_a^b f(x)dx \leq (\geq)\frac{b-L(a,b)}{b-a}f(b) + \frac{L(a,b)-a}{b-a}f(a).$$

In [7], the author proved the following identity and established some new Hermite-Hadamard-like type inequalities for the geometrically convex functions.

**Lemma 1.10.** *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable mapping on $I^\circ$, and $a,b \in I$,with $a < b$. If $f' \in L[a,b]$, then*

$$f\left(\sqrt{ab}\right) - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx =$$

$$\frac{(\ln b - \ln a)}{4}\left[a\int_0^1 t\left(\frac{b}{a}\right)^{\frac{t}{2}} f'\left(a^{1-t}(ab)^{\frac{t}{2}}\right)dt - b\int_0^1 t\left(\frac{a}{b}\right)^{\frac{t}{2}} f'\left(b^{1-t}(ab)^{\frac{t}{2}}\right)dt\right]$$

*and*

$$\frac{f(a)+f(b)}{2} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx$$

$$= \frac{(\ln b - \ln a)}{2}\left[a\int_0^1 t\left(\frac{b}{a}\right)^{t} f'\left(a^{1-t}b^t\right)dt - b\int_0^1 t\left(\frac{a}{b}\right)^{t} f'\left(b^{1-t}a^t\right)dt\right]$$

In this paper, we will give concepts GA-$s$-convex functions in the first and second sense and establish some new integral inequalities of Hermite-Hadamard like type for these classes of functions by using Lemma 1.10.

## 2. Definitions of GA-$s$-convex functions in the first and second sense

Now it is time to introduce two concepts, GA-$s$-convex functions in the first and second sense.

**Definition 2.1.** Let $0 < s \leq 1$. A function $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ is said to be a GA-$s$-convex (concave) function in the first sense on $I$ if

$$f(x^t y^{1-t}) \leq (\geq)\ t^s f(x) + (1-t^s) f(y)$$

holds for all $x,y \in I$ and $t \in [0,1]$.

**Definition 2.2** ([12, Definition 2.1])**.** Let $0 < s \leq 1$. A function $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ is said to be a GA-$s$-convex (concave) function in the second sense on $I$ if

$$f(x^t y^{1-t}) \leq (\geq)\ t^s f(x) + (1-t)^s f(y)$$

holds for all $x,y \in I$ and $t \in [0,1]$.

It is clear that when $s = 1$, GA-$s$-convex functions in the first and second sense become GA-convex functions.

## 3. Inequalities for GA-$s$-convex functions in the first and second sense

Let $u,n>0,\ m,r\geq 0$ and $q\geq 1$. Throughout this section we will take

$$C(u,m,n,r,q)=\int_0^1 t^m(1-t^n)^r u^{qt}\,dt.$$

Now we are in a position to establish some inequalities of Hermite–Hadamard type for GA-$s$-convex functions in the first and second sense

**Theorem 3.1.** *Let $0<s\leq 1$. Suppose that $f:I\subseteq\mathbb{R}_+\to\mathbb{R}$ is GA-$s$-convex function in the first sense and $a,b\in I$ with $a<b$. If $f\in L[a,b]$, then one has the inequalities:*

$$f\left(\sqrt{ab}\right)\leq\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}dx\leq\frac{f(a)+sf(b)}{s+1} \tag{2}$$

*Proof.* As $f$ is GA-$s$-convex function in the first sense, we have, for all $x,y\in I$

$$f\left(\sqrt{xy}\right)\leq\frac{1}{2^s}f(x)+\left(1-\frac{1}{2^s}\right)f(y). \tag{3}$$

Now, let $x=a^{1-t}b^t$ and $y=a^tb^{1-t}$ with $t\in[0,1]$. Then we get by (3) that:

$$f\left(\sqrt{ab}\right)\leq\frac{1}{2^s}f(a^{1-t}b^t)+\left(1-\frac{1}{2^s}\right)f(a^tb^{1-t})$$

for all $t\in[0,1]$. Integrating this inequality on $[0,1]$, we deduce the first part of (2).

Secondly, we observe that for all $t\in[0,1]$

$$f(a^tb^{1-t})\leq t^sf(a)+(1-t^s)f(b).$$

Integrating this inequality on $[0,1]$, we get

$$\int_0^1 f(a^tb^{1-t})dt\leq\frac{f(a)+sf(b)}{s+1}.$$

As the change of variable $x=a^tb^{1-t}$ gives us that

$$\int_0^1 f(a^tb^{1-t})dt=\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}dx,$$

the second inequality in (2) is proved. □

**Remark 3.2.** The constant $k=1/(s+1)$ for $s\in(0,1]$ is the best possible in the second inequality in (2). Indeed, as the mapping $f:[a,b]\to\mathbb{R}$ given $f(x)=s+1$, $0<a<b$, is GA-$s$-convex in the first sense and

$$\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx=s+1=\frac{f(a)+sf(b)}{s+1}$$

Similarly to Theorem 3.1, we will give the following theorem for GA-$s$-convex function in the second sense:

**Theorem 3.3.** *Suppose that $f:I\subseteq\mathbb{R}_+\to\mathbb{R}$ is GA-$s$-convex function in the second sense and $a,b\in I$ with $a<b$. If $f\in L[a,b]$, then one has the inequalities:*

$$2^{s-1}f\left(\sqrt{ab}\right)\leq\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx\leq\frac{f(a)+f(b)}{s+1} \tag{4}$$

*Proof.* As $f$ is GA-$s$-convex function in the second sense, we have, for all $x,y\in I$

$$f\left(\sqrt{xy}\right)\leq\frac{f(x)+f(y)}{2^s}. \tag{5}$$

Now, let $x=a^{1-t}b^t$ and $y=a^tb^{1-t}$ with $t\in[0,1]$. Then we get by (5) that:

$$f\left(\sqrt{ab}\right)\leq\frac{f(a^{1-t}b^t)+f(a^tb^{1-t})}{2^s}$$

for all $t\in[0,1]$. Integrating this inequality on $[0,1]$, we deduce the first part of (4).

Secondly, we observe that for all $t\in[0,1]$

$$f(a^tb^{1-t})\leq t^sf(a)+(1-t)^sf(b).$$

Integrating this inequality on $[0,1]$, we get

$$\int_0^1 f(a^tb^{1-t})dt\leq\frac{f(a)+f(b)}{s+1}.$$

As the change of variable $x=a^tb^{1-t}$ gives us that

$$\int_0^1 f(a^tb^{1-t})dt=\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx,$$

the second inequality in (4) is proved. □

**Theorem 3.4.** *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be differentiable on $I^\circ$, and $a,b \in I^\circ$ with $a < b$ and $f' \in L[a,b]$.*

*a) If $|f'|^q$ is GA-s-convex function in the second sense on $[a,b]$ for $q \geq 1$ and $s \in (0,1]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx\right| \leq \ln\left(\frac{b}{a}\right)\left(\frac{1}{2}\right)^{2-\frac{1}{q}}$$
$$\times\left[a\left\{C\left(\frac{b}{a},1,1,s,q\right)\left|f'(a)\right|^q + C\left(\frac{b}{a},s+1,1,0,q\right)\left|f'(b)\right|^q\right\}^{\frac{1}{q}}\right. \tag{6}$$
$$\left.+b\left\{C\left(\frac{a}{b},1,1,s,q\right)\left|f'(b)\right|^q + C\left(\frac{a}{b},s+1,1,0,q\right)\left|f'(a)\right|^q\right\}^{\frac{1}{q}}\right],$$

$$\left|f\left(\sqrt{ab}\right) - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx\right| \leq \ln\left(\frac{b}{a}\right)\left(\frac{1}{2}\right)^{3-\frac{1}{q}}$$
$$\times\left[a\left\{C\left(\frac{b}{a},1,1,s,\frac{q}{2}\right)\left|f'(a)\right|^q + C\left(\frac{b}{a},s+1,1,0,\frac{q}{2}\right)\left|f'\left(\sqrt{ab}\right)\right|^q\right\}^{\frac{1}{q}}\right. \tag{7}$$
$$\left.+b\left\{C\left(\frac{a}{b},1,1,s,\frac{q}{2}\right)\left|f'(b)\right|^q + C\left(\frac{a}{b},s+1,1,0,\frac{q}{2}\right)\left|f'\left(\sqrt{ab}\right)\right|^q\right\}^{\frac{1}{q}}\right].$$

*b) If $|f'|^q$ is GA-s-convex function in the first sense on $[a,b]$ for $q \geq 1$ and $s \in (0,1]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx\right| \leq \ln\left(\frac{b}{a}\right)\left(\frac{1}{2}\right)^{2-\frac{1}{q}}$$
$$\times\left[a\left\{C\left(\frac{b}{a},1,s,1,q\right)\left|f'(a)\right|^q + C\left(\frac{b}{a},s+1,1,0,q\right)\left|f'(b)\right|^q\right\}^{\frac{1}{q}}\right. \tag{8}$$
$$\left.+b\left\{C\left(\frac{a}{b},1,s,1,q\right)\left|f'(b)\right|^q + C\left(\frac{a}{b},s+1,1,0,q\right)\left|f'(a)\right|^q\right\}^{\frac{1}{q}}\right],$$

$$\left|f\left(\sqrt{ab}\right) - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx\right| \leq \ln\left(\frac{b}{a}\right)\left(\frac{1}{2}\right)^{3-\frac{1}{q}}$$
$$\times\left[a\left\{C\left(\frac{b}{a},1,s,1,\frac{q}{2}\right)\left|f'(a)\right|^q + C\left(\frac{b}{a},s+1,1,0,\frac{q}{2}\right)\left|f'\left(\sqrt{ab}\right)\right|^q\right\}^{\frac{1}{q}}\right. \tag{9}$$
$$\left.+b\left\{C\left(\frac{a}{b},1,s,1,\frac{q}{2}\right)\left|f'(b)\right|^q + C\left(\frac{a}{b},s+1,1,0,\frac{q}{2}\right)\left|f'\left(\sqrt{ab}\right)\right|^q\right\}^{\frac{1}{q}}\right].$$

*Proof.* $a)$ (1) Since $|f'|^q$ is GA-$s$-convex function in the second sense on $[a,b]$, from lemma 1.10 and power mean inequality, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx\right|$$
$$\leq \frac{\ln\left(\frac{b}{a}\right)}{2}\left[a\int_0^1 t\left(\frac{b}{a}\right)^t\left|f'\left(a^{1-t}b^t\right)\right|dt+b\int_0^1 t\left(\frac{a}{b}\right)^t\left|f'\left(b^{1-t}a^t\right)\right|dt\right]$$
$$\leq \frac{a\ln\left(\frac{b}{a}\right)}{2}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left(\frac{b}{a}\right)^{qt}\left|f'\left(a^{1-t}b^t\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{b\ln\left(\frac{b}{a}\right)}{2}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left(\frac{a}{b}\right)^{qt}\left|f'\left(b^{1-t}a^t\right)\right|^q dt\right)^{\frac{1}{q}} \tag{10}$$
$$\leq \frac{a\ln\left(\frac{b}{a}\right)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left(\frac{b}{a}\right)^{qt}\left((1-t)^s\left|f'(a)\right|^q+t^s\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}$$
$$+\frac{b\ln\left(\frac{b}{a}\right)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left(\frac{a}{b}\right)^{qt}\left((1-t)^s\left|f'(b)\right|^q+t^s\left|f'(a)\right|^q\right)dt\right)^{\frac{1}{q}}$$
$$\leq a\ln\left(\frac{b}{a}\right)\left(\frac{1}{2}\right)^{2-\frac{1}{q}}\left\{C\left(\frac{b}{a},1,1,s,q\right)\left|f'(a)\right|^q+C\left(\frac{b}{a},s+1,1,0,q\right)\left|f'(b)\right|^q\right\}^{\frac{1}{q}}$$
$$+b\ln\left(\frac{b}{a}\right)\left(\frac{1}{2}\right)^{2-\frac{1}{q}}\left\{C\left(\frac{a}{b},1,1,s,q\right)\left|f'(b)\right|^q+C\left(\frac{a}{b},s+1,1,0,q\right)\left|f'(a)\right|^q\right\}^{\frac{1}{q}}.$$

(2) Since $|f'|^q$ is GA-$s$-convex function in the second sense on $[a,b]$, from lemma 3.4 and power mean inequality, we have

$$\left|f\left(\sqrt{ab}\right)-\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx\right|$$
$$\leq \frac{\ln\frac{b}{a}}{4}\left[a\int_0^1 t\left(\frac{b}{a}\right)^{\frac{t}{2}}\left|f'\left(a^{1-t}(ab)^{\frac{t}{2}}\right)\right|dt+b\int_0^1 t\left(\frac{a}{b}\right)^{\frac{t}{2}}\left|f'\left(b^{1-t}(ab)^{\frac{t}{2}}\right)\right|dt\right]$$
$$\leq \frac{a\ln\frac{b}{a}}{4}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left(\frac{b}{a}\right)^{\frac{qt}{2}}\left|f'\left(a^{1-t}(ab)^{\frac{t}{2}}\right)\right|^q dt\right)^{\frac{1}{q}}+$$

$$+\frac{b\ln\frac{b}{a}}{4}\left(\int_0^1 t\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left(\frac{a}{b}\right)^{\frac{qt}{2}}\left|f'\left(b^{1-t}(ab)^{\frac{t}{2}}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq\frac{a\ln\frac{b}{a}}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left(\frac{b}{a}\right)^{\frac{qt}{2}}\left((1-t)^s\left|f'(a)\right|^q+t^s\left|f'\left(\sqrt{ab}\right)\right|^q\right)dt\right)^{\frac{1}{q}}$$

$$+\frac{b\ln\frac{b}{a}}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left(\frac{a}{b}\right)^{\frac{qt}{2}}\left((1-t)^s\left|f'(b)\right|^q+t^s\left|f'\left(\sqrt{ab}\right)\right|^q\right)dt\right)^{\frac{1}{q}}$$

$$\leq\ln\left(\frac{b}{a}\right)\left(\frac{1}{2}\right)^{3-\frac{1}{q}}\left[a\left\{C\left(\frac{b}{a},1,1,s,\frac{q}{2}\right)\left|f'(a)\right|^q\right.\right. \tag{11}$$

$$\left.+C\left(\frac{b}{a},s+1,1,0,\frac{q}{2}\right)\left|f'\left(\sqrt{ab}\right)\right|^q\right\}^{\frac{1}{q}}$$

$$\left.+b\left\{C\left(\frac{a}{b},1,1,s,\frac{q}{2}\right)\left|f'(b)\right|^q+C\left(\frac{a}{b},s+1,1,0,\frac{q}{2}\right)\left|f'\left(\sqrt{ab}\right)\right|^q\right\}^{\frac{1}{q}}\right],$$

$b)$ (1) Since $|f'|^q$ is GA-$s$-convex function in the first sense on $[a,b]$, from the inequality (10), we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}dx\right|$$

$$\leq\frac{a\ln\left(\frac{b}{a}\right)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left(\frac{b}{a}\right)^{qt}\left((1-t^s)\left|f'(a)\right|^q+t^s\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}$$

$$+\frac{b\ln\left(\frac{b}{a}\right)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left(\frac{a}{b}\right)^{qt}\left((1-t^s)\left|f'(b)\right|^q+t^s\left|f'(a)\right|^q\right)dt\right)^{\frac{1}{q}}$$

$$\leq a\ln\left(\frac{b}{a}\right)\left(\frac{1}{2}\right)^{2-\frac{1}{q}}\left\{C\left(\frac{b}{a},1,s,1,q\right)\left|f'(a)\right|^q+\right.$$

$$\left.+C\left(\frac{b}{a},s+1,1,0,q\right)\left|f'(b)\right|^q\right\}^{\frac{1}{q}}$$

$$+b\ln\left(\frac{b}{a}\right)\left(\frac{1}{2}\right)^{2-\frac{1}{q}}\left\{C\left(\frac{a}{b},1,s,1,q\right)\left|f'(b)\right|^q\right.$$

$$\left.+C\left(\frac{a}{b},s+1,1,0,q\right)\left|f'(a)\right|^q\right\}^{\frac{1}{q}}.$$

(2) Since $|f'|^q$ is GA-$s$-convex function in the first sense on $[a,b]$, the inequality (9) is easily obtained by using the inequality (11). □

If taking $s=1$ in Theorem 3.4, we can derive the following inequalities for GA-convex.

**Corollary 3.5.** *Let $f: I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be differentiable on $I^\circ$, and $a,b \in I^\circ$ with $a<b$ and $f' \in L[a,b]$. If $|f'|^q$ is GA-convex on $[a,b]$ for $q \geq 1$, then*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx\right| \leq \ln\left(\frac{b}{a}\right)\left(\frac{1}{2}\right)^{2-\frac{1}{q}} \\
&\times\left[a\left\{C\left(\frac{b}{a},1,1,1,q\right)\left|f'(a)\right|^q + C\left(\frac{b}{a},2,1,0,q\right)\left|f'(b)\right|^q\right\}^{\frac{1}{q}}\right. \qquad (12)\\
&\left.+b\left\{C\left(\frac{a}{b},1,1,1,q\right)\left|f'(b)\right|^q + C\left(\frac{a}{b},2,1,0,q\right)\left|f'(a)\right|^q\right\}^{\frac{1}{q}}\right],
\end{aligned}$$

$$\begin{aligned}
&\left|f\left(\sqrt{ab}\right) - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx\right| \leq \ln\left(\frac{b}{a}\right)\left(\frac{1}{2}\right)^{3-\frac{1}{q}} \\
&\times\left[a\left\{C\left(\frac{b}{a},1,1,1,\frac{q}{2}\right)\left|f'(a)\right|^q + C\left(\frac{b}{a},2,1,0,\frac{q}{2}\right)\left|f'\left(\sqrt{ab}\right)\right|^q\right\}^{\frac{1}{q}}\right. \qquad (13)\\
&\left.+b\left\{C\left(\frac{a}{b},1,1,1,\frac{q}{2}\right)\left|f'(b)\right|^q + C\left(\frac{a}{b},2,1,0,\frac{q}{2}\right)\left|f'\left(\sqrt{ab}\right)\right|^q\right\}^{\frac{1}{q}}\right],
\end{aligned}$$

*where*

$$\begin{aligned}
C\left(\frac{b}{a},1,1,1,q\right) &= \frac{L(a,b)}{qa^q(b-a)}\left[2b^q - 3L(a^q,b^q)\right],\\
C\left(\frac{b}{a},2,1,0,q\right) &= \frac{L(a,b)}{qa^q(b-a)}\left[2L(a^q,b^q) - b^q\right],\\
C\left(\frac{a}{b},1,1,1,q\right) &= \frac{L(a,b)}{qb^q(b-a)}\left[3L(a^q,b^q) - 2a^q\right],\\
C\left(\frac{a}{b},2,1,0,q\right) &= \frac{L(a,b)}{qb^q(b-a)}\left[a^q - 2L(a^q,b^q)\right].
\end{aligned}$$

If taking $q=1$ in Theorem 3.4, we can derive the following corollary.

**Corollary 3.6.** *Let $f: I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be differentiable on $I^\circ$, and $a,b \in I^\circ$ with $a<b$ and $f' \in L[a,b]$.*

*a) If $|f'|$ is GA-s-convex function in the second sense on $[a,b]$, $s\in(0,1]$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx\right|$$
$$\leq \frac{\ln\left(\frac{b}{a}\right)}{2}\left[\left(aC\left(\frac{b}{a},1,1,s,1\right)+bC\left(\frac{a}{b},s+1,1,0,1\right)\right)\left|f'(a)\right|\right.$$
$$\left.+\left(bC\left(\frac{a}{b},1,1,s,1\right)+aC\left(\frac{b}{a},s+1,1,0,1\right)\right)\left|f'(b)\right|\right]$$

$$\left|f\left(\sqrt{ab}\right)-\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx\right|$$
$$\leq \frac{\ln\left(\frac{b}{a}\right)}{4}\left[aC\left(\frac{b}{a},1,1,s,\frac{1}{2}\right)\left|f'(a)\right|+bC\left(\frac{a}{b},1,1,s,\frac{1}{2}\right)\left|f'(b)\right|\right.$$
$$\left.+\left(aC\left(\frac{b}{a},s+1,1,0,\frac{1}{2}\right)+bC\left(\frac{a}{b},s+1,1,0,\frac{1}{2}\right)\right)\left|f'\left(\sqrt{ab}\right)\right|\right].$$

*b) If $|f'|^q$ is GA-s-convex function in the first sense on $[a,b]$, $s\in(0,1]$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx\right|$$
$$\leq \frac{\ln\left(\frac{b}{a}\right)}{2}\left[\left(aC\left(\frac{b}{a},1,s,1,1\right)+bC\left(\frac{a}{b},s+1,1,0,1\right)\right)\left|f'(a)\right|\right.$$
$$\left.+\left(aC\left(\frac{b}{a},s+1,1,0,1\right)+bC\left(\frac{a}{b},1,s,1,1\right)\right)\left|f'(b)\right|\right],$$

$$\left|f\left(\sqrt{ab}\right)-\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx\right|$$
$$\leq \frac{\ln\left(\frac{b}{a}\right)}{4}\left[aC\left(\frac{b}{a},1,s,1,\frac{1}{2}\right)\left|f'(a)\right|+bC\left(\frac{a}{b},1,s,1,\frac{1}{2}\right)\left|f'(b)\right|\right.$$
$$\left.+\left(aC\left(\frac{b}{a},s+1,1,0,\frac{1}{2}\right)+bC\left(\frac{a}{b},s+1,1,0,\frac{1}{2}\right)\right)\left|f'\left(\sqrt{ab}\right)\right|\right].$$

**Theorem 3.7.** *Let $f: I\subseteq\mathbb{R}_+\to\mathbb{R}$ be differentiable on $I^\circ$, and $a,b\in I^\circ$ with $a<b$ and $f'\in L[a,b]$.*

*a) If* $|f'|^q$ *is GA-s-convex function in the second sense on* $[a,b]$ *for* $q>1$ *and* $s\in(0,1]$*, then*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}dx\right| &\leq \frac{\ln\left(\frac{b}{a}\right)}{2}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\\ &\times\left[a\left\{C\left(\frac{b}{a},0,1,s,q\right)\left|f'(a)\right|^q+C\left(\frac{b}{a},s,1,0,q\right)\left|f'(b)\right|^q\right\}^{\frac{1}{q}}\right.\\ &\left.+b\left\{C\left(\frac{a}{b},0,1,s,q\right)\left|f'(b)\right|^q+C\left(\frac{a}{b},s,1,0,q\right)\left|f'(a)\right|^q\right\}^{\frac{1}{q}}\right],\end{aligned}\tag{14}$$

$$\begin{aligned}\left|f\left(\sqrt{ab}\right)-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}dx\right| &\leq \frac{\ln\left(\frac{b}{a}\right)}{4}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\\ &\times\left[a\left\{C\left(\frac{b}{a},0,1,s,\frac{q}{2}\right)\left|f'(a)\right|^q+C\left(\frac{b}{a},s,1,0,\frac{q}{2}\right)\left|f'\left(\sqrt{ab}\right)\right|^q\right\}^{\frac{1}{q}}\right.\\ &\left.+b\left\{C\left(\frac{a}{b},0,1,s,\frac{q}{2}\right)\left|f'(b)\right|^q+C\left(\frac{a}{b},s,1,0,\frac{q}{2}\right)\left|f'\left(\sqrt{ab}\right)\right|^q\right\}^{\frac{1}{q}}\right].\end{aligned}\tag{15}$$

*b) If* $|f'|^q$ *is GA-s-convex function in the first sense on* $[a,b]$ *for* $q>1$ *and* $s\in(0,1]$*, then*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}dx\right| &\leq \frac{\ln\left(\frac{b}{a}\right)}{2}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\\ &\times\left[a\left\{C\left(\frac{b}{a},0,s,1,q\right)\left|f'(a)\right|^q+C\left(\frac{b}{a},s,1,0,q\right))\left|f'(b)\right|^q\right\}^{\frac{1}{q}}\right.\\ &\left.+b\left\{C\left(\frac{a}{b},0,s,1,q\right)\left|f'(b)\right|^q+C\left(\frac{a}{b},s,1,0,q\right)\left|f'(a)\right|^q\right\}^{\frac{1}{q}}\right]\end{aligned}\tag{16}$$

$$\begin{aligned}\left|f\left(\sqrt{ab}\right)-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}dx\right| &\leq \frac{\ln\left(\frac{b}{a}\right)}{4}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\\ &\times\left[a\left\{C\left(\frac{b}{a},0,s,1,\frac{q}{2}\right)\left|f'(a)\right|^q+C\left(\frac{b}{a},s,1,0,\frac{q}{2}\right)\left|f'\left(\sqrt{ab}\right)\right|^q\right\}^{\frac{1}{q}}\right.\\ &\left.+b\left\{C\left(\frac{a}{b},0,s,1,\frac{q}{2}\right)\left|f'(b)\right|^q+C\left(\frac{a}{b},s,1,0,q\right)\left|f'\left(\sqrt{ab}\right)\right|^q\right\}^{\frac{1}{q}}\right].\end{aligned}\tag{17}$$

*Proof.* $a)$ (1) Since $|f'|^q$ is GA-$s$-convex function in the second sense on $[a,b]$, from lemma 1.10 and Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx\right| \\
&\leq \frac{\ln\left(\frac{b}{a}\right)}{2}\left[a\int_0^1 t\left(\frac{b}{a}\right)^t\left|f'\left(a^{1-t}b^t\right)\right|dt+b\int_0^1 t\left(\frac{a}{b}\right)^t\left|f'\left(b^{1-t}a^t\right)\right|dt\right] \\
&\leq \frac{a\ln\left(\frac{b}{a}\right)}{2}\left(\int_0^1 t^{\frac{q}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\frac{b}{a}\right)^{qt}\left|f'\left(a^{1-t}b^t\right)\right|^q dt\right)^{\frac{1}{q}} \qquad (18)\\
&+\frac{b}{2}\ln\left(\frac{b}{a}\right)\left(\int_0^1 t^{\frac{q}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\frac{a}{b}\right)^{qt}\left|f'\left(b^{1-t}a^t\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{a\ln\left(\frac{b}{a}\right)}{2}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\frac{b}{a}\right)^{qt}\left((1-t)^s\left|f'(a)\right|^q+t^s\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}} \\
&+\frac{b}{2}\ln\left(\frac{b}{a}\right)\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\frac{a}{b}\right)^{qt}\left((1-t)^s\left|f'(b)\right|^q+t^s\left|f'(a)\right|^q\right)dt\right)^{\frac{1}{q}} \\
&\leq \frac{\ln\left(\frac{b}{a}\right)}{2}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left[a\left\{C\left(\frac{b}{a},0,1,s,q\right)\left|f'(a)\right|^q\right.\right. \\
&\left.+C\left(\frac{b}{a},s,1,0,q\right)\left|f'(b)\right|^q\right\}^{\frac{1}{q}} \\
&\left.+b\left\{C\left(\frac{a}{b},0,1,s,q\right)\left|f'(b)\right|^q+C\left(\frac{a}{b},s,1,0,q\right)\left|f'(a)\right|^q\right\}^{\frac{1}{q}}\right].
\end{aligned}$$

(2) Since $|f'|^q$ is GA-$s$-convex function in the first sense on $[a,b]$, from lemma 1.10 and Hölder inequality, we have

$$\begin{aligned}
&\left|f\left(\sqrt{ab}\right)-\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx\right| \\
&\leq \frac{\ln\frac{b}{a}}{4}\left[a\int_0^1 t\left(\frac{b}{a}\right)^{\frac{t}{2}}\left|f'\left(a^{1-t}(ab)^{\frac{t}{2}}\right)\right|dt+b\int_0^1 t\left(\frac{a}{b}\right)^{\frac{t}{2}}\left|f'\left(b^{1-t}(ab)^{\frac{t}{2}}\right)\right|dt\right]
\end{aligned}$$

$$\leq \frac{a\ln\frac{b}{a}}{4}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\frac{b}{a}\right)^{\frac{qt}{2}}\left((1-t)^s\left|f'(a)\right|^q+t^s\left|f'\left(\sqrt{ab}\right)\right|^q\right)dt\right)^{\frac{1}{q}}$$

$$+\frac{b\ln\frac{b}{a}}{4}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\frac{a}{b}\right)^{\frac{qt}{2}}\left((1-t)^s\left|f'(b)\right|^q+t^s\left|f'\left(\sqrt{ab}\right)\right|^q\right)dt\right)^{\frac{1}{q}}$$

$$\leq \frac{\ln\left(\frac{b}{a}\right)}{4}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}} \tag{19}$$

$$\times\left[a\left\{C\left(\frac{b}{a},0,1,s,\frac{q}{2}\right))\left|f'(a)\right|^q+C\left(\frac{b}{a},s,1,0,\frac{q}{2}\right)\left|f'\left(\sqrt{ab}\right)\right|^q\right\}^{\frac{1}{q}}\right.$$

$$\left.+b\left\{C\left(\frac{a}{b},0,1,s,\frac{q}{2}\right)\left|f'(b)\right|^q+C\left(\frac{a}{b},s,1,0,\frac{q}{2}\right)\left|f'\left(\sqrt{ab}\right)\right|^q\right\}^{\frac{1}{q}}\right].$$

b) (1) Since $|f'|^q$ is GA-$s$-convex function in the first sense on $[a,b]$, from the inequality (18), we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}dx\right|$$

$$\leq \frac{a\ln\left(\frac{b}{a}\right)}{2}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\frac{b}{a}\right)^{qt}\left((1-t^s)\left|f'(a)\right|^q+t^s\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}$$

$$+\frac{b\ln\left(\frac{b}{a}\right)}{2}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\frac{a}{b}\right)^{qt}\left((1-t^s)\left|f'(b)\right|^q+t^s\left|f'(a)\right|^q\right)dt\right)^{\frac{1}{q}}$$

$$\leq \frac{\ln\left(\frac{b}{a}\right)}{2}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left[a\left\{C\left(\frac{b}{a},0,s,1,q\right)\left|f'(a)\right|^q\right.\right.$$

$$\left.+C\left(\frac{b}{a},s,1,0,q\right)\left|f'(b)\right|^q\right\}^{\frac{1}{q}}$$

$$\left.+b\left\{C\left(\frac{a}{b},0,s,1,q\right)\left|f'(b)\right|^q+C\left(\frac{a}{b},s,1,0,q\right)\left|f'(a)\right|^q\right\}^{\frac{1}{q}}\right]$$

(2) Since $|f'|^q$ is GA-$s$-convex function in the first sense on $[a,b]$, the inequality (9) is easily obtained by using the inequality (19). □

If taking $s=1$ in Theorem 3.7, we can derive the following inequalities for GA-convex.

**Corollary 3.8.** *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be differentiable on $I^\circ$, and $a,b \in I^\circ$ with $a<b$ and $f' \in L[a,b]$. If $|f'|^q$ is GA-convex function in the second sense on $[a,b]$ for $q>1$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx\right| \le \frac{\ln\left(\frac{b}{a}\right)}{2}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}$$
$$\times\left[a\left\{C\left(\frac{b}{a},0,1,1,q\right)\left|f'(a)\right|^q+C\left(\frac{b}{a},1,1,0,q\right)\left|f'(b)\right|^q\right\}^{\frac{1}{q}}\right. \tag{20}$$
$$\left.+b\left\{C\left(\frac{a}{b},0,1,1,q\right)\left|f'(b)\right|^q+C\left(\frac{a}{b},1,1,0,q\right)\left|f'(a)\right|^q\right\}^{\frac{1}{q}}\right],$$

$$\left|f\left(\sqrt{ab}\right)-\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx\right| \le \frac{\ln\left(\frac{b}{a}\right)}{4}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}$$
$$\times\left[a\left\{C\left(\frac{b}{a},0,1,1,\frac{q}{2}\right)\left|f'(a)\right|^q+C\left(\frac{b}{a},1,1,0,\frac{q}{2}\right)\left|f'\left(\sqrt{ab}\right)\right|^q\right\}^{\frac{1}{q}}\right. \tag{21}$$
$$\left.+b\left\{C\left(\frac{a}{b},0,1,1,\frac{q}{2}\right)\left|f'(b)\right|^q+C\left(\frac{a}{b},1,1,0,\frac{q}{2}\right)\left|f'\left(\sqrt{ab}\right)\right|^q\right\}^{\frac{1}{q}}\right],$$

*where*

$$C\left(\frac{b}{a},0,1,1,q\right) = \frac{L(a,b)}{qa^q(b-a)}\left[L(a^q,b^q)-a^q\right],$$
$$C\left(\frac{b}{a},1,1,0,q\right) = \frac{L(a,b)}{qa^q(b-a)}\left[b^q-L(a^q,b^q)\right],$$

$$C\left(\frac{a}{b},0,1,1,q\right) = \frac{L(a,b)}{qb^q(b-a)}\left[b^q-L(a^q,b^q)\right],$$
$$C\left(\frac{a}{b},1,1,0,q\right) = \frac{L(a,b)}{qb^q(b-a)}\left[L(a^q,b^q)-a^q\right].$$

## 4. Application to Special Means

**Proposition 4.1.** *Let $0<a<b$, $n>0$, $q\ge 1$ and $nq\ne 1$. Then*

$$\left|A\left(a^{n+1},b^{n+1}\right)-L_n^n(a,b)L(a,b)\right| \le \frac{n+1}{q^{1/q}}\left(\frac{b-a}{L(a,b)}\right)^{1-\frac{1}{q}}\left(\frac{1}{2}\right)^{2-\frac{1}{q}}$$
$$\times\left[\left\{a^{nq}\left(2b^q-3L(a^q,b^q)\right)+b^{nq}\left(2L(a^q,b^q)-b^q\right)\right\}^{1/q}\right.$$
$$\left.+\left\{b^{nq}\left(3L(a^q,b^q)-2a^q\right)+a^{nq}\left(a^q-2L(a^q,b^q)\right)\right\}^{1/q}\right],$$

$$\left|G\left(a^{n+1},b^{n+1}\right)-L_n^n(a,b)L(a,b)\right|\leq \frac{n+1}{q^{1/q}}\left(\frac{b-a}{L(a,b)}\right)^{1-\frac{1}{q}}\left(\frac{1}{2}\right)^{3-\frac{2}{q}}$$
$$\times\left[\sqrt{a}\left\{a^{nq}\left(2b^{q/2}-3L(a^{q/2},b^{q/2})\right)+G^{nq}(a,b)\left(2L(a^{q/2},b^{q/2})-b^{q/2}\right)\right\}^{1/q}\right.$$
$$\left.+\sqrt{b}\left\{b^{nq}\left(3L(a^{q/2},b^{q/2})-2a^{q/2}\right)+G^{nq}(a,b)\left(a^{q/2}-2L(a^{q/2},b^{q/2})\right)\right\}^{1/q}\right],$$

*Proof.* Let
$$f(x)=\frac{x^{n+1}}{n+1},\ x>0.$$
Then $|f'(x)|^q=x^{nq}$ is a GA-convex function on $\mathbb{R}_+$. The assertion follows from the inequalities (12) and (13) in Corollary 3.5 for the function $f$. □

**Corollary 4.2.** *Under conditions of Proposition 4.1, when $q=1$ and $n\neq 1$, we have*

$$\left|A\left(a^{n+1},b^{n+1}\right)-L_n^n(a,b)L(a,b)\right|$$
$$\leq\frac{n+1}{2}\left\{a^n\left(a+2b-5L(a,b)\right)+b^n\left(5L(a,b)-2a-b\right)\right\},$$

$$\left|G\left(a^{n+1},b^{n+1}\right)-L_n^n(a,b)L(a,b)\right|$$
$$\leq\frac{n+1}{2}\left\{a^n\left(2G(a,b)-3\sqrt{a}L(\sqrt{a},\sqrt{b})\right)-2G^n(a,b)\left(\sqrt{b}-\sqrt{a}\right)(\sqrt{a},\sqrt{b})\right.$$
$$\left.+b^n\left(3\sqrt{b}L(\sqrt{a},\sqrt{b})-2G(a,b)\right)\right\}$$

**Proposition 4.3.** *Let $0<a<b$,and $q\geq 1$. Then*

$$|A(a,b)-L(a,b)|\leq\left[\ln\left(\frac{b}{a}\right)\right]^{1-\frac{1}{q}}\left(\frac{1}{2}\right)^{2-\frac{1}{q}}\left(\frac{1}{q}\right)^{\frac{1}{q}}$$
$$\times\left[\{b^q-L(a^q,b^q)\}^{\frac{1}{q}}+\{L(a^q,b^q)-a^q\}^{\frac{1}{q}}\right]$$

$$|G(a,b)-L(a,b)|\leq\left[\ln\left(\frac{b}{a}\right)\right]^{1-\frac{1}{q}}\left(\frac{1}{2}\right)^{3-\frac{1}{q}}\left(\frac{2}{q}\right)^{\frac{1}{q}}$$
$$\times\left[\sqrt{a}\left\{b^{q/2}-L(a^{q/2},b^{q/2})\right\}^{\frac{1}{q}}+\sqrt{b}\left\{L(a^{q/2},b^{q/2})-a^{q/2}\right\}^{\frac{1}{q}}\right].$$

*Proof.* The assertion follows from the inequalities (12) and (13) in Corollary 3.5 for $f(x)=x,\ x>0$. □

**Proposition 4.4.** *Let* $0 < a < b$, $n > 0$, $q > 1$ *and* $nq \neq 1$. *Then*

$$\left|A\left(a^{n+1},b^{n+1}\right) - L_n^n(a,b)\,L(a,b)\right| \leq \frac{n+1}{2q^{1/q}}\left(\frac{(q-1)(b-a)}{(2q-1)\,L(a,b)}\right)^{1-\frac{1}{q}}$$
$$\times\left[\left\{a^{nq}\left(L(a^q,b^q)-a^q\right)+b^{nq}\left(b^q-L(a^q,b^q)\right)\right\}^{1/q}\right.$$
$$\left.+\left\{b^{nq}\left(b^q-L(a^q,b^q)\right)+a^{nq}\left(L(a^q,b^q)-a^q\right)\right\}^{1/q}\right],$$

$$\left|G\left(a^{n+1},b^{n+1}\right) - L_n^n(a,b)\,L(a,b)\right| \leq \frac{(n+1)}{2^{2-1/q}q^{1/q}}\left(\frac{(q-1)(b-a)}{(2q-1)\,L(a,b)}\right)^{1-\frac{1}{q}}$$
$$\times\left[\sqrt{a}\left\{a^{nq}\left(L(a^{q/2},b^{q/2})-a^{q/2}\right)+G^{nq}(a,b)\left(b^{q/2}-L(a^{q/2},b^{q/2})\right)\right\}^{1/q}\right.$$
$$\left.+\sqrt{b}\left\{b^{nq}\left(b^{q/2}-L(a^{q/2},b^{q/2})\right)+G^{nq}(a,b)\left(L(a^{q/2},b^{q/2})-a^{q/2}\right)\right\}^{1/q}\right].$$

*Proof.* The assertion follows from the inequalities (20) and (21) in Corollary 3.8 for $f(x) = \frac{x^{n+1}}{n+1}$, $x > 0$. $\square$

**Proposition 4.5.** *Let* $0 < a < b$ *and* $q > 1$. *Then*

$$|A(a,b) - L(a,b)| \leq \ln\left(\frac{b}{a}\right)\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}} L^{\frac{1}{q}}(a^q,b^q)$$

$$|G(a,b) - L(a,b)| \leq \frac{\ln\left(\frac{b}{a}\right)}{2}\left(\frac{q-1}{(2q-1)}\right)^{1-\frac{1}{q}} L^{\frac{1}{q}}(a^{q/2},b^{q/2})A\left(\sqrt{a},\sqrt{b}\right).$$

*Proof.* The assertion follows from the inequalities (20) and (21) in Corollary 3.8 for $f(x) = x$, $x > 0$. $\square$

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