

# Hermite–Hadamard-type inequalities for generalized $s$ -convex functions on real linear fractal set $\mathbb{R}^\alpha$ ( $0 < \alpha < 1$ )

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**Abstract** In this paper, we establish the Hermite–Hadamard-type inequalities for the generalized  $s$ -convex functions in the second sense on real linear fractal set  $\mathbb{R}^\alpha$  ( $0 < \alpha < 1$ ).

**Keywords** Fractal set · Local fractional integral · Hermite–Hadamard-type inequality · Generalized  $s$ -convex function

## Introduction

The convex function plays an important role in the class mathematical analysis course and other fields. In [1], Hudzik and Maligranda introduced two kinds of  $s$ -convex functions in the space of European space  $\mathbb{R}$ . In addition, many important inequalities are established for the  $s$ -convex functions in  $\mathbb{R}$ . For example, the Hermite–Hadamard's inequality is one of the best known results in the literature, see [2–4] and so on.

In recent years, the fractal theory has received significantly remarkable attention [5]. The calculus on fractal set can lead to better comprehension for the various real-world models from the engineering and science [6].

On the fractal set, Mo et al. [7, 8] introduced the definition of the generalized convex function and established Hermite–Hadamard-type inequality. In [9], the authors

introduced two kinds of generalized  $s$ -convex functions on fractal sets  $\mathbb{R}^\alpha$  ( $0 < \alpha < 1$ ).

The definitions of the generalized  $s$ -convex functions are as follows:

**Definition 1.1** [9] Suppose that  $\mathbb{R}_+ = [0, \infty)$ . If the function  $f : \mathbb{R}_+ \rightarrow \mathbb{R}^\alpha$  satisfies the following inequality:

$$f(\lambda_1 u + \lambda_2 v) \leq \lambda_1^{zs} f(u) + \lambda_2^{zs} f(v), \quad (1.1)$$

for all  $u, v \in \mathbb{R}_+$  and all  $\lambda_1, \lambda_2 \geq 0$  with  $\lambda_1^s + \lambda_2^s = 1$  and  $0 < s < 1$ , then  $f$  is said to be a generalized  $s$ -convex function in the first sense. We denote this by  $f \in GK_s^1$ .

**Definition 1.2** [9] Suppose that  $\mathbb{R}_+ = [0, \infty)$ . If the function  $f : \mathbb{R}_+ \rightarrow \mathbb{R}^\alpha$  satisfies the following inequality:

$$f(\lambda_1 u + \lambda_2 v) \leq \lambda_1^{zs} f(u) + \lambda_2^{zs} f(v), \quad (1.2)$$

for all  $u, v \in \mathbb{R}_+$  and all  $\lambda_1, \lambda_2 \geq 0$  with  $\lambda_1 + \lambda_2 = 1$  and  $0 < s < 1$ , then  $f$  is said to be a generalized  $s$ -convex function in the second sense. We denote this by  $f \in GK_s^2$ .

Note that the generalized  $s$ -convex function in both sense is generalized convex function [9] for  $s = 1$ .

Inspired by [2, 3, 8], in this paper, we will establish the Hermite–Hadamard-type inequalities for the generalized  $s$ -convex functions.

## Preliminaries

Now, let us review the operations with real line number on fractal space. In addition, we will use the Gao–Yang–Kang's idea to describe the definitions of the local fractional derivative and local fractional integral [10–14].

Let  $a^\alpha, b^\alpha$  and  $c^\alpha$  belong to the set  $\mathbb{R}^\alpha$  ( $0 < \alpha < 1$ ) of real line numbers, then

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1.  $a^\alpha b^\alpha$  and  $a^\alpha + b^\alpha$  belong to the set  $\mathbb{R}^\alpha$ ;
2.  $a^\alpha + b^\alpha = (a + b)^\alpha = b^\alpha + a^\alpha = (b + a)^\alpha$ ;
3.  $a^\alpha + (b^\alpha + c^\alpha) = (a^\alpha + b^\alpha) + c^\alpha$ ;
4.  $a^\alpha b^\alpha = (ab)^\alpha = b^\alpha a^\alpha = (ba)^\alpha$ ;
5.  $a^\alpha (b^\alpha c^\alpha) = (a^\alpha b^\alpha) c^\alpha$ ;
6.  $a^\alpha (b^\alpha + c^\alpha) = a^\alpha b^\alpha + a^\alpha c^\alpha$ ;
7.  $0^\alpha + a^\alpha = a^\alpha + 0^\alpha = a^\alpha$  and  $1^\alpha \cdot a^\alpha = a^\alpha \cdot 1^\alpha = a^\alpha$ .

**Definition 2.1** ([10]) If the function  $f : [a, b] \rightarrow \mathbb{R}^\alpha$  satisfies the inequality

$$|f(x) - f(y)| < c|x - y|^\alpha, \quad (x, y \in [a, b]),$$

for  $c > 0$  and  $\alpha(0 < \alpha \leq 1)$ , then  $f$  is called a Hölder continuous function. In this case, we think that  $f$  is in the space  $C_\alpha[a, b]$ .

**Definition 2.2** [10] Let  $\Delta^\alpha(f(x) - f(x_0)) \cong \Gamma(1 + \alpha)(f(x) - f(x_0))$ . Then, the local fractional derivative of  $f$  of order  $\alpha$  at  $x = x_0$  is defined by

$$f^{(\alpha)}(x_0) = \frac{d^\alpha f(x)}{dx^\alpha} \Big|_{x=x_0} = \lim_{x \rightarrow x_0} \frac{\Delta^\alpha(f(x) - f(x_0))}{(x - x_0)^\alpha}.$$

If there exists  $f^{((k+1)\alpha)}(x) = \overbrace{D_x^\alpha \dots D_x^\alpha}^{k+1 \text{ times}} f(x)$  for any  $x \in I \subseteq \mathbb{R}$ , then we denoted  $f \in D_{(k+1)\alpha}(I)$ , where  $k = 0, 1, 2, \dots$

**Definition 2.3** [10] For  $f \in C_\alpha[a, b]$ , the local fractional integral of the function  $f$  is defined by

$$\begin{aligned} & {}_a I_b^{(\alpha)} f(x) \\ &= \frac{1}{\Gamma(1 + \alpha)} \int_a^b f(t) (dt)^\alpha \\ &= \frac{1}{\Gamma(1 + \alpha)} \lim_{\Delta t \rightarrow 0} \sum_{j=0}^N f(t_j) (\Delta t_j)^\alpha, \end{aligned}$$

where  $\Delta t_j = t_{j+1} - t_j$ ,  $\Delta t = \max\{\Delta t_1, \Delta t_2, \Delta t_j, \dots\}$  and  $[t_j, t_{j+1}]$ ,  $j = 0, \dots, N - 1$ ,  $t_0 = a$ ,  $t_N = b$ , is a partition of the interval  $[a, b]$ .

**Lemma 2.1** [10] Let  $f \in C_\alpha[g(a), g(b)]$  and  $g \in C_1[a, b]$ , then

$$g^{(a)} I_{g(b)}^{(\alpha)} f(x) = {}_a I_b^{(\alpha)} f(g(s)) [g'(s)]^\alpha.$$

**Lemma 2.2** [10]

1. Let  $f(x) = g^{(\alpha)}(x) \in C_\alpha[a, b]$ , then we have  ${}_a I_b^{(\alpha)} f(x) = g(b) - g(a)$ .
2. Let  $f(x), g(x) \in D_\alpha[a, b]$  and  $f^{(\alpha)}(x), g^{(\alpha)}(x) \in C_\alpha[a, b]$ , then we have

$${}_a I_b^\alpha f(x) g^{(\alpha)}(x) = f(x) g(x) \Big|_a^b - {}_a I_b^{(\alpha)} f^{(\alpha)}(x) g(x).$$

**Lemma 2.3** [10]

$$\frac{d^\alpha x^{k\alpha}}{dx^\alpha} = \frac{\Gamma(1 + k\alpha)}{\Gamma(1 + (k - 1)\alpha)} x^{(k-1)\alpha}.$$

From the above formula and Lemma 2.2, we have

$$\frac{1}{\Gamma(1 + \alpha)} \int_a^b x^{k\alpha} (dx)^\alpha = \frac{\Gamma(1 + k\alpha)}{\Gamma(1 + (k + 1)\alpha)} (b^{(k+1)\alpha} - a^{(k+1)\alpha}), k \in \mathbb{R}.$$

**Lemma 2.4** [10] (Generalized Hölder’s inequality) Let  $f, g \in C_\alpha[a, b]$  and  $p, q > 1$  with  $1/p + 1/q = 1$ . Then, it follows that

$$\begin{aligned} & \frac{1}{\Gamma(1 + \alpha)} \int_a^b |f(x)g(x)| (dx)^\alpha \leq \left( \frac{1}{\Gamma(1 + \alpha)} \int_a^b |f(x)|^p (dx)^\alpha \right)^{1/p} \\ & \times \left( \frac{1}{\Gamma(1 + \alpha)} \int_a^b |g(x)|^q (dx)^\alpha \right)^{1/q}. \end{aligned}$$

**Main results**

**Theorem 3.1** Let  $f : \mathbb{R}_+ \rightarrow \mathbb{R}^\alpha$  be a generalized  $s$ -convex function in the second sense for  $0 < s < 1$  and  $a, b \in [0, \infty)$  with  $a < b$ . Then, for  $f \in C_\alpha[a, b]$ , the following inequalities hold:

$$\frac{2^{(s-1)\alpha}}{\Gamma(1 + \alpha)} f\left(\frac{a+b}{2}\right) \leq \frac{{}_a I_b^{(\alpha)} f(x)}{(b-a)^\alpha} \leq \frac{\Gamma(1 + s\alpha)}{\Gamma(1 + (s+1)\alpha)} (f(a) + f(b)). \tag{3.1}$$

*Proof* Let  $x = a + b - t$ . Then, from Lemma 2.1, we have  $\frac{a+b}{2} I_b^{(\alpha)} f(x) = {}_a I_{\frac{a+b}{2}}^{(\alpha)} f(a + b - t)$ .

Since  $f$  is a generalized  $s$ -convex function in the second sense, then

$$\begin{aligned} {}_a I_b^{(\alpha)} f(x) &= {}_a I_{\frac{a+b}{2}}^{(\alpha)} [f(x) + f(a + b - x)] \\ &\geq 2^{s\alpha} {}_a I_{\frac{a+b}{2}}^{(\alpha)} f\left(\frac{a+b}{2}\right) \\ &= \frac{2^{(s-1)\alpha}}{\Gamma(1 + \alpha)} (b-a)^\alpha f\left(\frac{a+b}{2}\right). \end{aligned}$$

In the other hand, let  $x = b - (b - a)t$ ,  $0 \leq t \leq 1$ , then we get

$$\begin{aligned} {}_a I_b^{(\alpha)} f(x) &= (b-a)^\alpha {}_0 I_1^{(\alpha)} f[ta + (1-t)b] \\ &\leq (b-a)^\alpha {}_0 I_1^{(\alpha)} [t^{s\alpha} f(a) + (1-t)^{s\alpha} f(b)] \\ &= (b-a)^\alpha [f(a) {}_0 I_1^{(\alpha)} t^{s\alpha} + f(b) {}_0 I_1^{(\alpha)} (1-t)^{s\alpha}]. \end{aligned}$$

From Lemma 2.3, it is easy to see that

$${}_0 I_1^{(\alpha)} t^{s\alpha} = \frac{\Gamma(1 + s\alpha)}{\Gamma(1 + (s+1)\alpha)},$$

and

$${}_0I_1^{(\alpha)}(1-t)^{s\alpha} = \frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}.$$

Therefore

$${}_aI_b^{(\alpha)}f(x) \leq (b-a)^\alpha \frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)} (f(a)+f(b)).$$

Combining the above estimates, we obtain

$$\begin{aligned} \frac{2^{(s-1)\alpha}}{\Gamma(1+\alpha)} f\left(\frac{a+b}{2}\right) &\leq \frac{{}_aI_b^{(\alpha)}f(x)}{(b-a)^\alpha} \\ &\leq \frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)} (f(a)+f(b)). \end{aligned}$$

□

**Theorem 3.2** *Let  $I \subset \mathbb{R}$  be an interval, and  $I^0$  be the interior of  $I$ . Suppose that  $f : I \rightarrow \mathbb{R}^\alpha$  is a differentiable function on  $I^0$  such that  $f^{(\alpha)} \in C_\alpha[a, b]$ , where  $a, b \in I^0$  with  $a < b$ . If  $|f^{(\alpha)}|^q$  is a generalized  $s$ -convex function in the second sense on  $[a, b]$  for some fixed  $s \in (0, 1)$  and  $q \geq 1$ , then*

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2^\alpha} - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_aI_b^{(\alpha)}f(x) \right| \\ &\leq \frac{(b-a)^\alpha}{2^\alpha} \left( \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \right)^{\frac{q-1}{q}} \\ &\left[ \frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)} + \frac{\Gamma(1+\alpha)\Gamma(1+s\alpha)}{\Gamma(1+(s+2)\alpha)} \left( \left(\frac{1}{2}\right)^{s\alpha} - 2^\alpha \right) \right]^{\frac{1}{q}} \\ &\times \left[ |f^{(\alpha)}(a)|^q + |f^{(\alpha)}(b)|^q \right]^{\frac{1}{q}}. \end{aligned}$$

To show Theorem 3.2 is right, we need the following Lemma.

**Lemma 3.1** ([8]) *Let  $f : I \rightarrow \mathbb{R}^\alpha$ ,  $I \subset [0, \infty)$ . If  $f \in D_\alpha(I^0)$  and  $f^{(\alpha)} \in C_\alpha[a, b]$  for  $a, b \in I^0$  with  $a < b$ , then the following equality holds:*

$$\begin{aligned} &\frac{f(a)+f(b)}{2^\alpha} - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_aI_b^{(\alpha)}f(x) \\ &= \frac{(b-a)^\alpha}{2^\alpha} \frac{1}{\Gamma(1+\alpha)} \int_0^1 (1-2t)^\alpha f^{(\alpha)}(ta+(1-t)b) (dt)^\alpha. \end{aligned}$$

Now, let us give the proof of Theorem 3.2.

*Proof* From Lemma 3.1, it is obvious that

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2^\alpha} - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_aI_b^{(\alpha)}f(x) \right| \\ &\leq \frac{(b-a)^\alpha}{2^\alpha} \frac{1}{\Gamma(1+\alpha)} \int_0^1 |1-2t|^\alpha |f^{(\alpha)}(ta+(1-t)b)| (dt)^\alpha. \end{aligned} \tag{3.2}$$

Let us estimate

$$\frac{1}{\Gamma(1+\alpha)} \int_0^1 |1-2t|^\alpha |f^{(\alpha)}(ta+(1-t)b)| (dt)^\alpha,$$

for  $q = 1$  and  $q > 1$ .

**Case I.**  $q = 1$ .

Since  $|f^{(\alpha)}|$  is generalized  $s$ -convex on  $[a, b]$  in the second sense, we can know that for any  $t \in [0, 1]$

$$|f^{(\alpha)}(ta+(1-t)b)| \leq t^{s\alpha} |f^{(\alpha)}(a)| + (1-t)^{s\alpha} |f^{(\alpha)}(b)|.$$

Then, we have

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)} \int_0^1 |1-2t|^\alpha |f^{(\alpha)}(ta+(1-t)b)| (dt)^\alpha \\ &\leq \frac{1}{\Gamma(1+\alpha)} \int_0^1 |1-2t|^\alpha \left[ t^{s\alpha} |f^{(\alpha)}(a)| \right. \\ &\quad \left. + (1-t)^{s\alpha} |f^{(\alpha)}(b)| \right] (dt)^\alpha \\ &= \left[ |f^{(\alpha)}(a)| \frac{1}{\Gamma(1+\alpha)} \int_0^1 t^{s\alpha} |1-2t|^\alpha (dt)^\alpha + |f^{(\alpha)}(b)| \right. \\ &\quad \left. \frac{1}{\Gamma(1+\alpha)} \int_0^1 (1-t)^{s\alpha} |1-2t|^\alpha (dt)^\alpha \right]. \end{aligned} \tag{3.3}$$

From Lemmas 2.2 and 2.3, it is easy to see that

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)} \int_0^1 t^{s\alpha} |1-2t|^\alpha (dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)} \left[ \int_0^{\frac{1}{2}} t^{s\alpha} (1-2t)^\alpha (dt)^\alpha + \int_{\frac{1}{2}}^1 t^{s\alpha} (2t-1)^\alpha (dt)^\alpha \right] \\ &= \left[ \frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)} + \frac{\Gamma(1+\alpha)\Gamma(1+s\alpha)}{\Gamma(1+(s+2)\alpha)} \left( \left(\frac{1}{2}\right)^{s\alpha} - 2^\alpha \right) \right]. \end{aligned} \tag{3.4}$$

In addition, let  $1-t = x$ , then by Lemma 2.1 and (3.4), we have

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)} \int_0^1 (1-t)^{s\alpha} |1-2t|^\alpha (dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)} \int_0^1 x^{s\alpha} |1-2x|^\alpha (dx)^\alpha \\ &= \left[ \frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)} + \frac{\Gamma(1+\alpha)\Gamma(1+s\alpha)}{\Gamma(1+(s+2)\alpha)} \left( \left(\frac{1}{2}\right)^{s\alpha} - 2^\alpha \right) \right]. \end{aligned} \tag{3.5}$$

Thus, substituting (3.4) and (3.5) into (3.3), we have

$$\begin{aligned} & \frac{1}{\Gamma(1+\alpha)} \int_0^1 |1-2t|^\alpha |f^{(\alpha)}(ta+(1-t)b)| (dt)^\alpha \\ & \leq \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \left[ \frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)} + \frac{\Gamma(1+\alpha)\Gamma(1+s\alpha)}{\Gamma(1+(s+2)\alpha)} \right] \\ & \quad \times \left( \left(\frac{1}{2}\right)^{2s} - 2^{2s} \right) \times \left[ |f^{(\alpha)}(a)|^q + |f^{(\alpha)}(b)|^q \right]. \end{aligned} \tag{3.6}$$

Thus, from (3.2), we obtain

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2^\alpha} - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} I_b^{(\alpha)} f(x) \right| \\ & \leq \frac{(b-a)^\alpha}{2^\alpha} \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \left[ \frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)} \right. \\ & \quad \left. + \frac{\Gamma(1+\alpha)\Gamma(1+s\alpha)}{\Gamma(1+(s+2)\alpha)} \left( \left(\frac{1}{2}\right)^{2s} - 2^{2s} \right) \right] \\ & \quad \times \left[ |f^{(\alpha)}(a)|^q + |f^{(\alpha)}(b)|^q \right]. \end{aligned}$$

**Case II.**  $q > 1$ .

Using the generalized Hölder’s inequality (Lemma 2.4), we obtain

$$\begin{aligned} & \frac{1}{\Gamma(1+\alpha)} \int_0^1 |1-2t|^\alpha |f^{(\alpha)}(ta+(1-t)b)| (dt)^\alpha \\ & = \frac{1}{\Gamma(1+\alpha)} \\ & \quad \int_0^1 |1-2t|^{\alpha\frac{q-1}{q}} |1-2t|^{\alpha\frac{1}{q}} |f^{(\alpha)}(ta+(1-t)b)| (dt)^\alpha \\ & \leq \left( \frac{1}{\Gamma(1+\alpha)} \int_0^1 |1-2t|^\alpha (dt)^\alpha \right)^{\frac{q-1}{q}} \\ & \quad \times \left( \frac{1}{\Gamma(1+\alpha)} \int_0^1 |1-2t|^\alpha |f^{(\alpha)}(ta+(1-t)b)|^q (dt)^\alpha \right)^{\frac{1}{q}}. \end{aligned} \tag{3.7}$$

It is obvious that

$$\begin{aligned} & \frac{1}{\Gamma(1+\alpha)} \int_0^1 |1-2t|^\alpha (dt)^\alpha \\ & = \frac{1}{\Gamma(1+\alpha)} \int_0^{\frac{1}{2}} (1-2t)^\alpha (dt)^\alpha \\ & \quad + \frac{1}{\Gamma(1+\alpha)} \int_{\frac{1}{2}}^1 (2t-1)^\alpha (dt)^\alpha = \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}. \end{aligned} \tag{3.8}$$

Moreover, since  $|f^{(\alpha)}|^q$  is generalized  $s$  convex in the second sense on  $[a, b]$ , then

$$\begin{aligned} & \frac{1}{\Gamma(1+\alpha)} \int_0^1 |1-2t|^\alpha |f^{(\alpha)}(ta+(1-t)b)|^q (dt)^\alpha \\ & \leq \frac{1}{\Gamma(1+\alpha)} \int_0^1 |1-2t|^\alpha \\ & \quad \left( t^{2s} |f^{(\alpha)}(a)|^q + (1-t)^{2s} |f^{(\alpha)}(b)|^q \right) (dt)^\alpha \\ & = |f^{(\alpha)}(a)|^q \frac{1}{\Gamma(1+\alpha)} \int_0^1 |1-2t|^\alpha t^{2s} (dt)^\alpha \\ & \quad + |f^{(\alpha)}(b)|^q \frac{1}{\Gamma(1+\alpha)} \int_0^1 |1-2t|^\alpha (1-t)^{2s} (dt)^\alpha. \end{aligned}$$

From (3.3) and (3.4), it is easy to see that

$$\begin{aligned} & \frac{1}{\Gamma(1+\alpha)} \int_0^1 |1-2t|^\alpha t^{2s} (dt)^\alpha \\ & = \frac{1}{\Gamma(1+\alpha)} \int_0^1 |1-2t|^\alpha (1-t)^{2s} (dt)^\alpha \\ & = \left[ \frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)} + \frac{\Gamma(1+\alpha)\Gamma(1+s\alpha)}{\Gamma(1+(s+2)\alpha)} \left( \left(\frac{1}{2}\right)^{2s} - 2^{2s} \right) \right]. \end{aligned}$$

Therefore

$$\begin{aligned} & \frac{1}{\Gamma(1+\alpha)} \int_0^1 |1-2t|^\alpha |f^{(\alpha)}(ta+(1-t)b)|^q (dt)^\alpha \\ & = \left[ \frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)} + \frac{\Gamma(1+\alpha)\Gamma(1+s\alpha)}{\Gamma(1+(s+2)\alpha)} \left( \left(\frac{1}{2}\right)^{2s} - 2^{2s} \right) \right]^{\frac{1}{q}} \\ & \quad \left[ |f^{(\alpha)}(a)|^q + |f^{(\alpha)}(b)|^q \right]^{\frac{1}{q}}. \end{aligned} \tag{3.9}$$

Thus, substituting (3.8) and (3.9) into (3.7), we have

$$\begin{aligned} & \frac{1}{\Gamma(1+\alpha)} \int_0^1 |1-2t|^\alpha |f^{(\alpha)}(ta+(1-t)b)| (dt)^\alpha \\ & \leq \left( \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \right)^{\frac{q-1}{q}} \\ & \quad \left[ \frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)} + \frac{\Gamma(1+\alpha)\Gamma(1+s\alpha)}{\Gamma(1+(s+2)\alpha)} \left( \left(\frac{1}{2}\right)^{2s} - 2^{2s} \right) \right]^{\frac{1}{q}} \\ & \quad \left[ |f^{(\alpha)}(a)|^q + |f^{(\alpha)}(b)|^q \right]^{\frac{1}{q}}. \end{aligned}$$

Therefore, from (3.2), it follows that

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2^\alpha} - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^{(\alpha)} f(x) \right| \\ & \leq \frac{(b-a)^\alpha}{2^\alpha} \left( \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \right)^{\frac{q-1}{q}} \\ & \left[ \frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)} + \frac{\Gamma(1+\alpha)\Gamma(1+s\alpha)}{\Gamma(1+(s+2)\alpha)} \left( \left( \frac{1}{2} \right)^{\alpha s} - 2^\alpha \right) \right]^{\frac{1}{q}} \\ & \quad \times \left[ |f^{(\alpha)}(a)|^q + |f^{(\alpha)}(b)|^q \right]^{\frac{1}{q}}. \end{aligned}$$

Thus, we complete the proof of Theorem 3.2.  $\square$

**Theorem 3.3** Suppose that  $f: I \rightarrow \mathbb{R}^\alpha, I \subset [0, \infty)$  is a differentiable function on  $I^0$ , such that  $f^{(\alpha)} \in C_\alpha[a, b]$ , where  $a, b \in I$  with  $a < b$ . If  $|f^{(\alpha)}|^q$  is a generalized  $s$ -convex function in the second sense on  $[a, b]$  for some fixed  $s \in (0, 1)$  and  $q > 1$ , then

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2^\alpha} - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^{(\alpha)} f(x) \right| \\ & \leq \frac{(b-a)^\alpha}{2^\alpha} \left[ \frac{\Gamma(1+\frac{q}{q-1}\alpha)}{2^\alpha \Gamma(1+\frac{2q-1}{q-1}\alpha)} \right]^{\frac{q-1}{q}} \\ & \left( \frac{\Gamma(1+s\alpha)}{2^\alpha \Gamma(1+(s+1)\alpha)} \right)^{\frac{1}{q}} \\ & \quad \times \left[ \left( |f^{(\alpha)}(a)|^q + \left| f^{(\alpha)}\left(\frac{a+b}{2}\right) \right|^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left( |f^{(\alpha)}\left(\frac{a+b}{2}\right)|^q + |f^{(\alpha)}(b)|^q \right)^{\frac{1}{q}} \right]. \end{aligned}$$

*Proof* From Lemma 3.1, we have

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2^\alpha} - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a I_b^{(\alpha)} f(x) \right| \\ & \leq \frac{(b-a)^\alpha}{2^\alpha} \frac{1}{\Gamma(1+\alpha)} \int_0^1 |1-2t|^\alpha |f^{(\alpha)}(ta+(1-t)b)| (dt)^\alpha \\ & \leq \frac{(b-a)^\alpha}{2^\alpha} \left[ \frac{1}{\Gamma(1+\alpha)} \int_0^{\frac{1}{2}} (1-2t) |f^{(\alpha)}(ta+(1-t)b)| (dt)^\alpha \right. \\ & \quad \left. + \frac{1}{\Gamma(1+\alpha)} \int_{\frac{1}{2}}^1 (2t-1) |f^{(\alpha)}(ta+(1-t)b)| (dt)^\alpha \right]. \end{aligned} \tag{3.10}$$

Let us estimate

$$\frac{1}{\Gamma(1+\alpha)} \int_0^{\frac{1}{2}} (1-2t) |f^{(\alpha)}(ta+(1-t)b)| (dt)^\alpha$$

and

$$\frac{1}{\Gamma(1+\alpha)} \int_{\frac{1}{2}}^1 (2t-1) |f^{(\alpha)}(ta+(1-t)b)| (dt)^\alpha,$$

respectively.

Using the generalized Hölder’s inequality (Lemma 2.4), we obtain

$$\begin{aligned} & \frac{1}{\Gamma(1+\alpha)} \int_0^{\frac{1}{2}} (1-2t) |f^{(\alpha)}(ta+(1-t)b)| (dt)^\alpha \\ & \leq \left( \frac{1}{\Gamma(1+\alpha)} \int_0^{\frac{1}{2}} (1-2t)^{\frac{q}{q-1}} (dt)^\alpha \right)^{\frac{q-1}{q}} \\ & \quad \left( \frac{1}{\Gamma(1+\alpha)} \int_0^{\frac{1}{2}} |f^{(\alpha)}(ta+(1-t)b)|^{q\alpha} (dt)^\alpha \right)^{\frac{1}{q}}. \end{aligned} \tag{3.11}$$

It is easy to see that

$$\begin{aligned} & \frac{1}{\Gamma(1+\alpha)} \int_0^{\frac{1}{2}} (1-2t)^{\frac{q}{q-1}} (dt)^\alpha = \\ & \frac{1}{\Gamma(1+\alpha)} \int_{\frac{1}{2}}^1 (1-2t)^{\frac{q}{q-1}} (dt)^\alpha = \frac{\Gamma(1+\frac{q}{q-1}\alpha)}{2^\alpha \Gamma(1+\frac{2q-1}{q-1}\alpha)}. \end{aligned} \tag{3.12}$$

Let  $|f^{(\alpha)}(ta+(1-t)b)|^q = U(t)$ . It is easy to see that  $U(t)$  is a generalized  $s$ -convex function in the second sense. Thus, from the right-hand side of (3.1), it follows that

$$\begin{aligned} & \frac{1}{\Gamma(1+\alpha)} \int_0^{\frac{1}{2}} |f^{(\alpha)}(ta+(1-t)b)|^q (dt)^\alpha \\ & = \frac{1}{\Gamma(1+\alpha)} \int_0^{\frac{1}{2}} U(t) (dt)^\alpha \\ & \leq \left( \frac{1}{2} - 0 \right)^\alpha \frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)} \left( U(0) + U\left(\frac{1}{2}\right) \right) \\ & = \frac{\Gamma(1+s\alpha)}{2^\alpha \Gamma(1+(s+1)\alpha)} \left( \left| f^{(\alpha)}\left(\frac{a+b}{2}\right) \right|^q + |f^{(\alpha)}(b)|^q \right). \end{aligned} \tag{3.13}$$

Thus, substituting (3.12) and (3.13) into (3.11), we get

$$\begin{aligned} & \frac{1}{\Gamma(1+\alpha)} \int_0^{\frac{1}{2}} (1-2t) |f^{(\alpha)}(ta+(1-t)b)| (dt)^\alpha \\ & \leq \left( \frac{\Gamma(1+\frac{q}{q-1}\alpha)}{2^\alpha \Gamma(1+\frac{2q-1}{q-1}\alpha)} \right)^{\frac{q-1}{q}} \left( \frac{\Gamma(1+s\alpha)}{2^\alpha \Gamma(1+(s+1)\alpha)} \right)^{\frac{1}{q}} \\ & \quad \left( |f^{(\alpha)}(a)|^q + \left| f^{(\alpha)}\left(\frac{a+b}{2}\right) \right|^q \right)^{\frac{1}{q}}. \end{aligned} \tag{3.14}$$

Moreover

$$\begin{aligned} & \frac{1}{\Gamma(1+\alpha)} \int_{\frac{1}{2}}^1 (2t-1)^{\frac{\alpha}{q-1}} (dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)} \int_{\frac{1}{2}}^1 (1-2t)^{\frac{\alpha}{q-1}} (dt)^\alpha = \frac{\Gamma(1+\frac{q}{q-1}\alpha)}{2^\alpha \Gamma(1+\frac{2q-1}{q-1}\alpha)}. \end{aligned}$$

In addition, similar to the estimate of (3.13), we have

$$\begin{aligned} & \frac{1}{\Gamma(1+\alpha)} \int_{\frac{1}{2}}^1 |f^{(\alpha)}(ta+(1-t)b)|^q (dt)^\alpha \\ & \leq \frac{\Gamma(1+s\alpha)}{2^\alpha \Gamma(1+(s+1)\alpha)} \left( \left| f^{(\alpha)}\left(\frac{a+b}{2}\right) \right|^q + |f^{(\alpha)}(b)|^q \right). \end{aligned}$$

Therefore, it is analogous to the estimate of (3.11), we have

$$\begin{aligned} & \frac{1}{\Gamma(1+\alpha)} \int_0^{\frac{1}{2}} (1-2t) |f^{(\alpha)}(ta+(1-t)b)| (dt)^\alpha \\ & \leq \left( \frac{1}{\Gamma(1+\alpha)} \int_{\frac{1}{2}}^1 (2t-1)^{\frac{\alpha}{q-1}} (dt)^\alpha \right)^{\frac{q-1}{q}} \\ & \quad \left( \frac{1}{\Gamma(1+\alpha)} \int_{\frac{1}{2}}^1 |f^{(\alpha)}(ta+(1-t)b)|^{q\alpha} (dt)^\alpha \right)^{\frac{1}{q}} \\ & \leq \left( \frac{\Gamma(1+\frac{q}{q-1}\alpha)}{2^\alpha \Gamma(1+\frac{2q-1}{q-1}\alpha)} \right)^{\frac{q-1}{q}} \left( \frac{\Gamma(1+s\alpha)}{2^\alpha \Gamma(1+(s+1)\alpha)} \right)^{\frac{1}{q}} \\ & \quad \left( \left| f^{(\alpha)}\left(\frac{a+b}{2}\right) \right|^q + |f^{(\alpha)}(b)|^q \right)^{\frac{1}{q}}. \end{aligned} \tag{3.15}$$

Thus, combining (3.10), (3.14), and (3.15), we obtain

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2^\alpha} - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} I_b^{(\alpha)} f(x) \right| \\ & \leq \frac{(b-a)^\alpha}{2^\alpha} \left[ \frac{\Gamma(1+\frac{q}{q-1}\alpha)}{2^\alpha \Gamma(1+\frac{2q-1}{q-1}\alpha)} \right]^{\frac{q-1}{q}} \left( \frac{\Gamma(1+s\alpha)}{2^\alpha \Gamma(1+(s+1)\alpha)} \right)^{\frac{1}{q}} \\ & \quad \times \left[ \left( |f^{(\alpha)}(a)|^q + \left| f^{(\alpha)}\left(\frac{a+b}{2}\right) \right|^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left( \left| f^{(\alpha)}\left(\frac{a+b}{2}\right) \right|^q + |f^{(\alpha)}(b)|^q \right)^{\frac{1}{q}} \right]. \end{aligned}$$

□

Therefore, we complete the proof of Theorem 3.3.

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