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Hermite-Hadamard type inequalities for MT-convex functions via classical integrals and fractional integrals

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Abstract

Some inequalities of Hermite-Hadamard type for MT-convex functions via classical integrals and Riemann-Liouville fractional integrals are introduced, respectively, and applications for special means are given. Some error estimates for the trapezoidal formula are also obtained.

Keywords: MT-convex function, Hermite-Hadamard inequality, Hölder inequality, fractional integral, trapezoidal formula.
2010 MSC: 26A33, 33B15, 26A51, 26D10.

1. Introduction

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities for convex functions.

Theorem 1.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$, with $a < b$. Then:*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \le \frac{f(a)+f(b)}{2}. \tag{1.1}$$

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In [4], Dragomir and Agarwal established the following result connected with the right part of (1.1):

Theorem 1.2. *Let $f : I^o \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^o, a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)\mathrm{d}u\right| \le \frac{(b-a)(|f'(a)|+|f'(b)|)}{8}.$$

In [30] (see also [27, 29]), Tunc and Yidirim defined the following so-called MT-convex function:

Definition 1.3. A function: $I \subseteq \mathbb{R} \to \mathbb{R}$ is said to belong to the class of $\mathrm{MT}(I)$, if it is nonnegative and for all $x, y \in I$ and $t \in (0, 1)$ satisfies the following inequality:

$$f(tx + (1-t)y) \le \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{1.2}$$

Theorem 1.4. *Let $f \in MT(I), a, b \in I$ with $a < b$ and $f \in L_1[a, b]$. Then:*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \tag{1.3}$$

and

$$\frac{2}{b-a}\int_a^b \tau(x) f(x)\mathrm{d}x \le \frac{f(a)+f(b)}{2}, \tag{1.4}$$

where $\tau(x) = \frac{\sqrt{(b-x)(x-a)}}{b-a}, x \in [a, b]$.

For other recent results concerning Hermite-Hadamard type inequalities through various classes of convex functions, see [1, 3, 5, 7, 8, 9, 12, 10, 17, 20, 22, 24, 25, 28] and the references cited therein.

Fractional calculus [6, 18, 21] was introduced at the end of the nineteenth century by Liouville and Riemann, the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics. We recall some definitions and preliminary facts of fractional calculus theory which will be used in this paper.

Definition 1.5. Let $f \in L_1[a, b]$. The Riemann-Liouville integrals $J^\alpha_{a^+}(f)$ and $J^\alpha_{b^-}(f)$ of order $\alpha > 0$ with $a \ge 0$ are defined by

$$J^\alpha_{a^+} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\mathrm{d}t, x > a$$

and

$$J^\alpha_{b^-} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\mathrm{d}t, b > x,$$

where $\Gamma(\alpha) = \int_0^\infty e^{-u}u^{\alpha-1}\mathrm{d}u$. Here $J^0_{a^+} f(x) = J^0_{b^-} f(x) = f(x)$. In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Due to the wide application of fractional integrals, some authors extended to study fractional Hermite-Hadamard, Grüss, or Ostrowski type inequalities for functions of different classes, see [2, 11, 13, 14, 15, 16, 23, 26] where further references are listed.

The main aim of this paper is to establish some Hermite-Hadamard type inequalities for MT-convex functions via classical integrals and Riemann-Liouville fractional integrals, respectively. An interesting feature of our results is that they provide new estimates on these types of Hermite-Hadamard inequalities for MT-convex functions. Some applications for special means and for the error estimates of trapezoidal formula are also obtained.

2. Hermite-Hadamard type inequalities via classical integrals

In order to prove our main results, we need the following Lemma that has been obtained in [9]:

Lemma 2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^o, where $a, b \in I$ with $a < b$. If $f' \in L_1[a, b]$, then the following inequality holds:*

$$\frac{(b-x)f(b)+(x-a)f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(u)\mathrm{d}u = \frac{(x-a)^2}{b-a}\int_0^1 (t-1)f'(tx+(1-t)a)\mathrm{d}t + \frac{(b-x)^2}{b-a}\int_0^1 (1-t)f'(tx+(1-t)b)\mathrm{d}t, \tag{2.1}$$

for each $x \in [a, b]$.

We shall start with the following refinement of the Hermite-Hadamard type inequality.

Theorem 2.2. *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I^o such that $f' \in L_1[a, b]$ where $a, b \in I$. If $|f'|$ is MT-convex function on $[a, b]$ and $|f'(x)| \le M, x \in [a, b]$, then we have:*

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(u)\mathrm{d}u\right| \le \frac{M\pi[(x-a)^2+(b-x)^2]}{4(b-a)}, \tag{2.2}$$

for each $x \in [a, b]$.

Proof. Using Lemma 2.1 and MT-convexity of $|f'|$, it follows that

$$\begin{aligned}
&\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(u)\mathrm{d}u\right| \\
&\le \frac{(x-a)^2}{b-a}\int_0^1 (1-t)|f'(tx+(1-t)a)|\mathrm{d}t + \frac{(b-x)^2}{b-a}\int_0^1 (1-t)|f'(tx+(1-t)b)|\mathrm{d}t \\
&\le \frac{(x-a)^2}{b-a}\int_0^1 (1-t)\left[\frac{\sqrt{t}}{2\sqrt{1-t}}|f'(x)| + \frac{\sqrt{1-t}}{2\sqrt{t}}|f'(a)|\right]\mathrm{d}t \\
&\quad + \frac{(b-x)^2}{b-a}\int_0^1 (1-t)\left[\frac{\sqrt{t}}{2\sqrt{1-t}}|f'(x)| + \frac{\sqrt{1-t}}{2\sqrt{t}}|f'(b)|\right]\mathrm{d}t \\
&\le \frac{M[(x-a)^2+(b-x)^2]}{2(b-a)}\int_0^1 \left(t^{1/2}(1-t)^{1/2} + t^{-1/2}(1-t)^{3/2}\right)\mathrm{d}t.
\end{aligned}$$

With Euler Beta function defined by

$$\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t, x, y > 0,$$

the proof is completed. □

Remark 2.3. In Theorem 2.2, if we choose $x = (a+b)/2$, then we get

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)\mathrm{d}u\right| \le \frac{M\pi(b-a)}{8}. \tag{2.3}$$

The corresponding version for power of the absolute value of the first derivative is incorporated in the following results.

Theorem 2.4. *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I^o such that $f' \in L_1[a, b]$ where $a, b \in I$ with $a < b$. If $|f'|^q$ is MT-convex function on $[a, b], q > 1, p^{-1} + q^{-1} = 1$ and $|f'(x)| \le M, x \in [a, b]$, then we have:*

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u\right| \leq \frac{M}{(1+p)^{1/p}}\left(\frac{\pi}{2}\right)^{\frac{1}{q}}\frac{(x-a)^2+(b-x)^2}{(b-a)} \tag{2.4}$$

for each $x \in [a,b]$.

Proof. Suppose that $p > 1$. From Lemma 2.1 and using the Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u\right| \\
&\leq \frac{(x-a)^2}{b-a}\int_0^1(1-t)|f'(tx+(1-t)a)|\mathrm{d}t+\frac{(b-x)^2}{b-a}\int_0^1(1-t)|f'(tx+(1-t)b)|\mathrm{d}t \\
&\leq \frac{(x-a)^2}{b-a}\left(\int_0^1(1-t)^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1|f'(tx+(1-t)a)|^q\mathrm{d}t\right)^{\frac{1}{q}} \\
&\quad+\frac{(b-x)^2}{b-a}\left(\int_0^1(1-t)^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1|f'(tx+(1-t)b)|^q\mathrm{d}t\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $|f'|^q$ is MT-convex function and $|f'(x)| \leq M$, then we have

$$\int_0^1|f'(tx+(1-t)a)|^q\mathrm{d}t \leq \int_0^1\left[\frac{\sqrt{t}}{2\sqrt{1-t}}|f'(x)|^q+\frac{\sqrt{1-t}}{2\sqrt{t}}|f'(a)|^q\right]\mathrm{d}t=\frac{\pi}{4}\left[|f'(x)|^q+|f'(a)|^q\right] \leq \frac{\pi}{2}M^q$$

and similarly,

$$\int_0^1|f'(tx+(1-t)b)|^q\mathrm{d}t \leq \int_0^1\left[\frac{\sqrt{t}}{2\sqrt{1-t}}|f'(x)|^q+\frac{\sqrt{1-t}}{2\sqrt{t}}|f'(b)|^q\right]\mathrm{d}t=\frac{\pi}{4}\left[|f'(x)|^q+|f'(b)|^q\right] \leq \frac{\pi}{2}M^q.$$

Therefore, we have

$$\begin{aligned}
&\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u\right| \\
&\leq \frac{(x-a)^2}{b-a}\frac{1}{(1+p)^{1/p}}\left(\frac{\pi}{2}M^q\right)^{\frac{1}{q}}+\frac{(b-x)^2}{b-a}\frac{1}{(1+p)^{1/p}}\left(\frac{\pi}{2}M^q\right)^{\frac{1}{q}} \\
&=\frac{M}{(1+p)^{1/p}}\left(\frac{\pi}{2}\right)^{\frac{1}{q}}\frac{(x-a)^2+(b-x)^2}{(b-a)},
\end{aligned}$$

where $p^{-1}+q^{-1}=1$, which is required. □

Remark 2.5. In Theorem 2.4, if we choose $x=(a+b)/2$, then we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u\right| \leq \frac{M\pi^{\frac{1}{q}}}{(1+p)^{1/p}}\left(\frac{1}{2}\right)^{1+\frac{1}{q}}(b-a). \tag{2.5}$$

Theorem 2.6. *Let* $f : I \subset [0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* I^o *such that* $f' \in L_1[a,b]$ *where* $a,b \in I$ *with* $a < b$. *If* $|f'|^q$ *is MT-convex function on* $[a,b], q \geq 1$ *and* $|f'(x)| \leq M, x \in [a,b]$, *then the following inequality holds:*

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u\right| \leq M\left(\frac{1}{2}\right)^{1+\frac{1}{q}}\pi^{\frac{1}{q}}\frac{(x-a)^2+(b-x)^2}{(b-a)} \tag{2.6}$$

for each $x \in [a,b]$.

Proof. From Lemma 2.1 and using the Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u\right| \\
&\quad\le \frac{(x-a)^2}{b-a}\int_0^1(1-t)|f'(tx+(1-t)a)|\mathrm{d}t+\frac{(b-x)^2}{b-a}\int_0^1(1-t)|f'(tx+(1-t)b)|\mathrm{d}t \\
&\quad\le \frac{(x-a)^2}{b-a}\left(\int_0^1(1-t)\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)|f'(tx+(1-t)a)|^q\mathrm{d}t\right)^{\frac{1}{q}} \\
&\qquad+\frac{(b-x)^2}{b-a}\left(\int_0^1(1-t)\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)|f'(tx+(1-t)b)|^q\mathrm{d}t\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $|f'|^q$ is MT-convex function and $|f'(x)|\le M$, then we have

$$\begin{aligned}
\int_0^1(1-t)|f'(tx+(1-t)a)|^q\mathrm{d}t &\le \int_0^1\left[\frac{(1-t)\sqrt{t}}{2\sqrt{1-t}}|f'(x)|^q+\frac{(1-t)\sqrt{1-t}}{2\sqrt{t}}|f'(a)|^q\right]\mathrm{d}t \\
&=\frac{1}{2}|f'(x)|^q\int_0^1 t^{1/2}(1-t)^{1/2}\mathrm{d}t+\frac{1}{2}|f'(a)|^q\int_0^1 t^{-1/2}(1-t)^{3/2}\mathrm{d}t \\
&\le \frac{1}{2}M^q\frac{\pi}{8}+\frac{1}{2}M^q\frac{3\pi}{8}=\frac{\pi}{4}M^q
\end{aligned}$$

and

$$\int_0^1(1-t)|f'(tx+(1-t)b)|^q\mathrm{d}t\le\int_0^1\left[\frac{(1-t)\sqrt{t}}{2\sqrt{1-t}}|f'(x)|^q+\frac{(1-t)\sqrt{1-t}}{2\sqrt{t}}|f'(b)|^q\right]\mathrm{d}t\le\frac{\pi}{4}M^q.$$

Therefore, we have

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u\right|\le M\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{\pi}{4}\right)^{\frac{1}{q}}\frac{(x-a)^2+(b-x)^2}{(b-a)},$$

which is required. □

Remark 2.7. In Theorem 2.6, if we choose $x=(a+b)/2$, then we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u\right|\le M\pi^{\frac{1}{q}}\left(\frac{1}{2}\right)^{2+\frac{1}{q}}(b-a). \tag{2.7}$$

Theorem 2.8. *Let $f: I\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on I^o such that $f'\in L_1[a,b]$ where $a,b\in I$ with $a<b$. If $|f'|^q$ is MT-convex function on $[a,b], q\ge 1$ and $|f'(x)|\le M, x\in[a,b]$, then the following inequality holds:*

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u\right|\le M\left(\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(q+\frac{1}{2}\right)}{2\Gamma(q+1)}\right)^{\frac{1}{q}}\frac{(x-a)^2+(b-x)^2}{(b-a)} \tag{2.8}$$

for each $x\in[a,b]$.

Proof. From Lemma 2.1 and using the Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u\right| \\
&\quad\le \frac{(x-a)^2}{b-a}\int_0^1 1\cdot(1-t)|f'(tx+(1-t)a)|\mathrm{d}t+\frac{(b-x)^2}{b-a}\int_0^1 1\cdot(1-t)|f'(tx+(1-t)b)|\mathrm{d}t \\
&\quad\le \frac{(x-a)^2}{b-a}\left(\int_0^1 1\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)^q|f'(tx+(1-t)a)|^q\mathrm{d}t\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\frac{(b-x)^2}{b-a}\left(\int_0^1 1\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)^q|f'(tx+(1-t)b)|^q\mathrm{d}t\right)^{\frac{1}{q}}.$$

Since $|f'|^q$ is MT-convex function and $|f'(x)|\le M$, then we have

$$\begin{aligned}\int_0^1(1-t)^q|f'(tx+(1-t)a)|^q\mathrm{d}t &\le \int_0^1\left[\frac{(1-t)^q\sqrt{t}}{2\sqrt{1-t}}|f'(x)|^q+\frac{(1-t)^q\sqrt{1-t}}{2\sqrt{t}}|f'(a)|^q\right]\mathrm{d}t\\ &=\frac{1}{2}|f'(x)|^q\int_0^1 t^{\frac{1}{2}}(1-t)^{q-\frac{1}{2}}\mathrm{d}t+\frac{1}{2}|f'(a)|^q\int_0^1 t^{-\frac{1}{2}}(1-t)^{q+\frac{1}{2}}\mathrm{d}t\\ &\le\frac{1}{2}M^q\beta\left(\frac{3}{2},q+\frac{1}{2}\right)+\frac{1}{2}M^q\beta\left(\frac{1}{2},q+\frac{3}{2}\right)\\ &=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(q+\frac{1}{2}\right)}{2\Gamma(q+1)}M^q\end{aligned}$$

and

$$\begin{aligned}\int_0^1(1-t)^q|f'(tx+(1-t)b)|^q\mathrm{d}t &\le \int_0^1\left[\frac{(1-t)^q\sqrt{t}}{2\sqrt{1-t}}|f'(x)|^q+\frac{(1-t)^q\sqrt{1-t}}{2\sqrt{t}}|f'(b)|^q\right]\mathrm{d}t\\ &\le\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(q+\frac{1}{2}\right)}{2\Gamma(q+1)}M^q.\end{aligned}$$

Therefore, we have

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u\right|\le M\left(\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(q+\frac{1}{2}\right)}{2\Gamma(q+1)}\right)^{\frac{1}{q}}\frac{(x-a)^2+(b-x)^2}{(b-a)},$$

which is required. □

Remark 2.9. In Theorem 2.8, if we choose $x=(a+b)/2$, then we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u\right|\le M\left(\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(q+\frac{1}{2}\right)}{\Gamma(q+1)}\right)^{\frac{1}{q}}\left(\frac{1}{2}\right)^{1+\frac{1}{q}}(b-a). \tag{2.9}$$

3. Hermite-Hadamard type inequalities via fractional integrals

In this section, we apply the following fractional integral identity from Ozdemir *et al.* [19] to derive some new Hermite-Hadamard type inequalities for MT-convex functions.

Lemma 3.1. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I^o, the interior of I, where $a,b\in I$ with $a<b$. If $f'\in L_1[a,b]$, then for all $x\in[a,b]$ and $\alpha>0$ we have:*

$$\begin{aligned}&\frac{(x-a)^\alpha f(a)+(b-x)^\alpha f(b)}{b-a}-\frac{\Gamma(\alpha+1)}{b-a}[J_{x^-}^\alpha f(a)+J_{x^+}^\alpha f(b)]\\ &=\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1(t^\alpha-1)f'(tx+(1-t)a)\mathrm{d}t+\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1(1-t^\alpha)f'(tx+(1-t)b)\mathrm{d}t,\end{aligned}\tag{3.1}$$

where $\Gamma(\alpha)=\int_0^\infty e^{-u}u^{\alpha-1}\mathrm{d}u$ is the Euler Gamma function.

By using Lemma 3.1, one can extend to the following results.

Theorem 3.2. *Let $f: I\subset[0,\infty)\to\mathbb{R}$ be a differentiable function on I^o such that $f'\in L_1[a,b]$ where $a,b\in I$ with $a<b$. If $|f'|$ is MT-convex function on $[a,b]$ and $|f'(x)|\le M, x\in[a,b]$, then we have the following inequality for fractional integrals with $\alpha>0$:*

$$\left|\frac{(x-a)^\alpha f(a)+(b-x)^\alpha f(b)}{b-a}-\frac{\Gamma(\alpha+1)}{b-a}[J^\alpha_{x^-}f(a)+J^\alpha_{x^+}f(b)]\right|$$
$$\le \frac{M[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}]}{2(b-a)}\left[\pi-\frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\alpha+1)}\right], \tag{3.2}$$

where Γ is the Euler Gamma function.

Proof. From Lemma 3.1, property of the modulus and using the MT-convexity of $|f'|$, we have

$$\begin{aligned}
&\left|\frac{(x-a)^\alpha f(a)+(b-x)^\alpha f(b)}{b-a}-\frac{\Gamma(\alpha+1)}{b-a}[J^\alpha_{x^-}f(a)+J^\alpha_{x^+}f(b)]\right|\\
&\le \frac{(x-a)^{\alpha+1}}{b-a}\int_0^1|t^\alpha-1||f'(tx+(1-t)a)|\mathrm{d}t\\
&\quad+\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1|1-t^\alpha||f'(tx+(1-t)b)|\mathrm{d}t\\
&\le \frac{(x-a)^{\alpha+1}}{b-a}\int_0^1(1-t^\alpha)\left[\frac{\sqrt{t}}{2\sqrt{1-t}}|f'(x)|+\frac{\sqrt{1-t}}{2\sqrt{t}}|f'(a)|\right]\mathrm{d}t\\
&\quad+\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1(1-t^\alpha)\left[\frac{\sqrt{t}}{2\sqrt{1-t}}|f'(x)|+\frac{\sqrt{1-t}}{2\sqrt{t}}|f'(b)|\right]\mathrm{d}t\\
&\le \frac{M(x-a)^{\alpha+1}}{2(b-a)}\int_0^1(1-t^\alpha)\left(t^{1/2}(1-t)^{-1/2}+t^{-1/2}(1-t)^{1/2}\right)\mathrm{d}t\\
&\quad+\frac{M(b-x)^{\alpha+1}}{2(b-a)}\int_0^1(1-t^\alpha)\left(t^{1/2}(1-t)^{-1/2}+t^{-1/2}(1-t)^{1/2}\right)\mathrm{d}t\\
&=\frac{M[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}]}{2(b-a)}\int_0^1(1-t^\alpha)\left(t^{1/2}(1-t)^{-1/2}+t^{-1/2}(1-t)^{1/2}\right)\mathrm{d}t\\
&=\frac{M[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}]}{2(b-a)}\left[\pi-\frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\alpha+1)}\right],
\end{aligned}$$

where we have used the Beta function of Euler type, which is defined as

$$\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},\quad \forall\ x,y>0.$$

The proof is completed. □

Remark 3.3. In Theorem 3.2, if we choose $x=(a+b)/2$, we get

$$\left|\frac{(b-a)^{\alpha-1}}{2^{\alpha-1}}\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{b-a}\left[J^\alpha_{(\frac{a+b}{2})^-}f(a)+J^\alpha_{(\frac{a+b}{2})^+}f(b)\right]\right|$$
$$\le \frac{M(b-a)^\alpha}{2^{\alpha+1}}\left[\pi-\frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\alpha+1)}\right].$$

Remark 3.4. In Theorem 3.2, if we choose $\alpha=1$, we get the inequality in Theorem 2.2.

Theorem 3.5. *Let $f: I\subset[0,\infty)\to\mathbb{R}$ be a differentiable function on I^o such that $f'\in L_1[a,b]$, where $a,b\in I$ with $a<b$. If $|f'|^q$ is MT-convex function on $[a,b], q>1, p^{-1}+q^{-1}=1$ and $|f'(x)|\le M, x\in[a,b]$, then we have the following inequality for fractional integrals with $\alpha>0$:*

$$\left|\frac{(x-a)^\alpha f(a)+(b-x)^\alpha f(b)}{b-a}-\frac{\Gamma(\alpha+1)}{b-a}[J^\alpha_{x^-}f(a)+J^\alpha_{x^+}f(b)]\right|$$

$$\leq \frac{M[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}]}{b-a}\left(\frac{\pi}{2}\right)^{\frac{1}{q}}\left(\frac{\Gamma(1+p)\Gamma(\frac{1}{\alpha})}{\alpha\Gamma(1+p+\frac{1}{\alpha})}\right)^{\frac{1}{p}}, \tag{3.3}$$

where Γ *is the Euler Gamma function.*

Proof. From Lemma 3.1, property of the modulus and using the Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{(x-a)^{\alpha}f(a)+(b-x)^{\alpha}f(b)}{b-a}-\frac{\Gamma(\alpha+1)}{b-a}[J^{\alpha}_{x^-}f(a)+J^{\alpha}_{x^+}f(b)]\right| \\
&\leq \frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 |t^{\alpha}-1||f'(tx+(1-t)a)|\mathrm{d}t \\
&\quad + \frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 |1-t^{\alpha}||f'(tx+(1-t)b)|\mathrm{d}t \\
&\leq \frac{(x-a)^{\alpha+1}}{b-a}\left(\int_0^1 (1-t^{\alpha})^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1 |f'(tx+(1-t)a)|^q\mathrm{d}t\right)^{\frac{1}{q}} \\
&\quad + \frac{(b-x)^{\alpha+1}}{b-a}\left(\int_0^1 (1-t^{\alpha})^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1 |f'(tx+(1-t)b)|^q\mathrm{d}t\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $|f'|^q$ is MT-convex function and $|f'(x)| \leq M$, then we have

$$\begin{aligned}
\int_0^1 |f'(tx+(1-t)a)|^q\mathrm{d}t &\leq \int_0^1\left[\frac{\sqrt{t}}{2\sqrt{1-t}}|f'(x)|^q+\frac{\sqrt{1-t}}{2\sqrt{t}}|f'(a)|^q\right]\mathrm{d}t \\
&= \frac{\pi}{4}\left[|f'(x)|^q+|f'(a)|^q\right] \leq \frac{\pi}{2}M^q,
\end{aligned}$$

$$\begin{aligned}
\int_0^1 |f'(tx+(1-t)b)|^q\mathrm{d}t &\leq \int_0^1\left[\frac{\sqrt{t}}{2\sqrt{1-t}}|f'(x)|^q+\frac{\sqrt{1-t}}{2\sqrt{t}}|f'(b)|^q\right]\mathrm{d}t \\
&= \frac{\pi}{4}\left[|f'(x)|^q+|f'(b)|^q\right] \leq \frac{\pi}{2}M^q
\end{aligned}$$

and

$$\int_0^1 (1-t^{\alpha})^p\mathrm{d}t = \frac{1}{\alpha}\int_0^1 (1-s)^p s^{\frac{1}{\alpha}-1}\mathrm{d}s = \frac{\Gamma(1+p)\Gamma(\frac{1}{\alpha})}{\alpha\Gamma(1+p+\frac{1}{\alpha})}.$$

Hence we have

$$\begin{aligned}
&\left|\frac{(x-a)^{\alpha}f(a)+(b-x)^{\alpha}f(b)}{b-a}-\frac{\Gamma(\alpha+1)}{b-a}[J^{\alpha}_{x^-}f(a)+J^{\alpha}_{x^+}f(b)]\right| \\
&\leq \frac{(x-a)^{\alpha+1}}{b-a}\left(\frac{\Gamma(1+p)\Gamma(\frac{1}{\alpha})}{\alpha\Gamma(1+p+\frac{1}{\alpha})}\right)^{\frac{1}{p}}\left(\frac{\pi M^q}{2}\right)^{\frac{1}{q}} + \frac{(b-x)^{\alpha+1}}{b-a}\left(\frac{\Gamma(1+p)\Gamma(\frac{1}{\alpha})}{\alpha\Gamma(1+p+\frac{1}{\alpha})}\right)^{\frac{1}{p}}\left(\frac{\pi M^q}{2}\right)^{\frac{1}{q}},
\end{aligned}$$

which completes the proof. □

Remark 3.6. In Theorem 3.5, if we choose $x = (a+b)/2$, we get

$$\begin{aligned}
&\left|\frac{(b-a)^{\alpha-1}}{2^{\alpha-1}}\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{b-a}\left[J^{\alpha}_{(\frac{a+b}{2})^-}f(a)+J^{\alpha}_{(\frac{a+b}{2})^+}f(b)\right]\right| \\
&\leq \frac{M(b-a)^{\alpha}}{2^{\alpha}}\left(\frac{\pi}{2}\right)^{\frac{1}{q}}\left(\frac{\Gamma(1+p)\Gamma(\frac{1}{\alpha})}{\alpha\Gamma(1+p+\frac{1}{\alpha})}\right)^{\frac{1}{p}}.
\end{aligned}$$

Remark 3.7. In Theorem 3.5, if we choose $\alpha = 1$, we get the inequality in Theorem 2.4.

Theorem 3.8. *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable function on I^o such that $f' \in L_1[a, b]$ where $a, b \in I$ with $a < b$. If $|f'|^q$ is MT-convex function on $[a, b], q \geq 1$ and $|f'(x)| \leq M, x \in [a, b]$, then we have the following inequality for fractional integrals with $\alpha > 0$:*

$$\left|\frac{(x-a)^\alpha f(a) + (b-x)^\alpha f(b)}{b-a} - \frac{\Gamma(\alpha+1)}{b-a}[J^\alpha_{x^-} f(a) + J^\alpha_{x^+} f(b)]\right| \leq \frac{M[(x-a)^{\alpha+1} + (b-x)^{\alpha+1}]}{b-a}\left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}\left[\frac{\pi}{2} - \frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}{2\Gamma(\alpha+1)}\right]^{\frac{1}{q}}, \tag{3.4}$$

where Γ is the Euler Gamma function.

Proof. From Lemma 3.1, property of the modulus and using the Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{(x-a)^\alpha f(a) + (b-x)^\alpha f(b)}{b-a} - \frac{\Gamma(\alpha+1)}{b-a}[J^\alpha_{x^-} f(a) + J^\alpha_{x^+} f(b)]\right| \qquad (3.5)\\
&\leq \frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 |t^\alpha - 1||f'(tx+(1-t)a)|\mathrm{d}t + \frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 |1-t^\alpha||f'(tx+(1-t)b)|\mathrm{d}t\\
&\leq \frac{(x-a)^{\alpha+1}}{b-a}\left(\int_0^1 (1-t^\alpha)\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t^\alpha)|f'(tx+(1-t)a)|^q\mathrm{d}t\right)^{\frac{1}{q}}\\
&\quad + \frac{(b-x)^{\alpha+1}}{b-a}\left(\int_0^1 (1-t^\alpha)\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t^\alpha)|f'(tx+(1-t)b)|^q\mathrm{d}t\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $|f'|^q$ is MT-convex function and $|f'(x)| \leq M$, then we have

$$\begin{aligned}
\int_0^1 (1-t^\alpha)|f'(tx+(1-t)a)|^q\mathrm{d}t &\leq \int_0^1 (1-t^\alpha)\left[\frac{\sqrt{t}}{2\sqrt{1-t}}|f'(x)|^q + \frac{\sqrt{1-t}}{2\sqrt{t}}|f'(a)|^q\right]\mathrm{d}t \qquad (3.6)\\
&\leq \frac{M^q}{2}\int_0^1 (1-t^\alpha)\left(t^{1/2}(1-t)^{-1/2} + t^{-1/2}(1-t)^{1/2}\right)\mathrm{d}t\\
&= \frac{M^q}{2}\left(\pi - \frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\alpha+1)}\right)
\end{aligned}$$

and

$$\begin{aligned}
\int_0^1 (1-t^\alpha)|f'(tx+(1-t)b)|^q\mathrm{d}t &\leq \int_0^1 (1-t^\alpha)\left[\frac{\sqrt{t}}{2\sqrt{1-t}}|f'(x)|^q + \frac{\sqrt{1-t}}{2\sqrt{t}}|f'(b)|^q\right]\mathrm{d}t \qquad (3.7)\\
&\leq \frac{M^q}{2}\left(\pi - \frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\alpha+1)}\right).
\end{aligned}$$

If we use (3.6) and (3.7) in (3.5) we obtain the desired result. □

Remark 3.9. In Theorem 3.8, if we choose $x = (a+b)/2$, we get

$$\left|\frac{(b-a)^{\alpha-1}}{2^{\alpha-1}}\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{b-a}\left[J^\alpha_{(\frac{a+b}{2})^-} f(a) + J^\alpha_{(\frac{a+b}{2})^+} f(b)\right]\right| \leq \frac{M(b-a)^\alpha}{2^\alpha}\left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}\left[\frac{\pi}{2} - \frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}{2\Gamma(\alpha+1)}\right]^{\frac{1}{q}}.$$

Remark 3.10. In Theorem 3.8, if we choose $\alpha = 1$, we get the inequality in Theorem 2.6.

4. Applications to special means

Recall the following means which could be considered as extensions of arithmetic, logarithmic and generalized logarithmic from positive to real numbers.

(1) The arithmetic mean:

$$A = A(a,b) = \frac{a+b}{2};\ a,b \in \mathbb{R};$$

(2) The logarithmic mean:

$$L(a,b) = \frac{b-a}{\ln|b| - \ln|a|};\ |a| \neq |b|,\ ab \neq 0,\ a,b \in \mathbb{R};$$

(3) The generalized logarithmic mean:

$$L_n(a,b) = \left[\frac{b^{n+1} - a^{n+1}}{(b-a)(n+1)}\right]^{\frac{1}{n}};\ n \in \mathbb{Z}\backslash\{-1,0\},\ a,b \in \mathbb{R},\ a \neq b.$$

Now using the results of Section 2, we give some applications to special means of real numbers.

Proposition 4.1. *Let $a,b \in \mathbb{R}, 0 < a < b$ and $n \in \mathbb{Z}, |n| \geq 2$. Then, for all $q \geq 1$*

$$|A(a^n,b^n) - L_n^n(a,b)| \leq \frac{M\pi^{\frac{1}{q}}}{(1+p)^{1/p}} \left(\frac{1}{2}\right)^{1+\frac{1}{q}} (b-a), \tag{4.1}$$

$$|A(a^n,b^n) - L_n^n(a,b)| \leq M\pi^{\frac{1}{q}} \left(\frac{1}{2}\right)^{2+\frac{1}{q}} (b-a) \tag{4.2}$$

and

$$|A(a^n,b^n) - L_n^n(a,b)| \leq M \left(\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(q+\frac{1}{2}\right)}{\Gamma(q+1)}\right)^{\frac{1}{q}} \left(\frac{1}{2}\right)^{1+\frac{1}{q}} (b-a). \tag{4.3}$$

Proof. The assertion follows from Remark 2.5, 2.7 and 2.9 for $f(x) = x^n, x \in \mathbb{R}, n \in \mathbb{Z}, |n| \geq 2$. □

Proposition 4.2. *Let $a,b \in \mathbb{R}, 0 < a < b$. Then, for all $q \geq 1$*

$$|A(a^1,b^{-1}) - L^{-1}(a,b)| \leq \frac{M\pi^{\frac{1}{q}}}{(1+p)^{1/p}} \left(\frac{1}{2}\right)^{1+\frac{1}{q}} (b-a), \tag{4.4}$$

$$|A(a^1,b^{-1}) - L^{-1}(a,b)| \leq M\pi^{\frac{1}{q}} \left(\frac{1}{2}\right)^{2+\frac{1}{q}} (b-a) \tag{4.5}$$

and

$$|A(a^1,b^{-1}) - L^{-1}(a,b)| \leq M \left(\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(q+\frac{1}{2}\right)}{\Gamma(q+1)}\right)^{\frac{1}{q}} \left(\frac{1}{2}\right)^{1+\frac{1}{q}} (b-a). \tag{4.6}$$

Proof. The assertion follows from Remark 2.5, 2.7 and 2.9 for $f(x) = \frac{1}{x}$. □

5. Some error estimates for the Trapezoidal formula

Let d be a division $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ of the interval $[a,b]$ and consider the quadrature formula

$$\int_a^b f(x)\mathrm{d}x = T(f,d) + E(f,d), \tag{5.1}$$

where

$$T(f,d)=\sum_{i=0}^{n-1}\frac{f(x_i)+f(x_{i+1})}{2}(x_{i+1}-x_i)$$

for the trapezoidal version and $E(f,d)$ denotes the associated approximation error.

Proposition 5.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^o such that $f' \in L_1[a,b]$, where $a,b \in I$ with $a<b$ and $|f'|^q$ is MT-convex on $[a,b]$, where $p>1, p^{-1}+q^{-1}=1$. Then in (5.1), for every division d of $[a,b]$ and $|f'(x)| \le M, x \in [a,b]$, the trapezoidal error estimate satisfies*

$$|E(f,d)| \le M\left(\frac{1}{2}\right)^{1+\frac{1}{q}}\left(\frac{1}{1+p}\right)^{\frac{1}{p}}\pi^{\frac{1}{q}}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2. \tag{5.2}$$

Proof. On applying Remark 2.5 on the subinterval $[x_i,x_{i+1}](i=0,1,2,\ldots,n-1)$ of the division, we have

$$\left|\frac{f(x_i)+f(x_{i+1})}{2}-\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}}f(x)\mathrm{d}x\right| \le \left(\frac{1}{2}\right)^{1+\frac{1}{q}}\left(\frac{1}{1+p}\right)^{\frac{1}{p}}M\pi^{\frac{1}{q}}(x_{i+1}-x_i).$$

Hence in (5.1), we have

$$\begin{aligned}\left|\int_a^b f(x)\mathrm{d}x-T(f,d)\right| &= \left|\sum_{i=0}^{n-1}\left\{\int_{x_i}^{x_{i+1}}f(x)\mathrm{d}x-\frac{f(x_i)+f(x_{i+1})}{2}(x_{i+1}-x_i)\right\}\right| \\ &\le \sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}}f(x)\mathrm{d}x-\frac{f(x_i)+f(x_{i+1})}{2}(x_{i+1}-x_i)\right| \\ &\le M\left(\frac{1}{2}\right)^{1+\frac{1}{q}}\left(\frac{1}{1+p}\right)^{\frac{1}{p}}\pi^{\frac{1}{q}}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2,\end{aligned}$$

which completes the proof. □

Proposition 5.2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^o such that $f' \in L_1[a,b]$, where $a,b \in I$ with $a<b$ and $|f'|^q$ is MT-convex on $[a,b]$, where $q \ge 1$. Then in (5.1), for every division d of $[a,b]$ and $|f'(x)| \le M, x \in [a,b]$, the trapezoidal error estimate satisfies*

$$|E(f,d)| \le M\left(\frac{\pi}{4}\right)^{\frac{1}{q}}\left(\frac{1}{2}\right)^{2+\frac{1}{q}}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2. \tag{5.3}$$

Proof. The proof is similar to that of Proposition 5.1 and using Remark 2.7. □

Proposition 5.3. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^o such that $f' \in L_1[a,b]$, where $a,b \in I$ with $a<b$ and $|f'|^q$ is MT-convex on $[a,b]$, where $q \ge 1$. Then in (5.1), for every division d of $[a,b]$ and $|f'(x)| \le M, x \in [a,b]$, the trapezoidal error estimate satisfies*

$$|E(f,d)| \le M\left(\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(q+\frac{1}{2}\right)}{\Gamma(q+1)}\right)^{\frac{1}{q}}\left(\frac{1}{2}\right)^{1+\frac{1}{q}}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2. \tag{5.4}$$

Proof. The proof is similar to that of Proposition 5.1 and using Remark 2.9. □

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