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Hermite-Hadamard type inequalities for the *m*- and (α, m)-logarithmically convex functions

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Abstract. In the paper, the authors introduce concepts of m- and (α, m)-logarithmically convex functions and establish some Hermite-Hadamard type inequalities of these classes of functions.

1. Introduction

For convex functions, the following Hermite-Hadamard type inequalities were given in [8].

Theorem A ([8]). *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $a, b \in I$ with $a < b$. If $|f'(x)|^q$ for $q \ge 1$ is convex on $[a, b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{4}\left(\frac{|f'(a)|^q + |f'(b)|^q}{2}\right)^{1/q} \tag{1.1}$$

and

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{4}\left(\frac{|f'(a)|^q + |f'(b)|^q}{2}\right)^{1/q}. \tag{1.2}$$

The m-convex function was defined in [12] as follows.

Definition 1.1. *A function $f : [0, b] \to \mathbb{R}$ is said to be m-convex if*

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y) \tag{1.3}$$

holds for all $x, y \in [0, b]$, $t \in [0, 1]$, and $m \in (0, 1]$.

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In [1, p. 48, Theorem 2] and [2], the following Hermite-Hadamard type inequality for m-convex functions was proved.

Theorem B. *Let $f : [0,\infty) \to \mathbb{R}$ be m-convex and $m \in (0,1]$. If $f \in L([a,b])$ for $0 \le a < b < \infty$, then*

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le \min\left\{\frac{f(a)+mf(b/m)}{2}, \frac{mf(a/m)+f(b)}{2}\right\}. \tag{1.4}$$

The (α,m)-convex function was defined in [7] as follows.

Definition 1.2. *A function $f : [0,b] \to \mathbb{R}$ is said to be (α,m)-convex if*

$$f(tx+m(1-t)y) \le t^\alpha f(x) + m(1-t^\alpha)f(y) \tag{1.5}$$

is valid for all $x,y \in [0,b]$, $t \in [0,1]$, and $(\alpha,m) \in (0,1]\times(0,1]$.

For (α,m)-convex functions, the following Hermite-Hadamard type inequalities appeared in [5].

Theorem C ([5, Theorem 2.2])**.** *Let $I \supset [0,\infty)$ be an open real interval and let $f : I \to \mathbb{R}$ be a differentiable function on I such that $f' \in L([a,b])$ for $0 \le a < b < \infty$. If $|f'(x)|^q$ is m-convex on $[a,b]$ for some given numbers $m \in (0,1]$ and $q \in [1,\infty)$, then*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{4}\min\left\{\left(\frac{|f'(a)|^q + m|f'(b/m)|^q}{2}\right)^{1/q}, \left(\frac{m|f'(a/m)|^q + |f'(b)|^q}{2}\right)^{1/q}\right\}. \tag{1.6}$$

Theorem D ([5, Theorem 3.1])**.** *Let $I \supset [0,\infty)$ be an open real interval and let $f : I \to \mathbb{R}$ be a differentiable function on I such that $f' \in L([a,b])$ for $0 \le a < b < \infty$. If $|f'(x)|^q$ is (α,m)-convex on $[a,b]$ for $(\alpha,m) \in (0,1]\times(0,1]$ and $q \in [1,\infty)$, then*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| &\le \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-1/q} \\ &\quad\times \min\left\{\left[v_1|f'(a)|^q + v_2 m\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{1/q}, \left[v_2 m\left|f'\left(\frac{a}{m}\right)\right|^q + v_1|f'(b)|^q\right]^{1/q}\right\},\end{aligned} \tag{1.7}$$

where

$$v_1 = \frac{1}{(\alpha+1)(\alpha+2)}\left(\alpha + \frac{1}{2^\alpha}\right) \quad \text{and} \quad v_2 = \frac{1}{(\alpha+1)(\alpha+2)}\left(\frac{\alpha^2+\alpha+2}{2} - \frac{1}{2^\alpha}\right). \tag{1.8}$$

The aim of this paper is to introduce concepts of m- and (α,m)-logarithmically convex functions, and then to present some Hermite-Hadamard type inequalities for them.

2. Definitions and lemmas

Firstly we introduce concepts of m- and (α,m)-logarithmically convex functions.

Definition 2.1. *A function $f : [0,b] \to (0,\infty)$ is said to be m-logarithmically convex if the inequality*

$$f(tx+m(1-t)y) \le [f(x)]^t[f(y)]^{m(1-t)} \tag{2.1}$$

holds for all $x,y \in [0,b]$, $m \in (0,1]$, and $t \in [0,1]$.

Obviously, if putting $m = 1$ in Definition 2.1, then f is just the ordinary logarithmically convex function on $[0,b]$.

Definition 2.2. *A function $f:[0,b]\to(0,\infty)$ is said to be (α,m)-logarithmically convex if*

$$f(tx+m(1-t)y)\le[f(x)]^{t^\alpha}[f(y)]^{m(1-t^\alpha)} \tag{2.2}$$

holds for all $x,y\in[0,b]$, $(\alpha,m)\in(0,1]\times(0,1]$, and $t\in[0,1]$.

Clearly, when taking $\alpha=1$ in Definition 2.2, then f becomes the standard m-logarithmically convex function on $[0,b]$.

Secondly, we recite the following lemmas which will be used in proofs of our main results.

Lemma 2.1 ([3]). *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I° and $a,b\in I$ with $a<b$. If $f'\in L([a,b])$, then*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x=\frac{b-a}{2}\int_0^1(1-2t)f'(ta+(1-t)b)\,\mathrm{d}t. \tag{2.3}$$

Lemma 2.2 ([4]). *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I° and $a,b\in I$ with $a<b$. If $f'\in L([a,b])$, then*

$$f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x=(b-a)\left[\int_0^{1/2}tf'(ta+(1-t)b)\,\mathrm{d}t+\int_{1/2}^1(1-t)f'(ta+(1-t)b)\,\mathrm{d}t\right]. \tag{2.4}$$

3. Hermite-Hadamard type inequalities

In this section, we will present several Hermite-Hadamard type inequalities for the m- and (α,m)-logarithmically convex functions.

Theorem 3.1. *Let $I\supset[0,\infty)$ be an open real interval and let $f:I\to(0,\infty)$ be a differentiable function on I such that $f'\in L([a,b])$ for $0\le a<b<\infty$. If $|f'(x)|^q$ is (α,m)-logarithmically convex on $\left[0,\frac{b}{m}\right]$ for $(\alpha,m)\in(0,1]\times(0,1]$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\le\frac{b-a}{2}\left(\frac{1}{2}\right)^{1-1/q}\left|f'\left(\frac{b}{m}\right)\right|^m[E_1(\alpha,m,q)]^{1/q} \tag{3.1}$$

is valid for $q\ge1$, where

$$\mu=\frac{|f'(a)|}{|f'(b/m)|^m},\quad E_1(\alpha,m,q)=\begin{cases}\frac{1}{2}, & \mu=1,\\ F_1(\mu,\alpha q), & \mu<1,\\ \mu^{(1-\alpha)q}F_1(\mu,\alpha q), & \mu>1,\end{cases} \tag{3.2}$$

and

$$F_1(u,v)=\frac{1}{v^2\ln^2u}\left[v(u^v-1)\ln u-2\left(u^{v/2}-1\right)^2\right] \tag{3.3}$$

for $u,v>0$ and $u\ne1$.

Proof. When $q>1$, by Definition 2.2, Lemma 2.1, and Hölder inequality, we have

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|&=\frac{b-a}{2}\left|\int_0^1(1-2t)f'(ta+(1-t)b)\,\mathrm{d}t\right|\\&\le\frac{b-a}{2}\left(\int_0^1|1-2t|\,\mathrm{d}t\right)^{1-1/q}\left(\int_0^1|1-2t||f'(ta+(1-t)b)|^q\,\mathrm{d}t\right)^{1/q}\\&\le\frac{b-a}{2}\left(\frac{1}{2}\right)^{1-1/q}\left|f'\left(\frac{b}{m}\right)\right|^m\left(\int_0^1|1-2t|\mu^{qt^\alpha}\,\mathrm{d}t\right)^{1/q}.\end{aligned}$$

For $\mu = 1$, we have

$$\int_0^1 |1-2t|\mu^{qt^\alpha}\,\mathrm{d}\,t = \int_0^1 |1-2t|\,\mathrm{d}\,t = \frac{1}{2}.$$

For $\mu < 1$, we have $\mu^{qt^\alpha} \le \mu^{\alpha qt}$, thereby

$$\int_0^1 |1-2t|\mu^{qt^\alpha}\,\mathrm{d}\,t \le \int_0^1 |1-2t|\mu^{\alpha qt}\,\mathrm{d}\,t = \frac{\alpha q\mu^{\alpha q}\ln\mu - \alpha q\ln\mu - 2\mu^{\alpha q} + 4\mu^{\alpha q/2} - 2}{\alpha^2 q^2 \ln^2\mu}.$$

For $\mu > 1$, we have $\mu^{qt^\alpha} \le \mu^{q(\alpha t+1-\alpha)}$, thereby

$$\int_0^1 |1-2t|\mu^{qt^\alpha}\,\mathrm{d}\,t \le \mu^{q(1-\alpha)}\int_0^1 |1-2t|\mu^{\alpha qt}\,\mathrm{d}\,t = \mu^{q(1-\alpha)}\frac{\alpha q\mu^{\alpha q}\ln\mu - \alpha q\ln\mu - 2\mu^{\alpha q} + 4\mu^{\alpha q/2} - 2}{\alpha^2 q^2 \ln^2\mu}.$$

Thus, the inequality (3.1) follows.

When $q = 1$, we have

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| &= \frac{b-a}{2}\left|\int_0^1 (1-2t)f'(ta+(1-t)b)\,\mathrm{d}\,t\right| \\ &\le \frac{b-a}{2}\int_0^1 |1-2t||f'(a)|^{t^\alpha}\left|f'\left(\frac{b}{m}\right)\right|^{m(1-t^\alpha)}\mathrm{d}\,t \le \frac{b-a}{2}\left|f'\left(\frac{b}{m}\right)\right|^m E_1(\alpha,m,q).\end{aligned}$$

The proof of Theorem 3.1 is complete. □

Corollary 3.2. *Let $I \supset [0,\infty)$ be an open real interval and let $f : I \to (0,\infty)$ be a differentiable function on I such that $f' \in L([a,b])$ for $0 \le a < b < \infty$. If $|f'(x)|^q$ is m-logarithmically convex on $\left[0, \frac{b}{m}\right]$ for some given numbers $m \in (0,1]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-1/q}\left|f'\left(\frac{b}{m}\right)\right|^m [E_1(1,m,q)]^{1/q} \tag{3.4}$$

holds for $q \ge 1$, where E_1 is defined as in Theorem 3.1.

Theorem 3.3. *Let $I \supset [0,\infty)$ be an open real interval and let $f : I \to (0,\infty)$ be a differentiable function on I such that $f' \in L([a,b])$ for $0 \le a < b < \infty$. If $|f'(x)|^q$ is (α,m)-logarithmically convex on $\left[0, \frac{b}{m}\right]$ for $(\alpha,m) \in (0,1] \times (0,1]$, then*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-3/q}\left|f'\left(\frac{b}{m}\right)\right|^m E_2(\alpha,m,q) \tag{3.5}$$

is valid for $q \ge 1$, where μ is defined by (3.2) and

$$E_2(\alpha,m,q) = \begin{cases} 2\left(\dfrac{1}{8}\right)^{1/q}, & \mu = 1 \\ [F_2(\mu,\alpha q)]^{1/q} + [F_3(\mu,\alpha q)]^{1/q}, & 0 < \mu < 1 \\ \mu^{1-\alpha}\left\{[F_2(\mu,\alpha q)]^{1/q} + [F_3(\mu,\alpha q)]^{1/q}\right\}, & \mu > 1 \end{cases} \tag{3.6}$$

with

$$F_2(u,v) = \frac{1}{v^2\ln^2 u}\left(\frac{v}{2}u^{v/2}\ln u - u^{v/2} + 1\right) \quad \textit{and} \quad F_3(u,v) = \frac{1}{v^2\ln^2 u}\left(u^v - \frac{v}{2}u^{v/2}\ln u - u^{v/2}\right) \tag{3.7}$$

for $u, v > 0$ and $u \ne 1$.

Proof. When $q > 1$, by Definition 2.2, Lemma 2.2, and Hölder inequality yield

$$\begin{aligned}\left|f\Big(\frac{a+b}{2}\Big)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| &\le (b-a)\bigg[\int_0^{1/2} t|f'(ta+(1-t)b)|\,\mathrm{d}t+\int_{1/2}^1(1-t)|f'(ta+(1-t)b)|\,\mathrm{d}t\bigg]\\ &\le \frac{b-a}{4}\Big(\frac{1}{2}\Big)^{1-3/q}\bigg\{\bigg[\int_0^{1/2} t|f'(a)|^{qt^\alpha}\Big|f'\Big(\frac{b}{m}\Big)\Big|^{mq(1-t^\alpha)}\,\mathrm{d}t\bigg]^{1/q}+\bigg[\int_{1/2}^1(1-t)|f'(a)|^{qt^\alpha}\Big|f'\Big(\frac{b}{m}\Big)\Big|^{mq(1-t^\alpha)}\,\mathrm{d}t\bigg]^{1/q}\bigg\}\\ &= \frac{b-a}{4}\Big(\frac{1}{2}\Big)^{1-3/q}\Big|f'\Big(\frac{b}{m}\Big)\Big|^{m}\bigg\{\bigg(\int_0^{1/2} t\mu^{qt^\alpha}\,\mathrm{d}t\bigg)^{1/q}+\bigg[\int_{1/2}^1(1-t)\mu^{qt^\alpha}\,\mathrm{d}t\bigg]^{1/q}\bigg\}.\end{aligned}$$

If $\mu = 1$, we have

$$\bigg(\int_0^{1/2} t\mu^{qt^\alpha}\,\mathrm{d}t\bigg)^{1/q}+\bigg[\int_{1/2}^1(1-t)\mu^{qt^\alpha}\,\mathrm{d}t\bigg]^{1/q}=2\Big(\frac{1}{8}\Big)^{1/q}.$$

If $\mu < 1$, we obtain

$$\begin{aligned}\bigg(\int_0^{1/2} t\mu^{qt^\alpha}\,\mathrm{d}t\bigg)^{1/q}+\bigg[\int_{1/2}^1(1-t)\mu^{qt^\alpha}\,\mathrm{d}t\bigg]^{1/q} &\le \bigg(\int_0^{1/2} t\mu^{\alpha qt}\,\mathrm{d}t\bigg)^{1/q}+\bigg[\int_{1/2}^1(1-t)\mu^{\alpha qt}\,\mathrm{d}t\bigg]^{1/q}\\ &= \bigg[\frac{1}{\alpha^2q^2\ln^2\mu}\Big(\frac{\alpha q}{2}\mu^{\alpha q/2}\ln\mu-\mu^{\alpha q/2}+1\Big)\bigg]^{1/q}+\bigg[\frac{1}{\alpha^2q^2\ln^2\mu}\Big(\mu^{\alpha q}-\frac{\alpha q}{2}\mu^{\alpha q}\ln\mu-\mu^{\alpha q/2}\Big)\bigg]^{1/q}.\end{aligned}$$

If $\mu > 1$, then

$$\begin{aligned}\bigg(\int_0^{1/2} t\mu^{qt^\alpha}\,\mathrm{d}t\bigg)^{1/q}+\bigg[\int_{1/2}^1(1-t)\mu^{qt^\alpha}\,\mathrm{d}t\bigg]^{1/q} &\le \bigg(\int_0^{1/2} t\mu^{q(\alpha t+1-\alpha)}\,\mathrm{d}t\bigg)^{1/q}+\bigg[\int_{1/2}^1(1-t)\mu^{q(\alpha t+1-\alpha)}\,\mathrm{d}t\bigg]^{1/q}\\ &= \mu^{1-\alpha}\bigg\{\bigg[\frac{1}{\alpha^2q^2\ln^2\mu}\Big(\frac{\alpha q}{2}\mu^{\alpha q/2}\ln\mu-\mu^{\alpha q/2}+1\Big)\bigg]^{1/q}+\bigg[\frac{1}{\alpha^2q^2\ln^2\mu}\Big(\mu^{\alpha q}-\frac{\alpha q}{2}\mu^{\alpha q}\ln\mu-\mu^{\alpha q/2}\Big)\bigg]^{1/q}\bigg\}.\end{aligned}$$

Thus, the inequality (3.5) follows.

When $q = 1$, we have

$$\begin{aligned}\left|f\Big(\frac{a+b}{2}\Big)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| &\le \frac{b-a}{2}\bigg[\int_0^{1/2} t|f'(ta+(1-t)b)|\,\mathrm{d}t+\int_{1/2}^1(1-t)|f'(ta+(1-t)b)|\,\mathrm{d}t\bigg]\\ &\le (b-a)\bigg[\int_0^{1/2} t|f'(a)|^{t^\alpha}\Big|f'\Big(\frac{b}{m}\Big)\Big|^{m(1-t^\alpha)}\,\mathrm{d}t+\int_{1/2}^1(1-t)|f'(a)|^{t^\alpha}\Big|f'\Big(\frac{b}{m}\Big)\Big|^{m(1-t^\alpha)}\,\mathrm{d}t\bigg]\\ &\le (b-a)\Big|f'\Big(\frac{b}{m}\Big)\Big|^{m}E_2(\alpha,m,1).\end{aligned}$$

This completes the proof of Theorem 3.3. □

Corollary 3.4. *Let $I \supset [0,\infty)$ be an open real interval and let $f : I \to (0,\infty)$ be a differentiable function on I such that $f' \in L([a,b])$ for $0 \le a < b < \infty$. If $|f'(x)|^q$ is m-logarithmically convex on $\big[0,\frac{b}{m}\big]$ for some given numbers $m \in (0,1]$, then*

$$\left|f\Big(\frac{a+b}{2}\Big)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{4}\Big(\frac{1}{2}\Big)^{1-3/q}\Big|f'\Big(\frac{b}{m}\Big)\Big|^{m}E_2(1,m,q) \tag{3.8}$$

holds for $q \ge 1$, where E_2 is defined as in Theorem 3.3.

Theorem 3.5. *Let $f,g:[0,\infty)\to(0,\infty)$ such that $f,g\in L([a,b])$ for $0\le a<b<\infty$. If $f(x)$ is (α,m_1)-logarithmically convex and $g(x)$ is (α,m_2)-logarithmically convex on $[0,\frac{b}{m_i}]$ for $(\alpha,m_i)\in(0,1]\times(0,1]$ and $i=1,2$, then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x\le(b-a)\left[f\left(\frac{b}{m_1}\right)\right]^{m_1}\left[g\left(\frac{b}{m_2}\right)\right]^{m_2}E_3(\alpha),\tag{3.9}$$

where

$$\eta=f(a)g(a)\left[f\left(\frac{b}{m_1}\right)\right]^{-m_1}\left[g\left(\frac{b}{m_2}\right)\right]^{-m_2}\quad\text{and}\quad E_3(\alpha)=\begin{cases}1, & \eta=1,\\ \dfrac{\eta^\alpha-1}{\alpha\ln\eta}, & 0<\eta<1,\\ \dfrac{\eta^{1-\alpha}(\eta^\alpha-1)}{\alpha\ln\eta}, & \eta>1.\end{cases}\tag{3.10}$$

Proof. The (α,m)-logarithmic convexity of $f(x)$ and $g(x)$ yields

$$f\left(ta+m_1(1-t)\left(\frac{b}{m_1}\right)\right)\le[f(a)]^{t^\alpha}\left[f\left(\frac{b}{m_1}\right)\right]^{m_1(1-t^\alpha)}\quad\text{and}\quad g\left(ta+m_2(1-t)\left(\frac{b}{m_2}\right)\right)\le[g(a)]^{t^\alpha}\left[g\left(\frac{b}{m_2}\right)\right]^{m_2(1-t^\alpha)},$$

from which it follows that

$$\begin{aligned}\int_a^b f(x)g(x)\,\mathrm{d}x&=(b-a)\int_0^1 f(ta+1-tb)g(ta+1-tb)\,\mathrm{d}t\\&\le(b-a)\int_0^1[f(a)]^{t^\alpha}[g(a)]^{t^\alpha}\left[f\left(\frac{b}{m_1}\right)\right]^{m_1(1-t^\alpha)}\left[g\left(\frac{b}{m_2}\right)\right]^{m_2(1-t^\alpha)}\mathrm{d}t\\&=(b-a)\left[f\left(\frac{b}{m_1}\right)\right]^{m_1}\left[g\left(\frac{b}{m_2}\right)\right]^{m_2}\int_0^1\left\{f(a)g(a)\left[f\left(\frac{b}{m_1}\right)\right]^{m_1}\left[g\left(\frac{b}{m_2}\right)\right]^{m_2}\right\}^{t^\alpha}\mathrm{d}t.\end{aligned}$$

When $\eta=1$, we have $\int_0^1\eta^{t^\alpha}\,\mathrm{d}t=1$. When $\eta<1$, we have

$$\int_0^1\eta^{t^\alpha}\,\mathrm{d}t\le\int_0^1\eta^{\alpha t}\,\mathrm{d}t=\frac{\eta^\alpha-1}{\alpha\ln\eta}.$$

When $\eta>1$, we have

$$\int_0^1\eta^{t^\alpha}\,\mathrm{d}t\le\int_0^1\eta^{\alpha t+1-\alpha}\,\mathrm{d}t=\frac{\eta^{1-\alpha}(\eta^\alpha-1)}{\alpha\ln\eta}.$$

Theorem 3.5 is thus proved. □

Corollary 3.6. *Let $f,g:[0,\infty)\to(0,\infty)$ such that $f\cdot g\in L([a,b])$ for $0\le a<b<\infty$. If $f(x)$ is m_1-logarithmically convex and $g(x)$ is m_2-logarithmically convex on $[0,\frac{b}{m_i}]$ for $i=1,2$ and some given numbers $m_1,m_2\in(0,1]$, then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x\le(b-a)\left[f\left(\frac{b}{m_1}\right)\right]^{m_1}\left[g\left(\frac{b}{m_2}\right)\right]^{m_2}E_3(1),\tag{3.11}$$

where E_3 is defined as in Theorem 3.5.

Corollary 3.7. *Let $f,g:[0,\infty)\to(0,\infty)$ such that $f\cdot g\in L([a,b])$ for $0\le a<b<\infty$. If $f(x)$ and $g(x)$ are (α,m)-logarithmically convex on $\left[0,\frac{b}{m}\right]$ for $(\alpha,m)\in(0,1]\times(0,1]$, then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x\le(b-a)\left[f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right]^{m}E_3(\alpha),\tag{3.12}$$

where E_3 is defined as in Theorem 3.5.

Corollary 3.8. *Let* $f, g : [0, \infty) \to (0, \infty)$ *such that* $f \cdot g \in L([a, b])$ *for* $0 \le a < b < \infty$. *If* $f(x)$ *and* $g(x)$ *are* m*-logarithmically convex on* $\left[0, \frac{b}{m}\right]$ *for some given number* $m \in (0, 1]$, *then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}\,x \le (b-a)\left[f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right]^m E_3(1), \tag{3.13}$$

where E_3 *is defined as in Theorem 3.5.*

Remark 3.1. *In [6, 9–11, 13–16] the authors and their coauthors obtained some results on Hermite-Hadamard type inequalities for convex functions and for* m*- and* (α, m)*-geometrically convex functions.*

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