Hermite Reduction and Creative Telescoping for Hyperexponential Functions*

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ABSTRACT

We present a new reduction algorithm that simultaneously extends Hermite's reduction for rational functions and the Hermite-like reduction for hyperexponential functions. It yields a unique additive decomposition that allows to decide hyperexponential integrability. Based on this reduction algorithm, we design a new algorithm to compute minimal telescopers for bivariate hyperexponential functions. One of its main features is that it can avoid the costly computation of certificates. Its implementation outperforms Maple's function DEtools[Zeilberger]. We also derive an order bound on minimal telescopers that is tighter than the known ones.

Categories and Subject Descriptors

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General Terms

Algorithms, Theory

Keywords

Hermite reduction, Hyperexponential function, Telescoper

1. INTRODUCTION

Given a univariate rational function r, Hermite reduction in [17, 14, 6] finds rational functions r_1 and r_2 s.t. (i) $r = r_1 + r_2$, (ii) r_1 is rational integrable, (iii) r_2 is a proper fraction with a squarefree denominator. The additive decomposition is unique, and r is rational integrable if and only if $r_2 = 0$.

A univariate function is hyperexponential if its logarithmic derivative is rational. Exponential, radical and rational functions are hyperexponential. Rational Hermite reduction was extended to hyperexponential functions by Davenport in [11] and by Geddes, Le and Li in [12]. The former aims at solving Risch's equation; the latter is a differential analogue of the reduction algorithm for hypergeometric terms in [2]. For a given hyperexponential function H, the reduction algorithms in [11, 12] compute two hyperexponential functions H_1 and H_2 s.t. (i) $H = H_1 + H_2$, (ii) H_1 is hyperexponential integrable, (iii) H_2 is minimal in some sense. However, H_2 is not unique in general and it may be nonzero even when H is hyperexponential integrable. To decide the integrability of H, one additionally needs to compute polynomial solutions of a first-order linear differential equation.

The method of creative telescoping, developed initially for hyperexponential functions by Almkvist and Zeilberger [3], then extended by Chyzak to general holonomic functions [10], is nowadays an important automatic tool for computing definite integrals. Recently, it has also played an important role in the resolution of intriguing problems in enumerative combinatorics [15, 16]. For a bivariate hyperexponential function $H(x, y)$, the problem of creative telescoping is to find a nonzero operator $L(x, D_x)$ in $\mathbb{F}(x)\langle D_x\rangle$, the ring of linear differential operators over the rational-function field $\mathbb{F}(x)$, s.t.

$$L(x, D_x)(H) = D_y(G) \tag{1}$$

for some hyperexponential function G, where $D_x = \partial/\partial x$ and $D_y = \partial/\partial y$. The operator L above is called a *telescoper* for H, and G is the corresponding *certificate*. An algorithm for solving (1) is given in [3], which is based on a differential version of Gosper's algorithm. An algorithm for rational-function telescoping is given in [5], which is based on Hermite reduction. The latter separates the computation of telescopers from that of certificates, and has a lower complexity than the former for rational functions.

In the present paper, we develop a reduction algorithm which, given a univariate hyperexponential function H, constructs two hyperexponential functions H_1 and H_2 s.t. (i) $H = H_1 + H_2$, (ii) H_1 is hyperexponential integrable, and (iii) H_2 is either zero or not hyperexponential integrable. We

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show that H_2 in the above additive decomposition is unique in a certain technical sense and can be obtained without computing polynomial solutions of any differential equation. Our algorithm is based on the Hermite-like reduction in [12], a differential variant of the polynomial reduction in [2] and on the idea for reducing simple radicals in [18, Proposition 7]. The main new ingredient is property (iii), which is crucial in many applications. Using the reduction algorithm, we extend the rational telescoping algorithm in [5] to the hyperexponential case, and derive an order bound on the telescopers. The new telescoping algorithm avoids the costly computation of certificates, and the order bound is tighter than that obtained in [4] and [8].

The rest of the paper is organized as follows. We review the notion of hyperexponential functions and Hermite-like reduction in Sections 2 and 3, respectively. A new reduction algorithm is developed for hyperexponential functions in Section 4. After introducing kernel reduction in Section 5, we present a reduction-based telescoping algorithm for bivariate hyperexponential functions, and derive an upper bound on the order of minimal telescopers in Section 6. We briefly describe an implementation of the new telescoping algorithm, and present some experimental results in Section 7, which validate its practical relevance.

As a matter of notation, we let $\mathbb{F}$ be a field of characteristic zero and $\mathbb{F}(y)$ be the field of rational functions in y over $\mathbb{F}$. For a polynomial $p \in \mathbb{F}[y]$, we denote by $\deg(p)$ and $\mathrm{lc}(p)$ the degree and leading coefficient of p, respectively. Let D_y denote the usual derivation d/dy on $\mathbb{F}(y)$. Then $(\mathbb{F}(y), D_y)$ is a differential field.

2. HYPEREXPONENTIAL FUNCTIONS

Hyperexponential functions share the common properties of rational functions, simple radicals, and exponential functions. Together with hypergeometric terms, they are frequently viewed as a special and important class of "closed-form" solutions of linear differential and difference equations with polynomial coefficients.

Definition 1. *Let Φ be a differential field extension of $\mathbb{F}(y)$. A nonzero element $H \in \Phi$ is said to be* hyperexponential *over $\mathbb{F}(y)$ if its logarithmic derivative $D_y(H)/H$ is in $\mathbb{F}(y)$.*

The product of hyperexponential functions is also hyperexponential. Two hyperexponential functions H_1, H_2 are said to be *similar* if there exists $r \in \mathbb{F}(y)$ s.t. $H_1 = rH_2$. The sum of similar hyperexponential functions is still hyperexponential, provided that it is nonzero.

For brevity, we use the notation $\exp(\int f dy)$ to indicate a hyperexponential function whose logarithmic derivative is f. For a rational function $r \in \mathbb{F}(y)$, we have

$$r \exp\left(\int f\, dy\right) = \exp\left(\int \left(f + D_y(r)/r\right)\, dy\right).$$

A univariate hyperexponential function H is said to be *hyperexponential integrable* if it is the derivative of another hyperexponential function. For brevity, we say "integrable" instead of "hyperexponential integrable" in the sequel.

A hyperexponential function H can be expressed as a product $r \exp\left(\int f dy\right)$ for some $r, f \in \mathbb{F}(y)$. Assume that H is integrable. Then it is equal to $D_y(G)$ for some hyperexponential function G. A straightforward calculation shows that G is similar to H. In other words, $G = s \exp\left(\int f dy\right)$ for some $s \in \mathbb{F}(y)$. It follows that $H = D_y(G)$ if and only if

$$r = D_y(s) + f\, s. \tag{2}$$

Deciding the integrability of H amounts to finding a rational solution s s.t. the above equation holds.

3. HERMITE-LIKE REDUCTION

Reduction algorithms have been developed for computing additive decompositions of rational functions [17, 14], hypergeometric terms [1, 2], and hyperexponential functions [11, 12]. Those algorithms can be viewed as generalizations of Gosper's algorithm [13] and its differential analogue [3, §5].

For a hyperexponential function H, a reduction algorithm computes two hyperexponential functions H_1, H_2 s.t.

$$H = D_y(H_1) + H_2. \tag{3}$$

This implies that H, H_1 and H_2 are similar. So we may write $H = r \exp\left(\int f dy\right)$ and $H_i = r_i \exp\left(\int f dy\right)$, where r, r_i, f belong to $\mathbb{F}(y)$ and $i = 1, 2$. Then (3) translates into

$$r = D_y(r_1) + f\, r_1 + r_2. \tag{4}$$

A reduction algorithm for computing (3) amounts to choosing rational functions r, f and r_1 so that r_2 satisfies properties similar to those obtained in Hermite reduction for rational functions. There are at least two approaches to this end. One is given in [11], and the other in [12]. We review the latter, because the notion of differential-reduced rational functions plays a key role in Lemma 6 in Section 4.

Recall [12, §2] that a rational function $r = a/b \in \mathbb{F}(y)$ is said to be *differential-reduced* w.r.t. y if

$$\gcd\left(b, a - i\, D_y(b)\right) = 1 \quad \text{for all } i \in \mathbb{Z}.$$

By Lemma 2 in [12], r is differential-reduced if and only if none of its residues is an integer. The *differential rational canonical form* of a rational function f in $\mathbb{F}(y)$ is a pair (K, S) in $\mathbb{F}(y) \times \mathbb{F}(y)$ s.t. (i) K is differential-reduced; (ii) the denominator of S is coprime with that of K; and (iii) f is equal to $K + D_y(S)/S$. Every rational function has a unique canonical form in the sense that K is unique and S is unique up to a multiplicative constant in $\mathbb{F}$ [12, §3]. We call K and S the *kernel* and *shell* of f, respectively. They can be constructed by the method described in [12, §3].

Let H be a univariate hyperexponential function over $\mathbb{F}(y)$, in the form $\exp(\int f dy)$. Assume that K and S are the kernel and shell of f, respectively. Then $H = S \exp\left(\int K\, dy\right)$. Note that $K = 0$ if and only if H is a rational function, which is then equal to cS for some $c \in \mathbb{F}$.

Example 2. *Let $H = \sqrt{y^2+1}/(y-1)^2$. The logarithmic derivative of H is*

$$\frac{D_y H}{H} = \frac{D_y(1/(y-1)^2)}{1/(y-1)^2} + \frac{y}{y^2+1},$$

where $y/(y^2+1)$ is differential-reduced. The kernel and shell of $D_y(H)/H$ are $y/(y^2+1)$ and $1/(y-1)^2$, respectively. So $H = \exp\left(\int y/(y^2+1)\, dy\right)/(y-1)^2$.

For brevity, we make a notational convention.

Convention 3. *Let H denote a hyperexponential function whose logarithmic derivative has kernel K and shell S. Assume that K is nonzero, that is, H is not a rational function. Set $T = \exp\left(\int K\, dy\right)$. Moreover, write $K = k_1/k_2$, where k_1, k_2 are polynomials in $\mathbb{F}[y]$ with $\gcd(k_1, k_2) = 1$.*

The algorithm **ReduceCert** in [12] computes a rational function S_1 s.t.

$$S = D_y(S_1) + S_1K + \frac{a}{bk_2}, \quad (5)$$

where $a \in \mathbb{F}[y]$ and b is the squarefree part of the denominator of S. Thus, $\gcd(b, k_2) = 1$ by the definition of canonical forms. Note that a is not necessarily coprime with bk_2. As the algorithm **ReduceCert** only reduces the shell S, it is referred to as *shell reduction*. It follows from (5) that

$$H = D_y(S_1T) + \frac{a}{bk_2}T. \quad (6)$$

By Theorem 4 in [12], a/b belongs to $\mathbb{F}[y]$ if H is integrable.

Example 4. *Let H be the same hyperexponential function as in Example 2. Then $D_y(H)/H$ has kernel $K = y/(y^2+1)$ and shell $S = 1/(y-1)^2$. Shell reduction yields*

$$S = D_y(S_1) + S_1K + \frac{y}{(y-1)k_2},$$

where $S_1 = -1/(y-1)$ and $k_2 = y^2 + 1$. Then

$$H = D_y(S_1T) + \frac{yT}{(y-1)k_2}, \quad \text{where } T = \sqrt{y^2+1}.$$

By Theorem 4 in [12], H is not integrable.

Remark that a in (6) can be nonzero for an integrable H:

Example 5. *Let $H=y\exp(y)$ whose logarithmic derivative has kernel 1 and shell y, i.e., $H = y\exp\left(\int 1dy\right)$, for S_1=0. But H is integrable as it is equal to $D_y(y\exp(y) - \exp(y))$.*

Thus, shell reduction cannot be directly used to decide hyperexponential integrability, which is a difference to the rational case. To amend this, the solution proposed in [12, Algorithm **ReduceHyperexp**] was to find the polynomial solutions of an auxiliary first-order linear differential equation. In the following section, we show how this can be avoided and improved.

4. HERMITE REDUCTION FOR HYPEREXPONENTIAL FUNCTIONS

After the shell reduction described in (6), it remains to decide the integrability of $(a/bk_2)T$. In the rational case, i.e., when the kernel K is equal to zero, a in (6) can be chosen s.t. $\deg(a) < \deg(b)$, because all polynomials are rational integrable. But a hyperexponential function with a polynomial shell is not necessarily integrable. For example, $H = \exp\left(y^2\right)$.

We present a differential variant of [2, Theorem 7] to bound the degree of a in (5). The variant leads not only to a canonical additive decomposition of hyperexponential functions, but also a direct way to decide their integrability.

4.1 Polynomial reduction

With Convention 3, we define

$$\mathcal{M}_K = \{k_2D_y(p) + k_1p \mid p \in \mathbb{F}[y]\}.$$

It is an $\mathbb{F}$-linear subspace in $\mathbb{F}[y]$. We call $\mathcal{M}_K$ the *subspace for polynomial reduction w.r.t. K*. Moreover, let ϕ_K be the $\mathbb{F}$-linear map from $\mathbb{F}[y]$ to $\mathcal{M}_K$ that sends p to $k_2D_y(p)+k_1p$ for every $p \in \mathbb{F}[y]$. We call ϕ_K the *map for polynomial reduction w.r.t. K*.

Concerning the subspace $\mathcal{M}_K$ and the map ϕ_K, we have:

Lemma 6. *(i) If $k_2D_y(g) + k_1g \in \mathbb{F}[y]$ for some $g \in \mathbb{F}(y)$, then $g \in \mathbb{F}[y]$. (ii) The map ϕ_K is bijective.*

Proof. Assume that g has a pole. Without loss of generality, we assume that the pole is $y = 0$ and has order m, because the following argument is also applicable over the algebraic closure of $\mathbb{F}$. Expanding g around the origin yields

$$g = \frac{r}{y^m} + \text{terms of higher orders in } y,$$

where $r \in \mathbb{F} \setminus \{0\}$. It follows from $k_2D_y(g) + k_1g \in \mathbb{F}[y]$ that $y = 0$ is a pole of

$$\left(-\frac{mr}{y^{m+1}} + \text{higher terms}\right) + K\left(\frac{r}{y^m} + \text{higher terms}\right)$$

with order no more than that of K. This implies that y=0 is a simple pole of K with residue m, which is incompatible with K being differential-reduced. The first assertion holds.

The map ϕ_K is surjective by definition. If $\phi_K(p) = 0$ for some nonzero polynomial $p \in \mathbb{F}[y]$, then K equals $-D_y(p)/p$, which is nonzero since $K \neq 0$. So K is not differential-reduced, a contradiction. The second assertion holds. ∎

An $\mathbb{F}$-basis of $\mathcal{M}_K$ is called an *echelon basis* if distinct elements in the basis have distinct degrees. Echelon bases always exist and their degrees form a unique subset of $\mathbb{N}$. Let $\mathcal{B}$ be an echelon basis of $\mathcal{M}_K$. Define

$$\mathcal{N}_K = \operatorname{span}_{\mathbb{F}}\left\{y^\ell \mid \ell \in \mathbb{N} \text{ and } \ell \neq \deg(f) \text{ for all } f \in \mathcal{B}\right\}.$$

Then $\mathbb{F}[y] = \mathcal{M}_K \oplus \mathcal{N}_K$. We call $\mathcal{N}_K$ the *standard complement of $\mathcal{M}_K$*. Using an echelon basis of $\mathcal{M}_K$, one can reduce a polynomial p to a unique polynomial $\tilde{p} \in \mathcal{N}_K$ s.t. $p - \tilde{p} \in \mathcal{M}_K$.

In order to find an echelon basis of $\mathcal{M}_K$, we set $d_1 = \deg k_1$, $d_2 = \deg k_2$, $\tau_K = -\operatorname{lc}(k_1)/\operatorname{lc}(k_2)$, and $\mathcal{B} = \{\phi_K(y^n) \mid n \in \mathbb{N}\}$. By Lemma 6 (ii), $\mathcal{B}$ is an $\mathbb{F}$-basis of $\mathcal{M}_K$. Let p be a nonzero polynomial in $\mathbb{F}[y]$. We make the following case distinction.

Case 1. $d_1 \geq d_2$. Then

$$\phi_K(p) = \operatorname{lc}(k_1)\operatorname{lc}(p)y^{d_1+\deg p} + \text{lower terms}.$$

So $\mathcal{B}$ is an echelon basis, in which $\deg\phi_K(y^n) = d_1+n$ for all $n \in \mathbb{N}$. Accordingly, $\mathcal{N}_K$ is spanned by $1, y, \ldots, y^{d_1-1}$. It follows that $p \equiv q \bmod \mathcal{M}_K$ for some $q \in \mathcal{N}_K$ with $\deg q < d_1$.

Case 2. $d_1 = d_2 - 1$ and τ_K is not a positive integer. Then

$$\phi_K(p) = (\deg(p)\operatorname{lc}(k_2) + \operatorname{lc}(k_1))\operatorname{lc}(p)y^{d_1+\deg p} + \text{lower terms}. \quad (7)$$

Since τ_K is not a positive integer, $\deg\phi_K(y^n) = d_1 + n$. Thus, $\mathcal{M}_K$ and $\mathcal{N}_K$ have the same bases as in Case 1. Furthermore, $p \equiv q \bmod \mathcal{M}_K$ for some $q \in \mathcal{N}_K$ with $\deg q < d_1$.

Case 3. $d_1 < d_2 - 1$. If $\deg(p) > 0$, then

$$\phi_K(p) = \deg(p)\operatorname{lc}(k_2)\operatorname{lc}(p)y^{d_2+\deg(p)-1} + \text{lower terms}.$$

Otherwise, $\deg(p) = 0$ and $\phi_K(p) = k_1p$. Therefore, $\mathcal{B}$ is again an echelon basis, in which

$$\deg\phi_K(1) = d_1 \text{ and } \deg\phi_K(y^n) = d_2 + n - 1 \text{ for all } n \geq 1.$$

Accordingly, $\mathcal{N}_K$ has a basis $1, \ldots, y^{d_1-1}, y^{d_1+1}, \ldots, y^{d_2-1}$. It follows that there is $q \in \mathcal{N}_K$ s.t. $p \equiv q \bmod \mathcal{M}_K$, $\deg q < d_2$ and the coefficient of y^{d_1} in q is equal to zero.

Case 4. $d_1 = d_2 - 1$ and τ_K is a positive integer. It follows from (7) that $\deg\phi(y^n) = d_1 + n$ if $n \neq \tau_K$. Furthermore, for every polynomial p of degree τ_K, $\phi_K(p)$ is of degree less than $d_1+\tau_K$. So any echelon basis of $\mathcal{M}_K$ does not contain a polynomial of degree $d_1+\tau_K$. Set

$$\mathcal{B}' = \{\phi_K(y^n) \mid n \in \mathbb{N}, n \neq \tau_K\}.$$

Reducing $\phi_K(y^{\tau_K})$ by the polynomials in $\mathcal{B}'$, we obtain a polynomial a of degree less than d_1. Note that a is nonzero, because $\mathcal{B}$ is an $\mathbb{F}$-linearly independent set. Hence, $\mathcal{B}' \cup \{a\}$ is an echelon basis of $\mathcal{M}_K$. As a consequence, $\mathcal{N}_K$ has an $\mathbb{F}$-basis $\left\{1, y, \ldots, y^{\deg(a)-1}, y^{\deg(a)+1}, \ldots, y^{d_1-1}, y^{d_1+\tau_K}\right\}$. It follows that there exists $r \in \mathbb{F}[y]$ of degree less than d_1 s.t.

$$p \equiv s y^{d_1+\tau_K} + r \mod \mathcal{M}_K \quad \text{for some } s \in \mathbb{F}.$$

Moreover, $s y^{d_1+\tau_K} + r \in \mathcal{N}_K$, and r has at most d_1-1 terms.

Example 7. *Let $K = -6y^3/(y^4+1)$, which is differential-reduced. Then $\tau_K = 6$. According to Case 4, $\mathcal{M}_K$ has an echelon basis $\{y\} \cup \{(n-6)y^{n+3} + ny^{n-1} \mid n \in \mathbb{N}, n \neq 6\}$. Moreover, $\mathcal{N}_K$ has a basis $\{1, y^2, y^9\}$.*

The next lemma enables us to derive an order bound on telescopers for hyperexponential functions.

Lemma 8. *With Convention 3, we further let $d_1 = \deg(k_1)$ and $d_2 = \deg(k_2)$. Then there exists $\mathcal{P} \subset \{y^n \mid n \in \mathbb{N}\}$ with*

$$|\mathcal{P}| \leq \max(d_1, d_2 - 1)$$

s.t. every polynomial in $\mathbb{F}[y]$ can be reduced modulo $\mathcal{M}_K$ to an $\mathbb{F}$-linear combination of the elements in $\mathcal{P}$.

Proof. By the above case distinction, the dimension of $\mathcal{N}_K$ over $\mathbb{F}$ is at most $\max(d_1, d_2-1)$. The lemma follows. ∎

4.2 Hyperexponential integrability

With Convention 3, we further assume that the polynomials a and b are obtained by shell reduction in (6). So the decomposition (6) holds for the present section. Moreover, let $\mathcal{M}_K$ be the subspace for polynomial reduction w.r.t. K, and let $\mathcal{N}_K$ be its standard complement.

We are going to determine necessary and sufficient conditions on hyperexponential integrability. Since $\gcd(b, k_2)=1$,

$$\frac{a}{bk_2} = \frac{q}{b} + \frac{r}{k_2}, \tag{8}$$

where $q, r \in \mathbb{F}[y]$ and $\deg(q) < \deg(b)$. Using an echelon basis of $\mathcal{M}_K$, we compute two polynomials $u \in \mathcal{M}_K$ and $v \in \mathcal{N}_K$ s.t. $r = u + v$. By the definition of $\mathcal{M}_K$, there exists w in $\mathbb{F}[y]$ s.t. $u = k_2 D_y(w) + k_1 w$. By (8), we get

$$\frac{a}{bk_2} = \frac{q}{b} + \frac{k_2 D_y(w) + k_1 w + v}{k_2} = D_y(w) + Kw + \frac{q}{b} + \frac{v}{k_2}.$$

It follows from the equivalence of (4) and (3) that

$$\frac{a}{bk_2} T = D_y(wT) + \left(\frac{q}{b} + \frac{v}{k_2}\right) T. \tag{9}$$

The previous process for obtaining (9) is referred to as the *polynomial reduction for $(a/(bk_2))T$ w.r.t. K*, as it makes essential use of the subspaces $\mathcal{M}_K$ and $\mathcal{N}_K$. By (9) and (6),

$$H = D_y((S_1+w)T) + \left(\frac{q}{b} + \frac{v}{k_2}\right) T, \tag{10}$$

which motivates us to introduce a notion of *residual forms*.

Definition 9. *With Convention 3, we further let f be a rational function in $\mathbb{F}(y)$. Another rational function $r \in \mathbb{F}(y)$ is said to be a* residual form *of f w.r.t. K if there exist g in $\mathbb{F}(y)$ and q, b, v in $\mathbb{F}[y]$ s.t.*

$$f = D_y(g) + Kg + r \quad \text{and} \quad r = \frac{q}{b} + \frac{v}{k_2},$$

where b is squarefree, $\gcd(b, k_2) = 1$, $\deg(q) < \deg(b)$, and v is in the standard complement $\mathcal{N}_K$ of the subspace of polynomial reduction w.r.t. K. For brevity, we say that r is a residual form *w.r.t K if f is clear from the context.*

Remark 10. *The set of residual forms w.r.t. K is an $\mathbb{F}$-linear subspace of $\mathbb{F}(y)$ by the four conditions on b, k_2, q and v in the above definition.*

Residual forms are closely related to the integrability of hyperexponential functions.

Lemma 11. *With Convention 3, we further assume that r is a nonzero residual form w.r.t. K. Then the hyperexponential function rT is not integrable.*

Proof. Suppose on the contrary that rT is integrable. We let $\mathcal{M}_K$ be the subspace for polynomial reduction, and $\mathcal{N}_K$ its standard complement w.r.t. K. By the definition of residual forms, there exist $b, q \in \mathbb{F}[y]$ with b being squarefree and $v \in \mathcal{N}_K$ s.t.

$$\deg(q) < \deg(b), \ \gcd(b, k_2) = 1, \ \text{and } r = \frac{q}{b} + \frac{v}{k_2}. \tag{11}$$

Thus, r can be rewritten as $(k_2 q + bv)/(bk_2)$. It follows that

$$rT = \frac{k_2 q + bv}{b} \exp\left(\int \frac{k_1 - D_y(k_2)}{k_2} \, dy\right).$$

The pair $((k_2 q + bv)/b, (k_1 - D_y(k_2))/k_2)$ is an indecomposable pair according to Definition 2 in [12], since the rational function $(k_1 - D_y(k_2))/k_2$ is differential-reduced, k_2 and b are coprime, and b is squarefree. By Theorem 4 in [12], $(k_2 q + bv)/b$ is a polynomial in $\mathbb{F}[y]$. So $q = 0$ because $\gcd(b, k_2)=1$. It follows from the last equality in (11) that $(v/k_2)T$ is integrable. By (2), $v = k_2 D_y(s) + k_1 s$ for some $s \in \mathbb{F}(y)$. Since $v \in \mathbb{F}[y]$, $s \in \mathbb{F}[y]$ by Lemma 6 (i), and, thus, $v \in \mathcal{M}_K$ by the definition of $\mathcal{M}_K$ at the beginning of Section 4.1, which, together with $v \in \mathcal{N}_K$, implies that $v = 0$. Consequently, $r = 0$, a contradiction to the assumption that $r \neq 0$. ∎

The existence and uniqueness of residual forms are described below.

Lemma 12. *With Convention 3, we have that the shell S has a residual form w.r.t. the kernel K. If a rational function has two residual forms w.r.t. K, then they are equal.*

Proof. By (10), $S = D_y(S_1+w) + (S_1+w)K + q/b + v/k_2$. So $q/b + v/k_2$ is a required form.

Let r and r' be two residual forms of a rational function w.r.t. K. Then $D_y(f) + fK + r = D_y(f') + f'K + r'$ for some $f, f' \in \mathbb{F}(y)$. So $D_y(f - f') + (f - f')K + r - r' = 0$. Consequently, $(r - r')T$ is integrable by (2). We conclude that $r = r'$ by Remark 10 and Lemma 11. ∎

Below is the main result of the present section.

Theorem 13. *Let H be a hyperexponential function whose logarithmic derivative has kernel K and shell S. Then there is an algorithm for computing a rational function h in $\mathbb{F}(y)$ and a unique residual form r w.r.t. K s.t.*

$$H = D_y\left(h\exp\left(\int K\ dy\right)\right) + r\exp\left(\int K\ dy\right). \quad (12)$$

Moreover, H is integrable if and only if $r = 0$.

Proof. Let $T = \exp\left(\int K\, dy\right)$. Applying shell reduction to H w.r.t. K, we find a rational function S_1, and two polynomials a, b s.t. (5) holds. Then we apply polynomial reduction to $a/(bk_2)T$ to obtain the residual form $r = q/b + v/k_2$ s.t. (12) holds.

Suppose that there exists another decomposition

$$H = D_y\left(h'T\right) + r'T \quad (13)$$

for some $h' \in \mathbb{F}(y)$ and some residual form r' w.r.t. K. Then both r and r' are residual forms of S by (12), (13) and the fact $H = ST$. So $r = r'$ by Lemma 12.

If $r = 0$, then H is obviously integrable. Conversely, assume that H is integrable. Then rT is also integrable by (12). So $r = 0$ by Lemma 11. ∎

The reduction algorithm described in the proof of Theorem 13 decomposes a hyperexponential function into a sum of an integrable one and a non-integrable one in a canonical way. The given function is integrable if and only if the non-integrable part is trivial. As a byproduct, it decides hyperexponential integrability without computing a polynomial solution of any first-order linear differential equation, which enables us to construct telescopers for hyperexponential functions using merely linear algebra in Section 6. The algorithm will be referred to as *Hermite reduction for hyperexponential functions* in the sequel, because it extends all important features in Hermite reduction for rational functions to hyperexponential ones.

Example 14. *Let H be the same hyperexponential function as in Example 2. Then $K = y/(y^2+1)$ and $S = 1/(y-1)^2$. Set $T = \sqrt{y^2+1}$. By the shell reduction in Example 4,*

$$H = D_y\left(\frac{-1}{y-1}T\right) + \frac{y}{bk_2}T,$$

where $b = y - 1$ and $k_2 = y^2 + 1$. The polynomial reduction yields $(y/(bk_2))T = D_y(-T/2) + (1/(2b) + 1/(2k_2))\,T$. Combining the above equations, we decompose H as

$$H = D_y\left(\frac{-(y+1)}{2(y-1)}T\right) + \left(\frac{1}{2b} + \frac{1}{2k_2}\right)T.$$

Example 15. *Consider $H{=}y\exp(y)$ as given in Example 5. Since its logarithmic derivative has kernel $K{=}1$, the subspace $\mathcal{M}_K$ for polynomial reduction is equal to $\mathbb{F}[y]$. Thus, y is in $\mathcal{M}_K$ and H is integrable. More generally, $\mathcal{M}_K{=}\mathbb{F}[y]$ corresponds to the well-known fact that $p(y)\exp(y)$ is integrable for all $p \in \mathbb{F}[y] \setminus \{0\}$.*

5. KERNEL REDUCTION

Let $K = k_1/k_2$ be a nonzero differential-reduced rational function in $\mathbb{F}(y)$ with $\gcd(k_1, k_2) = 1$. We may want to reduce a hyperexponential function in the form

$$\frac{p}{k_2^m}\exp\left(\int K\ dy\right) \quad \text{for some } p \in \mathbb{F}[y] \text{ and } m \in \mathbb{N}.$$

One way would be to rewrite the above function as

$$p\exp\left(\int \frac{k_1 - mD_y(k_2)}{k_2}\ dy\right),$$

and proceed by polynomial reduction w.r.t. the new kernel $(k_1 - mD_y(k_2))/k_2$. However, it will prove to be more convenient in Section 6 to reduce the given function w.r.t. the initial kernel K. To this end, we introduce another type of reduction, based on ideas in [11, 18].

Lemma 16. *With Convention 3, we let $p \in \mathbb{F}[y]$ and $m \geq 1$. Then there exist $p_1, p_2 \in \mathbb{F}[y]$ s.t.*

$$\frac{p}{k_2^m} = D_y\left(\frac{p_1}{k_2^{m-1}}\right) + \frac{p_1}{k_2^{m-1}}K + \frac{p_2}{k_2}. \quad (14)$$

Proof. We proceed by induction on m. If $m = 1$, then taking $p_1 = 0$ and $p_2 = p$ yields the claimed form. Assume that $m > 1$. We first show that there exist $\tilde{p}_1, \tilde{p}_2 \in \mathbb{F}[y]$ s.t.

$$\frac{p}{k_2^m} = D_y\left(\frac{\tilde{p}_1}{k_2^{m-1}}\right) + \frac{\tilde{p}_1}{k_2^{m-1}}K + \frac{\tilde{p}_2}{k_2^{m-1}},$$

which is equivalent to

$$p = \tilde{p}_1(k_1 - (m-1)D_y(k_2)) + (D_y(\tilde{p}_1) + \tilde{p}_2)k_2.$$

Since k_1/k_2 is differential-reduced, there exist $u, v \in \mathbb{F}[y]$ s.t. $p = u(k_1 - (m-1)D_y(k_2)) + vk_2$ by the extended Euclidean algorithm. So we can take $\tilde{p}_1{=}u$ and $\tilde{p}_2{=}v{-}D_y(u)$. By the induction hypothesis, there exist $p_1', p_2' \in \mathbb{F}[y]$ s.t.

$$\frac{\tilde{p}_2}{k_2^{m-1}} = D_y\left(\frac{p_1'}{k_2^{m-2}}\right) + \frac{p_1'}{k_2^{m-2}}K + \frac{p_2'}{k_2}.$$

Setting $p_1 = p_1'k_2 + \tilde{p}_1$ and $p_2 = p_2'$ completes the proof. ∎

With Convention 3, we have

$$\frac{p}{k_2^m}T = D_y\left(\frac{p_1}{k_2^{m-1}}T\right) + \frac{p_2}{k_2}T$$

by Lemma 16. This reduction will be referred to as the *kernel reduction for $(p/k_2^m)T$ w.r.t. K.*

6. TELESCOPING VIA REDUCTIONS

Hermite reduction has been used to construct telescopers for bivariate rational functions in [5]. We extend the idea in [5] and apply Theorem 13 to develop a reduction-based telescoping method for bivariate hyperexponential functions. The method also yields an order bound on minimal telescopers, which is tighter than those given in [4, 8]

6.1 Creative telescoping for bivariate hyperexponential functions

A nonzero element H in some differential field extension of $\mathbb{F}(x, y)$ is said to be *hyperexponential* over $\mathbb{F}(x, y)$ if its logarithmic derivatives $D_x(H)/H$ and $D_y(H)/H$ are in $\mathbb{F}(x, y)$.

Set $f{=}D_x(H)/H$ and $g{=}D_y(H)/H$. Then $D_y(f){=}D_x(g)$ because D_x and D_y commute. Therefore, it is legitimate to denote H by $\exp(\int f\,dx + g\,dy)$. For every nonzero rational function $r \in \mathbb{F}(x, y)$,

$$rH = \exp\left(\int (f + D_x(r)/r)\,dx + (g + D_y(r)/r)\,dy\right).$$

The following fact is immediate from [12, Lemma 8].

Fact 17. *Let f and g be rational functions in $\mathbb{F}(x,y)$ satisfying $D_y(f) = D_x(g)$. Then the denominator of f divides that of g in $\mathbb{F}(x)[y]$.*

For a hyperexponential function H over $\mathbb{F}(x,y)$, the telescoping problem is to construct a linear ordinary differential operator $L(x, D_x)$ in $\mathbb{F}(x)\langle D_x \rangle$ s.t.

$$L(x, D_x)(H) = D_y(G)$$

for some hyperexponential function G over $\mathbb{F}(x,y)$. As in the rational case [5], we apply the Hermite reduction for univariate hyperexponential functions w.r.t. y to the derivatives $D_x^i(H)$ iteratively, and then find a linear dependency among the residual forms over $\mathbb{F}(x)$.

Lemma 18. *Let $H = \exp(\int f\,dx + g\,dy)$ be a hyperexponential function over $\mathbb{F}(x,y)$. Let K be the kernel and S the shell of g w.r.t. y. Then, for every $i \in \mathbb{N}$, the i-th derivative $D_x^i(H)$ can be decomposed into*

$$D_x^i(H) = D_y(u_i T) + r_i T, \tag{15}$$

where $u_i \in \mathbb{F}(x,y)$, $T = \exp(\int (f - D_x(S)/S)\,dx + K\,dy)$ and $r_i \in \mathbb{F}(x,y)$ is a residual form w.r.t. K. Moreover, let k_2 be the denominator of K, b the squarefree part of the denominator of S, and $\mathcal{N}_K$ the standard complement of the subspace for polynomial reduction w.r.t. K. Then

$$r_i = \frac{q_i}{b} + \frac{v_i}{k_2} \tag{16}$$

for some $q_i \in \mathbb{F}(x)[y]$ with $\deg_y q_i < \deg_y b$ and $v_i \in \mathcal{N}_K$.

Proof. We proceed by induction on i. If $i = 0$, then the assertion holds by Theorem 13.

Assume that $D_x^i(H)$ can be decomposed into (15) and assume that (16) holds. Moreover, let $\tilde{f} = f - D_x(S)/S$. Consider the $(i+1)$-th derivative $D_x^{i+1}(H)$. There exists a polynomial a in $\mathbb{F}(x)[y]$ s.t. $\tilde{f} = a/k_2$ by $D_y\left(\tilde{f}\right) = D_x(K)$ and Fact 17. A direct calculation leads to

$$D_x^{i+1}(H) = D_y(D_x(u_i T)) + \left(\frac{q_i a}{b k_2} + \frac{D_x(q_i)}{b} + \frac{D_x(v_i)}{k_2}\right) T + \left(\frac{-q_i D_x(b)}{b^2} + \frac{(a - D_x(k_2)) v_i}{k_2^2}\right) T.$$

Applying shell reduction to $\left(-q_i D_x(b)/b^2\right) T$ and kernel reduction to $\left((a - D_x(k_2)) v_i / k_2^2\right) T$ w.r.t. y, we get

$$\frac{-q_i D_x(b)}{b^2} = D_y\left(\frac{w_1}{b}\right) + \frac{w_1}{b} K + \frac{w_2}{b k_2},$$

$$\frac{(a - D_x(k_2)) v_i}{k_2^2} = D_y\left(\frac{p_1}{k_2}\right) + \frac{p_1}{k_2} K + \frac{p_2}{k_2},$$

where w_1, w_2, p_1 and p_2 are in $\mathbb{F}(x)[y]$. We then apply polynomial reduction to $\tilde{S} T$ w.r.t. K, where

$$\tilde{S} = \frac{w_2}{b k_2} + \frac{p_2}{k_2} + \frac{a q_i}{b k_2} + \frac{D_x(q_i)}{b} + \frac{D_x(v_i)}{k_2},$$

which leads to $\tilde{S} = D_y(w) + wK + (q_{i+1}/b + v_{i+1}/k_2)$, where w is in $\mathbb{F}(x,y)$ and $q_{i+1}/b + v_{i+1}/k_2$ is the residual form of $\tilde{S}$ w.r.t. K. It follows from a direct calculation that

$$D_x^{i+1}(H) = D_y(u_{i+1} T) + \left(\frac{q_{i+1}}{b} + \frac{v_{i+1}}{k_2}\right) T,$$

where $u_{i+1} = D_x(u_i) + u_i \tilde{f} + w_1/b + p_1/k_2 + w$. ∎

The main results in the present section are given below.

Theorem 19. *With the notation introduced in Lemma 18, we let $L = \sum_{i=0}^{\rho} e_i D_x^i$ with $e_0, \ldots, e_\rho \in \mathbb{F}(x)$, not all zero.*

(i) L is a telescoper for H if and only if $\sum_{i=0}^{\rho} e_i r_i = 0$.

(ii) The order of a minimal telescoper for H is no more than $\deg_y(b) + \max(\deg_y(k_1), \deg_y(k_2) - 1)$.

Proof. We regard hyperexponential functions involved in the proof as univariate ones in y. Moreover, let $u = \sum_{i=0}^{\rho} e_i u_i$ and $r = \sum_{i=0}^{\rho} e_i r_i$. By (15), we have

$$L(H) = D_y(uT) + rT. \tag{17}$$

If $r = 0$, then L is a telescoper by (17). Conversely, assume that L is a telescoper of H. Then rT is integrable w.r.t. y by (17). Since r is a residual form by Remark 10, it is equal to zero by Lemma 11. The first assertion holds.

Set $\lambda = \max(\deg_y(k_1), \deg_y(k_2) - 1)$. Let the residual form $r_i = q_i/b + v_i/k_2$ be as defined in (15) and (16). By Lemma 8, the v_i's have a common set $\mathcal{P}$ of supporting monomials with $|\mathcal{P}| \le \lambda$. Moreover, $\deg_y(q_i) < \deg_y(b)$ and $\gcd(b, k_2) = 1$. Therefore, the residual forms $r_0, \ldots, r_\rho$ are linearly dependent over $\mathbb{F}(x)$ if $\rho \ge \deg_y(b) + \lambda$. The second assertion holds ∎

Remark 20. *By Theorem 19, a linear dependency among the residual forms r_0, ..., r_σ, for minimal σ, gives rise to a minimal telescoper of H.*

With the notation introduced in Lemma 18, we outline a reduction-based telescoping algorithm for bivariate hyperexponential functions.

Algorithm. HermiteTelescoping: Given a bivariate hyperexponential function $H = \exp(\int f\,dx + g\,dy)$ over $\mathbb{F}(x,y)$, compute a minimal telescoper L and its certificate w.r.t. y.

1. Find the kernel K and shell S of $D_y(H)/H$ w.r.t. y. Set b to be the squarefree part of the denominator of S.
2. Decompose H into $H = D_y(u_0 T) + r_0 T$ using the Hermite reduction for hyperexponential functions given in Theorem 13. If $r_0 = 0$, return $(1, u_0 T)$.
3. Set $\rho := \deg_y(b) + \max(\deg_y(k_1), \deg_y(k_2) - 1)$.
4. For i from 1 to ρ do
   4.1. Compute (u_i, r_i) incrementally s.t.
   $$D_x^i(H) = D_y(u_i T) + r_i T$$
   by the shell, kernel and polynomial reductions described in Lemma 18.
   4.2. Find $\eta_j \in \mathbb{F}(x)$ s.t. $\sum_{j=0}^{i} \eta_j r_j = 0$ by solving a linear system over $\mathbb{F}(x)$. If there is a nontrivial solution, return $\left(\sum_{j=0}^{i} \eta_j D_x^j, \sum_{j=0}^{i} \eta_j u_j T\right)$.

Example 21. *Let $H = \sqrt{x - 2y}\,\exp(x^2 y)$. Then $D_x(H)/H$ and $D_y(H)/H$ are, respectively,*

$$f = \frac{1 + 4x^2 y - 8xy^2}{2(x - 2y)} \quad \text{and} \quad g = \frac{-1 + x^3 - 2x^2 y}{x - 2y}.$$

Since g is differential-reduced w.r.t. y, g is the kernel and 1 is the shell of $D_y(H)/H$ w.r.t. y. By Hermite reduction,

$$H = D_y\left(\frac{1}{x^2}H\right) + \frac{1}{x^2 k_2}H. \tag{18}$$

Applying D_x to the above equation yields

$$D_x(H) = D_y\left(\frac{-3x+8y+4x^3y-8x^2y^2}{2x^3(x-2y)}H\right) + rH,$$

where $r = (-5x+8y+4x^3y-8x^2y^2)/(2x^3k_2^2)$. The shell, kernel and polynomial reduction given in Lemma 18 yields

$$D_x(H) = D_y\left(\frac{2x^2y-3}{x^3}\cdot H\right) + \frac{3x^3-6}{2x^3k_2}H \tag{19}$$

Combining (18) and (19), we get $L = (6-3x^3)+2xD_x$ is a minimal telescoper for H and $G = (4y-3x)H$ is the corresponding certificate.

The algorithm above separates the computation of minimal telescopers from that of certificates. One may neglect the computation for certificates in the algorithm when they are irrelevant in applications. Moreover, one may opt for unnormalized certificates in the form wT, where $w = \sum_j w_j$ with $w_j \in \mathbb{F}(x,y)$ as described in step 4.2. Experiments carried out in Section 7 reveal that it is time-consuming to normalize w as a fraction p/q with $p, q \in \mathbb{F}[x,y]$ and $\gcd(p,q)=1$. In fact, unnormalized certificates are sufficient for many applications. For instance, we may want to compute $w(x,s)$ for $s \in F$ with $q(x,s) \neq 0$ when evaluating definite integrals. This can be achieved by unnormmalized certificates, because $w(x,s)$ equals the sum of all residues of $w_j/(y-s)$ at $y=s$.

Remark 22. *Another idea for computing a minimal telescoper of H is the following: We first compute a nonzero operator $L_1 \in \mathbb{F}(x)\langle D_x\rangle$ of minimal order s.t.*

$$L_1(H) = D_y(G_1) + (p/k_2)T$$

for some hyperexponential function G_1 and polynomial p. Note that such operators always exist, because $\deg_y q_i$ in (16) is less than $\deg_y b$. Then we apply the algorithm HermiteTelescoping to get a minimal telescoper L_2 for $(p/k_2)T$. One can show that L_2L_1 is a minimal telescoper of H.

Let $\ell_1 = \deg_y b$ and $\ell_2 = \max(\deg_y(k_1), \deg_y(k_2)-1)$, where b, k_1 and k_2 are given in Theorem 19. The algorithm HermiteTelescoping solves linear systems of at most $\ell_1+\ell_2$ equations over $\mathbb{F}(x)$ to obtain the minimal telescoper L, while an algorithm based on the idea given above solves linear systems of at most ℓ_1 equations to obtain L_1, and then solves linear systems of at most ℓ_2 equations to obtain L_2. However, the linear systems over $\mathbb{F}(x)$ corresponding to L_2 have coefficients of high degrees in x. In addition, it takes time to expand the product of L_2L_1. Preliminary experiments reveal that such an algorithm may outperform HermiteTelescoping in practice only when ℓ_2 is no more than three.

6.2 Comparison with the Apagodu-Zeilberger bound

Assume that H is a nonzero hyperexponential fundtion over $\mathbb{F}(x,y)$ of the form

$$H = u\exp\left(\frac{r_1}{r_2}\right)\prod_{i=1}^{m} p_i(x,y)^{c_i}, \tag{20}$$

where $u, r_1, r_2, p_1, \ldots, p_m$ are nonzero polynomials in $\mathbb{F}[x,y]$ and $c_1, \ldots, c_m$ are *distinct indeterminates*.

Theorem cAZ in [4] asserts that the order of minimal telescopers for H is bounded by

$$\alpha := \deg_y(r_2) + \max\left(\deg_y(r_1), \deg_y(r_2)\right) + \sum_{i=1}^{m}\deg_y(p_i) - 1.$$

Note that H can be viewed as a hyperexponential function over $\mathbb{F}(c_1,\ldots,c_m)(x,y)$. We now show that α given above is at least the order bound on minimal telescopers for H obtained from Theorem 19 (ii). The kernel and shell of the logarithmic derivative $D_y(H)/H$ are

$$K := D_y\left(\frac{r_1}{r_2}\right) + \sum_{i=1}^{m} c_i\frac{D_y(p_i)}{p_i} \quad \text{and} \quad S := u,$$

respectively, because K has no integral residue at any simple pole, S is a polynomial in $\mathbb{F}[x,y]$, and $D_y(H)/H$ is equal to $K + D_y(S)/S$. Let $K = k_1/k_2$ with $\gcd(k_1,k_2)=1$. A direct calculation leads to

$$\deg_y(k_1) \le \deg_y(r_1) + \deg_y(r_2) + \sum_{i=1}^{m}\deg_y(p_i) - 1,$$

and $\deg_y(k_2) \le 2\deg_y(r_2) + \sum_{i=1}^{m}\deg_y(p_i)$. By Theorem 19, the order of minimal telescopers for H is no more than $\max\left(\deg_y(k_1), \deg_y(k_2)-1\right)$, which is no more than α by the above two inequalities.

Indeed, the order bound in Theorem 19 (ii) may be smaller than that in Theorem cAZ.

Example 23. *Let $H = q^c\exp(a/q)$, where a, q are irreducible polynomials in $\mathbb{F}[x,y]$ with $\deg_y(a) < \deg_y(q)$, and c is a transcendental constant over $\mathbb{F}$. By Theorem cAZ, a minimal telescoper for H has order no more than $3\deg_y q - 1$. On the other hand, the kernel and shell of $D_y(H)/H$ are equal to $\left(D_y(a)q - aD_y(q) + cqD_y(q)\right)/q^2$ and 1, respectively. A minimal telescoper has order no more than $2\deg_y q - 1$ by Theorem 19 (ii).*

Without assuming that the exponents $c_1, \ldots, c_m$ in (20) are distinct indeterminates, Theorem 14 in [8] derives order and degree bounds for minimal telescopers, in which the order bound is the same as that in Theorem cAZ. Furthermore, Christopher's Theorem in [9, 7] states that a general hyperexponential function over $\mathbb{F}(x,y)$ can be written as:

$$\frac{u}{v}\exp\left(\frac{r_1}{r_2}\right)\prod_{i=1}^{m} p_i(x,y)^{c_i}, \tag{21}$$

where $u, v, r_1, r_2 \in \mathbb{F}[x,y]$, c_i is algebraic over $\mathbb{F}$, and p_i is in $\mathbb{F}(c_i)[x,y]$, $i = 1, \ldots, m$. So H given in (20) is a special instance of hyperexponential functions. In addition, it is easier to compute the kernel and shell w.r.t. y than to compute the decompositions (20) and (21) when a hyperexponential function is given by its logarithmic derivatives.

7. IMPLEMENTATION AND TIMINGS

We have implemented the algorithm HermiteTelescoping in the computer algebra system Maple 16. Our Maple code is available from

http://www.mmrc.iss.ac.cn/~zmli/HermiteCT.html

We now compare the performance of our algorithm with the Maple implementation DEtools[Zeilberger] of the telescoping algorithm in [3]. We take examples of the form

$$\frac{p}{q^m} \cdot \sqrt{\frac{a}{b}} \cdot \exp\left(\frac{u}{v}\right),$$

where $m \in \mathbb{N}$, $p, q, a, b, u, v \in \mathbb{Z}[x, y]$ are irreducible and their coefficients are randomly chosen. For simplicity, we choose $\lambda = \deg_y(p) = \deg_y(q)$, $\mu = \deg_y(a) = \deg_y(b)$, and $\nu = \deg_y(u) = \deg_y(v)$. The runtime comparison (in seconds) for different examples is shown in Table 1, in which

- ZT: the Maple function DEtools[Zeilberger].
- HT_un: the algorithm HermiteTelescoping, which returns telescopers and unnormalized certificates.
- HT_n: the algorithm HermiteTelescoping, which returns telescopers and normalized certificates.
- order: the order of the computed minimal telescoper.
- OOM: Maple runs out of memory.

(λ, μ, ν, m)	ZT	HT_un	HT_n	order
(2, 0, 2, 1)	2.16	2.01	3.80	5
(2, 0, 2, 2)	2.06	1.98	2.59	5
(3, 0, 2, 1)	8.68	6.54	14.01	6
(3, 0, 2, 2)	9.23	6.06	13.72	6
(6, 0, 1, 1)	44.04	24.39	70.49	7
(6, 0, 1, 2)	41.85	22.74	59.50	7
(2, 2, 2, 1)	1399.2	155.54	570.40	9
(2, 2, 2, 2)	1397.7	142.34	510.11	9
(3, 0, 3, 1)	151.84	44.07	120.44	8
(3, 0, 3, 2)	150.14	43.46	122.36	8
(3, 3, 0, 1)	206.90	46.15	165.67	8
(3, 3, 0, 2)	217.81	44.95	161.25	8
(3, 2, 1, 1)	300.93	60.33	184.71	8
(3, 2, 1, 2)	333.75	55.86	176.78	8
(3, 1, 3, 1)	OOM	361.79	1556.1	10
(3, 1, 3, 2)	OOM	370.18	1535.7	10

Table 1: Timings (in seconds) measured on a Mac OS X computer, 4Gb RAM, 3.06 GHz Core 2 Duo processor.

Our empirical results in the above table illustrate that HermiteTelescoping is markedly superior to Maple's function DEtools[Zeilberger] if it computes minimal telescopers and unnormalized certificates, and that it is either comparable to or faster than DEtools[Zeilberger] when it computes telescopers and normalized certificates.

Remark 24. *The orders of the computed minimal telescopers in our experiments equal the predicted order bounds given in Theorem 19.*

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