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Hermite–sinusoidal-Gaussian beams in complex optical systems

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Sinusoidal-Gaussian beams have recently been obtained as exact solutions of the paraxial wave equation for propagation in complex optical systems. Another useful set of beam solutions for Cartesian coordinate systems is based on Hermite–Gaussian functions. A generalization of these solution sets is developed here. The new solutions are referred to as Hermite–sinusoidal-Gaussian beams, because they are in the form of a product of Hermite-polynomial functions of either complex or real argument, sinusoidal functions of complex argument, and Gaussian functions of complex argument. These beams are valid for propagation through systems that can be represented in terms of complex beam matrices, and the previous beam solution sets are special cases of these more general results. Propagation characteristics and applications of these beams are discussed, including their use as a basis set for propagation of arbitrary electromagnetic beams. © 1998 Optical Society of America [S0740-3232(98)01904-8]

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1. INTRODUCTION

The propagation of electromagnetic beams in optical systems has long been of interest, and in some cases exact analytical solutions of the paraxial wave equation have been obtained. Such solutions are important because they require little or no numerical computation for their evaluation. The nature of the solutions that one obtains depends on the coordinate system that is employed. For many years the most general analytical solutions in Cartesian coordinates have been in the form of Hermite–Gaussian functions in which the arguments of both the Hermite-polynomial factor and the Gaussian factor are complex.^{1–10} Besides free-space and other lens and mirror elements, the complex Hermite–Gaussian beam solutions can also propagate in any media that can be characterized by only constant, linear, and quadratic transverse variations of the gain and the index of refraction in the vicinity of the beam. Thus they are valid for propagation through any systems that can be represented in terms of complex beam matrices. Some studies have allowed these beams to deviate from the axis of the complex optical system,^{6,10,11} and in the most general analyses misalignment of the component optical elements has also been permitted.^{12,13}

Recently, an alternative set of complex Cartesian beam solutions of the paraxial wave equation has been obtained in the form of sinusoidal-Gaussian beams.¹⁴ These solutions can also propagate through any systems that can be represented in terms of complex beam matrices, and they reduce to conventional rectangular waveguide modes in

the appropriate limit. Properties and applications of these beams are discussed here, including their use as a basis set for propagation of arbitrary electromagnetic beams and as solution modes in novel laser resonators.

As just noted, there are now two important beam solution sets for propagation with the use of Cartesian coordinates, and these include the Hermite–Gaussian and sinusoidal-Gaussian beams of complex argument. A new generalization and merging of these two solution sets is described here. The resulting solutions are referred to as Hermite–sinusoidal-Gaussian beams because they are in the form of a product of Hermite-polynomial functions of complex argument, sinusoidal functions of complex argument, and Gaussian functions of complex argument. They are also valid for propagation through systems that can be represented in terms of complex beam matrices, and the previous beam solution sets are special cases of these more general results. For example, Hermite–Gaussian beams for which the arguments of the Hermite polynomials are real are a special case of complex-argument Hermite–Gaussian beams,¹⁵ which are a special case of the Hermite–sinusoidal-Gaussian beams.

After a brief review of Hermite–Gaussian beam theory in Section 2, the basic derivation of the Hermite–sinusoidal-Gaussian modes is included in Section 3. Although other procedures are possible, the derivation here is set up as a generalization of the more familiar Hermite–Gaussian modes without reference back to the scalar wave equation. Section 4 includes a discussion of

how the less familiar sinusoidal-Gaussian beam solutions can be employed as a basis set for the propagation of an arbitrary electromagnetic field distribution.

2. HERMITE-GAUSSIAN BEAM THEORY

We begin with a brief summary of the equations governing conventional off-axis Hermite-Gaussian functions of complex argument. For the usual case of slowly varying complex wave number $k(x, y, z)$, the Maxwell-Heaviside equations reduce to the wave equation

$$\nabla^2 \mathbf{E}'(x, y, z) + k^2(x, y, z) \mathbf{E}'(x, y, z) = 0, \quad (1)$$

where \mathbf{E}' is the complex amplitude of the vector electric field \mathbf{E} . The wave number may have an imaginary part that is due to nonzero conductivity or out-of-phase components of the material polarization or magnetization. If needed, the weak z components of the fields may be found from the transverse components by means of the Maxwell-Heaviside equations.¹¹

In many practical situations the gain (or loss) and the index of refraction have at most quadratic variations in the vicinity of the propagating beam, and one can write

$$k^2(x, y, z) = k_0(z)[k_0(z) - k_{1x}(z)x - k_{1y}(z)y - k_{2x}(z)x^2 - k_{2y}(z)y^2]. \quad (2)$$

For an x -polarized wave propagating in the z direction, a useful substitution is

$$E'_x(x, y, z) = A(x, y, z) \exp\left[-i \int_0^z k_0(z') dz'\right], \quad (3)$$

and the x component of Eq. (1) reduces to the paraxial wave equation

$$\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} - 2ik_0 \frac{\partial A}{\partial z} - i \frac{dk_0}{dz} A - k_0(k_{1x}x + k_{1y}y + k_{2x}x^2 + k_{2y}y^2)A = 0, \quad (4)$$

where $A(x, y, z)$ is assumed to vary so slowly with z that its second derivative can be neglected.

A useful form of the solution to Eq. (4) for an astigmatic off-axis Gaussian beam is⁶

$$\begin{aligned} A_{m,n}(x, y, z) &= A_0 H_m[a_x(z)x + b_x(z)] H_n[a_y(z)y + b_y(z)] \\ &\times \exp\left\{-i \left[\frac{Q_x(z)x^2}{2} + \frac{Q_y(z)y^2}{2} \right. \right. \\ &\left. \left. + S_x(z)x + S_y(z)y + P(z) \right] \right\}, \quad (5) \end{aligned}$$

where H_m and H_n are Hermite polynomials of order m and n , respectively. The functions Q_x and Q_y are known as beam parameters, and S_x , S_y , and P are displacement and phase parameters. The functions $a_x(z)$, $b_x(z)$, $a_y(z)$, and $b_y(z)$ characterize the width and the displacement of the Hermite-polynomial factors of the solution. All of these z -dependent parameters are, in general, complex valued. If the input beam to an optical system of the type being considered (simple lenses, mirrors, lenslike

media, etc.) is of the form given in Eq. (5), the output beam will be of the same form. The propagation of such a beam through the system would be fully characterized by the above parameters.

One finds by direct substitution that Eq. (5) is an exact solution of Eq. (4) provided that the various parameters satisfy the following equations:

$$Q_x^2 + k_0 \frac{dQ_x}{dz} + k_0 k_{2x} = 0, \quad (6)$$

$$Q_y^2 + k_0 \frac{dQ_y}{dz} + k_0 k_{2y} = 0, \quad (7)$$

$$Q_x S_x + k_0 \frac{dS_x}{dz} + \frac{k_0 k_{1x}}{2} = 0, \quad (8)$$

$$Q_y S_y + k_0 \frac{dS_y}{dz} + \frac{k_0 k_{1y}}{2} = 0, \quad (9)$$

$$Q_x a_x + k_0 \frac{da_x}{dz} + i a_x^3 = 0, \quad (10)$$

$$Q_y a_y + k_0 \frac{da_y}{dz} + i a_y^3 = 0, \quad (11)$$

$$S_x a_x + k_0 \frac{db_x}{dz} + i a_x^2 b_x = 0, \quad (12)$$

$$S_y a_y + k_0 \frac{db_y}{dz} + i a_y^2 b_y = 0, \quad (13)$$

$$\begin{aligned} \frac{dP}{dz} &= -\frac{S_x^2 + S_y^2}{2k_0} - i \frac{Q_x + Q_y}{2k_0} - \frac{m a_x^2 + n a_y^2}{k_0} \\ &- \frac{i}{2k_0} \frac{dk_0}{dz}. \quad (14) \end{aligned}$$

This separation is accomplished by setting equal to zero the various terms in x^2 , y^2 , x , and y , and using the Hermite differential equation

$$\frac{d^2 H_m}{dx'^2} - 2x' \frac{dH_m}{dx'} + 2m H_m = 0. \quad (15)$$

For this form of the Hermite equation, x' would correspond to $a_x(z)x + b_x(z)$ in the above solution.

If the gain per wavelength is small, the significance of the Q parameters is contained in the relation

$$\frac{Q_x}{k_0} = \frac{1}{q_x} = \frac{1}{R_x} - i \frac{\lambda}{\pi w_x^2}, \quad (16)$$

where R_x and w_x are, respectively, the radius of curvature of the phase fronts and the $1/e$ amplitude spot size in the x direction. The ratio $d_{xa} = -S_{xi}/Q_{xi}$ is the displacement in the x direction of the amplitude center of the Gaussian part of the beam, and the ratio $d_{xp} = -S_{xr}/Q_{xr}$ is the displacement in the x direction of the phase center of the beam.¹¹ Here the subscripts i and r denote, respectively, the imaginary and the real parts of the parameters Q_x and S_x , and similar relations apply to the functions Q_y and S_y . The parameters a_x , b_x , a_y , and b_y characterize the higher-order amplitude and phase variations and displacements that are associated

with the polynomial factors in the solution, and the parameter P is an overall phase correction. The parameter a_x is sometimes recast as $\sqrt{2}/W_x$ (Refs. 12 and 13) or $\sqrt{2}/v_x$ (Ref. 15).

Thus far we have summarized the general Hermite-Gaussian beams in terms of a set of ordinary differential equations given above as Eqs. (6)–(14). The solutions of these equations are known for a variety of optical elements and systems. Our focus here will be on the astigmatic, complex, and misaligned systems that can be described in terms of generalized beam matrices of the form¹²

$$\begin{pmatrix} u_{x2} \\ (1/q_{x2})u_{x2} \\ S_{x2}u_{x2} \end{pmatrix} = \begin{bmatrix} A_x & B_x & 0 \\ C_x & D_x & 0 \\ G_x & H_x & 1 \end{bmatrix} \begin{pmatrix} u_{x1} \\ (1/q_{x1})u_{x1} \\ S_{x1}u_{x1} \end{pmatrix}, \quad (17)$$

where the subscripts 1 and 2 indicate the input and output parameters of the optical element, respectively. The A_x , B_x , C_x , and D_x elements in Eq. (17) are the usual complex matrix elements for an aligned system, and G_x and H_x allow the inclusion of displacements and misalignments in the formalism. Similar results apply for the y variations of the fields. The matrix in Eq. (17) has been obtained for a wide variety of optical elements.¹² Optical systems may be analyzed by multiplying the matrix representations of the optical elements in the reverse of the order in which those elements are encountered by an incident light beam.

We indicate first the beam transformation formulas for an arbitrary complex distributed lenslike medium. Dividing the second row of Eq. (17) by its first row yields the Kogelnik transformations:

$$\frac{1}{q_{x2}} = \frac{C_x + D_x/q_{x1}}{A_x + B_x/q_{x1}}, \quad (18)$$

$$\frac{1}{q_{y2}} = \frac{C_y + D_y/q_{y1}}{A_y + B_y/q_{y1}}. \quad (19)$$

Similarly, the displacement transformations are obtained by dividing the third row of Eq. (17) by its first row:¹²

$$S_{x2} = \frac{S_{x1} + G_x + H_x/q_{x1}}{A_x + B_x/q_{x1}}, \quad (20)$$

$$S_{y2} = \frac{S_{y1} + G_y + H_y/q_{y1}}{A_y + B_y/q_{y1}}. \quad (21)$$

Using these results, one may obtain transformation formulas for the other parameters of the beams:

$$a_{x2} = \frac{a_{x1}}{A_x + B_x/q_{x1}} \left(1 + \frac{2ia_{x1}^2}{k_{01}} \frac{B_x}{A_x + B_x/q_{x1}} \right)^{-1/2}, \quad (22)$$

$$a_{y2} = \frac{a_{y1}}{A_y + B_y/q_{y1}} \left(1 + \frac{2ia_{y1}^2}{k_{01}} \frac{B_y}{A_y + B_y/q_{y1}} \right)^{-1/2}, \quad (23)$$

$$b_{x2} = \left(b_{x1} - \frac{a_{x1}S_{x1}}{k_{01}} \frac{B_x}{A_x + B_x/q_{x1}} \right) \times \left(1 + \frac{2ia_{x1}^2}{k_{01}} \frac{B_x}{A_x + B_x/q_{x1}} \right)^{-1/2}, \quad (24)$$

$$b_{y2} = \left(b_{y1} - \frac{a_{y1}S_{y1}}{k_{01}} \frac{B_y}{A_y + B_y/q_{y1}} \right) \times \left(1 + \frac{2ia_{y1}^2}{k_{01}} \frac{B_y}{A_y + B_y/q_{y1}} \right)^{-1/2}, \quad (25)$$

$$P_2 - P_1 = \frac{i}{2} \ln(A_x D_x - B_x C_x) - \frac{i}{2} \ln(A_x + B_x/q_{x1}) - \frac{i}{2} \ln(A_y + B_y/q_{y1}) + \frac{i}{2} m \ln \left(1 + \frac{2ia_{x1}^2}{k_{01}} \frac{B_x}{A_x + B_x/q_{x1}} \right) + \frac{i}{2} n \ln \left(1 + \frac{2ia_{y1}^2}{k_{01}} \frac{B_y}{A_y + B_y/q_{y1}} \right) - \frac{B_x}{2k_{01}} \frac{(S_{x1} + G_x + H_x/q_{x1})^2}{A_x + B_x/q_{x1}} - \frac{B_y}{2k_{01}} \frac{(S_{y1} + G_y + H_y/q_{y1})^2}{A_y + B_y/q_{y1}} + \frac{H_x}{2k_{01}} (2S_{x1} + G_x + H_x/q_{x1}) + \frac{H_y}{2k_{01}} (2S_{y1} + G_y + H_y/q_{y1}) + \frac{1}{2k_{01}} \int_0^z \left(G_x \frac{dH_x}{dz'} - H_x \frac{dG_x}{dz'} \right) dz' + \frac{1}{2k_{01}} \int_0^z \left(G_y \frac{dH_y}{dz'} - H_y \frac{dG_y}{dz'} \right) dz'. \quad (26)$$

These transformation formulas are applicable to a wide range of optical elements and systems in addition to distributed lenslike media. With Eqs. (18)–(26), Eq. (5) is a complete solution for the propagation of a Hermite-Gaussian beam between two reference planes of a medium characterized by a matrix of the form given in Eq. (17).

3. DERIVATION OF THE BEAM MODES

The purpose of this section is to obtain a new set of Cartesian beam solutions of the paraxial wave equation in the form of Hermite-sinusoidal-Gaussian functions. Although one could obtain these solutions starting directly from the wave equation, there is a shortcut that reduces the mathematical effort required. Since the ordinary Hermite-Gaussian beam solutions are already known to form a complete set, any other solutions that might be possible must be expressible as a linear combination of the Hermite-Gaussian results. Our objective is to show that an appropriate superposition of these beams can correspond to the Hermite-sinusoidal-Gaussian solutions that are being emphasized here. In exploring this possibility, we will focus first on the field variations in the x direction.

The notation involved in introducing the sinusoidal variations in the x direction is simplified if the phase parameter is broken up into a part that involves the complex displacement parameter S_x and a part that does not. Thus we rewrite Eq. (26) in the form

$$P_2 = P_1 - \frac{B_x}{2k_{01}} \frac{(S_{x1} + G_x + H_x/q_{x1})^2}{A_x + B_x/q_{x1}} + \frac{H_x}{k_{01}} S_{x1} + P_0, \quad (27)$$

where the phase P_0 is given by

$$P_0 = \frac{i}{2} \ln(A_x D_x - B_x C_x) - \frac{i}{2} \ln(A_x + B_x/q_{x1}) - \frac{i}{2} \ln(A_y + B_y/q_{y1}) + \frac{i}{2} m \ln \left(1 + \frac{2ia_{x1}^2}{k_{01}} \frac{B_x}{A_x + B_x/q_{x1}} \right) + \frac{i}{2} n \ln \left(1 + \frac{2ia_{y1}^2}{k_{01}} \frac{B_y}{A_y + B_y/q_{y1}} \right) - \frac{B_y}{2k_{01}} \frac{(S_{y1} + G_y + H_y/q_{y1})^2}{A_y + B_y/q_{y1}} + \frac{H_x}{2k_{01}} (G_x + H_x/q_{x1}) + \frac{H_y}{2k_{01}} (2S_{y1} + G_y + H_y/q_{y1}) + \frac{1}{2k_{01}} \int_0^z \left(G_x \frac{dH_x}{dz'} - H_x \frac{dG_x}{dz'} \right) dz' + \frac{1}{2k_{01}} \int_0^z \left(G_y \frac{dH_y}{dz'} - H_y \frac{dG_y}{dz'} \right) dz'. \quad (28)$$

With this change of variables, a special superposition of two of the beams (an α beam and a β beam) given above as Eq. (5) can now be written in the form

$$A_{2,m,n}(x, y) = A_0 H_m(a_{x2}x + b_{x2}) H_n(a_{y2}y + b_{y2}) \times \exp \left[-i \left(\frac{Q_{x2}x^2}{2} + \frac{Q_{y2}y^2}{2} + S_{y2}y \right) \right] \times \left\{ \frac{1}{2} \exp[-i(S_{x2\alpha}x + P_{2\alpha})] + \frac{1}{2} \exp[-i(S_{x2\beta}x + P_{2\beta})] \right\}. \quad (29)$$

From Eq. (20) the transformation formulas for these new complex displacement parameters are

$$S_{x2\alpha} = \frac{S_{x1\alpha} + G_x + H_x/q_{x1}}{A_x + B_x/q_{x1}}, \quad (30)$$

$$S_{x2\beta} = \frac{S_{x1\beta} + G_x + H_x/q_{x1}}{A_x + B_x/q_{x1}}, \quad (31)$$

and from Eq. (27) the transformations for the new phase parameters are

$$P_{2\alpha} = P_{1\alpha} - \frac{B_x}{2k_{01}} \frac{(S_{x1\alpha} + G_x + H_x/q_{x1})^2}{A_x + B_x/q_{x1}} + \frac{H_x}{k_{01}} S_{x1\alpha} + P_0, \quad (32)$$

$$P_{2\beta} = P_{1\beta} - \frac{B_x}{2k_{01}} \frac{(S_{x1\beta} + G_x + H_x/q_{x1})^2}{A_x + B_x/q_{x1}} + \frac{H_x}{k_{01}} S_{x1\beta} + P_0. \quad (33)$$

To be specific, we now specify the initial values of the complex displacement and phase parameters in the forms

$$S_{x1\alpha} = S_{x1} - a'_{x1}, \quad (34)$$

$$S_{x1\beta} = S_{x1} + a'_{x1}, \quad (35)$$

$$P_{1\alpha} = P_1 - b'_{x1}, \quad (36)$$

$$P_{1\beta} = P_1 + b'_{x1}. \quad (37)$$

When Eqs. (30)–(37) are substituted into Eq. (29), one obtains

$$A_{2,m,n}(x, y) = A_0 H_m(a_{x2}x + b_{x2}) H_n(a_{y2}y + b_{y2}) \times \exp \left[-i \left(\frac{Q_{x2}x^2}{2} + \frac{Q_{y2}y^2}{2} + S_{y2}y \right) \right] \times \left\{ \frac{1}{2} \exp \left[-i \left[\frac{S_{x1} - a'_{x1} + G_x + H_x/q_{x1}}{A_x + B_x/q_{x1}} x + P_1 - b'_{x1} - \frac{B_x}{2k_{01}} \frac{(S_{x1} - a'_{x1} + G_x + H_x/q_{x1})^2}{A_x + B_x/q_{x1}} + \frac{H_x}{k_{01}} (S_{x1} - a'_{x1}) + P_0 \right] \right] + \frac{1}{2} \exp \left[-i \left[\frac{S_{x1} + a'_{x1} + G_x + H_x/q_{x1}}{A_x + B_x/q_{x1}} x + P_1 + b'_{x1} - \frac{B_x}{2k_{01}} \frac{(S_{x1} + a'_{x1} + G_x + H_x/q_{x1})^2}{A_x + B_x/q_{x1}} + \frac{H_x}{k_{01}} (S_{x1} + a'_{x1}) + P_0 \right] \right] \right\}. \quad (38)$$

After some rearranging Eq. (38) is

$$\begin{aligned}
 A_{2,m,n}(x, y) &= A_0 H_m(a_{x2}x + b_{x2}) H_n(a_{y2}y + b_{y2}) \\
 &\times \exp\left[-i\left(\frac{Q_{x2}x^2}{2} + \frac{Q_{y2}y^2}{2} + S_{x2}x + S_{y2}y + P_2\right)\right] \\
 &\times \exp\left(i\frac{a'_{x1}{}^2}{2k_{01}} \frac{B_x}{A_x + B_x/q_{x1}}\right) \\
 &\times \left\{\frac{1}{2} \exp\left[-i\left(\frac{-a'_{x1}}{A_x + B_x/q_{x1}}x - b'_{x1}\right)\right.\right. \\
 &\quad \left.+\frac{B_x}{k_{01}} \frac{S_{x1} + G_x + H_x/q_{x1}}{A_x + B_x/q_{x1}} a'_{x1} - \frac{H_x}{k_{01}} a'_{x1}\right] \\
 &\quad \left.+\frac{1}{2} \exp\left[+i\left(\frac{-a'_{x1}}{A_x + B_x/q_{x1}}x - b'_{x1}\right)\right.\right. \\
 &\quad \left.+\frac{B_x}{k_{01}} \frac{S_{x1} + G_x + H_x/q_{x1}}{A_x + B_x/q_{x1}} a'_{x1} - \frac{H_x}{k_{01}} a'_{x1}\right]\right\} \\
 &= A_0 H_m(a_{x2}x + b_{x2}) H_n(a_{y2}y + b_{y2}) \\
 &\times \exp\left[-i\left(\frac{Q_{x2}x^2}{2} + \frac{Q_{y2}y^2}{2} + S_{x2}x + S_{y2}y + P_2\right)\right] \\
 &\times \exp\left(i\frac{a'_{x1}{}^2}{2k_{01}} \frac{B_x}{A_x + B_x/q_{x1}}\right) \cos(a'_{x2}x + b'_{x2}), \quad (39)
 \end{aligned}$$

where S_{x2} and P_2 are given again by Eqs. (20) and (27), respectively, and we have identified the new parameters a'_{x2} and b'_{x2} with the formulas

$$\begin{aligned}
 a'_{x2} &= \frac{a'_{x1}}{A_x + B_x/q_{x1}}, \quad (40) \\
 b'_{x2} &= b'_{x1} - \frac{B_x}{k_{01}} \frac{S_{x1} + G_x + H_x/q_{x1}}{A_x + B_x/q_{x1}} a'_{x1} + \frac{H_x}{k_{01}} a'_{x1}. \quad (41)
 \end{aligned}$$

Thus we have obtained a new set of beam solutions that has the same form as that of our original set of Hermite-Gaussian beams, except that these original beams now have an extra evolving complex off-axis sinusoidal factor and an extra complex phase exponent. It is important to emphasize that these new factors to the Hermite-Gaussian beam solutions are the same as the corresponding terms in the sinusoidal-Gaussian beams that were obtained as direct solutions of the paraxial wave equation. Thus the new phase exponent in Eq. (39) was included previously in Eq. (35) of Ref. 14 (with γ_x set to unity), the coefficient a'_{x2} in Eq. (40) was given as Eq. (30) in the reference, and the term b'_{x2} in Eq. (41) is compatible for aligned systems ($G_x = 0, H_x = 0$) with the previous Eq. (33).

It is, of course, true that similar sinusoidal variations can also be found for the y direction and that sine-function or complex-exponential dependences can be used instead of cosine dependences. Thus we can write our general Hermite-sinusoidal-Gaussian solutions in the form

$$\begin{aligned}
 A_{2,m,n}(x, y) &= A_0 H_m(a_{x2}x + b_{x2}) H_n(a_{y2}y + b_{y2}) \\
 &\times \exp\left[-i\left(\frac{Q_{x2}x^2}{2} + \frac{Q_{y2}y^2}{2} + S_{x2}x + S_{y2}y + P_2\right)\right] \\
 &\times \exp\left(i\frac{a'_{x1}{}^2}{2k_{01}} \frac{B_x}{A_x + B_x/q_{x1}} + i\frac{a'_{y1}{}^2}{2k_{01}} \frac{B_y}{A_y + B_y/q_{y1}}\right) \\
 &\times \left(\begin{matrix} \sin(a'_{x2}x + b'_{x2}) \\ \cos(a'_{x2}x + b'_{x2}) \\ \exp(ia'_{x2}x + ib'_{x2}) \end{matrix}\right) \left(\begin{matrix} \sin(a'_{y2}y + b'_{y2}) \\ \cos(a'_{y2}y + b'_{y2}) \\ \exp(ia'_{y2}y + ib'_{y2}) \end{matrix}\right). \quad (42)
 \end{aligned}$$

The large parentheses in the last row of Eq. (42) represent a possible superposition of the functions that they enclose. Furthermore, if an i is factored out of a'_{x2} and b'_{x2} everywhere, the solutions can be written in terms of ordinary exponential or hyperbolic trigonometric functions, and the hyperbolic functions are expected to be at least as important in practice as the ordinary trigonometric functions shown. More complicated separations involving higher products of trigonometric and hyperbolic and exponential functions could also be readily obtained by using standard trigonometric identities. With this understanding, Eq. (42) is our general form for the Hermite-sinusoidal-Gaussian beams of complex argument in misaligned complex optical systems.

To visualize the shape of the Hermite-sinusoidal-Gaussian beams, we consider the special case $a_{x1} = \sqrt{2}/w_1$, where w_1 is the spot size of the beam at reference plane 1. If we further restrict our attention to the x variation of an on-axis beam, then an example of a beam field at this plane is

$$A_{1,m}(x, y) = A_0 H_m(\sqrt{2}x/w_1) \exp(-x^2/w_1^2) \cosh(a'_{x1}x). \quad (43)$$

With the change of variable $x' = x/w_1$, this field can be written in the normalized form

$$\begin{aligned}
 A_m(x, y) &= \frac{A_{1,m}(x, y)}{A_0} \\
 &= H_m(\sqrt{2}x') \exp(-x'^2) \cosh(a''x'), \quad (44)
 \end{aligned}$$

where the parameter change $a'' = a'_{x1}w_1$ has also been introduced. Finally, the intensity is often of more direct interest than the field, and from Eq. (44) the intensity is

$$\begin{aligned}
 I_m(x, y) &= A_m^2(x, y) \\
 &= H_m^2(\sqrt{2}x') \exp(-2x'^2) \cosh^2(a''x'). \quad (45)
 \end{aligned}$$

Equation (45) is plotted in Fig. 1 for the mode index $m = 4$ and various values of the coefficient a'' . With $a'' = 0$ it is clear from the figure, as from Eq. (45), that the intensity distribution reduces to the familiar H_4 Hermite-Gaussian form. For larger values of a'' , the $\cosh(a''x')$ function acts to concentrate the energy in the outer lobes of the beam. Thus the beam formulas given in this example would be useful for representing the propagation of a field distribution that at some reference plane has its energy concentrated in two widely spaced lobes.

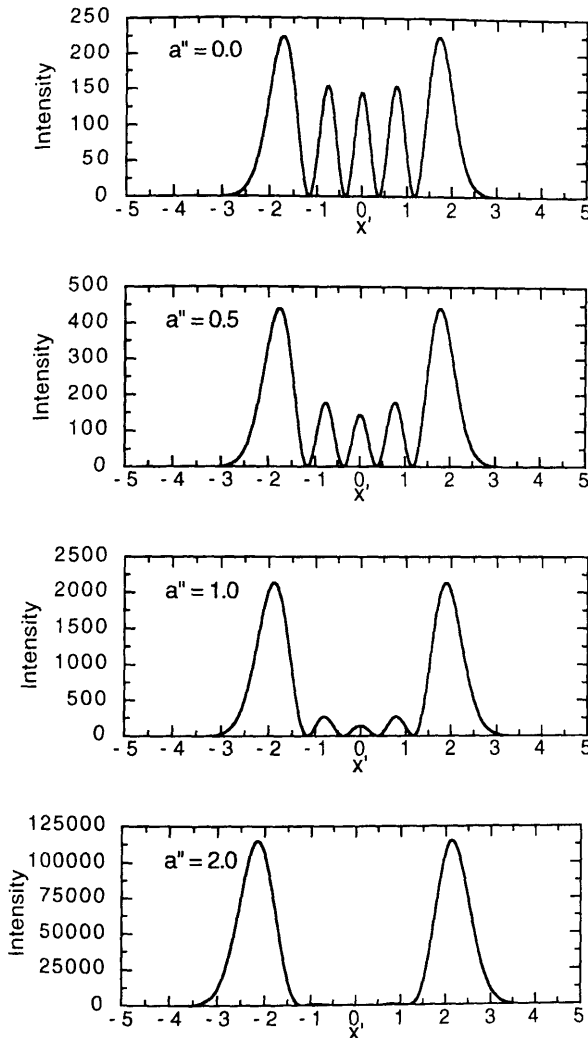


Fig. 1. Intensity plots of $m = 4$ Hermite-sinusoidal-Gaussian beam profiles from Eq. (45). The parameter a'' represents approximately the ratio of the width associated with the Hermite-Gaussian factors to the width associated with the sinusoidal portion of the beam, and in the plots a'' takes on the values 0.0, 0.5, 1.0, and 2.0.

4. SINUSOIDAL-GAUSSIAN EXPANSIONS

In the above analysis we have described a new way to obtain the sinusoidal-Gaussian beams that had been found recently as solutions of the paraxial wave equation, and a new and more general class of Hermite-sinusoidal-Gaussian beams has also been developed. For the remainder of this study, we focus on some of the practical applications of the sinusoidal-Gaussian beam solutions. It is not as obvious with the sine functions as with the more standard Hermite polynomials how these solutions can be used as a basis for the expansion of an arbitrary electromagnetic beam field, and such expansions are the subject of this discussion.

Classification of the beams described by Eq. (42) is somewhat different from the usual classification of conventional polynomial-Gaussian beams. In the polynomial- or Bessel-Gaussian beams, the governing beam formulas typically have one or more indices in addition to the other z -dependent beam parameters, and these indices are integer valued to ensure beam confine-

ment. For these beams questions of orthogonality, completeness, and beam expansions are addressed, assuming that the beam parameters have the same values for a particular family of beams and the beams within that family differ from each other only in having differing values of the integer indices. The beam parameters of a family, spot size for example, are each assigned a single arbitrary value at the reference plane of an expansion.

The sinusoidal-Gaussian beams, on the other hand, are confined for all values of the arguments of the sine functions, and thus they are not automatically associated with discrete indices. Sinusoidal functions of continuous argument become a set of discrete modes only when boundary conditions are imposed, and when that is done, integer indices are typically inserted within the arguments of the functions. In the case of freely propagating sinusoidal-Gaussian beams, there are no intrinsic boundaries in the transverse direction. Any transverse constraints that might lead to such indices must be imposed in some other way. For purposes of our beam expansions, it is necessary only that an expansion interval be chosen that is larger than the beam diameter at the expansion plane. This diameter must encompass both the actual beam and the off-axis Gaussian factor that has been chosen for its lowest-order representation. In this case the resulting family of complex sinusoidal factors is exactly the usual basis set for an ordinary Fourier-series expansion.

To see how an arbitrary field can be expanded in terms of sinusoidal-Gaussian beams, we first consider the general complex Fourier series. A function $f(x)$ can be expanded over the interval from $-L/2$ to $L/2$ in the series¹⁶

$$f(x) = \sum_n a_n \exp(i2\pi nx/L), \quad (46)$$

where the expansion coefficients are given by

$$a_n = \int_{-L/2}^{L/2} f(x) \exp(-i2\pi nx/L) dx. \quad (47)$$

As an illustration, we will express the complex expansion coefficients in terms of their magnitude and phase:

$$a_n = |a_n| \exp(i\phi_n). \quad (48)$$

With this substitution Eq. (46) is

$$\begin{aligned} f(x) &= \sum_n |a_n| \exp[i(2\pi nx/L + \phi_n)] \\ &= \sum_n |a_n| [\cos(2\pi nx/L + \phi_n) \\ &\quad + i \sin(2\pi nx/L + \phi_n)]. \end{aligned} \quad (49)$$

These trigonometric functions are in just the form given in Eq. (42) if one makes the reference plane identifications $a'_{x1} = 2\pi n/L$ and $b'_{x1} = \phi_n$.

With this background information on Fourier series, we can now indicate a procedure for expanding an arbitrary field distribution in a series of sinusoidal-Gaussian beams. The first step is to select the Gaussian beam parameters to be used in the expansion. For this purpose the off-axis Gaussian factor should represent an approximate fit to the beam being represented. The assumed

Gaussian factor should then be divided into the given beam, and the quotient should be expanded in a Fourier series. As noted above, it is necessary only that the expansion interval be large compared with the beam and its Gaussian approximation. Each term in such an expansion can be propagated analytically by using the formulas of Section 3.

It is, of course, also possible to use an expansion interval that is much larger than the beam diameter. This would, however, require the inclusion of a larger number of terms in the Fourier expansion. With an infinite expansion interval, the Fourier-series representation evolves into a continuous complex Fourier transform. Such a transform is never required in this method though, because the finite Gaussian beam factor width renders any larger expansion region unnecessary. Quite complicated beam profiles should be representable with only a few terms in the expansion.

To illustrate some of the concepts discussed above, we will briefly sketch a specific example. Consider the propagation of a conventional TEM_{m,n} Hermite-Gaussian beam through the hypothetical optical system shown in Fig. 2. In this system the lenses have a focal length of *f*, and the distance between each of the lenses and the transmission filter is also *f*. The transmission filter in this case is a thin element that has the amplitude transfer characteristic

$$T(x) = |\sin(2\pi x/L)|, \tag{50}$$

which is shown in Fig. 3. For this purpose we wish to represent the filter characteristic in a Fourier series, and thus Eq. (50) is written as¹⁷

$$T(x) = \frac{2}{\pi} - \frac{4}{\pi} \left[\frac{\cos(4\pi x/L)}{1 \times 3} + \frac{\cos(8\pi x/L)}{3 \times 5} + \frac{\cos(12\pi x/L)}{5 \times 7} + \dots \right]. \tag{51}$$

Similar methods would also be applicable with phase, rather than amplitude, filters.¹⁸

The first step in analyzing the transfer of a Hermite-Gaussian beam through the system shown in Fig. 1 is to propagate the beam from the input plane to the transmission filter. In this example the beam matrix for this purpose is

$$M_1 = \begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} = \begin{bmatrix} 0 & f \\ -1/f & 1 \end{bmatrix}. \tag{52}$$

The propagation methods for this region are already well known, and the initial Hermite-Gaussian beam will still be in the Hermite-Gaussian beam form. Next, to propagate the resulting beam through the transmission filter, the beam at the filter must be multiplied by *T(x)* as given in Eq. (50). Clearly, the result in this case will be a set of beams, each of which is a Hermite-sinusoidal-Gaussian in the form of Eq. (42). Thus Eq. (42) together with the propagation formulas presented in Sections 2 and 3 allow for the further propagation of the filtered beam. For the case shown in this example, the propagation to the output plane is governed by the matrix

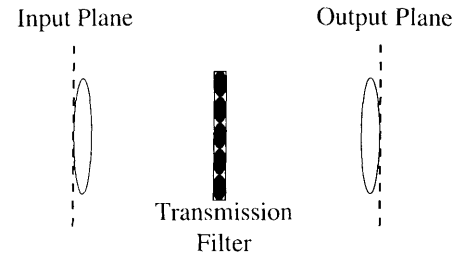


Fig. 2. Example of an optical system containing a periodic transmission filter, as discussed in Section 4. In this system the filter is centered between two identical lenses.

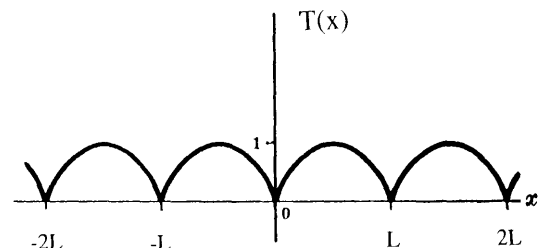


Fig. 3. Transmission characteristic $|\sin(xL/p)|$ for the amplitude filter in the system shown in Fig. 2.

$$M_2 = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & f \\ -1/f & 0 \end{bmatrix}. \tag{53}$$

At the output plane one can find the resultant field by adding up the individual complex Hermite-sinusoidal-Gaussian beam components. This general procedure would be applicable for any system containing a filter for which the transmission characteristic can be represented by a Fourier series. The method could be extended to systems with multiple filters, and in this case each component resulting from the expansion at one filter would itself need to be expanded on transmission through the next filter.

It may be noted here that the propagation of beams that have been transmitted through filters represented by their Fourier expansions is well known in studies of spatial modulation.¹⁸ In earlier treatments, however, the propagation of the filtered beams has been based on diffraction integral methods. With the procedure reported here, input Hermite-Gaussian beams are expanded by the Fourier-represented transmission filter into a set of components that can be propagated analytically. If the number of components needed for this expansion is not large, this procedure is an efficient alternative to the brute force diffraction calculations.

5. DISCUSSION

General Hermite-sinusoidal-Gaussian beam solutions of the paraxial wave equation have been developed for the propagation of off-axis electromagnetic waves through misaligned complex optical systems. These solutions include as special cases the Hermite-Gaussian beams of complex argument that have been studied previously and also the sinusoidal-Gaussian beams that have recently been reported. The generality of these new beams provides added flexibility as one seeks the simplest representation for a given propagation or resonator application.

In seeking to expand a field distribution in analytically propagatable beam functions, one would be led to consider beams that individually are as similar as possible to the field being expanded. This choice would tend to minimize the number of terms required for the expansion. For example, a field with most of its power in outer lobes might be well represented by a few higher-order Hermite-Gaussian functions, whereas a field emerging from a waveguide might couple most efficiently to a few sinusoidal-Gaussian functions.

Other considerations in choosing an expansion set might include mathematical simplicity or familiarity. Expansions in terms of Hermite-Gaussian solutions require some juggling of special functions, whereas sinusoidal-Gaussian expansions reduce to very basic Fourier series. Several aspects of field expansions in sinusoidal-Gaussian functions have been treated in Section 4. In particular, it has been indicated how a general sinusoidal-Gaussian beam solution can be reinterpreted as a set of discrete functions for use in such expansions.

The sinusoidal-Gaussian field solutions also lead to the possibility of new optical elements that do not have a simple *ABCD* matrix representation. For example, if a Gaussian beam is incident on an aperture having a cosh amplitude transmission function, one obtains a cosh-Gaussian beam, and such a beam can be propagated analytically through further *ABCD* elements. If the transmission element is not immediately in the form of a sinusoidal or hyperbolic-sinusoidal function, it can always be expanded in a series of such functions. In the example of Section 4, it was shown how an incident Hermite-Gaussian beam can be propagated through an optical system that contains a periodic amplitude transmission filter, and similar methods are applicable for phase filters. The transmission function of the filter is expanded in a Fourier series, and the output from the filter can be interpreted as a set of Hermite-sinusoidal-Gaussian beams. Each of these component beams can then be propagated analytically through a wide variety of optical elements by using the formulas developed in this study.

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