Hermitian and quaternionic Hermitian structures on tangent bundles

Rui Albuquerque*
rpa@dmat.uevora.pt

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Abstract

We review the theory of quaternionic Kähler and hyperkähler structures. Then we consider the tangent bundle of a Riemannian manifold M with a metric connection D (with torsion) and with its well estabilished canonical complex structure. With an extra almost Hermitian structure on M it is possible to find a quaternionic Hermitian structure on TM, which is quaternionic Kähler if, and only if, D is flat and torsion free. We also review the symplectic nature of TM. Finally a proper S^3-bundle of complex structures is introduced, expanding to TM the well known twistor bundle of M.

Key Words: torsion, quaternionic, Hermitian, Kähler, symplectic.

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1 Introduction

The subject of quaternionic Hermitian manifolds still conceals many mysteries for the working geometer. This article starts with a recreation of the main definitions regarding quaternionic Kähler structures and their almost immediate properties, pertaining holonomy reduction, which are used later in a particular context.

*Departamento de Matemática da Universidade de Évora and Centro de Investigação em Matemática e Aplicações (CIMA), Rua Romão Ramalho, 59, 7000 Évora, Portugal.

We develop the theory of complex and quaternionic structures on the tangent bundle of a Riemannian manifold M endowed with a metric connection D. It is well known by now how to define an orthogonal almost complex structure I on TM departing from such condition, a construction due to P. Dombrowsky. Such structures have also been studied in a more analytic perspective in [16]. Now, if we assume furthermore that the base manifold is almost Hermitian and take any compatible almost Hermitian D, then a sourceful of structures arise on the tangent bundle. We may consider new almost complex structures, orthogonal with respect to the naturally induced metric, as the above I, and in a way orthogonal to I.

Then TM also carries Hermitian and quaternionic Hermitian structures, and this work concentrates in deciding which conditions on the base space M must be satisfied in order to say wether they are integrable or symplectic and, respectively, quaternionic Kähler.

Our techniques involve the determination of the Levi-Civita connection of TM in order to describe the possible holonomy reductions. We hope this is important for other developments of the theory. Our results are confluent with some constructions in [5] and the study of quaternionic structures through geometry with torsion is indeed interesting, cf. [9].

2 Quaternionic Kähler structures

2.1 Definitions

By a quaternionic Hermitian module it is understood a real Euclidian vector space of dimension $4n$ together with a free action by isometries of the Lie group $Sp(1)$ of unit quaternions. This action is assumed to be on the right, as such is the canonical case of $\mathbb{H}^n$. On the Euclidian vector space we also have the left action of $SO(4n)$, which hence contains a copy of the unit quaternions. The automorphisms of the quaternionic Hermitian module constitute another subgroup $Sp(n) \subset SO(4n)$. An isometry $g \in Sp(n)$ if, and only if, $g(vw) = g(v)w$ for any vector v and any $w \in \mathbb{H}$. Hence there is a third resulting subgroup which is the product $Sp(n)Sp(1)$ and which we denote by $G(n)$. Since it is known that the fundamental group of $G(n)$ is $\mathbb{Z}_2$, while $Sp(n)$ is simplyconnected ([8]), we have $G(n) = Sp(n) \times_{\mathbb{Z}_2} Sp(1)$ due to the diagonal action of $\{\pm\mathrm{Id}\}$.

An oriented Riemannian $4n$-manifold M is said to be a *quaternionic Kähler* if its holonomy is inside $G(n)$, with an exception in the case $n = 1$ – cf. section 2.4. If such is the case, then there is a smooth quaternionic Hermitian structure on M, i.e. each tangent space T_xM admits a quaternionic Hermitian module structure smoothly varying with $x \in M$. The same is to say M admits a $G(n)$-structure.

Let us reflect upon the implications of the above condition. If the manifold has a $G(n)$-structure this means its frame bundle reduces to a principal $G(n)$-bundle, say P. Locally there exist quaternionic Hermitian frames[1] and thus there exists a local lift to

[1]These are vector sets $\{v_1, \ldots, v_n\}$ which generate T_xM under right multiplication by scalars in $\mathbb{H}$.

an $Sp(n)\times Sp(1)$-structure $\tilde{P}$. The real simple Lie group $Sp(n)$ is the same as $U(2n)\cap Sp(2n,\mathbb{C})$ (analyze the Lie algebras or simply cf. [8]) and hence it has an irreducible representation in $\mathbb{C}^{2n}$, giving rise, locally, to two Hermitian vector bundles:

$$E=\tilde{P}\times_{Sp(n)\times Sp(1)}\mathbb{C}^{2n}\qquad\text{and}\qquad H=\tilde{P}\times_{Sp(n)\times Sp(1)}\mathbb{C}^{2}\tag{2.1}$$

defined on every sufficiently small open subsets in M. One notes $TM\otimes_{\mathbb{R}}\mathbb{C}=E\otimes_{\mathbb{C}}H$, associated to P, in spite of E,H being not, in general, globally defined. Such is known as the E,H-formalism[2] (cf. [13]).

Recall the metric and the orthogonal complex structure $i1$ in $\mathbb{C}^2$ induce a symplectic 2-form ω_H. Then each $A\in\mathfrak{sp}(1)=\mathfrak{su}(2)=\mathfrak{so}(3)$ is determined by the symmetric 2-product $\omega_H(A\cdot,\cdot)$. In other words, the unit quaternions have Lie algebra (the purely imaginary part of $\mathbb{H}$) a real subspace of the complex vector space $S^2\mathbb{C}^2$, the symmetric complex bilinear forms of $\mathbb{C}^2$. For instance, the unit quaternions $a_11+a_2I+a_3J+a_4K,\ (a_1,\ldots,a_4)\in S^3$ may be represented by taking

$$I=\begin{bmatrix}0&1\\-1&0\end{bmatrix},\qquad J=\begin{bmatrix}i&0\\0&-i\end{bmatrix},\qquad K=IJ=-JI=\begin{bmatrix}0&-i\\-i&0\end{bmatrix}.\tag{2.2}$$

Indeed, $I,J,K\in\mathfrak{sp}(1)$.

As shown, a quaternionic Hermitian structure on a Riemannian manifold does not depend on the complex structure in which we decompose $\mathbb{H}$, but rather on having a real 3-dimensional vector subbundle of $\operatorname{End}TM$ over M, usually denoted Q, locally spanned by three anti-commuting orthogonal almost complex structures ($Q\otimes_{\mathbb{R}}\mathbb{C}=S^2H$). Reciprocally, this induces a $\mathfrak{sp}(1)\subset\mathfrak{so}(4n)$ associated smooth vector subbundle; hence, by the exponential map, a $Sp(1)$ action on each T_xM smoothly varying with x and therefore a quaternionic Hermitian structure on M. We have proved the known result that a $G(n)$ structure is equivalently given by a Q vector bundle as above.

Now the holonomy condition required for a quaternionic Kähler manifold corresponds, following the general theory of connections, to the $G(n)$-structure being parallel. The bundle of endomorphisms associated to $\mathfrak{g}=\mathfrak{sp}(n)\oplus\mathfrak{sp}(1)$ is closed under Levi-Civita covariant differentiation if, and only if, the same happens with the one associated with $\mathfrak{sp}(1)$, i.e. the rank 3 real vector bundle Q. Indeed, notice $\mathfrak{sp}(n)$ is the centralizer of $\mathfrak{sp}(1)$ in $\mathfrak{so}(4n)$ and we have

$$0=\nabla[\mathfrak{sp}(n),\mathfrak{sp}(1)]=[\nabla\mathfrak{sp}(n),\mathfrak{sp}(1)]+[\mathfrak{sp}(n),\nabla\mathfrak{sp}(1)].$$

Thus $\nabla\mathfrak{sp}(n)\subset\mathfrak{sp}(n)$ if, and only if, $\nabla\mathfrak{sp}(1)\subset\mathfrak{sp}(1)$.

Proposition 2.1 (cf. [13]). *An oriented Riemannian manifold M is quaternionic Kähler if, and only if, there exists a parallel vector subbundle $Q\subset\operatorname{End}TM$ locally spanned by three anti-commuting orthogonal almost complex structures.*

Their existence is proved by the methods in the appendix.

[2]Nothing as this happens in the geometry of a single almost complex structure, because $GL(1,\mathbb{C})\subset GL(n,\mathbb{C})$.

As we may check easily, if $q=(I,J,K)$ denotes a *quaternionic triple*, i.e. a local basis of Q of anti-commuting orthogonal almost complex structures, then

$$\nabla q = q\alpha + L \tag{2.3}$$

with $\alpha\in\Omega^1(\mathfrak{sp}(1))$ and $L_I,L_J,L_K\in\Omega^1(Q^\perp)$ (this is the orthogonal in $\mathfrak{so}(TM)$). Notice α is just a skew-symmetric matrix of 1-forms. The quaternionic Kähler condition is thus expressed by the equation $L=0$.

There is also another interesting invariant: any two quaternionic triples q,q' defined on open subsets U,U', respectively, and defining the same structure Q are related by a matrix function $a_{UU'}:U\cap U'\rightarrow SO(3)$, since any I,J,K are pairwise orthogonal and with norm $\sqrt{4n}$. Then in defining the 2-form $\omega_I(X,Y)=\langle IX,Y\rangle$ and ω_J,ω_K analogously, we get a well defined 4-form easily seen not to depend on the choice of q

$$\Omega=\omega_I\wedge\omega_I+\omega_J\wedge\omega_J+\omega_K\wedge\omega_K. \tag{2.4}$$

A straightforward computation yields, in the quaternionic Kähler case, $\mathrm{d}\Omega=0$. In general, we find $\mathrm{d}\Omega=\sum_i\omega_i\wedge\lambda_i$ with the given frame q, where $\omega_1=\omega_I,\ \lambda_1=\underset{X,Y,Z}{\circlearrowright\!\!\!\!\!\bigoplus}\langle L_I(X)Y,Z\rangle$, etc.

Finally let us recall a third approach to $G(n)$-structures. It is known that $G(n)$ is the set of isometries of a $4n$-dimensional Euclidian vector space for which a non-degenerate 4-form Ω defined by (2.4) remains invariant (cf. appendix). By a fundamental theorem of Riemannian geometry, the holonomy reduces to $G(n)$ if, and only if, $\nabla\Omega=0$. And it was proved in [15] that, when $n>2$, the equation $\mathrm{d}\Omega=0$ is also a sufficient condition for $G(n)$-holonomy.

2.2 Topology

There is a topological invariant of a quaternionic Hermitian structure which partly measures the obstruction to having globally defined three orthogonal almost complex structures. First notice we have a cohomology sequence associated to

$$\begin{array}{ccccccccc} 1 & \rightarrow & \mathbb{Z}_2 & \rightarrow & Sp(n)\times Sp(1) & \rightarrow & G(n) & \rightarrow & 1\\ & & & & \downarrow \mathrm{pr} & & \downarrow \mathrm{pr}' & & \\ 1 & \rightarrow & \mathbb{Z}_2 & \rightarrow & Sp(1) & \rightarrow & SO(3) & \rightarrow & 1 \end{array} \tag{2.5}$$

(there exists a projection pr'). A quaternionic Hermitian structure $P\in H^1(M,G(n))$, a principal $G(n)$-bundle over M, lifts to a global principal $Sp(n)\times Sp(1)$-bundle if, and only, if $\delta(P)$ vanishes. The coboundary homomorphism $\delta:H^1(M,G(n))\rightarrow H^2(M,\mathbb{Z}_2)$ follows from the long exact sequence associated to (2.5) as a sequence of sheaves of germs of group-valued smooth functions. Recall the second Stiefel-Whitney class $w_2(Q)$ corresponds with the obstruction on lifting the $SO(3)$-structure of $Q=\mathrm{pr}'(P)$ to an $Sp(1)$-structure. Moreover, any $Q\in H^1(M,Sp(1))$ raises to a structure P, as explained above through equations (2.1,2.2). We have thus proved $\delta(P)=w_2(Q)$. It measures the existence of E and H globally.

The picture may be resumed in the following way. Since any two quaternionic triples q, q' defined on open subsets U, U', respectively, are related by a matrix function $a_{UU'} : U \cap U' \to SO(3)$, a given family of quaternionic triples on an open covering of M gives a cocycle $Q \in H^1(M, SO(3))$; which arrives from a cocycle in $H^1(M, Sp(1))$ if, and only if, $\delta(P) = 0$.

2.3 Hyperkähler and locally hyperkähler

A given Riemannian holonomy is called *hyperkähler* if it reduces from $SO(4n)$ to $Sp(n) \subset G(n)$. In this case the existence of a covering of M by local quaternionic frames with transition functions in $Sp(n)$ only, is implied from the start (in particular $\delta(P) = 0$). From this we may construct a global quaternionic triple I, J, K and we observe that $\mathfrak{sp}(n) = \mathfrak{u}(2n, I) \cap \mathfrak{u}(2n, J)$ (a straightforward computation). Now the equation for holonomy reduction $\nabla \mathfrak{sp}(n) \subset \mathfrak{sp}(n)$ implies reduction to the unitary Lie algebra or simply $\nabla I \in \mathfrak{u}(2n, I)$ — which combined with $I^2 = -1$ gives $\nabla I = 0$. The same must hold for J. Reciprocally, from $\nabla I = \nabla J = 0$ we arrive to hyperkähler holonomy.

As it is well known, the condition is equivalent to the metric on M being Kähler with respect to each almost complex structure.

Some authors immediately attribute the name hyperkähler to a Riemannian manifold with a global quaternionic triple $q = (I, J, K)$ and such that all $\nabla I = \nabla J = \nabla K = 0$ (cf. [6]). Of course one of the three equations is superfluous.

The term *locally hyperkähler* is reserved for the case when only the reduced holonomy group is inside $Sp(n)$.

2.4 In dimension 4

In 4 real dimensions we have $Sp(1)Sp(1) = SO(4)$. Hence a Riemannian structure on an oriented manifold M is the same as a quaternionic Hermitian structure.

Every oriented Riemannian 4-manifold M has a unique parallel quaternionic Hermitian structure, since any triple I, J, K is identified to an orthonormal basis of the bundle Λ^2_+ of self-dual two forms and since $\nabla * = * \nabla$. If we select a vector field U with $\|U\| = 1$, then the quaternionic Hermitian module structure on TM, with $X_i = \lambda_i U + A_i,\ A_i \in U^\perp,\ i = 1, 2,$ is well known to be given by

$$X_1 \cdot X_2 = (\lambda_1\lambda_2 - \langle A_1, A_2\rangle)U + \lambda_1 A_2 + \lambda_2 A_1 + A_1 \times A_2$$

where $\langle A_1 \times A_2, A_3\rangle = \mathrm{vol}(U, A_1, A_2, A_3)$. Notice $\|X \cdot Y\| = \|X\|\|Y\|$. Then any almost complex structure $I = v\cdot : TM \to TM$ with $v \in U^\perp$, $\|v\| = 1$ and we easily find $\omega_I = U^b \wedge v^b + *(U^b \wedge v^b)$. This picture has led to the construction in [3] of G_2-structures on the 7-manifold which is the unit sphere tangent bundle of M.

As it was pointed in [13], we have a lift of a smooth quaternionic Hermitian structure on M to an $Sp(1) \times Sp(1)$-structure if and only if, M is spin. Hence, in this case, $w_2(Q) = w_2(M) = 0$.

In view of the above, we finally recall the exception in the definition of quaternionic Kähler 4-manifold: a Riemannian structure which is self dual and has the same curvature properties of any other quaternionic Kähler structure, namely it is Einstein.

For hyperkähler manifolds we have further strictness: such a 4-manifold is Ricci flat and has flat $\wedge^{\pm}$ bundles. This is a consequence of having three parallel self-dual 2-forms and hence $R* = *R$, from which $Ric = 0$ follows. If locally there exists one parallel unit vector field U, then the hyperkähler manifold is itself flat.

3 TM and its Levi-Civita connection

Let M be any Riemannian manifold and D any linear metric connection on M.

There exists a canonical *vertical* vector field ξ defined on the manifold TM:

$$\xi_v = v, \qquad \forall v \in TM, \tag{3.1}$$

under the identification of π^*TM with $\mathcal{V} = \ker(\mathrm{d}\pi : TTM \to \pi^*TM)$, where $\pi : TM \to M$ is the canonical projection. The connection D induces a splitting $TTM = \mathcal{H}^D \oplus \mathcal{V}$. Moreover, the tautological section ξ carries all the information to produce the splitting. This has already been thoroughly explained in the context of twistor bundles (cf. [2, 12]) or of the sphere tangent bundle (cf. [3]), where a similar canonical section ξ was defined.

In sum, it follows from the theory that $X \in \mathcal{H}^D \Leftrightarrow (\pi^*D)_X\xi = 0$. Essentially, one proves that ξ varies exactly on vertical directions.

Furthermore, for a given vector field $X \in \Omega(TM) = \mathfrak{X}_M$ and vector $v \in T_xM$, the vertical part of $\mathrm{d}X(v)$ is precisely D_vX. The theory gives us a projection map $\pi^*D.\xi$ and thus $(\mathrm{d}X(v))^v = \pi^*D_{\mathrm{d}X(v)}\xi = (X^*\pi^*D)_vX^*\xi = D_vX$.

Now, we may endow TM with a Riemaniann structure and an induced metric connection denoted D^*. Naturally, the metric is defined via the pull-back metric on $\pi^*TM = \mathcal{V}$ and the isometry $\mathrm{d}\pi_| : \mathcal{H}^D \to \pi^*TM$. The decomposition into horizontals and verticals is orthogonal and the metric connection D^*, in fact given by π^*D, preserves this splitting.

Let $R^* = \pi^*R^D = R^{\pi^*D}$ denote the curvature tensor of D^*. We have $R^*\xi \in \Omega^2(\mathcal{V})$. Notice we use $\cdot^{\,v}$, $\cdot^{\,h}$ to denote the vertical and horizontal parts, respectively, of a TTM valued tensor, but the identity $X^h = \mathrm{d}\pi(X)$ may appear as well.

Theorem 3.1. *The Levi-Civita connection ∇ of TM is given by*

$$\nabla_XY = D^*_XY - \frac{1}{2}R^*_{X,Y}\xi + A_XY + \tau_XY \tag{3.2}$$

where A, τ are $\mathcal{H}^D$-valued tensors defined by

$$\langle A_XY, Z^h\rangle = \frac{1}{2}\langle R^*_{X^h,Z^h}\xi, Y^v\rangle + \frac{1}{2}\langle R^*_{Y^h,Z^h}\xi, X^v\rangle \tag{3.3}$$

and

$$\tau(X,Y,Z) = \langle \tau_XY, Z^h\rangle = \frac{1}{2}\big(T(Y,X,Z) - T(Z,X,Y) + T(Y,Z,X)\big), \tag{3.4}$$

with $T(X,Y,Z) = \langle \pi^*T^D(X,Y), Z\rangle$, for any vector fields X, Y, Z over TM.

Proof. Let us first see the horizontal part of the torsion:

$$\begin{aligned} \mathrm{d}\pi(T^\nabla(X,Y)) &= D^*_X Y^h + A_X Y + \tau_X Y - D^*_Y X^h - A_Y X - \tau_Y X - \mathrm{d}\pi[X,Y] \\ &= \pi^* T^D(X,Y) + \tau_X Y - \tau_Y X, \end{aligned}$$

since this is how the torsion tensor of M lifts to π^*TM and since A is symmetric. Now we check the vertical part.

$$\begin{aligned} (T^\nabla(X,Y))^v &= D^*_X Y^v - \frac{1}{2} R^*_{X,Y}\xi - D^*_Y X^v + \frac{1}{2} R^*_{Y,X}\xi - [X,Y]^v \\ &= D^*_X D^*_Y \xi - R^*_{X,Y}\xi - D^*_Y D^*_X \xi - D^*_{[X,Y]}\xi \;=\; 0. \end{aligned}$$

∇ is a metric connection if, and only if, the difference with D^* is skew-adjoint. This is an easy straightforward computation: on one hand

$$\langle(\nabla - D^*)_X Y, Z\rangle = -\frac{1}{2}\langle R^*_{X,Y}\xi, Z^v\rangle + \frac{1}{2}\langle R^*_{X^h,Z^h}\xi, Y^v\rangle + \frac{1}{2}\langle R^*_{Y^h,Z^h}\xi, X^v\rangle + \tau(X,Y,Z).$$

On the other hand,

$$\langle(\nabla - D^*)_X Z, Y\rangle = -\frac{1}{2}\langle R^*_{X,Z}\xi, Y^v\rangle + \frac{1}{2}\langle R^*_{X^h,Y^h}\xi, Z^v\rangle + \frac{1}{2}\langle R^*_{Z^h,Y^h}\xi, X^v\rangle + \tau(X,Z,Y).$$

hence the condition is expressed simply by $\tau(X,Y,Z) = -\tau(X,Z,Y)$. This, together with $\pi^*T^D(X,Y) + \tau_X Y - \tau_Y X$, determines τ uniquely as the form given by (3.4). ■

We remark that from the formula it is clear that $\mathcal{H}^D$ corresponds with an integrable distribuition if, and only if, the Riemmannian manifold M is flat. Indeed, the vertical part of $[X,Y] = \nabla_X Y - \nabla_Y X$, for any pair of horizontal vector fields, is $R^*_{X,Y}\xi$.

Notice $R^*_{X,Y}\xi$ and $\tau(X,Y,Z)$ are null if one of the directions X,Y,Z is vertical. With $A_X Y$ the same happens if both X,Y are vertical or horizontal.

It is important to understand when the tensor τ vanishes. By a result of É. Cartan, cf. [1], it is known that the space of torsion tensors $\wedge^2 TM \otimes TM$ of a metric connection decomposes into irreducible subspaces like

$$\mathcal{A} \oplus \wedge^3 TM \oplus TM, \tag{3.5}$$

where $\wedge^3$ is the one for which $\langle T(X,Y), Z\rangle$ is completely skew-symmetric and where TM is the subspace a vectorial type torsions, i.e. for which there exists $V \in \mathfrak{X}_M$ such that $T(X,Y) = \langle V, X\rangle Y - \langle V, Y\rangle X$. $\mathcal{A}$ is an invariant subspace orthogonal to those two. We have the following result:

Proposition 3.1. $\tau = 0$ *if, and only if,* $T^D = 0$.

Proof. If $\tau = 0$, then $T(Y,X,Z) = T(Z,X,Y) + T(Z,Y,X)$; by the symmetries in X,Y this tensor vanishes. ■

3.1 A complex structure on TM

Let $\theta \in \operatorname{End} TTM$ be the map which sends $\mathcal{H}^D$ isomorphically onto $\mathcal{V}$, in view of each subspace T_vTM being identified with $T_{\pi(v)}M \oplus T_{\pi(v)}M$. We see θ as an endomorphism, imposing $\theta\mathcal{V} = 0$. With respect to the metric we defined above on TM the adjoint of θ verifies $\theta^t(\mathcal{V}) = \mathcal{H}^D,\ \theta^t(\mathcal{H}^D) = 0$. The following map

$$I = \theta^t - \theta \tag{3.6}$$

is a compatible almost complex structure on TM. Indeed, $\theta^t\theta = 1_{\mathcal{H}^D} \oplus 0,\ \theta\theta^t = 0 \oplus 1_{\mathcal{V}}$. For any metric connection, in general, we easily deduce $\nabla\theta^t = (\nabla\theta)^t$ and, for any compatible almost complex structure I,

$$\nabla_X\omega_I\,(Y,Z) = \langle(\nabla_X I)Y, Z\rangle. \tag{3.7}$$

For the moment we have $D^*\theta = 0$ and hence $D^*I = 0$.

Theorem 3.2. *(i) The following two assertions are equivalent: (TM, I) is a complex manifold; D is torsion free and flat. If any of these occur, then M is a flat Riemannian manifold and TM is Kähler flat.*
(ii) ω_I is closed if, and only if, D is torsion free.

Proof. On any Riemannian manifold a compatible almost complex structure is integrable if, and only if, $\nabla_u v$ is in the $+i$-eigenbundle of I for all u, v in this same eigenbundle (cf. [14]). The sufficiency of this condition is trivial to prove: if $\nabla_u v$ is in the $+i$-eigenbundle, then the same is true for $[u, v] = \nabla_u v - \nabla_v u$. The necessity comes from $\langle[u, v], w\rangle = 0$ implying $\langle\nabla_u v, w\rangle$ to be both a skew- and symmetric 3-*tensor*.

Let us prove (i). In our case, $Iu = iu$ is equivalent to $u = u^h + i\theta u^h$, i.e. the $+i$ eigenbundle $T'TM \simeq \mathcal{H}^D \otimes \mathbb{C}$. Indeed $(\theta^t - \theta)u = -\theta u^h + iu^h = iu$ and the dimensions agree. So we may take $u = X + i\theta X,\ v = Y + i\theta Y$, with $X, Y \in \mathcal{H}^D$ real horizontal vector fields. By (3.2)

$$\begin{aligned}\nabla_u v &= D^*_u v - \frac{1}{2}R^*_{u,v}\xi + A_u v + \tau_u v \\ &= D^*_u v - \frac{1}{2}R^*_{X,Y}\xi + i(A_X\theta Y + A_{\theta X}Y) + \tau_X Y.\end{aligned}$$

Now the condition resumes to

$$i\theta(i(A_X\theta Y + A_{\theta X}Y) + \tau_X Y) = -\frac{1}{2}R^*_{X,Y}\xi.$$

The imaginary part of this gives $\tau = 0$ or $T^D = 0$ by corollary 3.1. For the real part, doing the inner product with a vertical vector gives an equation which we may further simplify by the first Bianchi identity (D is torsion free). It yields the vanishing of the curvature tensor R^D. Therefore $\nabla I = D^*I = 0$ and the result follows.

Now we prove (ii) (which implies the second part of (i)). Consider a unitary frame on TM $e_1,\ldots,e_m,\theta e_1,\ldots,\theta e_m$ induced from an orthonormal frame on M. Let $e_{i+m}=\theta e_i$. By (3.7) and $[R^*_{e_i,\cdot}\xi,\theta]=0$, we have

$$\begin{aligned}\mathrm{d}\omega_I &= \sum_{i=1}^{2m}\nabla_i\omega_I\wedge e^i = \frac{1}{2}\sum_{i,j,k=1}^{2m}\langle\nabla_i(\theta^t-\theta)\,e_j,e_k\rangle e^{ijk} = -\sum\langle(\nabla_i\theta)e_j,e_k\rangle e^{ijk}\\ &= -\sum\langle(A+\tau)_{e_i}\theta e_j-\theta(A+\tau)_{e_i}e_j,e_k\rangle e^{ijk}.\end{aligned}$$

Since A is symmetric and τ_{ijk} vanishes when i,j or k is vertical, we get $\mathrm{d}\omega_I=$

$$\begin{aligned}&=-\sum_{i,j,k=1}^{2m}\langle A_{e_i}\theta e_j-\theta\tau_{e_i}e_j,e_k\rangle e^{ijk} = \sum_{i,j,k=1}^{m}-\frac{1}{2}\langle R^*_{ik}\xi,\theta e_j\rangle e^{ijk}+\tau_{ijk}e^{ijk+m} =\\ &=-\sum_{i<j<k}^{m}(\langle R^*_{ik}\xi,\theta e_j\rangle-\langle R^*_{jk}\xi,\theta e_i\rangle-\langle R^*_{ij}\xi,\theta e_k\rangle)e^{ijk}+\sum_{i<j}^{m}\sum_{k=1}^{m}(\tau_{ijk}-\tau_{jik})e^{ijk+m}.\end{aligned}$$

Since the skew-symmetric part in X,Y of $\tau(X,Y,Z)$ is the torsion of D, up to a constant, we must have 0 torsion and thence, by the Bianchi identity, the rest of $\mathrm{d}\omega_I$ vanishes as well. ∎

We remark the equivalence in part (i) of the theorem is due to P. Dombrowski, cf. [7], seemingly the first to discover and study the structure I.

Notice ω_I over TM looks very much the same as the natural closed symplectic structure on the co-tangent bundle T^*M of any smooth manifold. Up to the metric-induced isomorphism, we have proved these two are the same if, and only if, we consider the Levi-Civita connection of M.

3.1.1 A remark on complex structures on vector bundles

We recall here some details from the theory of holomorphic vector bundles. Let M be a complex manifold and $E\stackrel{\pi}{\to}M$ denote a complex vector bundle of rank k, so that it has a smooth complex structure $J=i$. Also let D denote a complex connection on E, i.e. one for which J is parallel.

Recall there exists a natural $\overline{\partial}^E$ operator on sections of E when this is holomorphic.

The following well known result is due to Koszul and Malgrange, cf. [10]. A vector bundle E admits a holomorphic structure such that $\overline{\partial}^E e=D''e:=\mathrm{pr}\circ De$, where e is any section and pr is the projection onto the $-i$ eigenbundle $T^*M^{(0,1)}\otimes E$, if, and only if, the $(0,2)$ part of the curvature R of D vanishes. Moreover the holomorphic structure is unique with such condition.

The proof is simple: if we write $E=P\times_{GL(k,\mathbb{C})}\mathbb{C}^k$ with P a principal bundle and use a global $\mathfrak{gl}(k,\mathbb{C})$-valued connection 1-form α to describe D and a local chart $z: U\to\mathbb{C}^n$ of M, then the components of α plus the components of $\pi^*\mathrm{d}z$ are sufficient to generate a subspace of, imposed, $(1,0)$- $GL(k,\mathbb{C})$-equivariant forms, and therefore a bundle compatible almost complex structure on P, and hence on E. By Newlander-Niremberg's celebrated theorem, such structure is integrable if, and only if, the subspace generates a d-closed ideal in the space of differential forms. This is equivalent to the vanishing of $(\mathrm{d}\alpha)^{(0,2)}=(\rho-\alpha\wedge\alpha)^{(0,2)}=\rho^{(0,2)}$ where ρ is the curvature form.

The uniqueness of the holomorphic structure *with* the condition $\overline{\partial}^E=D''$ follows, since it is known that it is univocally determined by the underlying almost complex structure and the latter is determined by π and α globally.

We may draw a further conclusion: the holomorphic structure of E is the same for all D for which $\rho^{(0,2)}=0$ *and* the connection 1-form is type $(1,0)$, $\alpha''=0$.

We remark that the uniqueness of D is sometimes mistakenly inferred in some of the literature, but it is not even the case in a Hermitian setting as the most trivial example will show; consider $M=\mathbb{C}$ and D nontrivial on the tangent bundle with canonical complex structure, $D=\mathrm{d}+\mu$, with μ any $i\mathbb{R}$-valued 1-form. Also $R^D=\overline{\partial}\mu-\partial\overline{\mu}$ is a pure imaginary 2-form which may well not vanish.

In the Hermitian case with *the* Hermitian connection, unique as Hermitian and type $(1,0)$ connection, we may say D is flat if, and only if, the connection 1-form is holomorphic. This is because the curvature can only be $(1,1)$, by the metric symmetries, and therefore $\rho=\overline{\partial}\alpha$.

Refering the naturally holomorphic tangent bundle of any complex manifold, furnished with a complex linear connection with $R^{(0,2)}=0$, we have a simple criteria to see if $\overline{\partial}^{TM}=D''$, and reciprocally: the torsion of D must be $(2,0)$. Essentially, this is because the torsion form coincides with $\alpha\wedge\mathrm{d}z$.

4 Natural complex structures on TM with almost Hermitian M

4.1 The second complex structure, a pair of them

Let $(M,\mathcal{J})$ be an almost Hermitian manifold of real dimension $m=2n$. Let D denote a linear Hermitian connection: a metric connection satisfying $D\mathcal{J}=0$. In the following we adopt the notation from the last section.

We may define two natural almost complex structures on TM, which we denote by J or $J^{\pm}$: admiting again the decomposition of TTM into $\mathcal{H}^D\oplus\mathcal{V}$ we write

$$J^{\pm}=\mathcal{J}\oplus\pm\mathcal{J}. \tag{4.1}$$

And let, as usual, $T'M$ denote the $+i$-eigenbundle of $\mathcal{J}$.

Theorem 4.1. *(i) J^+ is integrable if, and only if, $\mathcal{J}$ is integrable and the curvature of D verifies $R^D_{u,v}\overline{w}=0,\ \forall u,v,w\in T'M$.*
(ii) J^- is integrable if, and only if, $\mathcal{J}$ is integrable and $R^D_{u,v}w=0,\forall u,v,w\in T'M$.
(iii) $(TM,\omega_{J^{\pm}})$ is symplectic if, and only if, the Hermitian connection D is flat and its

torsion verifies

$$T \in [[\mathcal{A}]] \oplus [[\mathfrak{X}_M]]. \tag{4.2}$$

This meaning[3] *that: T has no totally skew-symmetric part, according to (3.5), and T is $(3,0)+(0,3)$ with respect to $\mathcal{J}$.*

Proof. Let u, v, w denote vectors in the $+i$-eigenbundle of J. The integrability equation is $(1+iJ)\nabla_u v = 0$, $\forall u, v$. Equivalently, since $\mathcal{J}, J$ are D, D^* parallel, respectively, we have

$$\text{(a)}\ \ (1 \pm i\mathcal{J})R^*_{u,v}\xi = 0 \qquad \text{and} \qquad \text{(b)}\ \ (1+i\mathcal{J})(A_u v + \tau_u v) = 0$$

according to vertical and horizontal types. So the two curvature conditions in (i) and (ii) correspond to (a). With respect to (b), in particular for $u, v \in \mathcal{H}^{D\,\prime}$ we must have $\tau_u v \in \mathcal{H}^{D\,\prime}$. By a straightforward argument as in corollary 3.1, this is the same as $\pi^* T^D(u,v) \in \mathcal{H}^{D\,\prime}$, or $\pi^*[\pi_* u, \pi_* v] \in \mathcal{H}^{D\,\prime}$ — corresponding on the base manifold M to the integrability of $\mathcal{J}$. For u, w horizontal and v vertical, since the metric on M is a (1,1) tensor, (b) reads equivalently as $\langle A_u v, w\rangle = 0$. Which is

$$\langle R^*_{u,w}\xi, v\rangle = \tfrac{1}{2}\langle (1 \mp i\mathcal{J})R^*_{u,w}\xi, v\rangle = 0,$$

due to (a). But this is always true since the projection $\frac{1}{2}(1 \pm i\mathcal{J})v = 0$.

Now let us see assertion (iii). We first compute,

$$\begin{aligned}
\langle (\nabla_X J)Y, Z\rangle &= \langle -\frac{1}{2}[R^*_{X,\cdot}\xi + A_X + \tau_X, J]Y, Z\rangle \\
&= \langle -\frac{1}{2}R^*_{X^h,\mathcal{J}Y^h}\xi \pm \frac{1}{2}\mathcal{J}R^*_{X^h,Y^h}\xi, Z^v\rangle + \langle \pm A_{X^h}\mathcal{J}Y^v + \\
&\qquad + A_{X^v}\mathcal{J}Y^h - \mathcal{J}A_X Y - \mathcal{J}\tau_{X^h}Y^h + \tau_{X^h}\mathcal{J}Y^h, Z^h\rangle.
\end{aligned}$$

We denote $R_{\alpha\beta\gamma} = \langle R^*_{e_\alpha,e_\beta}\xi, e_\gamma\rangle$, with $\mathcal{J}e_\alpha$ represented by $\hat{\alpha}$, for an orthonormal frame $e_1, \ldots, e_m, e_{1+m} = \theta e_1, \ldots, e_{m+m} = \theta e_m$ induced from an orthonormal frame of M. Now using the symmetry of A,

$$\begin{aligned}
\mathrm{d}\omega_J &= \sum_{i=1}^{2m} \nabla_i \omega_J \wedge e^i = \frac{1}{2}\sum_{i,j,k=1}^{2m} \langle (\nabla_i J)e_j, e_k\rangle e^{ijk} = \\
&= \sum_{i,j,k=1}^{m} -\frac{1}{4}R_{i\hat{j}k+m}e^{ijk+m} \mp \frac{1}{4}R_{ij\widehat{k+m}}e^{ijk+m} \pm \frac{1}{4}R_{ik\widehat{j+m}}e^{i,j+m,k} + \\
&\qquad + \frac{1}{4}R_{\hat{j}ki+m}e^{i+m,j,k} + \frac{1}{2}(\tau_{ij\hat{k}} + \tau_{i\hat{j}k})e^{ijk} \\
&= \sum_{i,j,k=1}^{m} \frac{1}{4}\big(-R_{i\hat{j}k+m} \mp R_{ij\widehat{k+m}} \mp R_{ij\widehat{k+m}} - R_{\hat{j}ik+m}\big)e^{ijk+m} + \frac{1}{2}(\tau_{ij\hat{k}} - \tau_{ik\hat{j}})e^{ijk}.
\end{aligned}$$

[3] We write $[[\mathcal{A}]] = \mathcal{A}' + \mathcal{A}''$ for a vector space of tensors on $T'M$ plus the conjugate of $\mathcal{A}'$.

Since τ_{ijk} is skew-symmetric in j,k, we get

$$\begin{aligned} \mathrm{d}\omega_J &= \sum_{i,j,k}^{m} \mp\frac{1}{2}R_{ij\widehat{k+m}}e^{ijk+m}+\tau_{ij\hat{k}}e^{ijk} = \\ &= \sum_{i<j}\sum_{k}\mp R_{ij\widehat{k+m}}e^{ijk+m}+2\sum_{i<j<k}(\tau_{ij\hat{k}}+\tau_{jk\hat{i}}+\tau_{ki\hat{j}})e^{ijk}. \end{aligned} \tag{4.3}$$

Now we are in position to prove (iii). To have $\mathrm{d}\omega_J=0$ the flatness of D is evident; the cyclic sum in i,j,k of $\tau_{ij\hat{k}}$ above implies

$$T_{ji\hat{k}}-T_{\hat{k}ij}+T_{j\hat{k}i}+T_{ik\hat{j}}-T_{\hat{i}jk}+T_{k\hat{i}j}+T_{kj\hat{i}}-T_{\hat{j}ki}+T_{i\hat{j}k}=0.$$

If i,j,k are indices of three vectors in $T'M$, then we simplify this to

$$T_{jik}-T_{kij}+T_{kji}=0$$

which is the totally skew part of T on $\otimes^3T'M$. If i,j represent vectors in $T'M$ and $k:=\overline{k}$ in $T''M$, then we find

$$\begin{aligned} -T_{ji\overline{k}}+T_{\overline{k}ij}-T_{j\overline{k}i}+T_{i\overline{k}j}-T_{ij\overline{k}}+T_{\overline{k}ij}+T_{\overline{k}ji}-T_{j\overline{k}i}+T_{ij\overline{k}} &= \\ T_{ij\overline{k}}-T_{i\overline{k}j}-3T_{j\overline{k}i} &= 0. \end{aligned}$$

Equivalently $3T_{j\overline{k}i}=T_{ij\overline{k}}-T_{i\overline{k}j}$ for *all* indices i,j,k. In repeating the equation, we deduce $9T_{j\overline{k}i}=3T_{ij\overline{k}}-T_{ji\overline{k}}+T_{j\overline{k}i}$ or $8T_{j\overline{k}i}=4T_{ij\overline{k}}$. Hence $T_{j\overline{k}i}$ is totally skew-symmetric and this same equation says it must be 0.

Taking conjugates, since T is real, we see both $T_{\overline{i}jk}$ and $T_{i\overline{j}\overline{k}}=0$. In particular, the whole skew-symmetric part of the torsion must vanish. This proves the result. ∎

Notice for the case J^+ we see in part (i) of the theorem that the integrability depends on $R^D_{\overline{u},\overline{v}}w=0,\ \forall u,v,w\in T'M$ (the conjugate of the written condition), just like Koszul-Malgrange's theorem prescribes when we see $E=T'M$ with complex structure $J=i$, cf. section 3.1.1. Moreover part (i) is stronger than this celebrated theorem since it does not assume integrability on the base space.

Let $\omega_{\mathcal{J}}$ denote the 2-form on M. It is easy to deduce the formula

$$\mathrm{d}\omega_{\mathcal{J}}(X,Y,Z)=\omega_{\mathcal{J}}(T(X,Y),Z)+\omega_{\mathcal{J}}(T(Y,Z),X)+\omega_{\mathcal{J}}(T(Z,X),Y),$$

therefore with little extra work we may show that T satisfies condition (4.2) if, and only if, $(M,\omega_{\mathcal{J}})$ is a symplectic manifold.

The condition found for the torsion in part (iii) is quite interesting if we confront with the “QKT-connections” studied in [9]; surprisingly those are required to have $T\in\wedge^3$ and to be type (1,2)+(2,1) with respect to $\mathcal{J}$.

4.2 The third complex structure on TM

This work would not be complete if we did not consider the following almost complex structure on the tangent bundle of the Riemannian manifold M. Consider the same setting as above and define J to be J^-. Consider also the complex structure I from section 3.1. Then $K=IJ=-JI$ is a new D^*-parallel almost complex structure, since $J\theta=-\theta J$, and hence we must do an analysis regarding complex and symplectic geometries just as previously.

Theorem 4.2. *(i) The following three are equivalent: K is integrable; D is flat and torsion free; $(M,\mathcal{J})$ is a flat Kähler manifold.*
(ii) (TM,ω_K) is symplectic if, and only if, D is torsion free. The same is to say $(M,\mathcal{J})$ is Kähler.

Proof. First we describe u in the $+i$-eigenbundle of K. In a decomposition $K(u^h+u^v)=iu^h+iu^v$, this translates in $u^v=i\mathcal{J}\theta u^h$. Thence we may write, $T'TM=\{u=X+i\mathcal{J}\theta X: X\in\mathcal{H}^D\}\otimes\mathbb{C}$. Now the integrability of K, as above, is given by $(1+iK)\nabla_u v=0,\ \forall u,v\in T'TM$. According to types this is simply

$$\text{(a)}\ \ (1+iK)R^*_{u,v}\xi=0\qquad\text{and}\qquad\text{(b)}\ \ (1+iK)(A_uv+\tau_uv)=0.$$

Taking $u\in T'TM$ and $v=Y+i\mathcal{J}\theta Y$ alike, we get from (a) the equation $(1+iK)R^*_{X,Y}\xi=0$ and so D is flat. From (b) the condition $\tau_XY=0$ follows. Now let us compute $\mathrm{d}\omega_K$. It could be seen by a formula, $\sum_{i,j,k=1}^{2m}\langle\nabla_i(J\theta e_j),e_k\rangle e^{ijk}$, but we shall follow the usual proceedre. First,

$$\begin{aligned}\langle(\nabla_XK)Y,Z\rangle &= \langle[-\frac{1}{2}R^*_{X,\cdot}\xi+A_X+\tau_X,K]Y,Z\rangle\\
&= \frac{1}{2}\langle R^*_{X^h,\theta^t\mathcal{J}Y^v}\xi,Z^v\rangle-\frac{1}{2}\langle\theta^t\mathcal{J}R^*_{X^h,Y^h}\xi,Z^h\rangle\\
&\quad-\langle A_{X^h}\theta\mathcal{J}Y^h+A_{X^v}\theta^t\mathcal{J}Y^v+\tau_{X^h}\theta^t\mathcal{J}Y^v,Z^h\rangle+\langle\theta\mathcal{J}(A_XY+\tau_XY),Z^v\rangle\\
&= \frac{1}{2}\langle R^*_{X^h,\theta^t\mathcal{J}Y^v}\xi+\theta\mathcal{J}(A+\tau)_XY,Z^v\rangle+\frac{1}{2}\langle R^*_{X^h,Y^h}\xi,\theta\mathcal{J}Z^h\rangle\\
&\quad-\frac{1}{2}\langle R^*_{X^h,Z^h}\xi,\theta\mathcal{J}Y^h\rangle-\frac{1}{2}\langle R^*_{\theta^t\mathcal{J}Y^v,Z^h}\xi,X^v\rangle-\tau(X^h,\theta^t\mathcal{J}Y^v,Z^h).\end{aligned}$$

Now with the notation of theorem 4.1, we have

$$\begin{aligned}2\mathrm{d}\omega_K &= \sum_{i,j,k=1}^{2m}\langle(\nabla_iK)e_j,e_k\rangle e^{ijk}\\
&= \sum_{i,j,k=1}^{m}\frac{1}{2}R_{i\hat{j}k+m}e^{i,j+m,k+m}+\frac{1}{2}R_{ij\widehat{k+m}}e^{ijk}-\frac{1}{2}R_{ik\widehat{j+m}}e^{ijk}\\
&\qquad\qquad-\frac{1}{2}R_{\hat{j}ki+m}e^{i+m,j+m,k}-\tau_{i\hat{j}k}e^{i,j+m,k}-\tau_{ij\hat{k}}e^{ijk+m}\\
&= \sum_{i,j,k=1}^{m}\frac{1}{2}\big(R_{i\hat{j}k+m}+R_{\hat{j}ik+m}\big)e^{i,j+m,k+m}+\frac{1}{2}R_{ij\widehat{k+m}}(e^{ijk}-e^{ikj})\\
&\qquad\qquad-\tau_{i\hat{j}k}(e^{i,j+m,k}-e^{ikj+m})\ =\ \sum R_{ij\widehat{k+m}}e^{ijk}+2\tau_{i\hat{j}k}e^{ikj+m}.\end{aligned}$$

Then by simple computation

$$\begin{aligned}\mathrm{d}\omega_K &= \sum_{i<j<k}^{m}(R_{ij\widehat{k+m}}+R_{jk\widehat{i+m}}+R_{ki\widehat{j+m}})e^{ijk}+\sum_{i<k}\sum_{j}(\tau_{i\hat{j}k}-\tau_{k\hat{j}i})e^{ikj+m} \\ &= \sum_{i<j<k}^{m}\circlearrowright\!\!\!\!\!\bigoplus_{ijk} R_{ij\widehat{k+m}}e^{ijk}+2\sum_{i<k}\sum_{j}T_{ik\hat{j}}e^{ikj+m}.\end{aligned}\tag{4.4}$$

The result now follows easily, since the vanishing of T implies Bianchi identity and already we had $\mathcal{J}R^*\xi=R^*\mathcal{J}\xi$. Finally if $T=0$ then D is the Levi-Civita connection and so $\mathcal{J}$ is integrable and henceforth Kähler. ∎

In some sense, the complex structure I plays a preponderant role. Notice (ii) above is also equivalent to (ii) from theorem 3.2.

5 Quaternionic Kähler structures on TM

In sections 3.1, 4.1 and 4.2 we saw how to define a quaternionic triple (I,J,K) over the tangent bundle of an almost Hermitian base $(M,\mathcal{J})$ of dimension $m=2n$. In order to decide if it corresponds to true $G(n)$ holonomy, at least in the case $n>2$, we must compute $\mathrm{d}\Omega$ where Ω is the 4-form defined in (2.4). To start with, let

$$e_1,\ldots,e_n,e_{n+1},\ldots,e_{2n},e_{2n+1},\ldots,e_{3n},e_{3n+1},\ldots,e_{4n}$$

be a frame on TM induced from a unitary frame of M: $e_{l+n}=\mathcal{J}e_l$, $e_{2n+i}=\theta e_i$, with $1\leq l\leq n,\ 1\leq i\leq 2n$. Then it is easy to deduce

$$\omega_I=-\sum e^{i,i+2n},\qquad \omega_J=\sum e^{l,l+n}-e^{l+2n,l+3n},\qquad \omega_K=e^{l+n,l+2n}-e^{l,l+3n}.$$

Theorem 5.1. *(TM,I,J,K) is a quaternionic Kähler manifold if, and only if, D is flat and torsion free.*

Proof. In the proof of theorem 3.2 we computed $\mathrm{d}\omega_I$. Using this and formulae (4.3) and (4.4) we deduce

$$\begin{aligned}\frac{1}{2}\mathrm{d}\Omega &= \mathrm{d}\omega_I\wedge\omega_I+\mathrm{d}\omega_J\wedge\omega_J+\mathrm{d}\omega_K\wedge\omega_K \\ &= \sum_{i<j<k}^{2n}\sum_{l=1}^{n}\circlearrowright\!\!\!\!\!\bigoplus_{ijk}\Big(R_{ijk+2n}(e^{ijkll+2n}+e^{ijkl+n,l+3n})+ \\ &\qquad +2\tau_{ij\hat{k}}(e^{ijkll+n}-e^{ijkl+2n,l+3n})+R_{ij\widehat{k+m}}(e^{ijkl+n,l+2n}-e^{ijkll+3n})\Big)+ \\ &\qquad +\sum_{i<j}^{2n}\sum_{k=2n+1}^{4n}\sum_{l=1}^{n}\Big(2T_{ijk-2n}(e^{ijk,l,l+2n}+e^{ijk,l+n,l+3n})+ \\ &\qquad +R_{ij\hat{k}}(e^{ijk,l,l+n}-e^{ijk,l+2n,l+3n})+2T_{ij\widehat{k-2n}}(e^{ijk,l+n,l+2n}-e^{ijk,l,l+3n})\Big)\end{aligned}$$

with notation given previously. It is easy to check $\mathrm{d}\Omega=0$ implies $R^D=0,T^D=0$. ∎

5.1 A family of quaternionic Kähler structures on TM.

Here we assume we have a $4n$ manifold endowed with a quaternionic triple $q=(\mathcal{J}_1,\mathcal{J}_2,\mathcal{J}_3)$; we are going to extend these endomorphisms to TTM in a canonical fashion as it was done in section 4.1, but now with a certain connection D known as the Obata connection. The following seems not to be so well known, hence we give a proof.

Proposition 5.1 (Obata)**.** *For every quaternionic Hermitian structure $\theta=(I,J,K)$ there is a metric connection D such that $DI=DJ=DK=0$.*

Proof. Let ∇ denote any metric connection and let $A_E=(\nabla E)E$, for any $E\in\mathrm{End}\,TM$. Then we have $[A_J,J]=(\nabla J)J^2-J(\nabla J)J=-\nabla J+(\nabla J)J^2=-2\nabla J$, proving we can always find a Hermitian connection: $(\nabla+\frac{1}{2}A_J)J=0$. It is easy to see that A_J is an $\mathfrak{so}(TM)$-valued 1-form. We also have

$$[A_J,I]=(\nabla J)JI-I(\nabla J)J=-(\nabla J)K+K\nabla J=[K,\nabla J]$$

and hence, letting $D=\nabla+\frac{1}{4}(A_I+A_J+A_K)$, we find

$$\begin{aligned} DI &= (\nabla+\tfrac{1}{2}A_I)I-\tfrac{1}{4}[A_I,I]+\tfrac{1}{4}[A_J,I]+\tfrac{1}{4}[A_K,I] \\ &= \tfrac{1}{4}\big(2\nabla I+K\nabla J-(\nabla J)K-J\nabla K+(\nabla K)J\big) \\ &= \tfrac{1}{4}\big(2\nabla I+\nabla(KJ)-\nabla(JK)\big)\ =\ 0. \end{aligned}$$

The same equation holds for J and K. ■

Now let $I_0=I$ be the endomorphism defined in 3.1 and let

$$I_i=\mathcal{J}_i\oplus-\mathcal{J}_i,\qquad \forall i=1,2,3, \tag{5.1}$$

as the case J^- in 4.1. Notice $I_3\neq I_1I_2=-I_2I_1$. However, the whole four I_i anti-commute with each other. Hence, for each point $(a,b)\in V_2^4$, the Stiefel manifold of pairs of orthonormal vectors $a,b\in\mathbb{R}^4$, we have a quaternionic triple $(I_a,I_b,I_{a,b})$ given by

$$I_x=x_0I_0+x_1I_1+x_2I_2+x_3I_3,\quad \forall x=a,b,\quad \text{and}\quad I_{a,b}=I_aI_b. \tag{5.2}$$

It is easy to verify $I_x^2=-1$ and $I_aI_b=-I_bI_a$. Also we let $\Omega_{a,b}=\omega_a^2+\omega_b^2+\omega_{a,b}^2$ where $\omega_a(X,Y)=\langle I_aX,Y\rangle$, etc.

We then have two extreme examples: $a=(1,0,0,0),\ b=(0,1,0,0)$ yield the case with which we started this section. Theorem 5.1 gives further information about Ω.

With $a=(0,1,0,0),\ b=(0,0,1,0)$ we have the other case, where the requirement of a quaternionic Hermitian base M is unavoidable. We have also done the computations of the respective $\mathrm{d}\Omega_{a,b}=0$ and the condition found was the same as for the first case: the very strict torsion free and flat metric connection D. The proof is very much alike using a quaternionic frame. Finally, due to the fact that every $a\in S^3$ is connected by a curve $e^{itx}e^{jty}e^{ktz}$ in $\mathbb{H}$ to $(1,0,0,0)$, it may be possible to prove that theorem 5.1 holds for every $(a,b)\in V_2^4$.

Recall that for every almost quaternionic Hermitian manifold $(M, Q =< q >)$, there is an associated twistor space $\mathcal{Z}(M) \subset Q$, an S^2-bundle of endomorphisms $a\mathcal{J}_1 + b\mathcal{J}_2 + c\mathcal{J}_3$, with $(a, b, c) \in S^2$, defining complex structures in each T_xM. Thus we have obtained a "Hopf-twistor" extension of such bundle associated to the tangent bundle.

5.2 Over a Riemann surface M

In order to speak of quaternionic Kähler structures on the tangent bundles of Riemannian manifolds, the cases $n = 1$ and 2 are missing. We concentrate on the case $n = 1$ and recall the desired condition now is the metric on TM to be self-dual and Einstein.

Let ξ be the canonical vector field (3.1) and let η be the unit vertical vector field such that $\{\frac{\xi}{c}, \eta\}_u = \{\frac{u}{c}, \eta_u\}$ is a direct orthonormal basis of $T_{\pi(u)}M$, $\forall u \in TM$, with $c_u = \|\xi_u\| = \|u\|$. Let D be the usual metric connection on M and denote by k the function $k(u) = \langle R_{\frac{u}{c},v}\frac{u}{c}, v\rangle$. We may also write $k = \frac{1}{c^2}\langle R_{\xi_h,\eta_h}\xi, \eta\rangle$ where ξ_h, η_h are such that their images under θ are ξ, η, respectively, θ being the map introduced in 3.1. Suppose the torsion of D is such that

$$T(\xi, \eta) = f_1\xi + f_2\eta$$

with f_1, f_2 real functions. Then the tensor defined in (3.4) satisfies

$$\tau.\xi_h = (f_1\xi_h^b + f_2\eta_h^b)\eta_h, \qquad \tau.\eta_h = -\frac{1}{c^2}(f_1\xi_h^b + f_2\eta_h^b)\xi_h.$$

A straightforward computation yields the following formulae for the Levi-Civita connection of TM:

$$\begin{array}{llll} \nabla_\xi\xi = \xi & \nabla_\xi\eta = 0 & \nabla_\xi\xi_h = \xi_h & \nabla_\xi\eta_h = 0 \\ \nabla_\eta\xi = \eta & \nabla_\eta\eta = -\frac{\xi}{c^2} & \nabla_\eta\xi_h = (1 + \frac{k}{2}c^2)\eta_h & \nabla_\eta\eta_h = -(\frac{1}{c^2} + \frac{k}{2})\xi_h \\ \nabla_{\eta_h}\xi = 0 & \nabla_{\eta_h}\eta = -\frac{k}{2}\xi_h & \nabla_{\eta_h}\xi_h = \frac{k}{2}c^2\eta + f_2\eta_h & \nabla_{\eta_h}\eta_h = -\frac{1}{c^2}f_2\xi_h \\ \nabla_{\xi_h}\xi = 0 & \nabla_{\xi_h}\eta = \frac{k}{2}c^2\eta_h & \nabla_{\xi_h}\xi_h = f_1c^2\eta_h & \nabla_{\xi_h}\eta_h = -\frac{k}{2}c^2\eta - f_1\xi_h \end{array} \tag{5.3}$$

From these and other identities such as $[\xi_h, \eta_h] = -c^2k\eta - f_1\xi_h - f_2\eta_h$ (the most relevant between the Lie bracket computations) we may compute the Riemannian curvature of TM. Notice k, f_1, f_2 only depend on $x \in M$. The upshot of these calculations is the following result: TM is Einstein if, and only if, $k = 0$ and

$$c^2\eta_h(f_1) - \xi_h(f_2) - c^2f_1^2 - f_2^2 = 0,$$

still an intriguing equation. In particular, we may conclude with a corollary when k is the Gauss curvature.

Corollary 5.1. *For a Riemann surface M, TM with its canonical metric is an Einstein manifold if, and only if, the Riemannian curvature of M is 0.*

6 Appendix

We prove here that $Sp(n)Sp(1)=G(n)$ is the (isotropy) subgroup of $SO(4n)$ which leaves invariant the 4-form Ω defined by the identity (2.4) on an Euclidian $4n$-vector space V.

The group $Sp(n)$ is by definition the subgroup of isometries of V which commute with the given quaternionic triple $q=(I,J,K)$.

If $g\in G(n)$ then $\forall X\in V,\ w\in\mathbb{H},\ g(Xw)=g(X)w'=g(X)ww''$ for some $w''\in S^3\subset\mathbb{H}$. This is, g preserves the quaternionic lines and reciprocally. Hence, to see $g^*\Omega=\Omega$ we are bound to prove it for $g\in Sp(1)$. Immediately we deduce

$$I^*\omega_I=\omega_I,\qquad I^*\omega_J=-\omega_J,\qquad I^*\omega_K=-\omega_K.$$

Since $I^*(\omega\wedge\omega)=I^*\omega\wedge I^*\omega$ and since all the same is true for J,K, we see

$$I^*\Omega=J^*\Omega=K^*\Omega=\Omega.$$

To prove the reciprocal we need a lemma: if Y,Y_1,Y_2,Y_3 is an orthonormal set such that $\Omega(Y,Y_1,Y_2,Y_3)=4$, then $Y_j\in\mathrm{span}\{IY,JY,KY\},\ \forall j=1,2,3$. Proof: let $Y_j=\alpha_jIY+\beta_jJY+\gamma_jKY+Z_j$ with Z_j orthogonal to the quaternionic line spanned by Y. Then it is easy to compute from identity (2.4)

$$4=\Omega(Y,Y_1,Y_2,Y_3)=4\det\begin{bmatrix}\alpha_1&\beta_1&\gamma_1\\ \alpha_2&\beta_2&\gamma_2\\ \alpha_3&\beta_3&\gamma_3\end{bmatrix}.$$

But since the Y_j are orthonormal, $\alpha_j^2+\beta_j^2+\gamma_j^2+|Z_j|^2=1$. Now these two equations yield $Z_j=0$, proving the lemma.

Finally, suppose $g\in SO(4n)$ and $g^*\Omega=\Omega$. Then take any quaternionic line, with an orthonormal basis X,X_1,X_2,X_3. We want to see the $Y_i=g(X_i)$ are all in the same quaternionic line. Since $\Omega(Y,Y_1,Y_2,Y_3)=\Omega(X,X_1,X_2,X_3)=4$, the lemma gives the result.

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