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Research Article

Hesitant Fuzzy Soft Set and Its Applications in Multicriteria Decision Making

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Molodtsov's soft set theory is a newly emerging mathematical tool to handle uncertainty. However, the classical soft sets are not appropriate to deal with imprecise and fuzzy parameters. This paper aims to extend the classical soft sets to hesitant fuzzy soft sets which are combined by the soft sets and hesitant fuzzy sets. Then, the complement, "AND", "OR", union and intersection operations are defined on hesitant fuzzy soft sets. The basic properties such as DeMorgan's laws and the relevant laws of hesitant fuzzy soft sets are proved. Finally, with the help of level soft set, the hesitant fuzzy soft sets are applied to a decision making problem and the effectiveness is proved by a numerical example.

1. Introduction

In the real world, there are many complicated problems in economics, engineering, environment, social science, and management science. They are characterized with uncertainty, imprecision, and vagueness. We cannot successfully utilize the classical methods to deal with these problems because there are various types of uncertainties involved in these problems. Moreover, though there are many theories, such as theory of probability, theory of fuzzy sets, theory of interval mathematics, and theory of rough sets to be considered as mathematical tools to deal with uncertainties, Molodtsov [1] pointed out all these theories had their own limitations. Moreover, in order to overcome these difficulties, Molodtsov [1] firstly proposed a new mathematical tool named soft set theory to deal with uncertainty and imprecision. This theory has been demonstrated to be a useful tool in many applications such as decision making, measurement theory, and game theory.

The soft set model can be combined with other mathematical models. Maji et al. [2] firstly presented the concept of fuzzy soft set by combining the theories of fuzzy set and soft set together. Furthermore, Maji et al. [3–5] established the notion of intuitionistic fuzzy soft set which was based on a combination of the intuitionistic fuzzy set [6, 7] and soft set models. By combining the interval-valued fuzzy set [8, 9] and soft set, Yang et al. [10] presented the concept of the interval-valued fuzzy soft set. Jiang et al. [11] initiated the concept of interval-valued intuitionistic fuzzy soft sets as an interval-valued fuzzy extension of the intuitionistic fuzzy soft set theory or an intuitionistic fuzzy extension of the interval-valued fuzzy soft set theory. Recently, Xiao et al. [12] presented the trapezoidal fuzzy soft set and Yang et al. [13] introduced the multifuzzy soft set, respectively, and applied them in decision making problems. Roy and Maji [14], Kong et al. [15], Feng et al. [16], and Jiang et al. [17] also applied the fuzzy soft set in the decision making problems. Jun [18] initiated the application of soft sets in BCK/BCI-algebras and introduced the concept of soft BCK/BCI-algebras. Furthermore, Jun and Park [19] and Jun et al. [20, 21] applied the soft sets in ideal theory of BCK/BCI-algebras and d-algebras. Feng et al. [22, 23] initiated the concept of rough soft sets, soft rough sets, and soft rough fuzzy sets.

Recently, in order to tackle the difficulty in establishing the degree of membership of an element in a set, Torra and Narukawa [24] and Torra [25] proposed the concept of a hesitant fuzzy set. This new extension of fuzzy set can handle the cases that the difficulty in establishing the membership

degree does not arise from a margin of error (as in intuitionistic or interval-valued fuzzy sets) or a specified possibility distribution of the possible values (as in type-2 fuzzy sets), but arises from our hesitation among a few different values [26]. Thus the hesitant fuzzy set can more accurately reflect the people's hesitancy in stating their preferences over objects, compared to the fuzzy set and its many classical extensions. The purpose of this paper is to extend the soft set model to the hesitant fuzzy set, and, thus, we establish a new soft set model named hesitant fuzzy soft set.

The rest of this paper is organized as follows. We first review some background on soft set, fuzzy soft set, and hesitant fuzzy set in Section 2. In Section 3, the concepts and operations of hesitant fuzzy soft set are proposed and their properties are discussed in detail. In Section 4, we apply the proposed hesitant fuzzy soft set to a decision making problem and give an explicit algorithm. Finally, we conclude in Section 5.

2. Preliminaries

2.1. Soft Sets. Suppose that U is an initial universe set, E is a set of parameters, $P(U)$ is the power set of U and $A \subset E$.

Definition 1 (see [1]). A pair (F, A) is called a soft set over U, where F is a mapping given by $F: A \to P(U)$.

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $e \in A$, $F(e)$ may be considered as the set of e-approximate elements of the soft set (F, A).

Example 2. Suppose that $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ is a set of houses and $A = \{e_1, e_2, e_3, e_4, e_5\}$ is a set of parameters, which stands for the parameters "cheap," "beautiful," "size," "location," and "surrounding environment," respectively. In this case, a soft set (F, A) can be defined as a mapping from parameter set A to the set of all subsets of U. In this way, the set of the houses with specific characteristics can be described.

Assume that $F(e_1) = \{h_2, h_4\}$, $F(e_2) = \{h_1, h_3\}$, $F(e_3) = \{h_3, h_4, h_5\}$, $F(e_4) = \{h_1, h_3, h_5\}$, and $F(e_5) = \{h_2\}$, which mean "houses (cheap)" whose function-value is the set $\{h_2, h_4\}$, "houses (beautiful)" whose function-value is the set $\{h_1, h_3\}$, "houses (big size)" whose function-value is the set $\{h_3, h_4, h_5\}$, "houses (in good location)" whose function-value is the set $\{h_1, h_3, h_5\}$, "houses (great surrounding environment)" whose function-value is the set $\{h_2\}$, respectively. Then we can view the soft set (F, A) as consisting of the following collection of approximations:

$$(F, A) = \{(e_1, \{h_2, h_4\}), (e_2, \{h_1, h_3\}), (e_3, \{h_3, h_4, h_5\}), (e_4, \{h_1, h_3, h_5\}), (e_5, \{h_2\})\}. \quad (1)$$

For the purpose of storing a soft set in a computer, we can represent the soft set defined in Example 2 in tabular form as Table 1 shows.

Table 1: The tabular representation of the soft set (F, A).

U	"Cheap"	"Beautiful"	"Size"	"Location"	"Great surrounding environment"
h_1	0	1	0	1	0
h_2	1	0	0	0	1
h_3	0	1	1	1	0
h_4	1	0	1	0	0
h_5	0	0	1	1	0
h_6	0	0	0	0	0

Table 2: The tabular representation of the fuzzy soft set $(\widetilde{F}, A)$.

U	"Cheap"	"Beautiful"	"Size"	"Location"	"Great surrounding environment"
h_1	0.2	0.6	0.4	0.3	0.6
h_2	0.5	0.5	0.6	0.2	0.3
h_3	0.3	0.6	0.8	0.5	0.5
h_4	0.3	0.7	0.3	0.7	0.4
h_5	0.4	0.4	0.4	0.5	0.7
h_6	0.6	0.3	0.7	0.8	0.3

2.2. Fuzzy Soft Sets

Definition 3 (see [2]). Let $\widetilde{P}(U)$ be the set of all fuzzy subsets of U, a pair $(\widetilde{F}, A)$ is called a fuzzy soft set over U, where $\widetilde{F}$ is a mapping given by $\widetilde{F}: A \to \widetilde{P}(U)$.

Example 4. Reconsider Example 2. In real life, the perception of the people is characterized by a certain degree of vagueness and imprecision; thus, when people consider if a house h_1 is cheap the information cannot be expressed with only two crisp numbers 0 and 1. Instead it should be characterized by a membership function $\mu_{\widetilde{A}}(x)$ which associates with each element a real number in the interval $[0, 1]$. Then, fuzzy soft set $(\widetilde{F}, A)$ can describe the characteristics of the house under the fuzzy information.

$$\begin{aligned}
\widetilde{F}(e_1) &= \left\{\frac{h_1}{0.2}, \frac{h_2}{0.5}, \frac{h_3}{0.3}, \frac{h_4}{0.3}, \frac{h_5}{0.4}, \frac{h_6}{0.6}\right\},\\
\widetilde{F}(e_2) &= \left\{\frac{h_1}{0.6}, \frac{h_2}{0.5}, \frac{h_3}{0.6}, \frac{h_4}{0.7}, \frac{h_5}{0.4}, \frac{h_6}{0.3}\right\},\\
\widetilde{F}(e_3) &= \left\{\frac{h_1}{0.4}, \frac{h_2}{0.6}, \frac{h_3}{0.8}, \frac{h_4}{0.3}, \frac{h_5}{0.4}, \frac{h_6}{0.7}\right\},\\
\widetilde{F}(e_4) &= \left\{\frac{h_1}{0.3}, \frac{h_2}{0.2}, \frac{h_3}{0.5}, \frac{h_4}{0.7}, \frac{h_5}{0.5}, \frac{h_6}{0.8}\right\},\\
\widetilde{F}(e_5) &= \left\{\frac{h_1}{0.6}, \frac{h_2}{0.3}, \frac{h_3}{0.5}, \frac{h_4}{0.4}, \frac{h_5}{0.7}, \frac{h_6}{0.3}\right\}.
\end{aligned} \quad (2)$$

Similarly, for the purpose of storing a fuzzy soft set in a computer, we could also represent the fuzzy soft set defined in Example 4 in Table 2.

2.3. Hesitant Fuzzy Sets

Definition 5 (see [25]). A hesitant fuzzy set (HFS) on U is in terms of a function that when applied to U returns a subset of $[0,1]$, which can be represented as the following mathematical symbol:

$$\widetilde{A} = \{\langle u, h_{\widetilde{A}}(u)\rangle \mid u \in U\}, \tag{3}$$

where $h_{\widetilde{A}}(u)$ is a set of values in $[0,1]$, denoting the possible membership degrees of the element $u \in U$ to the set $\widetilde{A}$. For convenience, we call $h_{\widetilde{A}}(u)$ a hesitant fuzzy element (HFE) and H the set of all HFEs.

Furthermore, Torra [25] defined the empty hesitant set and the full hesitant set.

Definition 6 (see [25]). Given hesitant fuzzy set $\widetilde{A}$, if $h(u) = \{0\}$ for all u in U, then $\widetilde{A}$ is called the null hesitant fuzzy set, denoted by $\widetilde{\phi}$. If $h(u) = \{1\}$ for all u in U, then $\widetilde{A}$ is called the full hesitant fuzzy set, denoted by $\widetilde{1}$.

Definition 7 (see [27]). For a HFE h, $s(h) = (1/l(h))\sum_{\gamma\in h}\gamma$ is called the score function of h, where $l(h)$ is the number of the values in h. For two HFEs h_1 and h_2, if $s(h_1) > s(h_2)$, then $h_1 > h_2$; if $s(h_1) = s(h_2)$, then $h_1 = h_2$.

Let h_1 and h_2 be two HFEs. It is noted that the number of values in different HFEs h_1 and h_2 are commonly different; that is, $l(h_1) \neq l(h_2)$. For convenience, let $l = \max\{l(h_1), l(h_2)\}$. To operate correctly, we should extend the shorter one until both of them have the same length when we compare them. To extend the shorter one, the best way is to add the same value several times in it [28]. In fact, we can extend the shorter one by adding any value in it. The selection of this value mainly depends on the decision makers' risk preferences. Optimists anticipate desirable outcomes and may add the maximum value, while pessimists expect unfavorable outcomes and may add the minimum value. For example, let $h_1 = \{0.1, 0.2, 0.3\}$, let $h_2 = \{0.4, 0.5\}$, and let $l(h_1) > l(h_2)$. To operate correctly, we should extend h_2 to $h_2' = \{0.4, 0.4, 0.5\}$ until it has the same length of h_1, the optimist may extend h_2 as $h_2' = \{0.4, 0.5, 0.5\}$ and the pessimist may extend it as $h_2' = \{0.4, 0.4, 0.5\}$. Although the results may be different if we extend the shorter one by adding different values, this is reasonable because the decision makers' risk preferences can directly influence the final decision. The same situation can also be found in many existing references [29–31]. In this paper, we assume that the decision makers are all pessimistic (other situations can be studied similarly).

We arrange the elements in $h_{\widetilde{A}}(u)$ in decreasing order, and let $h_{\widetilde{A}}^{\sigma(j)}(u)$ be the jth largest value in $h_{\widetilde{A}}(u)$.

Definition 8. Given two hesitant fuzzy sets $\widetilde{M} = \{\langle u, h_{\widetilde{M}}(u)\rangle \mid u \in U\}$ and $\widetilde{N} = \{\langle u, h_{\widetilde{N}}(u)\rangle \mid u \in U\}$, $\widetilde{M}$ is called the fuzzy subset of $\widetilde{N}$ if and only if $h_M^{\sigma(j)}(u) \leq h_N^{\sigma(j)}(u)$, for $\forall u \in U$ and $j = 1, 2, \ldots, l$, which can be denoted by $\widetilde{M} \widetilde{\subseteq} \widetilde{N}$.

Definition 9. $\widetilde{M}$ and $\widetilde{N}$ are two hesitant fuzzy sets, we call $\widetilde{M}$ and $\widetilde{N}$ is hesitant fuzzy equal if and only if

(1) $\widetilde{M} \widetilde{\subseteq} \widetilde{N}$

(2) $\widetilde{M} \widetilde{\supseteq} \widetilde{N}$

which can be denoted by $\widetilde{M} = \widetilde{N}$.

Given three HFEs, h, h_1, and h_2, Torra [25] and Torra and Narukawa [24] defined the following HFE operations:

(1) $h^c = \cup_{\gamma\in h}\{1-\gamma\}$,

(2) $h_1 \cup h_2 = \{h \in (h_1 \cup h_2) \mid h \geq \max(h_1^-, h_2^-)\}$,

(3) $h_1 \cap h_2 = \{h \in (h_1 \cup h_2) \mid h \leq \min(h_1^+, h_2^+)\}$,

where $h^-(u) = \min h(u)$ and $h^+(u) = \max h(u)$ are the lower bound and upper bound of the given hesitant fuzzy elements, respectively.

Therefore, giving two hesitant fuzzy soft sets $\widetilde{A}$ and $\widetilde{B}$, we can define the following operations:

(1) $\widetilde{A}^c = \{\langle u, h_{\widetilde{A}}^c(u)\rangle \mid u \in U\}$;

(2) $\widetilde{A} \cup \widetilde{B} = \{\langle u, h_{\widetilde{A}}(u) \cup h_{\widetilde{B}}(u)\rangle \mid u \in U\}$;

(3) $\widetilde{A} \cap \widetilde{B} = \{\langle u, h_{\widetilde{A}}(u) \cap h_{\widetilde{B}}(u)\rangle \mid u \in U\}$.

Moreover, for the aggregation of hesitant fuzzy information, Xia and Xu [27] defined the following new operations on HFEs h, h_1, and h_2:

(1) $h^\lambda = \cup_{\gamma\in h}\{\gamma^\lambda\}$;

(2) $\lambda h = \cup_{\gamma\in h}\{1-(1-\gamma)^\lambda\}$;

(3) $h_1 \oplus h_2 = \cup_{\gamma_1\in h_1, \gamma_2\in h_2}\{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\}$;

(4) $h_1 \otimes h_2 = \cup_{\gamma_1\in h_1, \gamma_2\in h_2}\{\gamma_1\gamma_2\}$.

3. Hesitant Fuzzy Soft Sets

3.1. The Concept of Hesitant Fuzzy Soft Sets

Definition 10. Let $\widetilde{H}(U)$ be the set of all hesitant fuzzy sets in U; a pair $(\widetilde{F}, A)$ is called a hesitant fuzzy soft set over U, where $\widetilde{F}$ is a mapping given by

$$\widetilde{F} : A \longrightarrow \widetilde{H}(U). \tag{4}$$

A hesitant fuzzy soft set is a mapping from parameters to $\widetilde{H}(U)$. It is a parameterized family of hesitant fuzzy subsets of U. For $e \in A$, $\widetilde{F}(e)$ may be considered as the set of e-approximate elements of the hesitant fuzzy soft set $(\widetilde{F}, A)$.

Example 11. Continue to consider Example 2. Mr. X evaluates the optional six houses under various attributes with hesitant fuzzy element; then, hesitant fuzzy soft set $(\widetilde{F}, A)$ can describe the characteristics of the house under the hesitant fuzzy information.

Table 3: The tabular representation of the hesitant fuzzy soft set $(\widetilde{F}, A)$.

U	"Cheap"	"Beautiful"	"Size"	"Location"	"Great surrounding environment"
h_1	{0.2, 0.3}	{0.4, 0.6, 0.7}	{0.2, 0.4}	{0.3, 0.5, 0.6}	{0.6}
h_2	{0.5, 0.6}	{0.5, 0.7, 0.8}	{0.6, 0.7}	{0.2}	{0.2, 0.3, 0.5}
h_3	{0.3}	{0.6, 0.8}	{0.8, 0.9}	{0.5}	{0.5, 0.7}
h_4	{0.3, 0.5}	{0.7, 0.9}	{0.3, 0.5}	{0.6, 0.7}	{0.2, 0.4}
h_5	{0.4, 0.5}	{0.3, 0.4, 0.5}	{0.4, 0.6}	{0.5, 0.6}	{0.5, 0.7}
h_6	{0.6, 0.7}	{0.3}	{0.7}	{0.8}	{0.3, 0.5}

Consider

$$\begin{aligned}
\widetilde{F}(e_1) &= \left\{\frac{h_1}{\{0.2,0.3\}}, \frac{h_2}{\{0.5,0.6\}}, \frac{h_3}{\{0.3\}}, \frac{h_4}{\{0.3,0.5\}}, \frac{h_5}{\{0.4,0.5\}}, \frac{h_6}{\{0.6,0.7\}}\right\},\\
\widetilde{F}(e_2) &= \left\{\frac{h_1}{\{0.4,0.6,0.7\}}, \frac{h_2}{\{0.5,0.7,0.8\}}, \frac{h_3}{\{0.6,0.8\}}, \frac{h_4}{\{0.7,0.9\}}, \frac{h_5}{\{0.3,0.4,0.5\}}, \frac{h_6}{\{0.3\}}\right\},\\
\widetilde{F}(e_3) &= \left\{\frac{h_1}{\{0.2,0.4\}}, \frac{h_2}{\{0.6,0.7\}}, \frac{h_3}{\{0.8,0.9\}}, \frac{h_4}{\{0.3,0.5\}}, \frac{h_5}{\{0.4,0.6\}}, \frac{h_6}{\{0.7\}}\right\},\\
\widetilde{F}(e_4) &= \left\{\frac{h_1}{\{0.3,0.5,0.6\}}, \frac{h_2}{\{0.2\}}, \frac{h_3}{\{0.5\}}, \frac{h_4}{\{0.6,0.7\}}, \frac{h_5}{\{0.5,0.6\}}, \frac{h_6}{\{0.8\}}\right\},\\
\widetilde{F}(e_5) &= \left\{\frac{h_1}{\{0.6\}}, \frac{h_2}{\{0.2,0.3,0.5\}}, \frac{h_3}{\{0.5,0.7\}}, \frac{h_4}{\{0.2,0.4\}}, \frac{h_5}{\{0.5,0.7\}}, \frac{h_6}{\{0.3,0.5\}}\right\}.
\end{aligned} \tag{5}$$

Similarly, we can also represent the hesitant fuzzy soft set in the form of Table 3 for the purpose of storing the hesitant fuzzy soft set in a computer.

Definition 12. Let $A, B \in E$. $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ are two hesitant fuzzy soft sets over U. $(\widetilde{F}, A)$ is said to be a hesitant fuzzy soft subset of $(\widetilde{G}, B)$ if

(1) $A \subseteq B$,

(2) For all $e \in A$, $\widetilde{F}(e) \widetilde{\subseteq} \widetilde{G}(e)$.

In this case, we write $(\widetilde{F}, A) \widetilde{\subseteq} (\widetilde{G}, B)$.

Example 13. Given two hesitant fuzzy soft sets $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$, $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ is the set of the optional houses. $A = \{e_1, e_2\} = \{\text{cheap}, \text{beautiful}\}$, $B = \{e_1, e_2, e_3\} = \{\text{cheap}, \text{beautiful}, \text{size}\}$, and

$$\begin{aligned}
\widetilde{F}(e_1) &= \left\{\frac{h_1}{\{0.2,0.3\}}, \frac{h_2}{\{0.5,0.6\}}, \frac{h_3}{\{0.3\}}, \frac{h_4}{\{0.3,0.5\}}, \frac{h_5}{\{0.4,0.5\}}, \frac{h_6}{\{0.6,0.7\}}\right\},\\
\widetilde{F}(e_2) &= \left\{\frac{h_1}{\{0.4,0.6,0.7\}}, \frac{h_2}{\{0.5,0.7,0.8\}}, \frac{h_3}{\{0.6,0.8\}}, \frac{h_4}{\{0.7,0.9\}}, \frac{h_5}{\{0.3,0.4,0.5\}}, \frac{h_6}{\{0.3\}}\right\},\\
\widetilde{G}(e_1) &= \left\{\frac{h_1}{\{0.3\}}, \frac{h_2}{\{0.6,0.8\}}, \frac{h_3}{\{0.4,0.5\}}, \frac{h_4}{\{0.3,0.4,0.5\}}, \frac{h_5}{\{0.5\}}, \frac{h_6}{\{0.7\}}\right\},\\
\widetilde{G}(e_2) &= \left\{\frac{h_1}{\{0.4,0.5,0.6,0.7\}}, \frac{h_2}{\{0.7,0.8\}}, \frac{h_3}{\{0.9\}}, \frac{h_4}{\{0.8,0.9\}}, \frac{h_5}{\{0.5\}}, \frac{h_6}{\{0.5\}}\right\},\\
\widetilde{G}(e_3) &= \left\{\frac{h_1}{\{0.2,0.4\}}, \frac{h_2}{\{0.6,0.7\}}, \frac{h_3}{\{0.8,0.9\}}, \frac{h_4}{\{0.3,0.5\}}, \frac{h_5}{\{0.4,0.6\}}, \frac{h_6}{\{0.7\}}\right\}.
\end{aligned} \tag{6}$$

Then, we have $(\widetilde{F}, A) \widetilde{\subseteq} (\widetilde{G}, B)$.

Definition 14. Two hesitant fuzzy soft sets $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ are said to be hesitant fuzzy soft equal if $(\widetilde{F}, A)$ is a hesitant fuzzy soft subset of $(\widetilde{G}, B)$ and $(\widetilde{G}, B)$ is a hesitant fuzzy soft subset of $(\widetilde{F}, A)$.

In this case, we write $(\widetilde{F}, A) \cong (\widetilde{G}, B)$.

Definition 15. A hesitant fuzzy soft set $(\widetilde{F}, A)$ is said to be empty hesitant fuzzy soft set, denoted by $\widetilde{\Phi}_A$, if $\widetilde{F}(e) = \widetilde{\phi}$ for all $e \in A$.

Definition 16. A hesitant fuzzy soft set $(\widetilde{F}, A)$ is said to be full hesitant fuzzy soft set, denoted by $\widetilde{U}_A$, if $\widetilde{F}(e) = \widetilde{1}$ for all $e \in A$.

Table 4: The result of "AND" operation on $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$.

U	e_1, e_1	e_1, e_2	e_1, e_3	e_2, e_1	e_2, e_2	e_2, e_3
h_1	{0.2, 0.3}	{0.2, 0.3}	{0.2, 0.3}	{0.3}	{0.4, 0.5, 0.6, 0.7}	{0.2, 0.4}
h_2	{0.5, 0.6}	{0.5, 0.6}	{0.5, 0.6}	{0.5, 0.6, 0.7, 0.8}	{0.5, 0.7, 0.8}	{0.5, 0.6, 0.7}
h_3	{0.3}	{0.3}	{0.3}	{0.4, 0.5}	{0.6, 0.8}	{0.6, 0.8}
h_4	{0.3, 0.5}	{0.3, 0.5}	{0.3, 0.5}	{0.3, 0.4, 0.5}	{0.7, 0.8, 0.9}	{0.3, 0.5}
h_5	{0.3, 0.4, 0.5}	{0.4, 0.5}	{0.4, 0.5}	{0.3, 0.4, 0.5}	{0.3, 0.4, 0.5}	{0.3, 0.4, 0.5}
h_6	{0.3}	{0.5}	{0.6, 0.7}	{0.3}	{0.3}	{0.3}

3.2. Operations on Hesitant Fuzzy Soft Sets

Definition 17. The complement of a hesitant fuzzy soft set $(\widetilde{F}, A)$ is denoted by $(\widetilde{F}, A)^c$ and is defined by

$$\left(\widetilde{F}, A\right)^c = \left(\widetilde{F}^c, A\right), \tag{7}$$

where $\widetilde{F}^c : A \to \widetilde{H}(U)$ is a mapping given by $\widetilde{F}^c(e) = (\widetilde{F}(e))^c$ for all $e \in A$.

Clearly, $(\widetilde{F}^c)^c$ is the same as $\widetilde{F}$ and $((\widetilde{F}, A)^c)^c = (\widetilde{F}, A)$.

It is worth noting that in the above definition of complement, the parameter set of the complement $(\widetilde{F}, A)^c$ is still the original parameter set A, instead of $\neg A$.

Example 18. Reconsider Example 11; the $(\widetilde{F}, A)^c$ can be calculated as follows:

$$\begin{aligned}
\widetilde{F}^c(e_1) &= \left\{\frac{h_1}{\{0.7, 0.8\}}, \frac{h_2}{\{0.4, 0.5\}}, \frac{h_3}{\{0.7\}}, \frac{h_4}{\{0.5, 0.7\}}, \frac{h_5}{\{0.5, 0.6\}}, \frac{h_6}{\{0.3, 0.4\}}\right\},\\
\widetilde{F}^c(e_2) &= \left\{\frac{h_1}{\{0.3, 0.4, 0.6\}}, \frac{h_2}{\{0.2, 0.3, 0.5\}}, \frac{h_3}{\{0.2, 0.4\}}, \frac{h_4}{\{0.1, 0.3\}}, \frac{h_5}{\{0.5, 0.6, 0.7\}}, \frac{h_6}{\{0.7\}}\right\},\\
\widetilde{F}^c(e_3) &= \left\{\frac{h_1}{\{0.6, 0.8\}}, \frac{h_2}{\{0.3, 0.4\}}, \frac{h_3}{\{0.1, 0.2\}}, \frac{h_4}{\{0.5, 0.7\}}, \frac{h_5}{\{0.4, 0.6\}}, \frac{h_6}{\{0.3\}}\right\},\\
\widetilde{F}^c(e_4) &= \left\{\frac{h_1}{\{0.4, 0.5, 0.7\}}, \frac{h_2}{\{0.8\}}, \frac{h_3}{\{0.5\}}, \frac{h_4}{\{0.3, 0.4\}}, \frac{h_5}{\{0.4, 0.5\}}, \frac{h_6}{\{0.2\}}\right\},\\
\widetilde{F}^c(e_5) &= \left\{\frac{h_1}{\{0.4\}}, \frac{h_2}{\{0.5, 0.7, 0.8\}}, \frac{h_3}{\{0.3, 0.5\}}, \frac{h_4}{\{0.6, 0.8\}}, \frac{h_5}{\{0.3, 0.5\}}, \frac{h_6}{\{0.5, 0.7\}}\right\}.
\end{aligned} \tag{8}$$

Definition 19. The AND operation on two hesitant fuzzy soft sets $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ which is denoted by $(\widetilde{F}, A) \wedge (\widetilde{G}, B)$ is defined by $(\widetilde{F}, A) \wedge (\widetilde{G}, B) = (\widetilde{J}, A \times B)$, where $\widetilde{J}(\alpha, \beta) = \widetilde{F}(\alpha) \cap \widetilde{G}(\beta)$, for all $(\alpha, \beta) \in A \times B$.

Definition 20. The OR operation on the two hesitant fuzzy soft sets $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ which is denoted by $(\widetilde{F}, A) \vee (\widetilde{G}, B)$ is defined by $(\widetilde{F}, A) \vee (\widetilde{G}, B) = (\widetilde{O}, A \times B)$, where $\widetilde{O}(\alpha, \beta) = \widetilde{F}(\alpha) \cup \widetilde{G}(\beta)$, for all $(\alpha, \beta) \in A \times B$.

Example 21. The results of "AND" and "OR" operations on the hesitant fuzzy soft sets $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ in Example 13 are shown in Tables 4 and 5, respectively.

Theorem 22 (De Morgan's laws of hesitant fuzzy soft sets). *Let $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ be two hesitant fuzzy soft sets over U; we have*

(1) $((\widetilde{F}, A) \wedge (\widetilde{G}, B))^c = (\widetilde{F}, A)^c \vee (\widetilde{G}, B)^c$,

(2) $((\widetilde{F}, A) \vee (\widetilde{G}, B))^c = (\widetilde{F}, A)^c \wedge (\widetilde{G}, B)^c$.

Proof. (1) Suppose that $(\widetilde{F}, A) \wedge (\widetilde{G}, B) = (\widetilde{J}, A \times B)$. Therefore $((\widetilde{F}, A) \wedge (\widetilde{G}, B))^c = (\widetilde{J}, A \times B)^c = (\widetilde{J}^c, A \times B)$. Similarly, $(\widetilde{F}, A)^c \vee (\widetilde{G}, B)^c = (\widetilde{F}^c, A) \vee (\widetilde{G}^c, B) = (\widetilde{O}, A \times B)$. Now take $(\alpha, \beta) \in A \times B$; therefore,

$$\begin{aligned}
&\widetilde{J}^c(\alpha, \beta)\\
&= \left(\widetilde{J}(\alpha, \beta)\right)^c = \left(F(\alpha) \cap G(\beta)\right)^c\\
&= \left\{\left\langle u, h_{\widetilde{A}}(u) \cap h_{\widetilde{B}}(u)\right\rangle \mid u \in U\right\}^c\\
&= \left\{\left\langle u, \left\{h \in \left(h_{\widetilde{A}} \cup h_{\widetilde{B}}\right) \mid h \le \min\left(h^+_{\widetilde{A}}, h^+_{\widetilde{B}}\right)\right\}\right\rangle \mid u \in U\right\}^c\\
&= \left\{\left\langle u, \left\{1 - h \mid h \in \left(h_{\widetilde{A}} \cup h_{\widetilde{B}}\right) \wedge h \le \min\left(h^+_{\widetilde{A}}, h^+_{\widetilde{B}}\right)\right\}\right\rangle \mid u \in U\right\}\\
&= \left\{\left\langle u, \left\{1 - h \mid (1 - h) \in \left(1 - h_{\widetilde{A}}\right) \cup \left(1 - h_{\widetilde{B}}\right) \wedge (1 - h) \ge \max\left(\left(1 - h_{\widetilde{A}}\right)^-, \left(1 - h_{\widetilde{B}}\right)^-\right)\right\}\right\rangle \mid u \in U\right\}\\
&= \left\{\left\langle u, \left\{(1 - h) \in \left(1 - h_{\widetilde{A}}\right) \cup \left(1 - h_{\widetilde{B}}\right) \mid (1 - h) \ge \max\left(\left(1 - h_{\widetilde{A}}\right)^-, \left(1 - h_{\widetilde{B}}\right)^-\right)\right\}\right\rangle \mid u \in U\right\},
\end{aligned}$$

Table 5: The result of "OR" operation on $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$.

U	e_1, e_1	e_1, e_2	e_1, e_3	e_2, e_1	e_2, e_2	e_2, e_3
h_1	{0.3}	{0.4, 0.5, 0.6, 0.7}	{0.2, 0.3, 0.4}	{0.4, 0.6, 0.7}	{0.4, 0.5, 0.6, 0.7}	{0.4, 0.6, 0.7}
h_2	{0.6, 0.8}	{0.7, 0.8}	{0.6, 0.7}	{0.6, 0.7, 0.8}	{0.7, 0.8}	{0.6, 0.7, 0.8}
h_3	{0.4, 0.5}	{0.9}	{0.8, 0.9}	{0.6, 0.8}	{0.9}	{0.8, 0.9}
h_4	{0.3, 0.4, 0.5}	{0.8, 0.9}	{0.3, 0.5}	{0.7, 0.9}	{0.8, 0.9}	{0.7, 0.9}
h_5	{0.5}	{0.5}	{0.4, 0.5, 0.6}	{0.5}	{0.5}	{0.4, 0.5, 0.6}
h_6	{0.7}	{0.6, 0.7}	{0.7}	{0.7}	{0.5}	{0.7}

$$\begin{aligned}
&\widetilde{O}(\alpha, \beta) \\
&\quad = F^c(\alpha) \cup G^c(\beta) \\
&\quad = \left\{\left\langle u, h_{\widetilde{A}}(u)\right\rangle \mid u \in U\right\}^c \cup \left\{\left\langle u, h_{\widetilde{B}}(u)\right\rangle \mid u \in U\right\}^c \\
&\quad = \left\{\left\langle u, 1 - h_{\widetilde{A}}(u)\right\rangle \mid u \in U\right\} \cup \left\{\left\langle u, 1 - h_{\widetilde{B}}(u)\right\rangle \mid u \in U\right\} \\
&\quad = \left\{\left\langle u, \left(1 - h_{\widetilde{A}}\right) \cup \left(1 - h_{\widetilde{B}}\right)\right\rangle \mid u \in U\right\} \\
&\quad = \left\{\left\langle u, \left\{(1-h) \in \left(1 - h_{\widetilde{A}}\right) \cup \left(1 - h_{\widetilde{B}}\right) \mid (1-h)\right.\right.\right. \\
&\qquad \left.\left.\left. \geq \max\left(\left(1 - h_{\widetilde{A}}\right)^-, \left(1 - h_{\widetilde{B}}\right)^-\right)\right\}\right\rangle \mid u \in U\right\}.
\end{aligned} \tag{9}$$

Hence, $\widetilde{J}^c(\alpha, \beta) = \widetilde{O}(\alpha, \beta)$ is proved.

(2) Similar to the above process, it is easy to prove that $((\widetilde{F}, A) \vee (\widetilde{G}, B))^c = (\widetilde{F}, A)^c \wedge (\widetilde{G}, B)^c$. □

Theorem 23. *Let $(\widetilde{F}, A)$, $(\widetilde{G}, B)$, and $(\widetilde{J}, C)$ be three hesitant fuzzy soft sets over U. Then the associative law of hesitant fuzzy soft sets holds as follows:*

$$\begin{aligned}
&\left(\widetilde{F}, A\right) \wedge \left(\left(\widetilde{G}, B\right) \wedge \left(\widetilde{J}, C\right)\right) = \left(\left(\widetilde{F}, A\right) \wedge \left(\widetilde{G}, B\right)\right) \wedge \left(\widetilde{J}, C\right), \\
&\left(\widetilde{F}, A\right) \vee \left(\left(\widetilde{G}, B\right) \vee \left(\widetilde{J}, C\right)\right) = \left(\left(\widetilde{F}, A\right) \vee \left(\widetilde{G}, B\right)\right) \vee \left(\widetilde{J}, C\right).
\end{aligned} \tag{10}$$

Proof. For all $\alpha \in A$, $\beta \in B$, and $\delta \in C$, because $\widetilde{F}(\alpha) \cap (\widetilde{G}(\beta) \cap \widetilde{J}(\delta)) = (\widetilde{F}(\alpha) \cap \widetilde{G}(\beta)) \cap \widetilde{J}(\delta)$, we can conclude that $(\widetilde{F}, A) \wedge ((\widetilde{G}, B) \wedge (\widetilde{J}, C)) = ((\widetilde{F}, A) \wedge (\widetilde{G}, B)) \wedge (\widetilde{J}, C)$ holds.

Similarly, we can also conclude that $(\widetilde{F}, A) \vee ((\widetilde{G}, B) \vee (\widetilde{J}, C)) = ((\widetilde{F}, A) \vee (\widetilde{G}, B)) \vee (\widetilde{J}, C)$. □

Remark 24. The distribution law of hesitant fuzzy soft sets does not hold. That is,

$$\begin{aligned}
&\left(\widetilde{F}, A\right) \wedge \left(\left(\widetilde{G}, B\right) \vee \left(\widetilde{J}, C\right)\right) \\
&\quad \neq \left(\left(\widetilde{F}, A\right) \wedge \left(\widetilde{J}, C\right)\right) \vee \left(\left(\widetilde{G}, B\right) \wedge \left(\widetilde{J}, C\right)\right), \\
&\left(\widetilde{F}, A\right) \vee \left(\left(\widetilde{G}, B\right) \wedge \left(\widetilde{J}, C\right)\right) \\
&\quad \neq \left(\left(\widetilde{F}, A\right) \vee \left(\widetilde{J}, C\right)\right) \wedge \left(\left(\widetilde{G}, B\right) \vee \left(\widetilde{J}, C\right)\right).
\end{aligned} \tag{11}$$

For all $\alpha \in A$, $\beta \in B$, and $\delta \in C$, because $\widetilde{F}(\alpha) \cap (\widetilde{G}(\beta) \cup \widetilde{J}(\delta)) \neq (\widetilde{F}(\alpha) \cap \widetilde{G}(\beta)) \cup (\widetilde{F}(\alpha) \cap \widetilde{J}(\delta))$ for the hesitant fuzzy sets, we can conclude that $(\widetilde{F}, A) \wedge ((\widetilde{G}, B) \vee (\widetilde{J}, C)) \neq ((\widetilde{F}, A) \wedge (\widetilde{J}, C)) \vee ((\widetilde{G}, B) \wedge (\widetilde{J}, C))$. Similarly, we can also conclude that $(\widetilde{F}, A) \vee ((\widetilde{G}, B) \wedge (\widetilde{J}, C)) \neq ((\widetilde{F}, A) \vee (\widetilde{J}, C)) \wedge ((\widetilde{G}, B) \vee (\widetilde{J}, C))$.

Table 6: The Union of hesitant fuzzy soft sets $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$.

U	e_1	e_2	e_3
h_1	{0.3}	{0.4, 0.5, 0.6, 0.7}	{0.2, 0.4}
h_2	{0.6, 0.8}	{0.7, 0.8}	{0.6, 0.7}
h_3	{0.4, 0.5}	{0.9}	{0.8, 0.9}
h_4	{0.3, 0.4, 0.5}	{0.8, 0.9}	{0.3, 0.5}
h_5	{0.5}	{0.5}	{0.4, 0.6}
h_6	{0.7}	{0.5}	{0.7}

Definition 25. Union of two hesitant fuzzy soft sets $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ over U is the hesitant fuzzy soft set $(\widetilde{J}, C)$, where $C = A \cup B$, and for all $e \in C$,

$$\begin{aligned}
\widetilde{J}(e) &= \widetilde{F}(e), \quad \text{if } e \in A - B, \\
&= \widetilde{G}(e), \quad \text{if } e \in B - A, \\
&= \widetilde{F}(e) \cup \widetilde{G}(e), \quad \text{if } e \in A \cap B.
\end{aligned} \tag{12}$$

We write $(\widetilde{F}, A) \widetilde{\cup} (\widetilde{G}, B) = (\widetilde{J}, C)$.

Example 26. Reconsider Example 13, the Union of hesitant fuzzy soft sets $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ is shown in Table 6.

Definition 27. Intersection of two hesitant fuzzy soft sets $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ with $A \cap B \neq \phi$ over U is the hesitant fuzzy soft set $(\widetilde{J}, C)$, where $C = A \cap B$, and for all $e \in C$, $\widetilde{J}(e) = \widetilde{F}(e) \cap \widetilde{G}(e)$.

We write $(\widetilde{F}, A) \widetilde{\cap} (\widetilde{G}, B) = (\widetilde{J}, C)$.

Example 28. Reconsider Example 13, the Intersection of hesitant fuzzy soft sets $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ is shown in Table 7.

Theorem 29. *Let $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ be two hesitant fuzzy soft sets over U. Then,*

(1) $(\widetilde{F}, A) \widetilde{\cup} (\widetilde{F}, A) = (\widetilde{F}, A)$,
(2) $(\widetilde{F}, A) \widetilde{\cap} (\widetilde{F}, A) = (\widetilde{F}, A)$,
(3) $(\widetilde{F}, A) \cup \widetilde{\Phi}_A = (\widetilde{F}, A)$,
(4) $(\widetilde{F}, A) \cap \widetilde{\Phi}_A = \widetilde{\Phi}_A$,
(5) $(\widetilde{F}, A) \cup \widetilde{U}_A = \widetilde{U}_A$,

Table 7: The Intersection of hesitant fuzzy soft sets $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$.

U	e_1	e_2
h_1	{0.2, 0.3}	{0.4, 0.5, 0.6, 0.7}
h_2	{0.5, 0.6}	{0.5, 0.7, 0.8}
h_3	{0.3}	{0.6, 0.8}
h_4	{0.3, 0.5}	{0.7, 0.8, 0.9}
h_5	{0.3, 0.4, 0.5}	{0.3, 0.4, 0.5}
h_6	{0.3}	{0.3}

(6) $(\widetilde{F}, A) \widetilde{\cap} \widetilde{U}_A = (\widetilde{F}, A)$,

(7) $(\widetilde{F}, A) \widetilde{\cup} (\widetilde{G}, B) = (\widetilde{G}, B) \widetilde{\cup} (\widetilde{F}, A)$,

(8) $(\widetilde{F}, A) \widetilde{\cap} (\widetilde{G}, B) = (\widetilde{G}, B) \widetilde{\cap} (\widetilde{F}, A)$.

Theorem 30. *Let $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ be two hesitant fuzzy soft sets over U. Then,*

(1) $((\widetilde{F}, A) \widetilde{\cup} (\widetilde{G}, B))^c \widetilde{\subseteq} (\widetilde{F}, A)^c \widetilde{\cup} (\widetilde{G}, B)^c$,

(2) $(\widetilde{F}, A)^c \widetilde{\cap} (\widetilde{G}, B)^c \widetilde{\subseteq} ((\widetilde{F}, A) \widetilde{\cap} (\widetilde{G}, B))^c$.

Proof. (1) Suppose that $(\widetilde{F}, A) \widetilde{\cup} (\widetilde{G}, B) = (\widetilde{J}, C)$, where $C = A \cup B$, and for all $e \in C$,

$$\begin{aligned} \widetilde{J}(e) &= \widetilde{F}(e), \quad \text{if } e \in A - B, \\ &= \widetilde{G}(e), \quad \text{if } e \in B - A, \\ &= \widetilde{F}(e) \cup \widetilde{G}(e), \quad \text{if } e \in A \cap B. \end{aligned} \tag{13}$$

Thus $((\widetilde{F}, A) \cup (\widetilde{G}, B))^c = (\widetilde{J}, C)^c = (\widetilde{J}^c, C)$, and

$$\begin{aligned} \widetilde{J}^c(e) &= \widetilde{F}^c(e), \quad \text{if } e \in A - B, \\ &= \widetilde{G}^c(e), \quad \text{if } e \in B - A, \\ &= \widetilde{F}^c(e) \cap \widetilde{G}^c(e), \quad \text{if } e \in A \cap B. \end{aligned} \tag{14}$$

Similarly, suppose that $(\widetilde{F}, A)^c \widetilde{\cup} (\widetilde{G}, B)^c = (\widetilde{F}^c, A) \widetilde{\cup} (\widetilde{G}^c, B) = (\widetilde{O}, C)$, where $C = A \cup B$, and for all $e \in C$,

$$\begin{aligned} \widetilde{O}(e) &= \widetilde{F}^c(e), \quad \text{if } e \in A - B, \\ &= \widetilde{G}^c(e), \quad \text{if } e \in B - A, \\ &= \widetilde{F}^c(e) \cup \widetilde{G}^c(e), \quad \text{if } e \in A \cap B. \end{aligned} \tag{15}$$

Obviously, for all $e \in C$, $\widetilde{J}^c(e) \widetilde{\subseteq} \widetilde{O}(e)$. Part (1) of Theorem 30 is proved.

(2) Similar to the above process, it is easy to prove that $(\widetilde{F}, A)^c \widetilde{\cap} (\widetilde{G}, B)^c \widetilde{\subseteq} ((\widetilde{F}, A) \widetilde{\cap} (\widetilde{G}, B))^c$. □

Theorem 31. *Let $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ be two hesitant fuzzy soft sets over U. Then,*

(1) $(\widetilde{F}, A)^c \widetilde{\cap} (\widetilde{G}, B)^c \widetilde{\subseteq} ((\widetilde{F}, A) \widetilde{\cup} (\widetilde{G}, B))^c$,

(2) $((\widetilde{F}, A) \widetilde{\cap} (\widetilde{G}, B))^c \widetilde{\subseteq} (\widetilde{F}, A)^c \widetilde{\cup} (\widetilde{G}, B)^c$.

Proof. (1) Suppose that $(\widetilde{F}, A) \widetilde{\cup} (\widetilde{G}, B) = (\widetilde{J}, C)$, where $C = A \cup B$, and for all $e \in C$,

$$\begin{aligned} \widetilde{J}(e) &= \widetilde{F}(e), \quad \text{if } e \in A - B, \\ &= \widetilde{G}(e), \quad \text{if } e \in B - A, \\ &= \widetilde{F}(e) \cup \widetilde{G}(e), \quad \text{if } e \in A \cap B. \end{aligned} \tag{16}$$

Then $((\widetilde{F}, A) \cup (\widetilde{G}, B))^c = (\widetilde{J}, C)^c = (\widetilde{J}^c, C)$, and

$$\begin{aligned} \widetilde{J}^c(e) &= \widetilde{F}^c(e), \quad \text{if } e \in A - B, \\ &= \widetilde{G}^c(e), \quad \text{if } e \in B - A, \\ &= \widetilde{F}^c(e) \cap \widetilde{G}^c(e), \quad \text{if } e \in A \cap B. \end{aligned} \tag{17}$$

Similarly, suppose $(\widetilde{F}, A)^c \widetilde{\cap} (\widetilde{G}, B)^c = (\widetilde{F}^c, A) \widetilde{\cap} (\widetilde{G}^c, B) = (\widetilde{O}, J)$, where $J = A \cap B$. Then, for all $e \in J$, $\widetilde{O}(e) = F^c(e) \cap G^c(e)$.

Hence, it is obvious that $J \subseteq C$ and, for all $e \in J$, $\widetilde{O}(e) = \widetilde{J}^c(e)$; thus, $(\widetilde{F}, A)^c \widetilde{\cap} (\widetilde{G}, B)^c \widetilde{\subseteq} ((\widetilde{F}, A) \widetilde{\cup} (\widetilde{G}, B))^c$ is proved.

(2) Similar to the above process, it is easy to prove that $((\widetilde{F}, A) \widetilde{\cap} (\widetilde{G}, B))^c \widetilde{\subseteq} (\widetilde{F}, A)^c \widetilde{\cup} (\widetilde{G}, B)^c$. □

Theorem 32. *Let $(\widetilde{F}, A)$ and $(\widetilde{G}, A)$ be two hesitant fuzzy soft sets over U; then,*

(1) $((\widetilde{F}, A) \widetilde{\cup} (\widetilde{G}, A))^c = (\widetilde{F}, A)^c \widetilde{\cap} (\widetilde{G}, A)^c$,

(2) $((\widetilde{F}, A) \widetilde{\cap} (\widetilde{G}, A))^c = (\widetilde{F}, A)^c \widetilde{\cup} (\widetilde{G}, A)^c$.

Proof. (1) Suppose that $(\widetilde{F}, A) \widetilde{\cup} (\widetilde{G}, A) = (\widetilde{J}, A)$, and for all $e \in A$,

$$\widetilde{J}(e) = \widetilde{F}(e) \cup \widetilde{G}(e). \tag{18}$$

Then $((\widetilde{F}, A) \cup (\widetilde{G}, A))^c = (\widetilde{J}, A)^c = (\widetilde{J}^c, A)$, and for all $e \in A$,

$$\widetilde{J}^c(e) = \widetilde{F}^c(e) \cap \widetilde{G}^c(e). \tag{19}$$

Similarly, suppose that $(\widetilde{F}, A)^c \widetilde{\cap} (\widetilde{G}, A)^c = (\widetilde{F}^c, A) \widetilde{\cap} (\widetilde{G}^c, A) = (\widetilde{O}, A)$, and for all $e \in A$,

$$\widetilde{O}(e) = \widetilde{F}^c(e) \cap \widetilde{G}^c(e). \tag{20}$$

Hence, $\widetilde{O}(e) = \widetilde{J}^c(e)$; the $((\widetilde{F}, A) \widetilde{\cup} (\widetilde{G}, A))^c = (\widetilde{F}, A)^c \widetilde{\cap} (\widetilde{G}, A)^c$ is proved.

(2) Similar to the above process, it is easy to prove that $((\widetilde{F}, A) \widetilde{\cap} (\widetilde{G}, A))^c = (\widetilde{F}, A)^c \widetilde{\cup} (\widetilde{G}, A)^c$. □

4. Application of Hesitant Fuzzy Soft Sets

Roy and Maji [14] presented an algorithm to solve the decision making problems according to a comparison table from the fuzzy soft set. Later, Kong et al. [15] pointed out that the algorithm in Roy and Maji [14] was incorrect by giving a counterexample. Kong et al. [15] also presented a modified algorithm which was based on the comparison of choice values of different objects [13]. However, Feng et al. [16] further explored the fuzzy soft set based decision making problems

Table 8: The tabular representation of the hesitant fuzzy soft set $(\widetilde{F}, A)$ in Example 35.

U	e_1	e_2	e_3	e_4	e_5
x_1	{0.5, 0.6}	{0.3, 0.5}	{0.3}	{0.3, 0.4}	{0.6, 0.9}
x_2	{0.7}	{0.4}	{0.2, 0.4, 0.5}	{0.4, 0.5}	{0.5, 0.6}
x_3	{0.4, 0.6}	{0.6, 0.7}	{0.5, 0.6}	{0.6}	{0.2, 0.3, 0.5}
x_4	{0.3, 0.5}	{0.4, 0.6}	{0.5}	{0.6, 0.7}	{0.2, 0.4, 0.5}
x_5	{0.5, 0.6, 0.8}	{0.3}	{0.3, 0.4}	{0.2, 0.3, 0.5}	{0.7, 0.8}

more deeply and pointed out that the concept of choice values was not suitable to solve the decision making problems involving fuzzy soft sets, though it was an efficient method to solve the crisp soft set based decision making problems. They proposed a novel approach by using the level soft sets to solve the fuzzy soft set based decision making problems. Thus, some decision making problems that cannot be solved by the methods in Roy and Maji [14] and Kong et al. [15] can be successfully solved by this approach. In the following, we will apply this level soft sets method to hesitant fuzzy soft set based decision making problems.

Let $U = \{u_1, u_2, \ldots, u_n\}$ and $(\widetilde{F}, A)$ be a hesitant fuzzy soft set over U. For every $e \in A$, $\widetilde{F}(e) = \{u_1/h_{\widetilde{A}}(u_1), u_2/h_{\widetilde{A}}(u_2), \ldots, u_n/h_{\widetilde{A}}(u_n)\}$, we can calculate the score of each hesitant fuzzy element by Definition 7. Thus, we define an induced fuzzy set with respect to e in U as $\mu_{\widetilde{F}(e)} = \{u_1/s(h_{\widetilde{A}}(u_1)), u_2/s(h_{\widetilde{A}}(u_2)), \ldots, u_n/s(h_{\widetilde{A}}(u_n))\}$. By using this method, we can change a hesitant fuzzy set $\widetilde{F}(e)$ into an induced fuzzy set $\mu_{\widetilde{F}(e)}$. Furthermore, once the induced fuzzy soft set of a hesitant fuzzy soft set has been obtained, we can determine the optimal alternative according to Feng's [16] algorithm. The explicit algorithm of the decision making based on hesitant fuzzy soft set is given as follows.

Algorithm 33. Consider the following.

(1) Input the hesitant fuzzy soft set $(\widetilde{F}, A)$.

(2) Compute the induced fuzzy soft set $\Delta_{\widetilde{F}} = (\widetilde{\Gamma}, A)$.

(3) Input a threshold fuzzy set $\lambda : A \to [0, 1]$ (or give a threshold value $t \in [0, 1]$; or choose the midlevel decision rule; or choose the top-level decision rule) for decision making.

(4) Compute the level soft set $L(\Delta_{\widetilde{F}}; \lambda)$ of $\Delta_{\widetilde{F}}$ with respect to the threshold fuzzy set λ (or the t-level soft set $L(\Delta_{\widetilde{F}}; t)$; or the mid-level soft set $L(\Delta_{\widetilde{F}}; \text{mid})$; or the top-level soft set $L(\Delta_{\widetilde{F}}; \max)$).

(5) Present the level soft set $L(\Delta_{\widetilde{F}}; \lambda)$ (or $L(\Delta_{\widetilde{F}}; t)$; or $L(\Delta_{\widetilde{F}}; \text{mid})$; or $L(\Delta_{\widetilde{F}}; \max)$) in tabular form and compute the choice value c_i of u_i, for all i.

(6) The optimal decision is to select $u_j = \text{argmax}_i c_i$.

(7) If there are more than one u_js, then any one of u_j may be chosen.

Remark 34. In order to get a unique optimal choice according to the above algorithm, the decision makers can go back to the third step and change the threshold (or decision criteria) in case that there is more than one optimal choice that can be obtained in the last step. Moreover, the final optimal decision can be adjusted according to the decision makers' preferences.

Table 9: The tabular representation of the induced fuzzy soft set $\Delta_{\widetilde{F}} = (\widetilde{\Gamma}, A)$ in Example 35.

U	e_1	e_2	e_3	e_4	e_5
x_1	0.55	0.4	0.3	0.35	0.75
x_2	0.7	0.4	0.37	0.45	0.55
x_3	0.5	0.65	0.55	0.6	0.33
x_4	0.4	0.5	0.5	0.65	0.37
x_5	0.63	0.3	0.35	0.33	0.75

We adopt the following example to illustrate the idea of algorithm given above.

Example 35. Consider a retailer planning to open a new store in the city. There are five sites, x_i $(i = 1, 2, \ldots, 5)$, to be selected. Five attributes are considered: market (e_1), traffic (e_2), rent price (e_3), competition (e_4), and brand improvement (e_5). Suppose that the retailer evaluates the optional five sites under various attributes with hesitant fuzzy element; then, hesitant fuzzy soft set $(\widetilde{F}, A)$ can describe the characteristics of the sites under the hesitant fuzzy information, which is shown in Table 8.

In order to select the optimal sites to open the store based on the above hesitant fuzzy soft set, according to Algorithm 33, we can calculate the score of each hesitant fuzzy element and obtain the induced fuzzy soft set $\Delta_{\widetilde{F}} = (\widetilde{\Gamma}, A)$, which is shown as in Table 9.

As an adjustable approach, one can use different rules (or the thresholds) to get different decision results, according to his or her preference. For example, we use the midlevel decision rule in our paper and the midlevel threshold fuzzy set of $\Delta_{\widetilde{F}}$ is as $\widetilde{\text{mid}}_{\Delta_{\widetilde{F}}} = \{(e_1, 0.556), (e_2, 0.45), (e_3, 0.414), (e_4, 0.476), (e_5, 0.55)\}$. Furthering, we can get the midlevel soft set $L(\Delta_{\widetilde{F}}; \text{mid})$ of $\Delta_{\widetilde{F}}$, whose tabular form is as Table 10 shows. The choice values of each site are also shown in the last column of Table 10.

According to Table 10, the maximum choice value is 3 and so the optimal decision is to select x_3. Therefore, the retailer should select site 3 as the best site to open a store.

Table 10: The tabular representation of the midlevel soft set $L(\Delta_{\tilde{F}}; \text{mid})$ with choice values in Example 35.

U	e_1	e_2	e_3	e_4	e_5	Choice value
x_1	0	0	0	0	1	1
x_2	1	0	0	0	0	1
x_3	0	1	1	1	0	3
x_4	0	1	0	1	0	2
x_5	1	0	0	0	1	2

5. Conclusion

In this paper we consider the notion of hesitant fuzzy soft sets which combine the hesitant fuzzy sets and soft sets. Then we define the complement, "AND", "OR", union and intersection operations on hesitant fuzzy soft sets. The basic properties such as De Morgan's laws and the relevant laws of hesitant fuzzy soft sets are proved. Finally, we apply it to a decision making problem with the help of level soft set.

Our research can be explored deeply in two directions in the future. Firstly, we can combine other membership functions and soft sets to make novel soft sets which have different forms; Secondly, we can not only explore the application of hesitant fuzzy soft set in decision making more deeply, but also apply the hesitant fuzzy soft set in many other areas such as forecasting and data analysis.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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