Heterotic Vortex Strings

David Tong

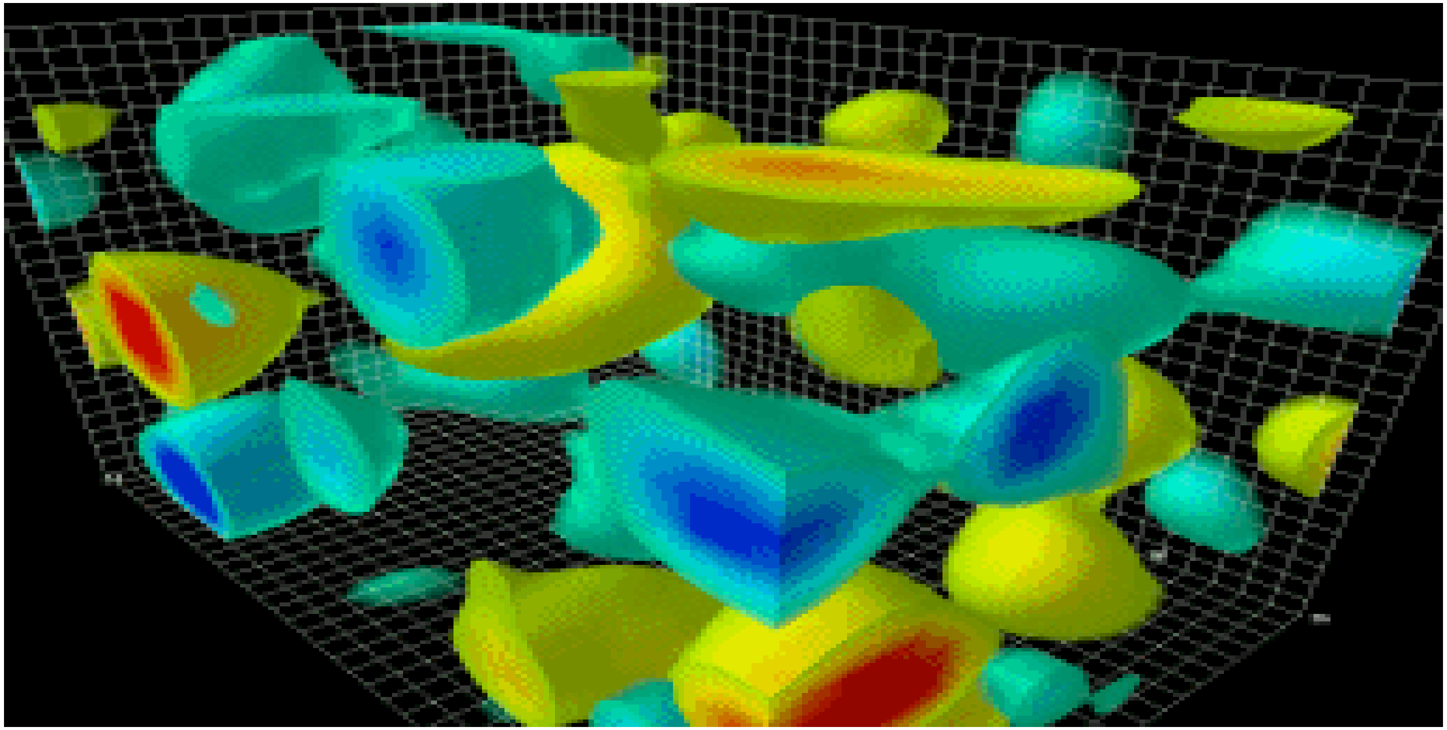
UNIVERSITY OF CAMBRIDGE

The Take-Home Message

- For 25 years we've known that 4d non-abelian gauge theories share certain features with 2d sigma-models
  - Asymptotic freedom
  - Confinement
  - Dynamically generated mass gap
  - Anomalies
  - Instantons
  - Chiral Symmetry Breaking
  - Large N limits
- In fact, there are *quantitative* links between the two. The relationship is derived through the dynamics of solitonic vortex strings.

A Cartoon of the Basic Idea

- Take a strongly coupled theory with U(N) gauge group and some fundamental scalar fields.



Graphics by Leinweber

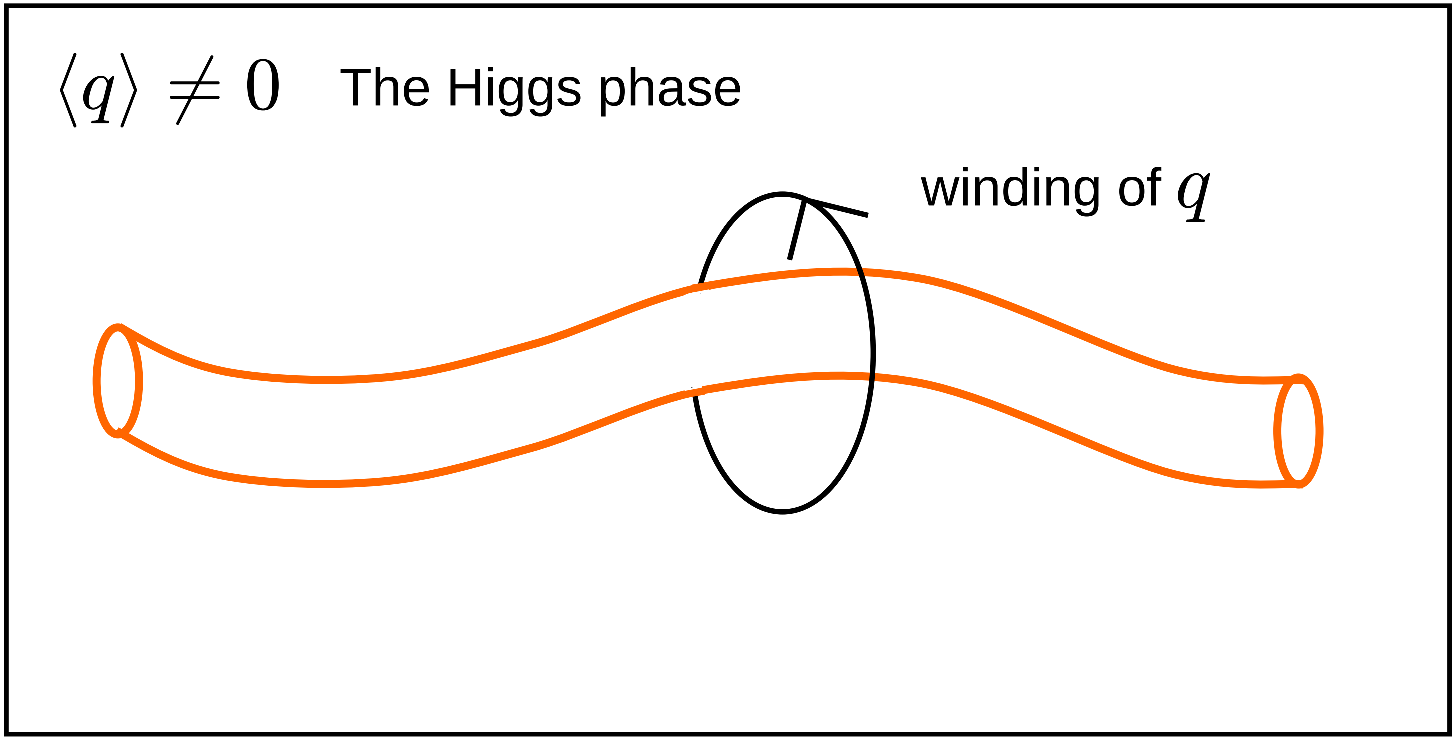
A Cartoon of the Basic Idea

- Deform the theory by inducing an expectation value for the scalar fields
  - If the gauge group is completely broken, the theory now lies in the weakly coupled Higgs phase

$\langle q \rangle \neq 0$ The Higgs phase

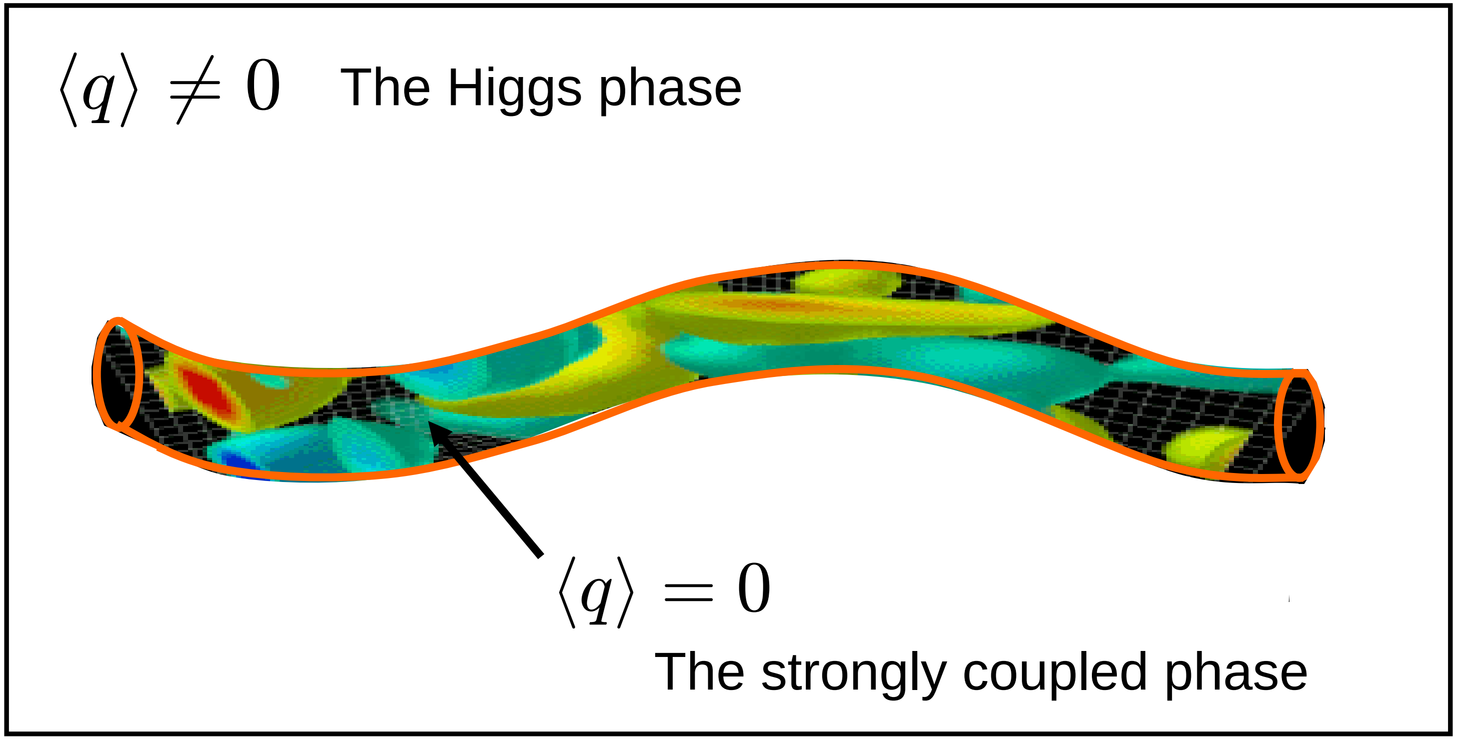
A Cartoon of the Basic Idea

- The theory now admits vortex strings, supported by the phase of the scalar winding at infinity



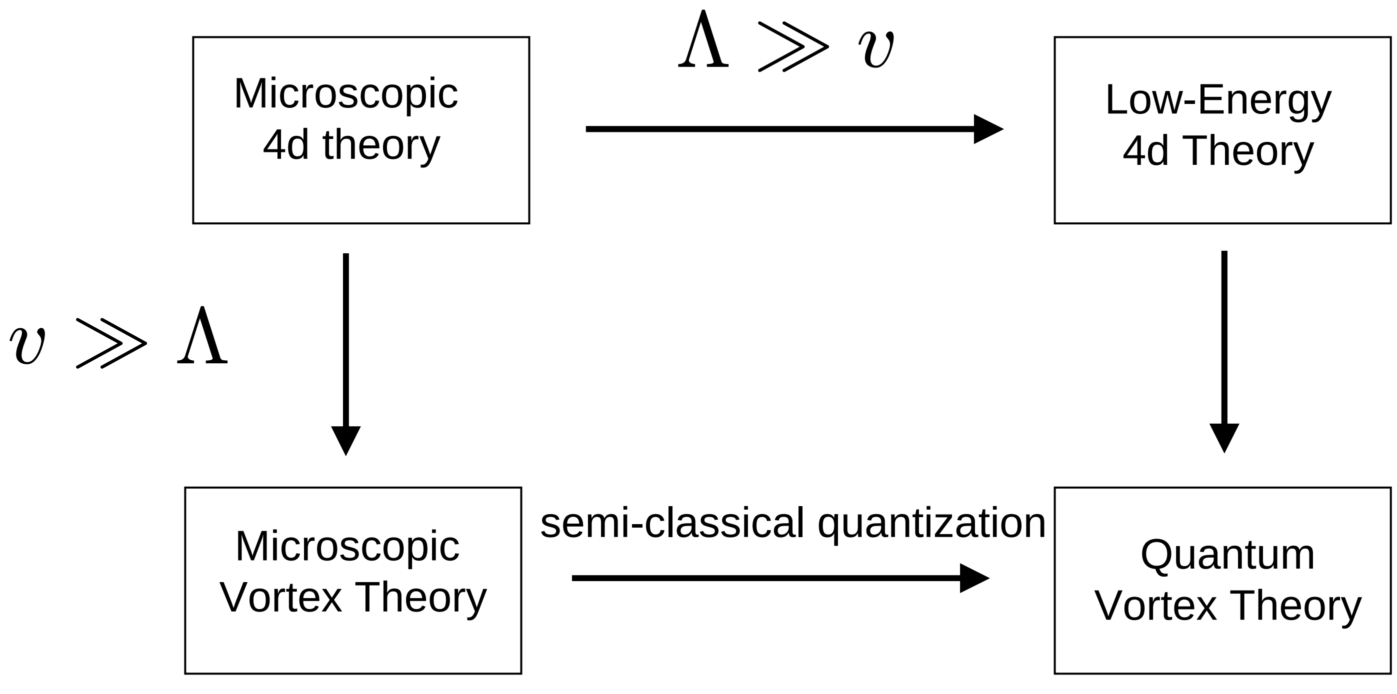
A Cartoon of the Basic Idea

- The interior of the vortex string is a strongly coupled system
  - The vortex string knows about the original 4d gauge theory.



Two Paths to Soliton Quantization

Two scales: Λ the strong coupling scale
v the symmetry breaking scale



The Results of this Technique

By studying the worldsheet theory of the vortex string, we can reconstruct the following results about four dimensional gauge theories:

$$4\text{d } \mathcal{N} = 2 \text{ Theories: } (N_f \geq N_c)$$

- Exact BPS Mass Spectrum, including all quantum effects
- The Dimensions of Chiral Primary Operators at Superconformal Points
- The Seiberg-Witten Curve as the twisted superpotential of the worldsheet theory.

The Results of this Technique

$4\text{d}\ \mathcal{N} = 1$ Theories: $(N_f = N_c)$

- Quantum Numbers of Spectrum: i.e. Confinement
- $\det M - B\tilde{B} = \Lambda^{2N}$

The purpose of this talk is to describe these results for theories with N=1 supersymmetry: the vortex strings have N=(0,2) supersymmetry on their worldsheet. They are called *heterotic vortex strings.*

The Basic Theory

Starting point: d=3+1 with U(N) gauge group and N_f =N fundamental flavours.

$$L = \frac{1}{4e^2}\,\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu} + \sum_{i=1}^{N_f} |\mathcal{D}q_i|^2$$
$$-\frac{e^2}{2}\,\mathrm{Tr}\,(\sum_i q_i \otimes q_i^\dagger - v^2)^2$$

We write $q^a{}_i$ where a=1,…,N is the colour index, and i=1,…,N_f is the flavour index.

The 4d Theory

- **Vacuum:** The ground state is unique (up to a gauge transformation)

$$q^a{}_i = v\,\delta^a{}_i$$

- **Spectrum:** The theory has a mass gap,with

$$m_\gamma = m_q \sim ev$$

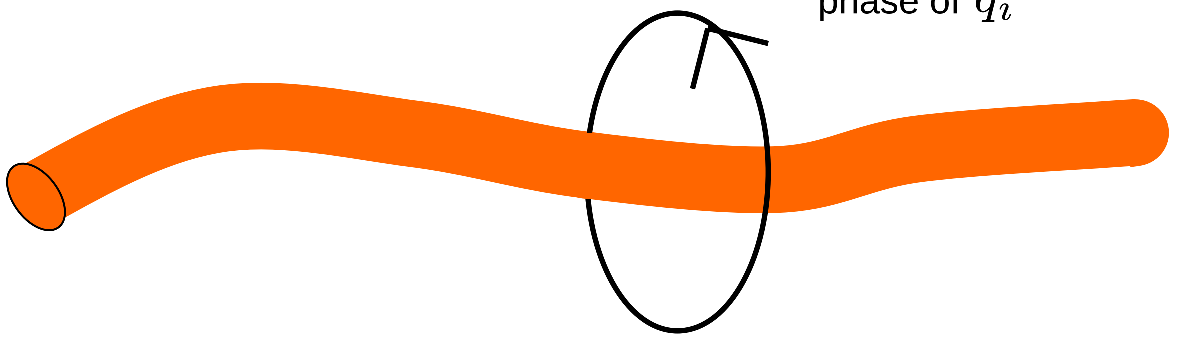
- **Symmetries:** The theory lies in the “colour-flavour” locked phase

$$U(N)\times SU(N_f) \rightarrow SU(N)_{\rm diag}$$

Note that overall U(1) is broken: $\Pi_1(U(1)) \cong \mathbf{Z}$ $\Longrightarrow$ Vortices

Vortices

Broken U(1) gauge symmetry $\Longrightarrow$ Vortices



$$(B_3)^a{}_b = e^2\left(\sum_i q_i^a q_{ib}^\dagger - v^2 \delta^a{}_b\right)$$

$$(\mathcal{D}_z q_i)^a = 0$$

$z = x_1 + ix_2$

$$T_{\rm vortex} = 2\pi v^2$$

Nielsen and Olesen, '73

Vortex Moduli Space

Suppose we have an Abelian vortex solution $B_\star$,$q_\star$. We can trivially embed this in the non-Abelian theory.

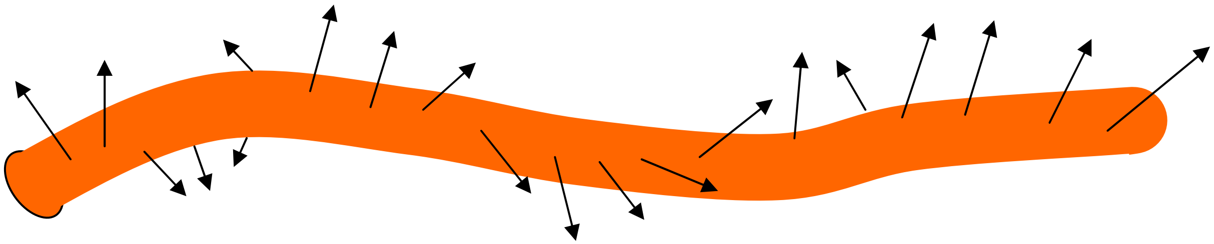
$$B = \begin{pmatrix} B_\star & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \qquad q = \begin{pmatrix} q_\star & & & \\ & v & & \\ & & \ddots & \\ & & & v \end{pmatrix}$$

Different embeddings $\Longrightarrow$ moduli space of vortex

$$SU(N)_{\rm diag}/SU(N-1)\times U(1) \cong \mathbf{CP}^{N-1}$$

Vortex Dynamics

The low energy dynamics of an infinite, straight vortex string is the d=1+1 sigma model with target space $\mathbf{C} \times \mathbf{CP}^{N-1}$



Size of $\mathbf{CP}^{N-1}$ is

$$r = \frac{2\pi}{e^2}$$

This means that when the 4d theory is weakly coupled, the 2d theory is also weakly coupled.

More General Vortex Dynamics

4d Gauge Theory	2d Sigma Model
U(N) Gauge Theory	CP^{N-1} Sigma-Model
Add Charged Fermions	Fermion Zero Modes
Add Charged Bosons	Further Bosonic Zero Modes
Change Scalar VEVs	Induce Potentials on Target Space
Add Interactions (e.g Yukawa)	Add Interactions (e.g. 4-fermi)

An Example: Supersymmetric Theories

N=2 4d theory $\Longrightarrow$ N=(2,2) sigma model
N=1 4d theory $\Longrightarrow$ N=(0,2) sigma model

$$L_{(0,2)} = \sum_{i=1}^{N} |\mathcal{D}\phi_i|^2 - D\left(\sum_i |\phi_i|^2 - r\right)$$

$$+2i(\bar{\psi}_{-i}\mathcal{D}_+\psi_{-i} + \bar{\psi}_{+i}\mathcal{D}_-\psi_{+i}) + \bar{\phi}_i\psi_{+i}\zeta_-$$

$$L_{(2,2)} = L_{(0,2)} + |\sigma|^2|\phi_i|^2 + \bar{\phi}_i\psi_{-i}\zeta_+ + \bar{\psi}_{-i}\sigma\psi_{+i} + \text{h.c}$$

$$\sum_i |\phi_i|^2 = r \text{ mod } \phi_i \to e^{i\alpha}\phi_i \quad \Longrightarrow \quad \mathbf{CP}^{N-1}$$

Review of N=1 SQCD

$N_f = N_c$: The quantum deformed moduli space

$$V_{4d} = \frac{e^2}{2}\mathrm{Tr}(q_i q_i^\dagger - \tilde{q}_i^\dagger \tilde{q}_i)^2$$

Classically the theory has flat directions parameterized by

Mesons: $M_{ij} = \tilde{Q}_i Q_j$

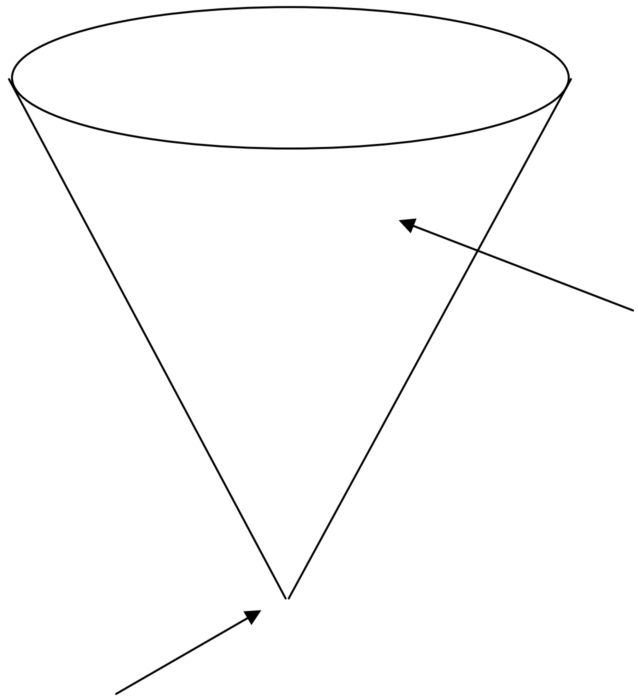
Baryons: $B = \epsilon_{a_1 \ldots a_N} Q_{i_1}^{a_1} \ldots Q_{i_N}^{a_N}$ $\tilde{B} = \epsilon_{a_1 \ldots a_N} \tilde{Q}_{i_1}^{a_1} \ldots \tilde{Q}_{i_N}^{a_N}$

These are not all independent, but satisfy the constraint

$$\det M - B\tilde{B} = 0$$

Classical Moduli Space of Vacua

$$\det M - B\tilde{B} = 0$$



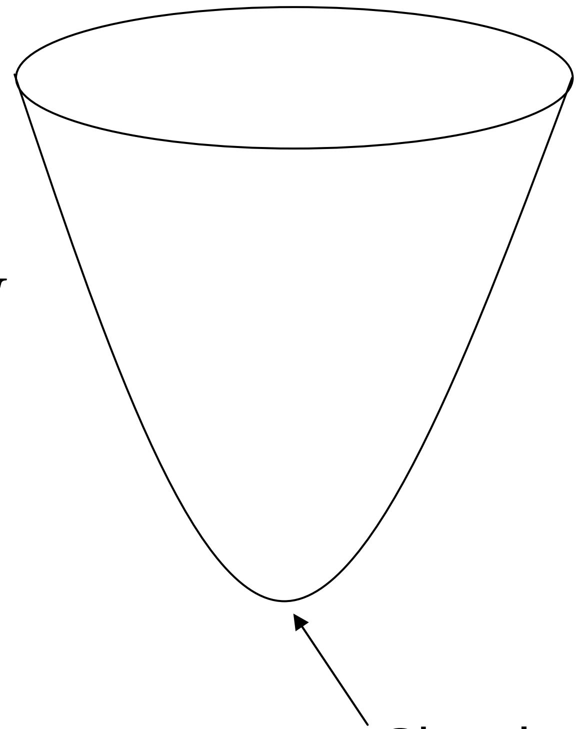
At a smooth point, we have N^2+1 massless particles

Singular point at $B = \tilde{B} = 0$ and $\mathrm{rank}(M) < N - 2$, the symmetry breaking is less than maximal $\Longrightarrow$ new massless gluons

Quantum Moduli Space

Seiberg, ‘94

$$\det M - B\tilde{B} = \Lambda^{2N}$$



Singularity is resolved, reflecting confinement and the fact that gluons get a mass

What Does This Mean for Vortices?

Gauge $U(1)_B$ and introduce FI parameter $v \ll \Lambda$

$\Longrightarrow$ extra D-term constraint $|B|^2 - |\tilde{B}|^2 = v^2$

BPS Vortex Equations in this theory are $F_{12}^B = e^2(|B|^2 - |\tilde{B}|^2 - v^2)$

$$\mathcal{D}_z B = \mathcal{D}_z \tilde{B} = 0$$

<u>Key Question:</u> When do these equations have solutions?

<u>Key Answer:</u> When $\tilde{B} = 0$

Classically, BPS vortices exist when $\det M = 0$
Quantum mechanically, BPS vortices exist when $\det M = \Lambda^{2N}$

Susy Breaking on the Worldsheet

We want to reproduce these effects from the $\mathbf{CP}^{N-1}$ sigma model, valid in the regime $v \gg \Lambda$

Claim The (0,2) $\mathbf{CP}^{N-1}$ model breaks supersymmetry by giving a vev to the auxiliary D-term.

Proof Work at large N. Integrate out ϕ_i to get effective potential for D, which is minimized at

$$D = \Lambda_{2d}^2$$

Susy Restoration on the Worldsheet

We need to understand how turning on meson vevs M affects the vortex

Claim $$\delta L_{\rm vortex} = \bar{\phi}_i \frac{M^\dagger_{ij} M_{jk}}{v^2} \phi_k + \bar{\psi}_{i-} \frac{M_{ij}}{v} \psi_{j+}$$

Justification: When the meson vev is turned on, classically we must have $\tilde{Q} \neq 0$. But vortex equations $\mathcal{D}_z Q_i = \mathcal{D}_z \tilde{Q}_i = 0$ have solutions only when $\tilde{Q}$ is not sourced. This means the vortex must lie in a part of the gauge group away from $\tilde{Q}$, reducing the available moduli.

Repeat quantum calculation to find: $\det \left(\frac{M^\dagger M}{v^2} + D \right) = \Lambda_{2d}^{2N}$

The quantum theory has susy ground state when $\det M = v^N \Lambda_{2d}^N = \Lambda_{4d}^{2N}$
This agrees with Seiberg's deformation.

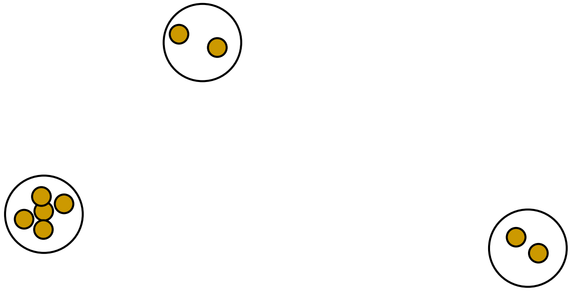
Summary

4d Quantum Deformation of Moduli Space ⟺ 2d Susy Breaking and Susy Restoration

We can also see qualitative agreement between other non-BPS quantities……like the spectrum

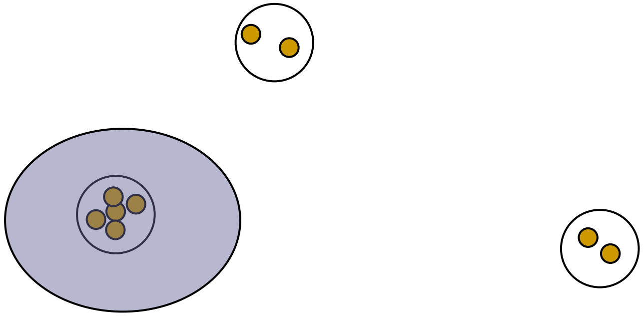
Why Would the Spectra Agree?

Start in the strongly coupled 4d theory with $v^2 = 0$
Spectrum = mesons and baryons



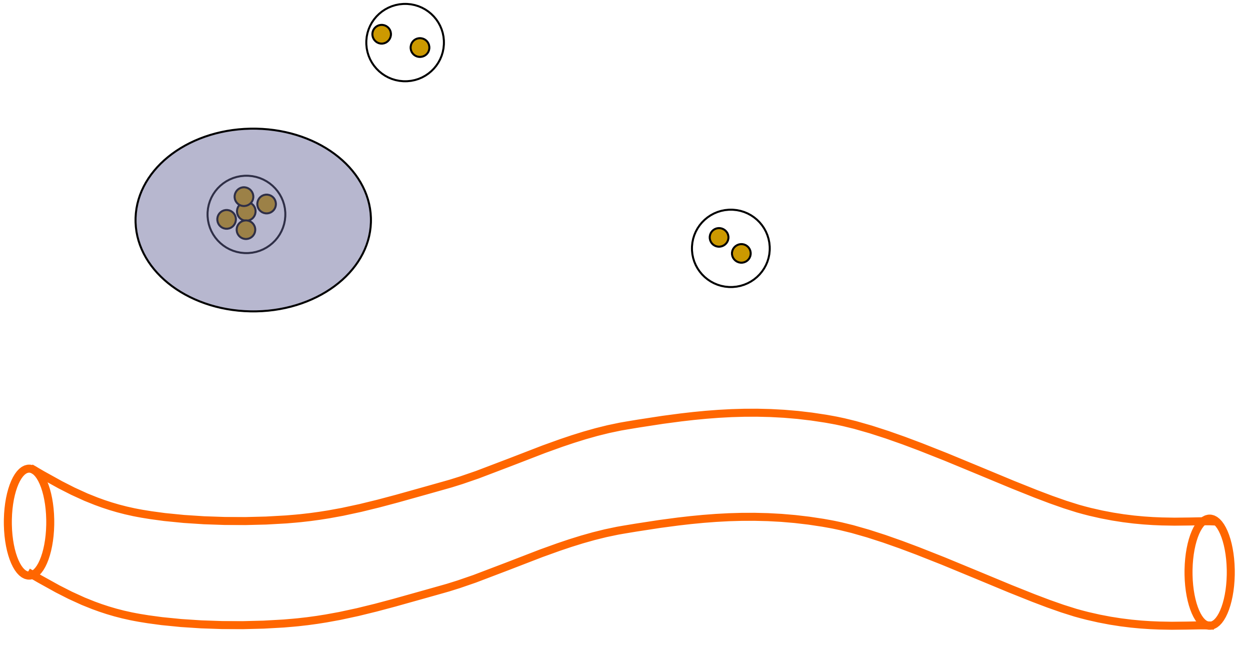
Why Would the Spectra Agree?

Gauge $U(1)_B$ and Higgs at scale $v \ll \Lambda$
Baryons are screened; mesons left largely unaffected.



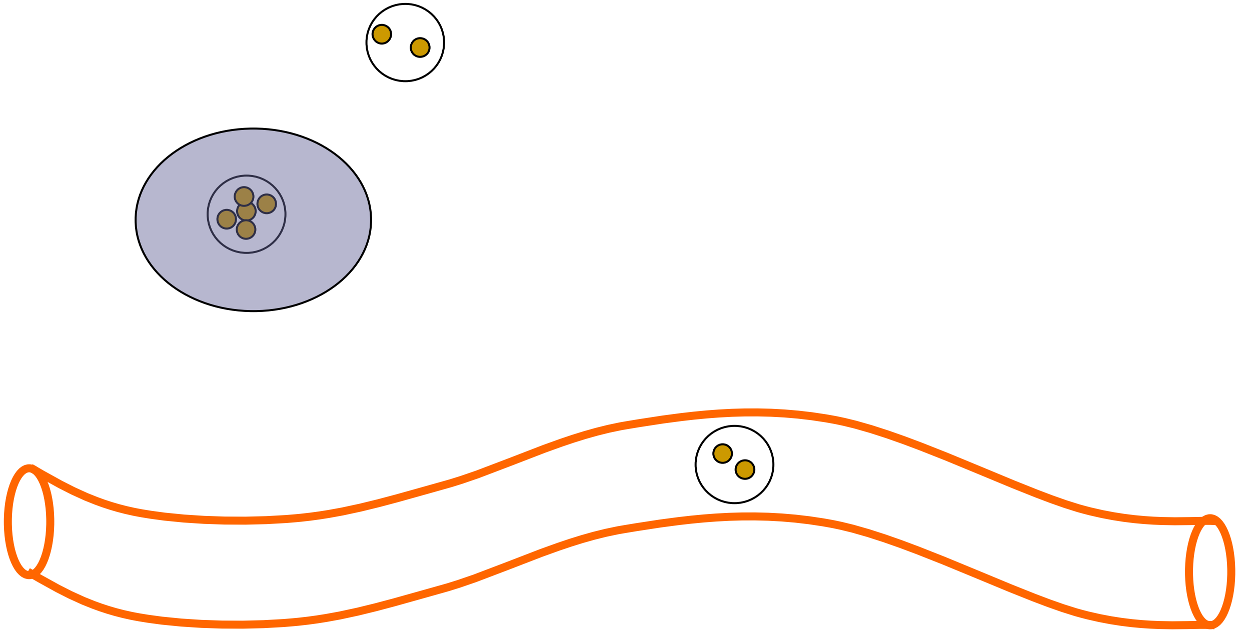
Why Would the Spectra Agree?

Introduce a vortex string. Some of the meson will form bound states with the string.



Why Would the Spectra Agree?

Now increase the ratio v/Λ . Those bound states which remain light (i.e. of order Λ) must show up as internal excitations of the 2d sigma-model.



Spectrum of the Bosonic Sigma Model

Witten, '78

Write the $\mathbf{CP}^{N-1}$ sigma model using an auxiliary U(1) gauge field.

$$L_{\rm vortex} = \sum_{i=1}^{N} |\mathcal{D}\phi_i|^2 - D\left(\sum_i |\psi_i|^2 - r\right)$$

$$\sum_i |\phi_i|^2 = r \ \mathrm{mod}\ \phi_i \to e^{i\alpha}\psi_i \quad \Longrightarrow \quad \mathbf{CP}^{N-1}$$

Integrate out ϕ to generate a kinetic term for the gauge field: $\frac{1}{\Lambda_{2d}^2} F_{01}^2$

This gives rise to a Coulomb force between ϕ particles

But in 2d, the Coulomb force is linearly confining. The physical states transform in the singlet and adjoint representation of the SU(N) symmetry.

Spectrum of the N=(2,2) Sigma Model

The story is the same as the bosonic case. There is a linearly confining Coulomb force, but the fermions drastically change the spectrum. The U(1) R-symmetry

$$\psi_{\pm i} \to e^{i\alpha}\psi_{\pm i}$$

is broken to Z_{2N} by an anomaly in the quantum theory. Further, the condensate

$$\langle \bar{\psi}_{+i}\psi_{-i}\rangle \sim \Lambda^2_{2d}$$

spontaneously breaks Z_{2N} to Z_2, ensuring N isolated vacua.

The kinks interpolating between these vacua transform in the fundamental representation N of SU(N): the theory no longer confines. This is important in matching spectrum to 4d N=2 theory.

Spectrum of the N=(0,2) Sigma Model

Do we get a condensate? Might think so, but in fact it is forbidden by the Coleman-Mermin-Wagner theorem. The condensate would break the $SU(N)_L \times SU(N)_R$ chiral symmetry of the theory.

This means there is a single vacuum, and the theory consists of singlet, adjoint and bi-fundamental reps of $SU(N)_L \times SU(N)_R$

However, the 1/N expansion does predict a condensate. Resolution to this was given by Witten (in the context of the Thirring model) 30 years ago: the theory lies in the Kosterlitz-Thouless phase

$$\langle \bar{\psi}_{i+}\psi_{-i}(x)\ \bar{\psi}_{i+}\psi_{-i}(0)\rangle \sim \Lambda_{2d}^4/x^{1/N}$$

There are massless particles, transforming in the bi-fundamental representation, which give rise to this long-range correlation.

(see also Fukawa et al., hep-th/0609049)

Comparison to 4d Spectrum

- 4d Theory has chiral superfields Q and $\tilde{Q}$, in (anti)-fundamental representation of SU(N) gauge group.

- Physical spectrum consists of singlets under the gauge group, and various multiplets under $SU(N)_L \times SU(N)_R$ flavor group.

  - Meson Spectrum:
    - Massless bi-fundamental: $\tilde{Q}_i Q_j$
    - Massive singlet and adjoint $Q_i^\dagger Q_j$ and $\tilde{Q}_i \tilde{Q}_j^\dagger$

    } In agreement with the spectrum of the vortex theory

  - Baryon Spectrum: Slew of tensor reps under flavor symmetry. Not seen in the vortex theory.

Summary and Future Directions

- Quantitative agreement between 2d sigma models and 4d gauge dynamics
  - N=2 Gauge Theories = N=(2,2) sigma models
    - Exact agreement between BPS mass spectra
    - Agreement between superconformal theories
  - N=1 Gauge Theories = N=(0,2) sigma models
    - Baryon vevs = worldsheet supersymmetry breaking
    - qualitative agreement between spectra
- Open Questions: N=1 Gauge Theories with $N_f > N_c$.
  - Conformal Window? Seiberg Duality?