Heuristic approach to the Schwarzschild geometry

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Abstract. In this article I will present a simple Newtonian heuristic for "deriving" a weak-field approximation for the spacetime geometry of a point particle. The heuristic is based on Newtonian gravity, the notion of local inertial frames [the Einstein equivalence principle], plus the use of Galilean coordinate transformations to connect the freely falling local inertial frames back to the "fixed stars". Because of the heuristic and quasi-Newtonian manner in which the spacetime geometry is obtained, we are only justified in expecting it to be a weak-field approximation to the true spacetime geometry. However, in the case of a spherically symmetric point mass the result is an *exact* solution of the vacuum Einstein field equations — it is the Schwarzschild geometry in Painlevé–Gullstrand coordinates.

This result is much stronger than the well-known result of Michell and Laplace whereby a Newtonian argument correctly estimates the value of the Schwarzschild radius — using the heuristic of this article one obtains the entire Schwarzschild geometry. Unfortunately the heuristic construction does not seem to generalize; it does not give the correct result for the Reissner–Nordström geometry (though it gets rather close), and does not seem capable of generating the Kerr geometry. Thus it is at this stage still somewhat unclear as to whether there is anything deeper to the heuristic than a remarkable but fortuitous coincidence.

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1. Introduction

The heuristic construction presented in this article arose from combining three quite different trains of thought:

- For an undergraduate course, I wanted to develop a reasonably clean motivation for looking at the Schwarzschild geometry suitable for students who had *not* seen any formal differential geometry. These students had however been exposed to Taylor and Wheeler's "*Spacetime Physics*" [1], so they had seen a considerable amount of Special Relativity, including the Minkowski space invariant interval. They had also already been exposed to the notion of local inertial frames [local "free-float" frames], which notion is equivalent to introduction of the Einstein equivalence principle. But there is no justification in the framework of [1] for introducing the Schwarzschild geometry.
- Additionally, I was of course aware of the Newtonian idea of a "dark star"; this notion going back to the Reverend John Michell (1783) [2, 3], and popularized by Pierre Simon Maquis de Laplace (1799) [4], who noted that in Newtonian physics the escape velocity from the surface of a star can exceed the speed of light when
$$\frac{1}{2}\, v_{\mathrm{escape}}^2 = \frac{GM}{R} > \frac{1}{2}\, c^2. \tag{1}$$
That is, in Newtonian physics, (adopting the "corpuscular model" [5]), light cannot escape from the surface of a star once
$$R < R_{\mathrm{escape}} = \frac{2GM}{c^2}, \tag{2}$$
and this critical radius is (in suitable coordinates) *exactly* the same as the Schwarzschild radius of General Relativity.
- Finally, from exposure to the "analogue models" of General Relativity [6, 7, 8, 9], I was aware of the large number of different ways in which effective Lorentzian spacetime geometries can arise in quite different physical systems. In particular, Bondi accretion [10] (spherically symmetric accretion onto a gravitating point mass) leads to an "acoustic geometry" qualitatively similar to the Schwarzschild geometry.

By combining these ideas I found it was possible to develop a good heuristic for the weak-field metric, which can be presented at a level appropriate for undergraduate students. (Though some of the technical comments made below are definitely not appropriate at this level.) The remarkable feature of this heuristic is that for the Schwarzschild geometry it happens to be *exact*. This appears to be a coincidence, but is a good way of introducing students who may not intend to specialize in General Relativity to the notion of a black hole.

2. The heuristic

2.1. Free float frames:

Start with a mass M which has Newtonian gravitational potential

$$\Phi = -\frac{GM}{r}. \tag{3}$$

Take a collection of local inertial frames [local free-float frames] that are stationary out at infinity, and drop them. In the Newtonian approximation these local free-float frames pick up a speed

$$\vec{v} = -\sqrt{\frac{2GM}{r}}\,\hat{r}. \tag{4}$$

In the local free-float frames, physics looks simple, and the invariant interval is simply given by the standard Special Relativity result

$$\mathrm{d}s_{FF}^2 = -c^2\,\mathrm{d}t_{FF}^2 + \mathrm{d}x_{FF}^2 + \mathrm{d}y_{FF}^2 + \mathrm{d}z_{FF}^2, \tag{5}$$

where I want to emphasise that these are locally defined free-fall coordinates.

2.2. Rigid frame:

Let us now try to relate these freely falling local inertial coordinates to a rigidly defined surveyor's system of coordinates that is tied down at spatial infinity — that is, we want a coordinate system connected to the "fixed stars". Call these coordinates t_{rigid}, x_{rigid}, y_{rigid}, and z_{rigid}. Since we know the speed of the freely falling system with respect to the rigid system, and we assume velocities are small, we can write an approximate Galilean transformation:

$$\mathrm{d}t_{FF} = \mathrm{d}t_{rigid}; \tag{6}$$

$$\mathrm{d}\vec{x}_{FF} = \mathrm{d}\vec{x}_{rigid} - \vec{v}\,\mathrm{d}t_{rigid}. \tag{7}$$

Warning: Most relativists will quite justifiably be concerned by the suggestion that there is a rigid background to refer things to. The only reason we have for even hoping to get away with this is because all of the discussion is at this stage in the slow-speed weak-field approximation. For students with a Special Relativity background who have not been exposed to the mathematics of differential geometry the existence of these rigid coordinates is "obvious" and it is only the mathematically sophisticated students that have problems here. □

2.3. Approximate "metric":

Substituting, we find that in terms of the rigid coordinates the spacetime interval takes the form

$$\mathrm{d}s_{rigid}^2 = -c^2\mathrm{d}t_{rigid}^2 + ||\mathrm{d}\vec{x}_{rigid} - \vec{v}\,\mathrm{d}t_{rigid}||^2. \tag{8}$$

Expanding

$$\mathrm{d}s^2_{rigid} = -[c^2 - v^2]\mathrm{d}t^2_{rigid} - 2\vec{v} \cdot \mathrm{d}\vec{x}\ \mathrm{d}t_{rigid} + ||\mathrm{d}\vec{x}_{rigid}||^2. \tag{9}$$

That is

$$\mathrm{d}s^2_{rigid} = -\left[c^2 - \frac{2GM}{r}\right]\mathrm{d}t^2_{rigid} + 2\sqrt{\frac{2GM}{r}}\ \mathrm{d}r_{rigid}\ \mathrm{d}t_{rigid} + ||\mathrm{d}\vec{x}_{rigid}||^2. \tag{10}$$

This is only an approximation — we have used Newton's gravity, Galileo's relativity, and the notion of local inertial frames. There is no fundamental reason to believe this spacetime metric once GM/r becomes large.

2.4. The miracle:

Dropping the subscript "rigid", the invariant interval

$$\mathrm{d}s^2 = -\left[c^2 - \frac{2GM}{r}\right]\mathrm{d}t^2 + 2\sqrt{\frac{2GM}{r}}\ \mathrm{d}r\ \mathrm{d}t + ||d\vec{x}||^2 \tag{11}$$

is an *exact* solution of Einstein's equations of general relativity, $R_{ab} = 0$. It is the Schwarzschild solution in disguise. In spherical polar coordinates we have

$$\mathrm{d}s^2 = -\left[c^2 - \frac{2GM}{r}\right]\mathrm{d}t^2 + 2\sqrt{\frac{2GM}{r}}\ \mathrm{d}r\ \mathrm{d}t + \mathrm{d}r^2 + r^2\left[\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right]. \tag{12}$$

This is *one* representation of the space-time geometry of a Schwarzschild black hole, in a particular coordinate system (the Painlevé–Gullstrand coordinates) [11, 12, 13]. There are many other coordinate systems you could use.

Warning: I emphasise that this is a *heuristic* that happens to give the exact result. I do not view this as a rigorous derivation of the Schwarzschild geometry from Newtonian physics, and on this issue disagree with reference [14]. The heuristic does however provide a good motivation for being interested in this specific geometry, even if for pedagogical reasons you do not yet have the full vacuum Einstein equations available. □

Exercise: More advanced students could at this stage be asked to (1) find the coordinate transformation required to bring the above into standard curvature coordinate form (see for instance [8]); and/or (2) use a symbolic algebra package to verify that the Ricci tensor is zero. □

2.5. Schwarzschild radius:

You can now easily see that something interesting happens at

$$\frac{2GM}{r_S} = c^2; \qquad r_S = \frac{2GM}{c^2}; \tag{13}$$

where we essentially recover the observations of Reverend John Michell (1783) and Pierre Simon Marquis de Laplace (1799). In Einstein's gravity the coefficient of $\mathrm{d}t^2_{\mathrm{rigid}}$ goes to zero at the Schwarzschild radius; in Newton's gravity the escape velocity

$$v_{\mathrm{escape}} = \sqrt{\frac{2GM}{R}} \tag{14}$$

reaches the speed of light once $R = r_S$.

Warning: This is a good point at which to introduce the difference between "coordinate velocity" and "physical velocity". For ingoing and outgoing null rays

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\sqrt{\frac{2GM}{r}} \mp c. \tag{15}$$

At the event horizon the coordinate velocity of the infalling local inertial frames (relative to the "fixed" coordinates) exceeds the speed of light — but this is perfectly acceptable in General Relativity as it is only a statement about coordinate systems, not a statement about physical objects. The coordinate velocity of the outgoing light ray goes to zero. In addition, all physical velocities are limited by the speed of light and must lie in or on the light cone defined by the spacetime metric. □

Warning: Some students will at this stage take the notion of a "gravitational aether" a little too seriously. This is the major drawback of this heuristic, which can best be ameliorated by pointing out that this heuristic is not fundamental physics. The heuristic does not work well for the Reissner–Nordström black hole and fails utterly for the Kerr black hole. □

3. Discussion

Overall, I feel that the benefits of this heuristic outweigh the risks — once the specific spacetime geometry has been motivated in this way, students not intending to specialize in General Relativity can simply be told that this is the Schwarzschild solution, and the properties of this spacetime investigated in the usual manner [15, 16, 17]. Two key points are:

- This sort of argument should work generically for weak fields.
- That it is exact for Schwarzschild seems to be an accident.

I expand on these points below. Some of the issues raised below are very definitely nontrivial and not suitable for an undergraduate audience. Suitably modified, some points may be of interest for mathematically sophisticated students who do not have a significant physics background.

3.1. Spherical symmetry:

That the heuristic presented above, or some variant thereof, has some chance of working for general spherically symmetric geometries, can be seen by appropriately choosing the coordinates. Spherical symmetry by itself is enough to yield [17]

$$\mathrm{d}s^2 = -a(r,t)\ \mathrm{d}t^2 + 2b(r,t)\ \mathrm{d}r\,\mathrm{d}t + c(r,t)\ \mathrm{d}r^2 + d(r,t)\ \mathrm{d}\Omega^2. \tag{16}$$

The usual procedure at this point is to use the coordinate freedom in the r-t plane to eliminate the off-diagonal term, and also to normalize the $\mathrm{d}\Omega^2$ coefficient, to locally obtain

$$\mathrm{d}s^2 = -a(r,t)\ \mathrm{d}t^2 + c(r,t)\ \mathrm{d}r^2 + r^2\ \mathrm{d}\Omega^2. \tag{17}$$

Warning: Coordinate arguments will only tell you that you can do this in suitably defined local coordinate patches; that global coordinate systems of this type exist for stars is a deep result that requires some assumptions about the the regularity of the centre, the nature of matter and dynamical information from the Einstein equations — specifically if the null energy condition holds then there are no "wormhole throats" and the coordinate r is continuously increasing as one moves away from the center [18, 19]. □

In contrast, one could use the $\mathrm{d}\Omega^2$ coefficient to define a new r-coordinate and then use the remaining coordinate freedom in the r-t plane to set [20, 21, 22, 23]

$$\mathrm{d}s^2 = -a(r,t)\ \mathrm{d}t^2 + 2b(r,t)\ \mathrm{d}r\,\mathrm{d}t + \mathrm{d}r^2 + r^2\ \mathrm{d}\Omega^2. \tag{18}$$

One then defines functions $N(r,t)$ and $\beta(r,t)$ so that

$$\mathrm{d}s^2 = -[N(r,t)^2 - \beta(r,t)^2]\ \mathrm{d}t^2 + 2\beta(r,t)\ \mathrm{d}r\,\mathrm{d}t + \mathrm{d}r^2 + r^2\ \mathrm{d}\Omega^2. \tag{19}$$

The interpretation is that in spherical symmetry one can always [patchwise] choose coordinates to make space [*not spacetime*] flat. In the language of the ADM decomposition (see for instance [19, 24]), you bury all of the spacetime curvature in the lapse and shift functions, $N(r,t)$ and $\beta(r,t)$. The heuristic argument above consists of setting $N(r,t) = c^2$ and $\beta(r,t) = -\sqrt{2GM/r}$, but we now see that by choosing suitable ansatze for the lapse and shift we would be able to fit arbitrary spherically symmetric spacetimes.

Coordinates of this type are known as Painlevé–Gullstrand coordinates [11, 12, 13] and have many pedagogically and computationally useful properties [20, 21, 22, 23]. A particularly nice feature is that infalling observers cross the event horizon in finite coordinate time, so that one does not have to confront the pseudo-paradox encountered in standard coordinates where one has to wait an infinite amount of coordinate time (but finite proper time) in order for a test particle to reach the event horizon.

Historically the Painlevé–Gullstrand coordinates were developed in an attempt to show there was something wrong with the Schwarzschild coordinates [11, 12]. (More recently, see also [14].) However, as emphasised by Lemaître [13], these are just specific coordinates [albeit somewhat unusual ones] and their adoption or rejection cannot affect the underlying physics.

The heuristic applied to a generic spherically symmetric field yields

$$\mathrm{d}s^2 = -\left[c^2 + 2\Phi(r)\right]\mathrm{d}t^2 + 2\sqrt{-2\Phi(r)}\;\mathrm{d}r\;\mathrm{d}t + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \tag{20}$$

If there is a well defined surface beyond which the object is vacuum, then in that region Newtonian physics gives $\Phi(r) = -GM/r$ and so our heuristic reproduces the Birkhoff theorem [24]. But in general, in Newtonian gravity the gravitational acceleration in a situation with spherical symmetry is

$$\vec{g} = -\frac{Gm(r)}{r^2}\,\hat{r}. \tag{21}$$

Integrating, this now implies

$$\Phi(r) = \int g\;\mathrm{d}r = -\frac{Gm(r)}{r} + G\int \rho(r)\;r\;\mathrm{d}r \tag{22}$$

As long as the density falls off sufficiently rapidly at spatial infinity, $\rho(r) \to C/r^{3+\epsilon}$, the second term is sub-dominant near spatial infinity, $\int \rho(r)\;r\;\mathrm{d}r \to C/r^{1+\epsilon}$, and we can (in the weak field limit) write

$$\mathrm{d}s^2 = -\left[c^2 - \frac{2Gm(r)}{r}\right]\mathrm{d}t^2 + 2\sqrt{\frac{2Gm(r)}{r}}\;\mathrm{d}r\;\mathrm{d}t + \mathrm{d}r^2 + r^2\left[\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right]. \tag{23}$$

This geometry, while reasonably general, is *not* the most general weak-field metric possible in General Relativity. For this reason our heuristic will not be able to exactly reproduce all spherically symmetric geometries. [You could also come to a similar conclusion, but without some of the interesting intermediate results, by noting that the general spherically symmetric geometry is specified by two arbitrary functions $N(r,t)$ and $\beta(r,t)$ whereas the heuristic depends on only one arbitrary function $\Phi(r,t)$.]

3.2. Reissner–Nordström geometry:

The exact Reissner–Nordström geometry [24] corresponds to the choice $N(r,t) = c^2$ and $\beta(r,t) = -\sqrt{2GM/r - Q^2/r^2}$ so that

$$\mathrm{d}s^2 = -\left[c^2 - \frac{2GM}{r} + \frac{Q^2}{r^2}\right]\mathrm{d}t^2 + 2\sqrt{\frac{2GM}{r} - \frac{Q^2}{r^2}}\;\mathrm{d}r\;\mathrm{d}t + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \tag{24}$$

Unfortunately, while we can put the Reissner–Nordström geometry into the Painlevé–Gullstrand *form* appropriate for our heuristic analysis, the precise details do not quite work out correctly. For a charged particle surrounded by an electric field we could argue that the equivalence of mass and energy requires

$$\rho = M\;\delta^3(\vec{x}) + \frac{1}{8\pi}E^2 = M\;\delta^3(\vec{x}) + \frac{1}{8\pi}\frac{Q^2}{r^4}, \tag{25}$$

so that

$$m(r) = M - \frac{1}{2}\frac{Q^2}{r}. \tag{26}$$

Unfortunately this now implies

$$\Phi = \int g\;\mathrm{d}r = G\int\left[\frac{M}{r^2} - \frac{1}{2}\frac{Q^2}{r^3}\right]\;\mathrm{d}r = -G\left[\frac{M}{r} - \frac{Q^2}{4r^2}\right], \tag{27}$$

and the coefficient of the Q^2 term does not match the exact Reissner–Nordström geometry, being off by a factor of 2. That is, the heuristic argument yields

$$\mathrm{d}s^2 = -\left[c^2 - \frac{2GM}{r} + \frac{Q^2}{2r^2}\right]\mathrm{d}t^2 + 2\sqrt{\frac{2GM}{r} - \frac{Q^2}{2r^2}}\;\mathrm{d}r\;\mathrm{d}t + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \tag{28}$$

instead of the exact result of equation (24). Though the heuristic does not exactly reproduce the Reissner–Nordström geometry, it does get remarkably close. There is a completely *ad hoc* "fix": Simply assert that in spherically symmetric General Relativity you should [in arbitrarily strong fields] replace the Newtonian potential by

$$\Phi \to \frac{Gm(r)}{r}. \tag{29}$$

While this last statement is "true" in certain situations its derivation requires the full Einstein equations, which is exactly what we were trying to avoid.

3.3. Kerr geometry:

The heuristic approach definitely fails for the Kerr geometry — most fundamentally because the Kerr geometry is not spherically symmetric. More technically, the Painlevé–Gullstrand coordinates require the existence of flat spatial slices, and the Kerr geometry does not possess such a slicing. In fact the Kerr geometry does not even possess a conformally flat spatial slicing [25]. The closest that one seems to be able to get to Painlevé–Gullstrand coordinates seems to be Doran's form of the metric [26], for which a brief computation shows that $N(r,\theta) = c^2$; the lapse function is a *constant* independent of position. Unfortunately the spatial slices in Doran's coordinates are very definitely not flat. More critically I have not been able to find any useful set of coordinates that would make the Kerr geometry amenable to treatment along the lines of the heuristic approach considered above. For this reason, among others, the heuristic approach should not be thought of as fundamental physics.

3.4. Bondi acoustic geometry:

A particularly nice feature of the heuristic analysis is the clean relationship with the acoustic geometry occurring in Bondi accretion [10]. Consider a fluid with a linear equation of state

$$\rho(p) = \rho_0 + \frac{p}{c_s^2}; \qquad p = (\rho - \rho_0)\, c_s^2; \tag{30}$$

undergoing spherically symmetric accretion onto a compact object [10]. Here c_s is the speed of sound, assumed constant. Then as long as backpressure can be neglected, the infalling matter satisfies $v = -\sqrt{2GM/r}\;\hat{r}$. Sound waves travelling on the background of this infalling matter will then travel at speed

$$\left|\left| -\sqrt{2GM/r}\hat{r} + c_s\,\hat{n} \right|\right| \tag{31}$$

with respect to the fixed stars. This situation is tailor-made for application of the acoustic geometry formalism [8, 9], and as long as the backpressure is negligible the effective acoustic geometry is exactly the Schwarzschild geometry with the speed of light replaced by the speed of sound; that is, with the substitution $c \to c_s$.

3.5. Spatially flat geometries:

Recently Nurowski, Schücking, and Trautman have used metrics with flat spatial slices (which include as a subset all of the spherically symmetric geometries in Painlevé–Gullstrand coordinates) to investigate general relativistic spacetimes with close Newtonian analogues [27]. That approach, since it starts from the full Einstein equations, is in some sense the converse of the heuristic developed here. Metrics with flat spatial slices also occur ubiquitously in the various "analogue model" geometries, not just the spherically symmetric ones. A necessarily incomplete set of references includes [6, 7, 8, 9, 28, 29]. The class of spatially flat geometries appear to be of interest in its own right, even if it is not general enough to contain the Kerr geometry.

3.6. Open questions:

- Is there something more fundamental going on here that we do not yet understand? Or is it all just a glorious accident?
- Is it possible, despite my negative comments above, to get a modified version of this argument that works "cleanly" for the Reissner–Nordström geometry?
- Is it possible, despite my very negative comments above, to get a *drastically* modified version of this argument that works "cleanly" for the Kerr geometry?

3.7. Summary:

The basic heuristic discussed in the first few pages of this article can easily be explained to undergraduate students who have no intention of specializing in General Relativity, and can be used to motivate interest the Schwarzschild geometry and black hole physics. The remarkable feature of the heuristic is that it leads directly to an exact solution of the full Einstein equations — the Schwarzschild geometry in Painlevé–Gullstrand coordinates. As we have seen in the commentary, this leads naturally to a number of rather technical issues and questions hiding in this rather innocent looking heuristic.

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[1] E.F. Taylor and J.A. Wheeler, *Spacetime Physics*, (Freeman, New York, 1992).

[2] Reverend John Michell, FRS, "On the Means of discovering the Distance, Magnitude, etc. of the Fixed Stars, in consequence of the Diminution of the Velocity of their Light, in case such a Diminution should be found to take place in any of them, and such other Data should be procured from Observations, as would be farther necessary for that Purpose", Philosophical Transactions of the Royal Society of London **74** (1784) 35–57. (Warning: The correct spelling is Michell, not Mitchell, though usage is somewhat inconsistent.) The original reference is difficult to obtain and I provide a brief quotation:

> If the semi-diameter of a sphere of the same density as the Sun in the proportion of five hundred to one, and by supposing light to be attracted by the same force in proportion to its [mass] with other bodies, all light emitted from such a body would be made to return towards it, by its own proper gravity.

It is also interesting to note that Reverend John Michell, though most commonly thought of as a geologist, had another major influence on the field of gravitation as the inventor of the [Cavendish] torsion balance — subsequently used by Cavendish in his experimental determination of Newton's constant.

[3] For some recent comments on the historical connections between Michell, Cavendish, and Laplace, see:
D. Lynden-Bell, "Why Do Disks Form Jets?", arXiv:astro-ph/0203480;
Published in *The cental kiloparsec of starbursts and AGN: the La Palma Connection*, ASP conference series, **249** (2001). Edited by J.H. Knapen, J.E. Beckman, I. Shlosman, and T.J. Mahoney.

[4] Pierre Simon Marquis de Laplace, *Exposition du Systeme du Monde*, 1796. The intimate connection between the work of Michell and Laplace can clearly be seen from the following quotation:

> A luminous star, of the same density as the earth, and whose diameter should be two hundred and fifty times larger than that of the Sun, would not, in consequence of its attraction, allow any of its [light] rays to arrive at us; it is therefore possible that the largest luminous bodies in the universe may, through this cause, be invisible.

An easy-to-find English translation of an essay giving the technical justification for this statement is available as Appendix A of S.W. Hawking and G.F.R. Ellis, *The large scale structure of spacetime*, (Cambridge, England, 1972).

[5] Isaac Newton, *Optics*, 1704:

> *Query 1:* And do not Bodies act upon Light at a distance and, by their action, bend its Rays, and is not this action strongest at the least distance?

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