Hidden structures of knot invariants

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Alexey Sleptsov

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Alexey Sleptsov

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Faculteit der Natuurwetenschappen, Wiskunde en Informatica

Hidden structures of knot invariants

Alexey Sleptsov

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Chapter 1

Introduction

1.1 Knots

1.1.1 How to distinguish knots

Knot theory is an area of low-dimensional topology. Topology studies properties of geometric objects preserved under continuous deformations.

Definition 1.1.1. ([1]) A knot *is an embedding of the circle into the three-dimensional Euclidean space* $\mathcal{K} : S^1 \hookrightarrow \mathbb{E}^3$.

In other words, a knot is a closed curve without self-intersections in the 3-space, see Figure 1.1 for particular examples of knots. Usually, one takes $\mathbb{R}^3$ for

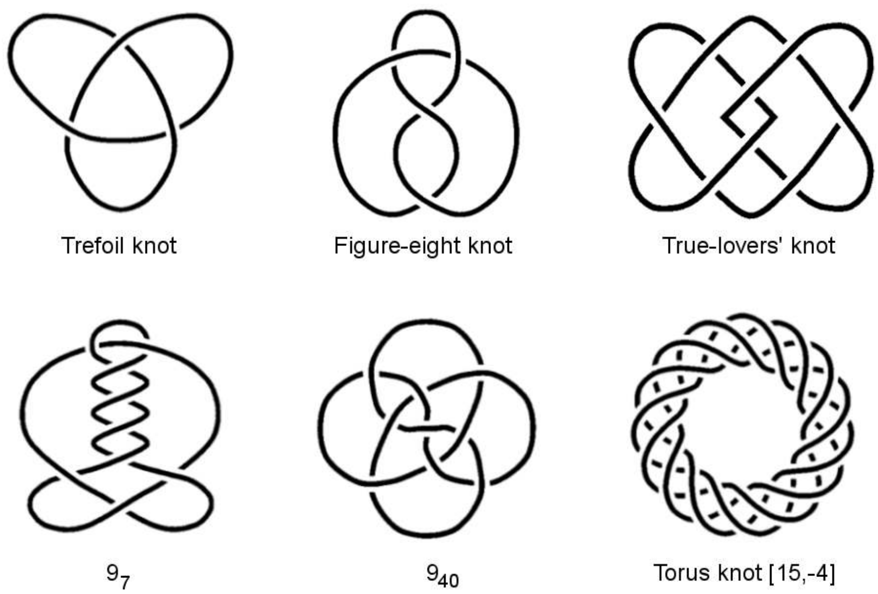


Figure 1.1: Examples of knots

the ambient 3-space. We do not distinguish between a knot and any continuous deformations of this knot which can be performed without self-intersections. All these deformed curves are considered to be one and the same knot. We can think about a knot as if it is made from easily deformable rubber, which we cannot cut and glue. Such deformation are called *ambient isotopies*, which are a special type of homotopy. *A homotopy* of a space $X \subset \mathbb{E}^3$ is a continuous map $h : X \times [0,1] \to \mathbb{E}^3$. If h_t is one-to-one for all $t \in [0,1]$, then h is called an *isotopy*.

Definition 1.1.2. ([2]) *Two knots $\mathcal{K}_1$ and $\mathcal{K}_2$ are* ambient isotopic *if there is an isotopy $h : \mathbb{E}^3 \times [0,1] \to \mathbb{E}^3$ such that $h(\mathcal{K}_1, 0) = \mathcal{K}_1$ and $h(\mathcal{K}_1, 1) = \mathcal{K}_2$.*

Thus, we consider ambient isotopy as an equivalence relation on knots, that is, two knots are equivalent if they can be deformed into one another. We refer to each equivalence class of knots as a knot type, and equivalent knots have the same type. However to avoid the abuse of terminology it is very common to apply the word "knot" to mean the whole equivalence class i.e. to a knot type, or a particular representative member which we are interested in. For example, when we say that two knots are different, we actually mean that they are inequivalent, i.e. have different types.

The simplest knot of all is the unknotted circle, which we call the trivial knot or *the unknot*, see Figure 1.2. If a knot has the same type as the trivial knot, we say it is unknotted.

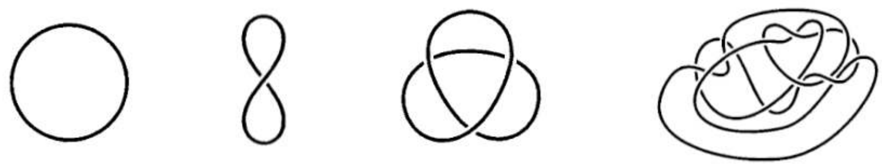


Figure 1.2: Examples of unknots

If you look carefully at Figure 1.2 and use physical intuition of deformable rubber, you will understand that these knots are equivalent. This simple exercise naturally leads us to the first significant scientific question in the knot theory:

How to distinguish knots?

Indeed, let us have two knots $\mathcal{K}_1$ and $\mathcal{K}_2$, for example, as in Figure 1.3. How do we know they are actually different or the same? Also we have not yet proved that there exist any other knots besides the unknot. Maybe every projection of a knot at the any figure above could simply be a messy projection of the unknot. To answer this question we shall find such properties of a knot, which depend only on the equivalence class of the knot. This idea gives rise to a theory of knot invariants, a major part of the knot theory.

Graphically we represent knots by means of knot diagrams. A knot diagram is a plane closed curve, which can have only double points (crossings) as singularities, together with the chosen overcrossing string and undercrossing string at each crossing. Thus, knot diagram can be considered as a projection of a knot along some "vertical" direction, overcrossings and undercrossings indicate which string is

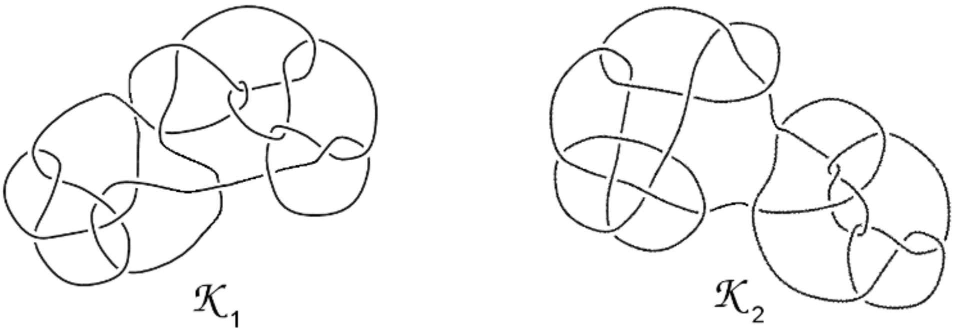


Figure 1.3: Two arbitrary knots

"higher" and which one is "lower". We refer to a deformation of a knot projection as a planar isotopy or an isotopy in the plane if it deforms the projection plane as if it were made of rubber with the projection drawn upon it Figure 1.4 [3]. We deform the knot only within the projection plane.

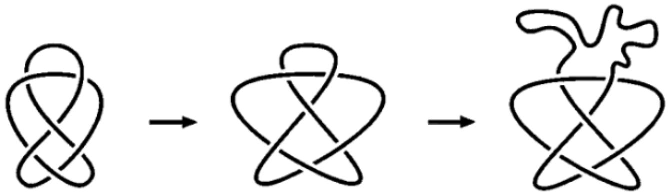


Figure 1.4: Planar isotopies

Proposition 1.1.3. (Reidemeister, [1]) *Two knots $\mathcal{K}_1$ and $\mathcal{K}_2$, are equivalent if and only if diagram of $\mathcal{K}_1$ can be transformed into a diagram of $\mathcal{K}_2$ by a sequence of ambient isotopies of the plane and local moves of the following three types:*

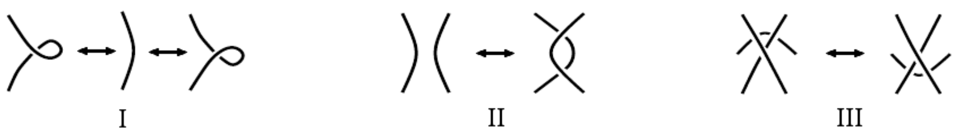


Figure 1.5: *Reidemeister moves*

Notice that although each of these moves changes the projection of the knot, it does not change the knot represented by the projection. Each such move is an ambient isotopy. For example, two projections in Figure 1.6 (taken from [3]), the left-most one and the right-most one, correspond to the same knot. Therefore, according to the Reidemeister theorem, there is a series of Reidemeister moves and planar isotopies that takes us from the first projection to the second. In Figure 1.6 we see one example of such series of moves, which demonstrates this equivalence.

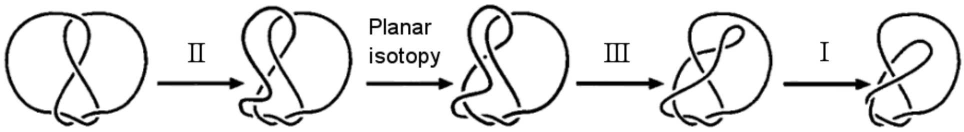


Figure 1.6: Example of Reidemeister moves

The problem of determining whether two projections represent the same knot is not such an easy one as one might have hoped. We just check whether or not there is a series of Reidemeister moves to take us from the one projection to the other, but there is no limit on the number of Reidemeister moves. If the two given projections have 10 crossings each, it might happen that in the process of performing the Reidemeister moves the number of crossings necessarily increases to 57, before the projection is simplified back down to 10 crossings.

1.1.2 The problem of classification

We begin with a discussion of some types of knots, which can be useful for the present thesis.

Alternating knots. We refer to a knot with a projection that has crossings, which alternate between over and under on travelling along the knot as an alternating knot. Otherwise a knot is called non-alternating. The trefoil knot and the figure-eight knot are alternating.

Connected sum. If we have two knots, we can define a new knot obtained by removing a small arc from each knot and then connecting the four loose ends by two new arcs. It must be done with some caution: for projections we assume they do not overlap and we avoid removing or adding any crossings as in Figure 1.7 ([3]). We refer to the resulting knot as *the composition* of the two knots or *the connected sum* of the two knots, denoted by $\mathcal{K}_1\#\mathcal{K}_2$.

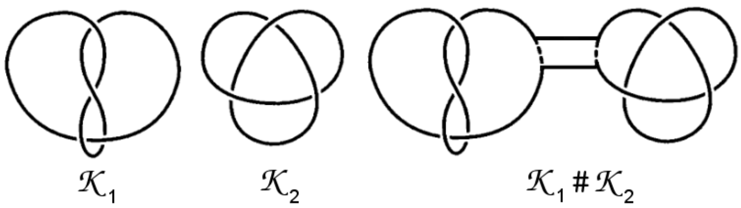


Figure 1.7: The composition $\mathcal{K}_1\#\mathcal{K}_2$ of two knots $\mathcal{K}_1$ and $\mathcal{K}_2$.

We refer to a knot as *composite knot* if it can be represented as a composition of two knots, neither of which is the trivial knot. We refer to a knot as *prime knot*, if it is not a composition of any two nontrivial knots. Note that the composition of knot $\mathcal{K}$ with the unknot is again $\mathcal{K}$. From this point of view the knots are analogous to the positive integers, where we call an integer composite if it is a product of positive integers, neither of which is equal to 1 and we call it prime otherwise. If we multiply an integer by 1, we get the same integer back again. The knots which make up the composite knot are called *factor knots*. In the Figure both the trefoil knot and the figure-eight knot are prime knots.

The unknot is not a composite knot, because it is not possible to take the composition of two nontrivial knots and obtain the unknot. We can use integers analogy here again: this result is analogous to the fact that the integer 1 is not the product of two positive integers, each greater than 1. Moreover, just as an integer factors into unique set of prime numbers, a composite knot factors into unique set of prime knots. Tables of knots, e.g. the Rolfsen table [80], list only the prime knots and do not include any composite knots. They are similar to tables of prime numbers.

Mirror knots. *A mirror knot* is a knot obtained by changing every crossing in the given knot to the opposite crossing. A knot which is equivalent to its mirror image is called *amphicheiral* or *achiral*, otherwise *non-amphicheiral* or *chiral*. Despite a knot and its mirror image are distinct knots unless the knot is amphicheiral, knot tables do not list both a knot and its mirror image, only one from this pair. The simplest example of amphicheiral knot is figure-eight knot. One can prove it with the help of Reidemeister moves.

Links. Up to now we have considered embeddings of a single circle, i.e. restricted our attention to single knotted loops. However, there is a natural generalisation of this idea so that we consider embeddings of collections of circles and, hence, we obtain a set of knotted loops.

Definition 1.1.4. ([2]) A link *is a finite disjoint union of knots:* $L = \mathcal{K}_1 \bigcup \ldots \bigcup \mathcal{K}_n$*. Each knot* $\mathcal{K}_i$ *is called a component of the link. The number of components of a link* L *is called the multiplicity of the link, and is denoted by* $\mu(L)$*. A subset of the components embedded in the same way is called a sublink.*

We refer to a set of l disjoint circles embedded in a plane as the trivial link of multiplicity l.

Some simple examples of links are shown in Figure 1.8. Each of the five 2-component links has two trivial knots as components, but these five links are different. Thus, the notation $\mathcal{K}_1 \bigcup \ldots \bigcup \mathcal{K}_n$ lists the component parts and does not indicate how they are put together, hence it is not enough to completely describe a link.

The Thistlethwaite link table [80] is the analog to the Rolfsen knot table. There the following notation is used: a label of the form $L_i a_j$ or $L_i n_j$ indicates the jth link with i crossings of its minimal plane projection; the label a (n) indicates alternating (non-alternating) link. Also some other catalogues use labels of the form N^{μ}_{m}, which indicate the mth link with μ components with N crossings. Obviously, the set of links contains the set of knots.

Torus knots. Some of the simplest knots are torus knots, the ones which can be embedded into the surface of a standard torus in $\mathbb{R}^3$. They can be easily described parametrically.

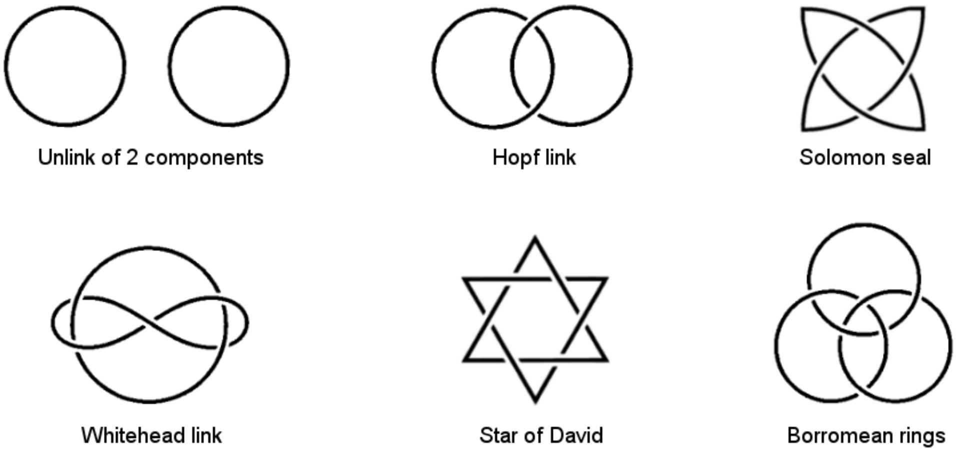


Figure 1.8: Examples of links

The $T[m,n]$ torus knot is a knot obtained by winding a loop over one cycle of the torus m times and over the other n times. If integers m and n are coprime, than it is a knot, otherwise it is a link. A torus knot is trivial if and only if either m or n is equal to 1 or -1. The simplest nontrivial example is the $T[2,3]$-torus knot, also known as the trefoil knot. The simplest nontrivial example of a link is the $T[2,2]$-torus link, also known as the Hopf link. Each nontrivial torus knot is prime and chiral. The $T[m,n]$-torus knot is equivalent to the $T[n,m]$-torus knot. The $T[m,-n]$-torus knot is the mirror image of the $T[m,n]$-torus knot. We have already had some pictures of torus knots: the first and the last knots on Figure 1.1 are torus knots, on Figure 1.8 we can see the Hopf link.

Classification. The classification of objects of study is a basic problem in any branch of mathematics. Usually a classification is a list of objects, which contains all possibilities without repetition. We can use different criteria to create such a list and the usefulness of the list depends on the criteria. Usually we can produce an algorithm that will list all possible objects, but this list will contain duplicates. For a creature of the knot catalogue it lists knot diagrams with increasing numbers of crossings. There naturally arises the problem to identify which diagrams represented the same knot. Till 1980's it was a big trouble, because there were no appropriate invariants to distinguish knot diagrams. The Reidemeister moves can be successfully used only in the case of positive solution, because there is a finite number of steps while in the case of negative solution the process will never terminate. As a special case one can consider the question about the knot triviality problem: find an algorithm which applies Reidemeister moves, simplifying the diagram at each step, and continue until it has no crossings; if there is no more possible simplification then the diagram cannot be trivial. However, Figure 1.9 (taken from [2]) shows diagrams of the trivial knot, which break this approach: any Reidemeister moves increases the number of crossings. Nevertheless there was invented an algorithm to solve the knot triviality problem by Wolfgang Haken, but it is based on the structure of the knot exterior – a compact 3-manifold. However the problem of detecting triviality with the help of

Reidemeister moves is still open.

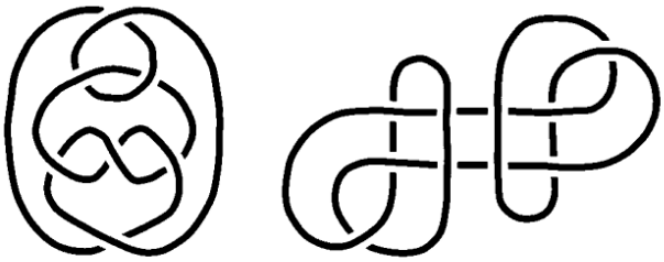


Figure 1.9: Awkward diagrams of the trivial knot: any Reidemeister moves increase the number of crossings.

Anyway an algorithm for detecting knot equivalence gives us only a mere classification list with no underlying structure. In some sense we need a construction of the "moduli space" of knots, and for this reason we have to create the complete set of knot invariants.

1.1.3 Knot invariants

Knot invariants are central objects of the study in knot theory. Here we briefly discuss some invariants.

Link, unknotting and crossing numbers.

Definition 1.1.5. ([2]) A link invariant *is a function from the set of links to some other set whose value depends only on the equivalence class of the link. Any representative from the class can be chosen to calculate the invariant. There is no restriction on the kind of objects in the target space. For example, they could be integers, polynomials, matrices or groups.*

One of the the simplest link invariants is *the multiplicity* of a link denoted by $\mu(L)$, which is the number of components of L.

Many invariants are related to geometric and topological properties of links and measure their complexity in various ways. Some of them are easy to define and very hard to calculate. One of the oldest link invariants is the unknotting number.

Definition 1.1.6. ([2]) The unknotting number *is the minimal number of times that a link must pass through itself to be transformed into a trivial link. This number is denoted by* $u(L)$.

Although this invariant is one of the most obvious measures of knot complexity, it is very difficult to calculate it. A non-trivial knot $\mathcal{K}$ which can be unknotted with only one pass has $u(\mathcal{K}) = 1$ (see Figure 1.10 for an example of such knot).

Any knot can be represented by a plane diagram in infinitely many ways; for this reason the following invariant was introduced.

Figure 1.10: The knot 7_2 becomes the unknot.

Definition 1.1.7. *We refer to the minimal number of crossings in a plane diagram of* $\mathcal{K}$ *as* the crossing number $c(\mathcal{K})$ *of a knot* $\mathcal{K}$*.*

If $c(\mathcal{K}) \leq 2$, then knot $\mathcal{K}$ is trivial. Therefore, to draw a diagram of a nontrivial knot the minimal number of crossings is required at least 3.

Braid index. *A braid* is a set of n strings, all of which are attached to a horizontal bar at the top and at the bottom as in Figure 1.11 [3]. Each string intersects any horizontal plane between the two bars exactly once, i.e. each string always goes in a downward direction while we are moving along it from the top bar to the bottom bar.

Figure 1.11: A braid.

We can always pull the bottom bar around and glue it to the top bar, so that the resulting strings form a knot or link. It is called the closure of the braid (see Figure 1.12 [3]). Thus, every braid corresponds to a particular knot and we have a *closed braid representation* of the knot.

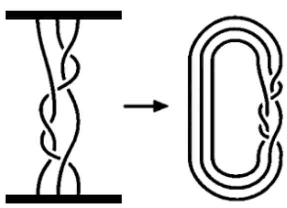


Figure 1.12: The closure of a braid.

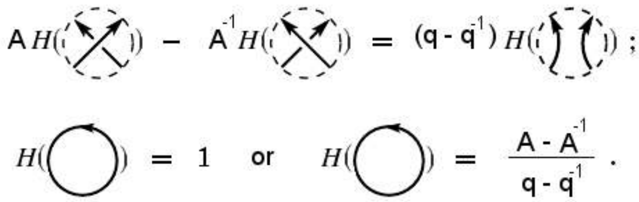
Every knot or link is a closed braid as was proven by Alexander. As in the case of plane diagrams, we are interested in representing the link by the braid with as few strings as possible.

Definition 1.1.8. The braid index *of a link is the minimal number of strings in a braid corresponding to a closed braid representation of the link.*

For example, the braid index of the unknot is 1 and of the trefoil is 2. The braid index is an invariant for knots and links.

Polynomial invarants. The most important knot invariants, from my point of view, are polynomial invariants taking values in the rings of polynomials in one or several variables with integer coefficients. The first discovered polynomial invariant was the Alexander polynomial $\Lambda(\mathcal{K})$ introduced in 1928. Then in 1970 Conway found a simple recursive construction of the Alexander polynomial. Then in 1985 Jones invented the Jones polynomial, which generalises Alexander polynomial. Very soon the HOMFLY polynomial (sometimes called HOMFLY-PT) was discovered which generalises the Jones polynomial. There is much speculation that the HOMFLY polynomial is the cenral object in knot theory of the present days. My point of view is the same, hence, we have devoted our efforts to studies of the HOMFLY polynomials and actually this thesis is devoted to them.

Definition 1.1.9. The HOMFLY polynomial *is the Laurent polynomial in two variables A and q with integer coefficients satisfying the following skein relation and the initial condition:*



The first initial condition corresponds to the so-called normalized HOMFLY polynomial, while the second one corresponds to the non-normalized HOMFLY. For normalized HOMFLY polynomial we use notation $H^{\mathcal{K}}$, while for non-normalized we use $\mathcal{H}^{\mathcal{K}}$. The HOMFLY polynomial unifies the quantum $\mathfrak{sl}_N$ polynomial invarants of $\mathcal{K}$ which are denoted by $H_N^{\mathcal{K}}(q)$ or $\mathcal{H}_N^{\mathcal{K}}(q)$ and are equal to $H^{\mathcal{K}}(A=q^N,q)$ or $\mathcal{H}^{\mathcal{K}}(A=q^N,q)$. To abuse the terminology we below use only notation "HOMFLY polynomial" considering $A=q^N$ correspondingly.

The HOMFLY polynomial is not a complete invariant for knots, because it cannot distinguish all knots. In particular, a pair of mutant knots always have the same HOMFLY polynomial (Figure 1.13 from [3]).

Figure 1.13: Two mutant knots have the same polynomial.

After the HOMFLY polynomials were introduced, it soon became clear that they are the first members of a whole family of knot polynomial invariants called quantum invariants. The original idea of quantum invariants was proposed by A. Schwarz [4] and E. Witten in the paper [5] in 1989. This approach came from physics, namely from quantum field theory, and was not completely justified from the mathematical point of view. However Reshetikhin and Turaev soon gave mathematically impeccable definition of quantum invariants of knots [74, 75, 76]. They used quantum groups, which were introduced shortly before by Drinfeld in [73, 72]. Actually, a quantum group is a family of Hopf algebras, depending on a complex parameter q and satisfying certain axioms. The quantum group $U_q\mathfrak{g}$ of a semisimple Lie algebra $\mathfrak{g}$ is a deformation of the universal enveloping algebra of $\mathfrak{g}$ (corresponding the value $q = 1$) in the class of Hopf algebras [1]. HOMFLY (Jones) polynomial coincides, up to normalization, with the quantum invariant corresponding to the Lie algebra $g = sl_N(sl_2)$ in its standard two-dimensional representation. However this approach allows to consider any irreducible representations and construct corresponding polynomial invariants. For the Lie algebra $\mathfrak{su}_N$ they are called *colored HOMFLY polynomials*.

1.1.4 Relation with Quantum Field Theory

It is believed that the path-integral representation for knot invariants arising from topological quantum field theory (TQFT) gives the most profound and general description of knot invariants. Ideologically, it means that all possible descriptions of knot invariants can be derived from this representation by utilizing different methods of path-integral calculus. For example, the usage of certain non-perturbative methods leads to the well known description of polynomial knot invariants through the "skein relations" [5]. The perturbative computations naturally lead to the numerical Vassiliev Invariants [60]. In the last case we obtain the formulae for Vassiliev invariants in the form of "Feynman integrals". For a recent comprehensive treatise on Vassiliev invariants see [1].

Despite beauty and simplicity of this picture many problems in the theory of knot invariants remain unsolved. Currently more mathematical descriptions of knot invariants are known than can be derived from path-integral. One such problem is the derivation of quantum group invariants from the path-integral representation. The main ingredient in the theory of these invariants is the universal quantum R-matrix defined for integrable quantum deformation of a Lie group. The appearance of quantum groups in the path-integral representation looks mysterious and we lack the derivation of the corresponding R-matrix (object with noncommutative matrix elements should appear from classical integral) from path-integral. These problems were discussed in details in [61, 62].

The second interesting problem (which as we believe is closely connected to the first one) concerns the combinatorial description of the numerical Vassiliev invariants of knots. At the moment, there exist three different descriptions of Vassiliev invariants: through the generalized Gauss integrals [63], through Kontsevich integral [64], and finally there are combinatorial formulae for Vassiliev invariants of orders 2, 3 and 4, [65]-[68]. The first two descriptions can be easily derived from the path-integral but (surprisingly) we lack such a derivation for combinatorial formulae. The Gauss integral representation for Vassiliev invariants comes from the perturbative computations of path-integral in the covariant Lorentz gauge. Similarly, the usage of non-covariant holomorphic gauge (which is sometimes referred to as light-cone gauge) computation of path-integral leads to Kontsevich integral [70].

The main difference between the non-covariant gauges (temporal or holomorphic gauge) and the covariant Lorentz gauge is that the Feynman integrals in the former case can be naturally "localized", in the sense that only a finite number of special points on the knots contribute to the integrals. In the Lorentz gauge the Feynman integrals have a form of multiple 3-d integrals (see (3.36),(3.37) as an example) and all points of the knot enter the integral equally. On the contrary, in the holomorphic gauge only special points contribute to the Feynman integrals. Summed in all orders of perturbation theory these contributions lead to definition of polynomial knot invariants through simple crossing operator and Drinfeld associator with rational zeta-function coefficients [71]. We discuss this localization process in detail in section 3.1.1.

The temporal gauge is distinguished among all gauges. The Feynman integrals here have ultra-local form – only crossing points of two-dimensional projection of the knot contribute to the answer. This is exactly what happens in the quantum group description of knot invariants where the crossing points contribute as a universal quantum R-matrix. On the other hand, the combinatorial formulae for Vassiliev invariants also are based on the information from these crossing points only. All these observations make it natural to argue that the Feynman integrals arising from the general path-integral representation in the temporal gauge should give the universal combinatorial formulae for Vassiliev invariants, and summed in all orders of perturbation theory they should give the perturbative expansion of the universal quantum R-matrix for $\hbar$-deformed gauge group of the path integral. These questions and structures arising from the path-integral representation of knot invariants in the temporal gauge are discussed in section 3.5.

Recently new polynomial invariants of knots appeared, the so-called "superpolynomials" [24], which are generalizations of HOMFLY polynomials. However, there is still no related QFT-like formulation. Probably this interpritation should involve something like "Chern-Simons theory with a quantum group for its gauge group" which is still not constructed.

1.2 Main results

In this section we overview our main results presented in the thesis without going into details. The theorem number corresponds to the section number, where this theorem is introduced and considered; same for conjectures.

Theorem 2.3.5. *Colored HOMFLY polynomial of a knot $\mathcal{K}$ belongs to the algebra of shifted symmetric functions Λ^*.*

Theorem 2.4.7. *Ooguri-Vafa partition function of a HOMFLY polynomial is a Hurwitz partition function.*

Theorem 3.4.1. *(Conditional) Higher special polynomials are related with Vassiliev invarants as follows:*

$$\sigma_{\Delta}^{\mathcal{K}}(g) = \sum_{k\geq 0} \alpha^k \sum_{m=1}^{\mathcal{N}_{u+k}} c_{u+k,m,k}(\Delta)\mathcal{V}^c_{u+k,m}.$$

The theorem is proven under assumption on the explicit form of genus expansion for HOMFLY polynomial (conjecture 2.4.3 below).

Theorem 3.4.2. *Vassiliev invariants of a knot $\mathcal{K}$ with r strands satisfy the following relation*

$$\det M_{j_1\ldots j_{r+1}} = 0,$$

where M is a matrix of Vassiliev invarants and any positive integers j_k.

Beyond these points there are some numerical results in the thesis. We have calculated Vassiliev invariants up to order 6 (inclusive) for knots with crossing numbers ≤ 14. There are around 60 000 such knots, complete results are available on the website [80]. One can download packages with these Vassiliev invarants from this website or refer to the section "Vassiliev invariants" for particular knots from the Rolfsen table. In section 3.4.2 for illustrative purpose we list Vassiliev invariants for knots with crossing numbers ≤ 7. Futhermore, with the help of these results we check that there are no universal relations on the Vassiliev invariants with order ≤ 6.

Also we have some conjectures based on explicit calculations done for a lot of particular knots.

Conjecture 2.4.3. *Large N expansion (or genus expansion) for HOMFLY* *polynomials is given by*

$$\begin{aligned}
H_R^{\mathcal{K}}(q,A) &= \exp\left(\sum_{\Delta} \hbar^{|\Delta|+l(\Delta)-2}\; S_{\Delta}^{\mathcal{K}}\big(A^2,\hbar^2\big)\; \varphi_R(\Delta)\right), \\
S_{\Delta}^{\mathcal{K}}(A^2,\hbar^2) &= \sum_{g} \sigma_{\Delta}^{\mathcal{K}}(g)\,\hbar^{2g},
\end{aligned}$$

where the sum goes over all Young diagrams Δ *and* $q = \exp(\hbar/2)$.

Conjecture 4.3.1. *DAHA-superpolynomials belong to the* Λ_β^* *algebra of functions symmetric in variables* $\mu_i = R_i - \beta i$.

Conjecture 4.3.2. *As for multiplicative basis* $T_k^\beta(R)$ *of the algebra* Λ_β^* *we consider*

$$T_k^\beta(R) = \sum_{i,j}\Big((j-1)-\beta(i-1)\Big)^{k-1}.$$

Conjecture 4.3.3. *Large* N *expansion for DAHA-superpolynomials is given by the following expression:*

$$\begin{aligned}
P_R^{\mathcal{K}}(q,t,A) &= \exp\Big\{\sum_{\Delta} \hbar^{|\Delta|+l(\Delta)-2}\cdot \mathcal{S}_{\Delta}^{\mathcal{K}}(\hbar^2,\beta,A)\cdot T_{\Delta}^{\beta}(R)\Big\}, \\
\mathcal{S}_{\Delta}^{\mathcal{K}}(\hbar^2,\beta,A) &= \sum_{n=0}^{\infty} \hbar^{2n} s_{\Delta}^{\mathcal{K}}(n).
\end{aligned}$$

For loop expansion of superpolynomials we consider thin knots and thick knots separately.

Conjecture 4.3.5. *In the case of thin knots the loop expansion of superpolynomials has the form*

$$P_R^{\mathcal{K}}(A,q,t) = \sum_{i=0}^{\infty} \hbar^i \sum_{j=1}^{\mathcal{N}_i^\beta} D_{i,j}^{(R)} \mathcal{V}_{i,j}^{\mathcal{K}},$$

where $D_{i,j}^{(R)}$ *are beta-deformations of trivalent diagrams,* $\mathcal{V}_{i,j}^{\mathcal{K}}$ *are exactly the same Vassiliev invariants as for the loop expansion of HOMFLY polynomials.*

Conjecture 4.3.6. *In the case of thick knots the loop expansion of superpolynomials has the form*

$$P_R^{\mathcal{K}}(A,q,t) = \sum_{i=0}^{\infty} \hbar^i \sum_{j=1}^{\mathcal{N}_i^\beta} D_{i,j}^{(R)} \mathcal{V}_{i,j}^{\mathcal{K}} \quad + \quad (\beta-1)\cdot \sum_{i=0}^{\infty} \hbar^i \sum_{j=1}^{\mathcal{M}_i^\beta} \Xi_{i,j}^{(R)} \rho_{i,j}^{\mathcal{K}},$$

where the first sum is the same as for the thin knots, while the second sum is different: $\Xi^{(R)}_{i,j}$ *are some other group factors and* $\rho^{\mathcal{K}}_{i,j}$ *are some other numbers different from the Vassiliev invariants of the first sum.*

These results are mostly based on the following papers by the author of the thesis:

1. P. Dunin-Barkowski, A. Sleptsov, A. Smirnov, *Kontsevich Integral for Knots and Vassiliev Invariants*, International Journal of Modern Physics A 28.17 (2013), arXiv:1112.5406 [hep-th]

2. A.Mironov, A.Morozov, A.Sleptsov, *On genus expansion of knot polynomials and hidden structure of Hurwitz tau-functions*, The European Physical Journal C 73 (2013) 2492 , arXiv:1304.7499 [hep-th]

3. A.Mironov, A.Morozov, A.Sleptsov, *On genus expansion of superpolynomials*, arXiv:1310.7622 [hep-th]

Particular examples of higher special polynomials and superpolynomials were taken from:

1. P. Dunin-Barkowski, A. Mironov, A. Morozov, A. Sleptsov, A. Smirnov, *Superpolynomials for toric knots from evolution induced by cut-and-join operators*, Journal of High Energy Physics 03 (2013) 021, arXiv:1106.4305 [hep-th]

2. A.Mironov, A.Morozov, A.Sleptsov, *Genus expansion of HOMFLY polynomials*, Theor.Math.Phys. 177 (2013) 179-221, arXiv:1303.1015 [hep-th]

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Chapter 2

Relation between HOMFLY polynomials and Hurwitz numbers

In the begining of this chapter we recall some basic facts about Hurwitz theory and theory of symmetric functions. Then we define colored HOMFLY polynomials via Chern-Simons theory and prove that they belong to the algebra of shifted symmetric functions. With the help of this result we, first, make a conjecture about explicit form of the large N expansion of HOMFLY polynomials. Second, we prove that the generating function for HOMFLY polynomial of a given knot in all representations is equal to generating function of Hurwitz numbers. Then in the rest of the chapter we discuss the large N expansion in the context of KP integrability.

2.1 Hurwitz theory

The Hurwitz theory for a curve X studies the enumeration of covers of X with particular ramifications, which are determined by the profile of the cover over the branching points. We consider covers

$$\pi : C \to X, \tag{2.1}$$

where C is a complex curve of genus $g(C)$ and X is a complex curve of genus $g(X)$. Denote the degree of π by d. We refer to the partition $\Delta \vdash d$ obtained from multiplicities $\pi^{-1}(x)$ as the profile of π over a point $x \in X$.

A partition Δ of d is defined as a sequence of integers

$$\Delta = \{\Delta_1 \geq \Delta_2 \geq \cdots \geq \Delta_l\} \tag{2.2}$$

where $|\Delta| = \sum_i \Delta_i = d$, $l(\Delta)$ is the length of the partition and $m_i(\Delta)$ denotes the multiplicity of the part i. The profile of π over x is the partition $\{1^d\}$ if and only if π is unramified over x.

Let us consider two covers $\pi : C \to X$, $\pi' : C' \to X$. We call them isomorphic if there exists an isomorphism of curves $h : C \to C'$, which satisfies $\pi' \circ h = \pi$.

In Hurwitz theory we consider covers up to isomorphism, because then there are finitely many Hurwitz covers of X of genus g, degree d and monodromy Δ^i at x_i.

Hurwitz numbers $\mathrm{Cov}_d(\Delta^1, \dots, \Delta^k)$ count the number of Hurwitz covers π with prescribed data. To calculate Hurwitz numbers one can use the Frobenius formula [43], but, first, we need to introduce more definitions.

Let S_d be the symmetric group, R be an irreducible representation of S_d of dimension $\dim R$. Also let us introduce dimensional factor $d_R := \dfrac{\dim R}{d!}$.

For each $k \geq 1$ the kth power sum is

$$p_k = \sum_i y_i^k, \tag{2.3}$$

$$p_\Delta = \prod_i p_{\Delta_i}. \tag{2.4}$$

In accordance with the Frobenius theorem, the linear group character χ_R is an antisymmetric polynomial of the monomials p_Δ:

$$\chi_R = \sum_{|\Delta|=|R|} |Aut(\Delta)|^{-1} \Phi_R^\Delta p_\Delta, \tag{2.5}$$

where $|Aut(\Delta)|$ counts the order of the automorphism group of the Young diagram Δ and the transition matrix Φ_R^Δ is the character of the symmetric group [41, 42]. We use, however, differently normalized characters $\varphi_R(\Delta)$ so that

$$\chi_R(p) = \sum_{|\Delta|=|R|} d_R \varphi_R(\Delta) p_\Delta. \tag{2.6}$$

Then the characters $\varphi_R(\Delta)$ are related by the Frobenius formula with the Hurwitz numbers [43] as follows:

$$\mathrm{Cov}_d(\Delta_1, \dots, \Delta_k) = \sum_R d_R^2 \varphi_R(\Delta_1) \dots \varphi_R(\Delta_k) \delta_{|R|,d}. \tag{2.7}$$

Now we extend the definition of Hurwitz numbers valid in the degree 0 case and in case the ramification conditions Δ satisfy $|\Delta| \neq d$.

Definition 2.1.1. ([48]) *The Hurwitz numbers* Cov_d *are defined for all degrees* $d \geq 0$ *and all partitions* Δ^i *by the following rules:*

1. $\mathrm{Cov}_0(\emptyset, \dots, \emptyset) = 1$, *where* $\emptyset$ *denotes the empty partition;*
2. *if* $|\Delta^i| > d$ *for some* i, *then the Hurwitz number vanishes;*
3. *if* $|\Delta^i| \leq d$ *for all* i, *then*

$$\mathrm{Cov}_d(\Delta_1, \dots, \Delta_k) = \prod_{i=1}^k \binom{m_1(\lambda^i)}{m_1(\Delta^i)} \mathrm{Cov}_d(\lambda^1, \dots, \lambda^k) \tag{2.8}$$

where λ^i *be the partition of size* d *obtained from* Δ^i *by adding* $d - |\Delta^i|$ *parts of size 1.*

Thus, we extend the definition of the symmetric group characters $\varphi_R(\Delta)$ to bigger diagrams R with $|R| > |\Delta|$ in the following way:

$$\varphi_R([\Delta, \underbrace{1, ..., 1}_{k}]) \equiv \begin{cases} 0 & \text{for} \quad |\Delta| + k > |R| \\ \varphi_R([\Delta, \underbrace{1, \ldots, 1}_{|R|-|\Delta|}])\binom{k}{|R|-|\Delta|} & \text{for} \quad |\Delta| + k \leq |R| \end{cases} \tag{2.9}$$

Here Δ is a Young diagram that does not contain units, $1 \notin \Delta$. With this extension one can lift the requirement that all $|\Delta^i| = d$ in (2.7).

2.1.1 Hurwitz partition function

Let us consider the following Hurwitz numbers

$$\mathrm{Cov}_d(\lambda^1, \lambda^2 | \underbrace{\Delta^1 ... \Delta^1}_{n_1}, \underbrace{\Delta^2 ... \Delta^2}_{n_2}, \ldots, \underbrace{\Delta^k ... \Delta^k}_{n_k}), \tag{2.10}$$

where λ^1, λ^2 are two distinguished diagrams, $\Delta^i \neq \Delta^j$ if $i \neq j$ and k is the highest possible number (it is finite, because $|\Delta|$ is not bigger then d due to the definition of extended Hurwitz numbers).

Definition 2.1.2. *Hurwitz partition function for Hurwitz numbers (2.10) is defined as follows:*

$$Z(p, \bar{p} | w_\Delta) = \sum_d \sum_{\lambda^1, \lambda^2} \sum_{\{n_i\}} \mathrm{Cov}_d(\lambda^1, \lambda^2 | \underbrace{\Delta^1 ... \Delta^1}_{n_1}, \ldots, \underbrace{\Delta^k ... \Delta^k}_{n_k}) p_{\lambda^1} \bar{p}_{\lambda^2} \frac{w_{\Delta^1}^{n_1}}{n_1!} \ldots \frac{w_{\Delta^k}^{n_k}}{n_k!} \tag{2.11}$$

Then using formulas (2.6) and (2.7) we immediately get the following proposition.

Proposition 2.1.3. *Hurwitz partition function is equal to*

$$Z(p, \bar{p} | w_\Delta) = \sum_R \chi_R(p) \chi_R(\bar{p}) \exp \left\{ \sum_\Delta w_\Delta \varphi_R(\Delta) \right\}. \tag{2.12}$$

Of course, it is not the only way to define generating function of Hurwitz numbers, see [44] for other generating functions related to partitions.

There is a well-known fact about one particular example of functions $Z(p, \bar{p} | w_\Delta)$.

Proposition 2.1.4. *Hurwitz partition function with $w_\Delta = 0$, $\forall \Delta$, is a solution (tau-function) of the Kadomtsev-Petviashvili hierarchy. This particular solution, denoted by $\tau_0(p, \bar{p})$, is called the trivial tau-function and equals*

$$\tau_0(p, \bar{p}) = \sum_R \chi_R(p) \chi_R(\bar{p}) = \exp \left(\sum_k \frac{p_k \bar{p}_k}{k} \right). \tag{2.13}$$

Usually more involved functions are given as an action of some operator on the trivial τ-function, especially to study integrable properties. In the next paragraph we represent the Hurwitz partition function in a such way.

2.1.2 Cut-and-join operators

Definition 2.1.5. *Cut-and-join operators $\hat{W}_\Delta$ are defined as differential operators acting on the arbitrary time-variables $\{p_k\}$ and have characters $\chi_R(p)$ as their eigenfunctions and $\varphi_R(\Delta)$ as the corresponding eigenvalues:*

$$\hat{W}_\Delta \chi_R = \varphi_{_R}(\Delta)\chi_R. \tag{2.14}$$

In the simplest case of $\Delta = [2]$ we get the standard cut-and-join operator [45]

$$\hat{W}_{[2]} = \frac{1}{2}\sum_{a,b=1}^{\infty}\left((a+b)p_a p_b \frac{\partial}{\partial p_{a+b}} + ab p_{a+b}\frac{\partial^2}{\partial p_a \partial p_b}\right). \tag{2.15}$$

Detailed construction of cut-and-join operators can be found in [93].

Proposition 2.1.6. *As an immediate corollary of (2.14) and (2.12) we have that the Hurwitz partition function can be obtained as an action of cut-and-join operators on the trivial KP τ-function:*

$$Z(p,\bar{p}|w_\Delta) = \exp\left(\sum_\Delta w_\Delta \hat{W}(\Delta)\right)\tau_0(p,\bar{p}). \tag{2.16}$$

2.2 Shifted symmetric functions

Definition 2.2.1. *A polynomial in N variables $R_1,\dots,R_N$ is said to be* a shifted symmetric, *if it becomes symmetric in the new variables $r_i = R_i - i,\ i = 1,\dots,N$.*

Let $\Lambda^*(N)$ denote the algebra of shifted symmetric polynomials in N variables. Define the projection $\Lambda^*(N) \to \Lambda^*(N-1)$ as the restriction of the polynomials in $R_1,\dots,R_n$ to the hyperplane $R_N = 0$.

Definition 2.2.2. *The projective limit of the sequence of filtered algebras $\Lambda^*(1) \leftarrow \Lambda^*(2)\dots$ is called* the algebra *of shifted symmetric functions and is denoted by Λ^*.*

The algebra Λ^* is similar to the algebra Λ of symmetric functions. In particular, the graded algebra $\mathrm{gr}\Lambda^*$ is naturally isomorphic to Λ.

Definition 2.2.3. *For any positive integer k, the corresponding shifted symmetric power sum $C(k)$ is defined as*

$$C_R(k) = \sum_{i=1}^{l(R)}\left(R_i - i + \frac{1}{2}\right)^k - \left(-i+\frac{1}{2}\right)^k. \tag{2.17}$$

The shifted symmetric power sum $C(k)$ will play a central role in our study. The shift $1/2$ can be replaced with any other constant, this induces a linear transformation of the set of $C(k)$, the particular choice of $1/2$ being more convenient for many purposes, including application to the large N expansion.

Now to clarify group-theoretic sense of $C_R(k)$ let $U(gl(N))$ be the universal enveloping algebra of the Lie algebra $gl(N)$ and let $Z(gl(N))$ be the center of $U(gl(N))$. The Harish-Chandra homomorphism establishes an isomorphism of filtered algebras $Z(gl(N)) \to \Lambda^*(N)$ such that the eigenvalue of an element $C \in Z(gl(N))$ in the irreducible representation R equals the value of the corresponding polynomial at the point $R_1, \dots, R_N$. In other words, $C_R(k)$ are eigenvalues of Casimir operators $\hat{C}(k)$:

$$\hat{C}(k) \;=\; C_R(k)\hat{I}. \tag{2.18}$$

It follows from the Schur lemma that for the Casimir operator a prime character χ_R of the linear group is its eigenfunction:

$$\hat{C}(k)\,\chi_R = C_R(k)\chi_R. \tag{2.19}$$

So, due to $C(k)$ are power sums, then they form multiplicative basis of the algebra Λ^*.

Another example of shifted symmetric function is the function $\varphi_R(\Delta)$, arising in the character formulas for Hurwitz numbers (2.7):

$$\varphi(\Delta) \in \Lambda^*. \tag{2.20}$$

Finally, we have very important proposition.

Proposition 2.2.4. ([46]). *The vector space spanned by the functions $\varphi(\Delta)$ coincides with the algebra generated by the functions $C(1), C(2), \dots$.*

Thus, the set $\{C(k)\}$ is multiplicative basis and $\{\varphi(\Delta)\}$ is additive basis of Λ^*. Also it is convenient to introduce multiplicative combinations of Casimir operators $\hat{C}(\Delta)$ labeled by partitions (Young diagrams) $\Delta = \{\Delta_1 \geq \Delta_2 \geq \dots \geq \Delta_l\}$:

$$\hat{C}(\Delta) \;=\; \prod_{j=1}^{l(\Delta)} \hat{C}(\delta_i), \tag{2.21}$$

$$\hat{C}(\Delta)\,\chi_R \;=\; C_R(\Delta)\chi_R. \tag{2.22}$$

Then the identification of the highest degree term of $\varphi(\Delta)$ ([47, 46]) gives:

$$\varphi(\Delta) = \frac{1}{\prod \Delta_i} C(\Delta) + \dots . \tag{2.23}$$

where the dots stand for terms of degree lower than $|\Delta|$. The combinatorial interrelation between the two linear bases $\{C(\Delta)\}$ and $\{\varphi(\Delta)\}$ of Λ^* is a fundamental aspect of the algebra Λ^*. These two bases define the Gromov-Witten/Hurwitz correspondence [48].

2.2.1 From Hurwitz to KP partition functions and renormalization group

Let us comment more on the two bases, multiplicative C_k and additive $\varphi_R(\Delta)$. Note that the generic Hurwitz exponential spanned by the additive basis

$$F_R = \exp\Big\{\sum_\Delta \mathcal{C}_\Delta \varphi_R(\Delta)\Big\} \tag{2.24}$$

gives rise to the generating function which is *not* a KP τ-function [116]. However, if the exponential is spanned by the linear basis (2.17),

$$F_R = \exp\Big\{\sum_k t_k C_R(k)\Big\} \tag{2.25}$$

it induces the Toda lattice τ-function [116, 50] with respect to times $p_k^{(1,2)}$.

Below we would like to informally discuss the physics side the interplay between multiplicative and additive bases.

When dealing with these Hurwitz exponentials, one may keep in mind the following analogy with the renormalization group (RG) and completeness of basis [15]. Let us consider a quantum field theory partition function

$$Z(G;\varphi_0;t) = \int_{\mathcal{A};\varphi_0} D\phi \exp\left(\frac{1}{2}\phi G\phi + A(t;\phi)\right) \tag{2.26}$$

which depends on: (a) the background fields φ_0; (b) the coupling constants t and (c) the metric G.

The coupling constants parameterize the shape of the action

$$A(t;\phi) = \sum_{n\in B} t^{(n)}\mathcal{O}_n(\phi), \tag{2.27}$$

where the sum goes over some complete set B of functions $\mathcal{O}_n(\phi)$, not obligatory finite or even discrete. The space $\mathcal{M} \subset \mathrm{Fun}(\mathcal{A})$ of actions parameterized by the coupling constants $t^{(n)}$, is referred to as the moduli space of theories. The actions usually take values in numbers or, more generally, in certain rings, perhaps, non-commutative. The space $\mathrm{Fun}(\mathcal{A})$ of all functions of ϕ is always a ring, but this needs not be true about the moduli space $\mathcal{M}$, which could be as small a subset as one likes. However, the interesting notion of partition function arises only if the completeness requirement is imposed on $\mathcal{M}$. There are two different degrees of completeness, relevant for discussions of partition functions. In the first case (strong completeness), the functions $\mathcal{O}_n(\phi)$ form a *linear* basis in $\mathrm{Fun}(\mathcal{A})$, then $\mathcal{M}$ is essentially the same as $\mathrm{Fun}(\mathcal{A})$ itself. In the second case (weak completeness), the functions $\mathcal{O}_n$ generate $\mathrm{Fun}(\mathcal{A})$ as a ring, i.e. an arbitrary function of ϕ can be decomposed into a sum of *multiplicative* combinations of $\mathcal{O}_n$'s. In the case of strong completeness, the notion of RG is absolutely straightforward, but there is no clear idea how RG can be formulated in the case of weak completeness (which is more relevant for most modern considerations).

In the strongly complete case, the non-linear (in coupling derivatives) equation, even if occurs, can be always rewritten as a linear equation. In fact, one can easily make a weakly complete model strongly complete, by adding all the newly emerging operators to the action $A(t;\phi)$, then, if the product $\mathcal{O}_m\mathcal{O}_n$ is added with the coefficient $t^{(m,n)}$, one has an identity $\partial^2 Z/\partial^{(m)}\partial t^{(n)} = \partial Z/\partial t^{(m,n)}$.

The core of the analogy is that $\varphi_R(\Delta)$ (linear basis) corresponds to a strongly complete set and $C_R(k)$ (multiplicative basis) corresponds to a weakly complete set of operators. Moreover, as we saw one can lift the multiplicative basis to the linear basis as the weak completeness lift to the strong completeness, i.e. via introducing new operators (2.21).

2.3 HOMFLY polynomials

In the previous sections we defined algebra Λ^* of shifted symmetric functions and constructed two linear bases of Λ^*, namely $\{C(\Delta)\}$ and $\{\varphi(\Delta)\}$. Here we define HOMFLY polynomials and explain why they are shifted symmetric functions.

2.3.1 Chern-Simons approach

This approach is also reviewed in the next chapter, now we need only general definitons. Let A be a connection on $\mathbb{R}^3$ taking values in some representation R of a Lie algebra $\mathfrak{su}(N)$, i.e., in components:

$$\mathcal{A} = A_i^a(x)\, T^a\, dx^i, \tag{2.28}$$

where T^a are the generators of $\mathfrak{su}(N)$. Let curve C in $\mathbb{R}^3$ gives a particular realization of knot $\mathcal{K}$. Consider the holonomy of A along C, it can be represented as the path-ordered exponential:

$$\begin{aligned}\Gamma(C,A) &= P\exp\oint_C A = \\ &= 1 + \oint_C A_i^a(x)T^a dx^i + \oint_C A_{i_1}^{a_1}(x_1)dx_1^i \int_0^{x_1} A_{i_2}^{a_2}(x_2)\, T^{a_1}T^{a_2} dx_2^i + \ldots\end{aligned} \tag{2.29}$$

Definition 2.3.1. *The Wilson loop along a contour C is defined as the trace of holonomy of the gauge connection $\mathcal{A}$ along this contour:*

$$W_R(C,\mathcal{A}) = \mathrm{tr}\,_R\Gamma(C,\mathcal{A}). \tag{2.30}$$

Definition 2.3.2. *HOMFLY polynomial for a knot $\mathcal{K}$ is the vacuum expectation value of the Wilson loop in* 3*d Chern-Simons theory with the gauge group $SU(N)$:*

$$\mathcal{H}_R^{\mathcal{K}} = \langle W_R(\mathcal{K},\mathcal{A})\rangle_{\mathrm{CS}}, \tag{2.31}$$

where brackets $\langle\ldots\rangle_{\mathrm{CS}}$ denote averaging over all connections $A_i^a(x)$ with Chern-Simons weight.

So far, expilicit form of this weight is not important as well as details of the averaging. Their specifications we discuss in the section (3.1.1), while here it is important only the point that the averaging is going over $A_i(x)$.

Since holonomy $\Gamma(C, \mathcal{A})$ is a group element of $SU(N)$, then $W_R(\mathcal{K}, \mathcal{A}) = \operatorname{tr}_R(U)$, $U \in SU(N)$. Thus, we have

$$\mathcal{H}_R^{\mathcal{K}} = \langle \operatorname{tr}_R(U) \rangle_{\mathrm{CS}} = \langle \chi_R(U) \rangle_{\mathrm{CS}}. \tag{2.32}$$

HOMFLY polynomial $\mathcal{H}_R^{\mathcal{K}}$ defined in a such way for any knot $\mathcal{K}$ actually is not a polynomial but a rational function, so-called non-normalized HOMFLY. To get a polynomial one needs to normalize $\mathcal{H}_R$ on the HOMFLY of unknot, which is equal to the Schur function $\chi_R(p)$, calculated at the special point (topological locus)

$$p_k^* = \frac{A^k - A^{-k}}{q^k - q^{-k}}. \tag{2.33}$$

Thus, for the normalized HOMFLY polynomial we have the following definition.

Definition 2.3.3. *The normalized HOMFLY polynomial is defined as follows:*

$$H_R^{\mathcal{K}} = \frac{\langle \chi_R(U) \rangle_{\mathrm{CS}}}{\chi_R(p^*)}. \tag{2.34}$$

Below we will usually omit "normalized" and use only "HOMFLY polynomial" for the normalized case.

Let us apply formula (2.6) $\chi_R(p) = \sum\limits_{|\Delta|=|R|} d_R\, \varphi_R(\Delta)\, p_\Delta$ in this case and use the fact CS averaging is over U only:

$$H_R^{\mathcal{K}} = \frac{\Big\langle \sum\limits_{|\Delta|=|R|} d_R\, \varphi_R(\Delta)\, p_\Delta(U) \Big\rangle_{\mathrm{CS}}}{\sum\limits_{|\Delta|=|R|} d_R\, \varphi_R(\Delta)\, p_\Delta^*} = \frac{\sum\limits_{|\Delta|=|R|} \varphi_R(\Delta) \Big\langle\, p_\Delta(U)\, \Big\rangle_{\mathrm{CS}}}{\sum\limits_{|\Delta|=|R|} \varphi_R(\Delta)\, p_\Delta^*} \tag{2.35}$$

Since $\varphi_R(\Delta) \in \Lambda^*$, it is clear from (2.35) that $H_R^{\mathcal{K}}$ is a symmetric function in $\{r_i\}$. Moreover, the numerator is a shifted symmetric polynomial and the denominator is a shifted symmetric polynomial. Let us rewrite the denominator as follows:

$$\sum_{|\Delta|=|R|} \varphi_R(\Delta)\, p_\Delta^* = \frac{\chi_R(p^*)}{d_R} \tag{2.36}$$

and proof the following lemma.

Lemma 2.3.4.

$$\frac{\chi_R(p^*)}{d_R} \neq 0, \quad \text{while } r_i = 0,\ \forall i \tag{2.37}$$

The dimensional factor d_R is given by the following formula:

$$d_R = \frac{\prod\limits_{i<j=1}^{|R|} (r_i - r_j)}{\prod\limits_{i=1}^{|R|} (r_i + |R|)!} \tag{2.38}$$

Characters can be expressed through the eigenvalues of matrix X, $p_k = \operatorname{tr} X^k$ by the second Weyl determinant formula

$$\chi_R[X] = \frac{\det_{i,j} x_i^{r_j}}{\det_{i,j} x_i^{-j}}. \tag{2.39}$$

From this formula it is obviously that $\chi_R[X]$ have zeros while $r_i = r_j$, $\forall i \neq j$, furthermore multiplicities of zeros equal to 1 (take corresponding derivative). Expanding the function $\det\limits_{i,j} x_i^{m_j}$ as series in $\{r_i\}$ it is easy to prove by induction that the lowest term has exactly such degree as Vandermonde, hence, it is proportional to $\prod\limits_{i<j=1}^{|R|} (r_i - r_j)$. Therefore, (2.37) is not equal to zero at the point $(r_1 = 0, r_2 = 0, \dots)$.

Thus, the denominator of (2.35) does not have a pole at the same point. Therefore, HOMFLY (2.35) is a power series in $\{r_i\}$ (without negative degrees). So, we get the following theorem.

Theorem 2.3.5. *The normalized HOMFLY polynomial $H_R^{\mathcal{K}}$ for an arbitrary knot $\mathcal{K}$ and for an irreducible representation R is a shifted symmetric function.*

Remark. Note that now we proved this theorem only for positive integers values of N. To complete the proof for any real N we have to extend the LHS and the RHS of equation (2.35) to any real N. It will be done after explicit definition of the Chern-Simons averaging $\langle \dots \rangle_{CS}$ in section 3.1, see remark after proposition 3.1.4.

Corollary 2.3.6. *The normalized HOMFLY polynomial $H_R^{\mathcal{K}}$ as a function of $\{r_i\}$ can be expanded in basis $\{\varphi(\Delta)\}$ or $\{C(\Delta)\}$.*

In the next section we consider the case when the expansion in $\{\varphi(\Delta)\}$ for HOMFLY appears in a natural way.

2.4 Large N expansion

There are several important perturbative expansions of the HOMFLY polynomials: the "volume" expansion, large N expansion, loop expansion, all of them are giving rise to very interesting invariants. We do not consider the "volume" expansion here, because it deals with the limit $|R| \to \infty$, and we are going to study the R-dependence. In this section we consider the large N expansion and discuss its properties. In the next chapter we consider Vassiliev expansion.

In fact, the variable A is defined as an (exponentiated) 't Hooft's coupling constant $A = \exp\left(\frac{N\hbar}{2}\right)$, while $q = \exp\left(\frac{\hbar}{2}\right)$, and so one can consider a limit $\hbar \to 0,\ N \to \infty$. This limit is well-defined for normalized polynomials.

Definition 2.4.1. *The special polynomial $\sigma_R^{\mathcal{K}}(A)$ is defined as the limit $q \to 1$ of the normalized HOMFLY polynomial $H^{\mathcal{K}} \in M_R(A,q)$:*

$$\sigma_R^{\mathcal{K}}(A) = \lim_{q\to 1} H_R^{\mathcal{K}}(A,q). \tag{2.40}$$

We refer to this limit as the 't Hooft's planar limit *or* the large N limit*.*

In this limit HOMFLY polynomial for a link (a multi-trace operator) decomposes into a product of averages over its particular components. If the cabling technique (see [102, 103] for a review and modern applications) is used to represent the colored knot as a link made from $|R|$ copies the same knot in the fundamental representation, this implies the factorization of the special polynomials [94, 104, 105, 106]. In other words we have the following proposition.

Proposition 2.4.2. *The colored special polynomial depends on the irreducible representation R as follows:*

$$\sigma_R^{\mathcal{K}}(A) = \left(\sigma_{[1]}^{\mathcal{K}}(A)\right)^{|R|}. \tag{2.41}$$

This is the starting point, zero order, of the *large N expansion* (also known as the AMM/CEO topological recursion [107]). Note that equation (2.41) imply

$$\sigma_R^{\mathcal{K}}(A) = \exp\left(\log\sigma_{[1]}^{\mathcal{K}}(A)\cdot|R|\right) = \exp\left(\log\sigma_{[1]}^{\mathcal{K}}(A)\cdot\varphi_R([1])\right), \tag{2.42}$$

i.e. we see that zero order of expansion is given by the exponential, which degree is divided into two terms. First term describes dependence on the knot $\mathcal{K}$, while second term describes dependence on the representation R and is given by the character of symmetric group. From the other hand we know from Theorem 2.3.5 and Corollary 2.3.6 that HOMFLY polynomial can be written as follows:

$$H_R^{\mathcal{K}}(q,A) = \exp\left(\sum_{\Delta} w_{\Delta}^{\mathcal{K}}\,\varphi_R(\Delta)\right). \tag{2.43}$$

Thus, based on (2.42) and (2.43) we formulate the following conjecture.

Conjecture 2.4.3. *Large N expansion for HOMFLY polynomials is given by*

$$\begin{aligned} H_R^{\mathcal{K}}(q,A) &= \exp\left(\sum_{\Delta} \hbar^{|\Delta|+l(\Delta)-2}\ S_{\Delta}^{\mathcal{K}}\big(A^2,\hbar^2\big)\ \varphi_R(\Delta)\right), \qquad (2.44)\\ S_{\Delta}^{\mathcal{K}}(A^2,\hbar^2) &= \sum_{g}\sigma_{\Delta}^{\mathcal{K}}(g)\,\hbar^{2g}, \end{aligned}$$

where the sum goes over all Young diagrams Δ, as usual, $|\Delta|$ and $l(\Delta)$ denote respectively the numbers of boxes and lines in Δ,

$$\hbar = \log q, \tag{2.45}$$

$\sigma_\Delta^\mathcal{K}(n)$ are polynomials in A, and $\varphi_R(\Delta)$ are the symmetric group characters.

At the moment it is important only that [93, 108]

$$\varphi_R(\Delta) = 0 \quad \text{for} \quad |\Delta| > |R| \tag{2.46}$$

The coefficients of the $\hbar^2$-series $S_\Delta(A^2, \hbar^2)$ are polynomials in A (the $\hbar^0$ term in $S_{[1]}$ would be logarithmic). They depend on a knot $\mathcal{K}$ and in the next chapter we give an explicit formulas for them in terms of Vassiliev invariants.

Remark. Note that all explicitly known HOMFLY polynomials satisfy the statement of Conjecture 2.4.3.

Remark. We should make an immediate important comment about the expansion (2.44). Note that the first and the second orders of the expansion are completely defined only by (anti)symmetric representations, i.e. the knowledge of the HOMFLY polynomials in these representations fixes the first two corrections to the special polynomial in any representation. Same thing happens for the third order in spite of the term $\varphi_{_R}([2,1])$. Namely, this term can be determined by symmetric representation [3] because $\varphi_{_{[3]}}([2,1]) \neq 0$.

However in the fourth order it is no longer true. The point is that $\varphi_{_R}([3])$ and $\varphi_{_R}([1,1,1])$ are not linearly independent for (anti)symmetric representations, while both are present in the fourth order. Appearance of non-symmetric representation is related with the well known fact that HOMFLY polynomials in (anti)symmetric representations can not distinguish mutant knots, however they do so in non-symmetric representations. For instance, the (mirrored) Conway knot $K11n34$ and the (mirrored) Kinoshita-Terasaka knot $K11n42$ are a mutant pair of knots with 11 intersections, so they are notoriously difficult to tell apart and are distinguished only by the HOMFLY polynomials starting from representation [21] (or by the framed Vassiliev invariant of type 11).

2.4.1 HOMFLY polynomial as a W-transform of the character

The most spectacular feature of the large N expansion (2.44) is that the dependence on the representation R is fully encoded in the extended symmetric group characters $\varphi_R(\Delta)$. As we discussed above they are the eigenvalues of the generalized cut-and-join operators

$$\hat{W}(\Delta)\,\chi_R \;=\; \varphi_R(\Delta)\,\chi_R \tag{2.47}$$

Proposition 2.4.4. *(see [112, 93, 108]) These operators form a commutative algebra:*

$$\hat{W}(\Delta_1)\hat{W}(\Delta_2) = \sum_\Delta C^\Delta_{\Delta_1\Delta_2}\hat{W}(\Delta) \tag{2.48}$$

Here we list some explicit examples of the structure constants in (2.48): a multiplication table restricted to the case when $|\Delta| \leq 4$.

$$\hat{W}_{[1]}\hat{W}_{[1]} = \hat{W}_{[1]} + 2\hat{W}_{[1,1]},$$

$$\hat{W}_{[1]}\hat{W}_{[2]} = 2\hat{W}_{[2]} + \hat{W}_{[2,1]},$$
$$\hat{W}_{[1]}\hat{W}_{[1,1]} = 2\hat{W}_{[1,1]} + 3\hat{W}_{[1,1,1]},$$

$$\hat{W}_{[1]}\hat{W}_{[3]} = 3\hat{W}_{[3]} + \hat{W}_{[3,1]},$$
$$\hat{W}_{[1]}\hat{W}_{[2,1]} = 3\hat{W}_{[2,1]} + 2\hat{W}_{[2,1,1]},$$
$$\hat{W}_{[1]}\hat{W}_{[1,1,1]} = 3\hat{W}_{[1,1,1]} + 4\hat{W}_{[1,1,1,1]},$$

$$\hat{W}_{[1]}\hat{W}_{[4]} = 4\hat{W}_{[4]} + \hat{W}_{[4,1]},$$
$$\hat{W}_{[1]}\hat{W}_{[3,1]} = 4\hat{W}_{[3,1]} + 2\hat{W}_{[3,1,1]},$$
$$\hat{W}_{[1]}\hat{W}_{[2,2]} = 4\hat{W}_{[2,2]} + \hat{W}_{[2,2,1]},$$
$$\hat{W}_{[1]}\hat{W}_{[2,1,1]} = 4\hat{W}_{[2,1,1]} + 3\hat{W}_{[2,1,1,1]},$$
$$\hat{W}_{[1]}\hat{W}_{[1,1,1,1]} = 4\hat{W}_{[1,1,1,1]} + 5\hat{W}_{[1,1,1,1,1]},$$

$$\hat{W}_{[1,1]}\hat{W}_{[2]} = \hat{W}_{[2]} + 2\hat{W}_{[2,1]} + \hat{W}_{[2,1,1]},$$
$$\hat{W}_{[1,1]}\hat{W}_{[1,1]} = \hat{W}_{[1,1]} + 6\hat{W}_{[1,1,1]} + 6\hat{W}_{[1,1,1,1]},$$
$$\hat{W}_{[2]}\hat{W}_{[2]} = \hat{W}_{[1,1]} + 3\hat{W}_{[3]} + 2\hat{W}_{[2,2]},$$

$$\hat{W}_{[1,1]}\hat{W}_{[3]} = 3\hat{W}_{[3]} + 3\hat{W}_{[3,1]} + \hat{W}_{[3,1,1]},$$
$$\hat{W}_{[1,1]}\hat{W}_{[2,1]} = 3\hat{W}_{[2,1]} + 6\hat{W}_{[2,1,1]} + \hat{W}_{[2,1,1,1]},$$
$$\hat{W}_{[1,1]}\hat{W}_{[1,1,1]} = 3\hat{W}_{[1,1,1]} + 12\hat{W}_{[1,1,1,1]} + 10\hat{W}_{[1,1,1,1,1]},$$
$$\hat{W}_{[2]}\hat{W}_{[3]} = \hat{W}_{[3,2]} + 4\hat{W}_{[4]} + 2\hat{W}_{[2,1]}$$
$$\hat{W}_{[2]}\hat{W}_{[2,1]} = 2\hat{W}_{[2,2,1]} + 3\hat{W}_{[3,1]} + 4\hat{W}_{[2,2]} + 3\hat{W}_{[3]} + 3\hat{W}_{[1,1,1]}$$
$$\hat{W}_{[2]}\hat{W}_{[1,1,1]} = \hat{W}_{[2,1]} + 2\hat{W}_{[2,1,1]} + \hat{W}_{[2,1,1,1]},$$

$$\ldots$$

Furthermore, in accordance with (2.47) the eigenvalues $\varphi_R(\Delta)$ satisfy the same algebra (2.48):

$$\varphi_R(\Delta_1)\varphi_R(\Delta_2) = \sum_{\Delta} C^{\Delta}_{\Delta_1\Delta_2}\varphi_R(\Delta) \tag{2.49}$$

The structure constants in this relation do not depend on R, which is not so obvious if one extracts $\varphi_R(\Delta)$ from the character expansion (2.6).

Making use of the fact (2.47) and (2.44) we get the following proposition.

Proposition 2.4.5. *HOMFLY polynomial is given by the action of a W-evolution operator on the character χ_R*

$$H_R^{\mathcal{K}}(q,A)\chi_R = \Big(\sigma_{\Box}(A)\Big)^{|R|} \exp\left(\sum_{\Delta} w_{\Delta}^{\mathcal{K}}\,\hat{W}(\Delta)\right)\chi_R \tag{2.50}$$

where the coefficients $w_{\Delta}^{\mathcal{K}}$, yet another set of time-variables, depend on the knot and on the $\hbar$-variable:

$$w_{\Delta}^{\mathcal{K}} = \hbar^{|\Delta|+l(\Delta)-2}\ S_{\Delta}^{\mathcal{K}}\big(A^2,\hbar^2\big) \tag{2.51}$$

2.4.2 Ooguri-Vafa partition function as a Hurwitz tau-function

Now one can naturally consider the generating function of the *non-normalized* HOMFLY polynomials and study its properties.

Definition 2.4.6. *The generating function for the non-normalized HOMFLY polynomials for a given knot in all representations is defined as follows:*

$$Z_{OV}(\bar{p}|\mathcal{K}) = \sum_R \chi_R(\bar{p})\mathcal{H}_R^{\mathcal{K}} = \sum_R \chi_R(\bar{p})\chi_R(p^*)H_R^{\mathcal{K}} \tag{2.52}$$

This is exactly the Ooguri-Vafa partition function which was considered in [92] in the context of duality of the Chern-Simons theory and topological string on the resolved conifold.

With the help of formula (2.43) we immediately get the following form of the Ooguri-Vafa partition function:

$$Z_{OV}(\,\bar{p}\,|\,\mathcal{K}\,) = \sum_R \chi_R(\bar{p})\chi_R(p^*)\exp\left(\sum_{\Delta} w_{\Delta}^{\mathcal{K}}\,\varphi_R(\Delta)\right). \tag{2.53}$$

Recalling Proposition (2.1.3)

$$Z(p,\bar{p}|w_\Delta) = \sum_R \chi_R(p)\chi_R(\bar{p})\exp\left\{\sum_{\Delta} w_\Delta\varphi_R(\Delta)\right\}, \tag{2.54}$$

we see that the Ooguri-Vafa partition function of given knot $\mathcal{K}$ is equal to the Hurwitz partition function with the corresponding parameters $w_{\Delta}^{\mathcal{K}}$ and $p_k = p_k^*$. Thus, for different knots Hurwitz covers are same, while parameters $w_{\Delta}^{\mathcal{K}}$ of partition function are different. They are labeled by Young diagrams Δ and depend on variables q and A.

Theorem 2.4.7. *Ooguri-Vafa partition function is the Hurwitz partition function (2.12) where $p_k = p_k^*$:*

$$Z_{OV}(\,\bar{p}\,|\,\mathcal{K}\,) = \sum_R \chi_R(\bar{p})\chi_R(p^*)\exp\left(\sum_{\Delta} w_{\Delta}^{\mathcal{K}}\,\varphi_R(\Delta)\right). \tag{2.55}$$

Note that, conversely, $Z_{OV}(\,\bar{p}\,|\,\mathcal{K}\,)$ can be naturally extended to $Z_{OV}(p,\bar{p}|\mathcal{K})$ by replacing p_k^* with arbitrary variables p_k.

Corollary 2.4.8. *Ooguri-Vafa partition function can be obtained as an action of cut-and-join operators on the trivial KP τ-function:*

$$Z(\bar{p}|\mathcal{K})_{OV} = \exp\left(\sum_{\Delta} w_{\Delta}^{\mathcal{K}}\,\hat{W}(\Delta)\right)\tau_0^{\mathcal{K}}\{\bar{p}\}, \tag{2.56}$$

where

$$\tau_0^{\mathcal{K}}\{\bar{p}\} = \exp\left(\sum_k \frac{\sigma_{\Box}^{\mathcal{K}}(A)}{k} p_k^* \bar{p}_k\right) \tag{2.57}$$

We used here the celebrated Cauchy formula

$$\sum_R \mu^{|R|}\chi_R\{p\}\chi_R\{\bar{p}\} = \exp\left(\sum_k \frac{\mu^k p_k \bar{p}_k}{k}\right). \tag{2.58}$$

The main implication of (2.56) is that the Ooguri-Vafa partition function is actually a Hurwitz tau-function (2.16)

$$\tau_H\{w|\bar{p}\} = \exp\left(\sum_{\Delta} w_{\Delta}\hat{W}(\Delta)\right)\tau_0\{\bar{p}\}, \qquad \tau_0\{\bar{p}\} = \exp\left(\sum_k c_k \bar{p}_k\right) \tag{2.59}$$

taken at a particular knot-dependent value of the Hurwitz time-variables w_{Δ}. In accordance with Conjecture 2.4.3 they are equal to

$$w_{\Delta}^{\mathcal{K}}(q,A) = \hbar^{|\Delta|+l(\Delta)-2}\, S_{\Delta}^{\mathcal{K}}(A^2,\hbar^2). \tag{2.60}$$

Plücker relations

Let us consider generating function of the form

$$\tau(\bar{p}) = \sum_R \xi_R \chi_R(\bar{p}), \tag{2.61}$$

where $\{\bar{p}_k\}$ is a set of time-variables in Schur polynomial, ξ_R are arbitrary variables independent on $\{\bar{p}\}$. Then $\tau(\bar{p})$ solves the KP hierarchy if and only if the coefficients ξ_R are subject to the relation [53]:

$$\xi\begin{pmatrix} i_1 ... \check{i}_\mu ... \check{i}_\nu ... i_r \\ j_1 ... \check{j}_\mu ... \check{j}_\nu ... j_r \end{pmatrix} \xi\begin{pmatrix} i_1 ... i_r \\ j_1 ... j_r \end{pmatrix} - \xi\begin{pmatrix} i_1 ... \check{i}_\mu ... i_r \\ j_1 ... \check{j}_\mu ... j_r \end{pmatrix} \xi\begin{pmatrix} i_1 ... \check{i}_\nu ... i_r \\ j_1 ... \check{j}_\nu ... j_r \end{pmatrix} + \\ + \xi\begin{pmatrix} i_1 ... \check{i}_\mu ... i_r \\ j_1 ... \check{j}_\nu ... j_r \end{pmatrix} \xi\begin{pmatrix} i_1 ... \check{i}_\nu ... i_r \\ j_1 ... \check{j}_\mu ... j_r \end{pmatrix} = 0 \tag{2.62}$$

where we have put $\xi\begin{pmatrix} i_1\dots i_r \\ \\ j_1\dots j_r\end{pmatrix} = \xi_R$ for a Young diagram R, column (i,j) is a hook diagram with row i and column $j+1$, the diagram is represented by a set of hooks. For illustrative purpose we list here some examples of Plücker relations:

$$\begin{aligned}
\xi_{[22]}\xi_{[0]} - \xi_{[21]}\xi_{[1]} + \xi_{[2]}\xi_{[11]} &= 0 \\
\\
\xi_{[32]}\xi_{[0]} - \xi_{[31]}\xi_{[1]} + \xi_{[3]}\xi_{[11]} &= 0 \\
\xi_{[221]}\xi_{[0]} - \xi_{[211]}\xi_{[1]} + \xi_{[2]}\xi_{[111]} &= 0 \\
\\
\xi_{[42]}\xi_{[0]} - \xi_{[41]}\xi_{[1]} + \xi_{[4]}\xi_{[11]} &= 0 \\
\xi_{[33]}\xi_{[0]} - \xi_{[31]}\xi_{[2]} + \xi_{[3]}\xi_{[21]} &= 0 \\
\xi_{[321]}\xi_{[0]} - \xi_{[311]}\xi_{[1]} + \xi_{[3]}\xi_{[111]} &= 0 \\
\xi_{[222]}\xi_{[0]} - \xi_{[211]}\xi_{[11]} + \xi_{[21]}\xi_{[111]} &= 0 \\
\xi_{[2211]}\xi_{[0]} - \xi_{[2111]}\xi_{[1]} + \xi_{[2]}\xi_{[1111]} &= 0 \\
&\dots
\end{aligned} \tag{2.63}$$

Example 2.4.9. *As a particular example let us consider trivial tau-function (2.13):*

$$\tau_0(p,\bar p) = \sum_R \chi_R(p)\chi_R(\bar p). \tag{2.64}$$

Since Schur polynomials satisfy Plücker relations themselves, then in this case $\xi_R = \chi_R(p)$ *or* $\xi_R = \chi_R(\bar p)$. *Therefore,* $\tau_0(p,\bar p)$ *is a tau-function KP in variables* $\{p_k\}$ *and* $\{\bar p_k\}$. *Thus, considering (2.52) for the unknot we get*

$$Z_{OV}^{unknot} = \sum_R \chi_R(\bar p)\chi_R(p^*) = \tau_0(\bar p). \tag{2.65}$$

Example 2.4.10. *Now let us consider Ooguri-Vafa partition function for arbitrary knot, but for large N limit. Then from formulas (2.50) and (2.52) we get the following*

$$Z_{OV}^{\mathcal{K}} = \sum_R \chi_R(\bar p)\chi_R(p^*)\sigma_R^{\mathcal{K}}(A). \tag{2.66}$$

Since in the large N limit $\sigma_R^{\mathcal{K}}(A) = \left(\sigma_\Box^{\mathcal{K}}(A)\right)^{|R|}$, *then*

$$Z_{OV}^{\mathcal{K}} = \sum_R \chi_R(\bar p)\,\xi_R^{\mathcal{K}}, \tag{2.67}$$

$$\xi_R^{\mathcal{K}} = \chi_R(p^*)\left(\sigma_\Box^{\mathcal{K}}(A)\right)^{|R|}. \tag{2.68}$$

It is clear from (2.62) that Plücker relations are homogeneous in R. Therefore, terms $\left(\sigma_\Box^{\mathcal{K}}(A)\right)^{|R|}$ *can be factored out from each relation, terms* $\chi_R(p^*)$ *satisfy relations, hence, Ooguri-Vafa partition function (2.67) is a KP tau-function again, moreover, for each knot separately.*

2.4.3 Linear vs non-linear evolution

Any system of commuting operators, like $\{\hat{W}(\Delta)\}$ can be considered as a system of Hamiltonians of an integrable system, and matrix elements of the evolution operator

$$\hat{\mathcal{U}}\{\beta\} = \exp\left(\sum_{\Delta} \beta_{\Delta} \hat{W}(\Delta)\right) \tag{2.69}$$

generate an object deserving the name of tau-function.

However, there is a question of whether the Hamiltonians are independent. As well known in the theory of renormalization group [113], it is important to distinguish between *linearly* and *algebraically* independent generators: the best possible example is provided by the multi- and single-trace operators in matrix models. In ordinary integrable systems evolutions are always generated by algebraically independent (single-trace) operators, while their non-linear combinations (multi-trace operators) are not included into the definition of the tau-function.

However, an exact relation of the evolution generated by linearly independent operators and that generated by the algebraically independent operators, remains an important open question in the theory of integrable hierarchies, closely related (but not equivalent) to the problem of *equivalent hierarchies* [114].

In our case, the $\hat{W}(\Delta)$-operators are all linearly independent, while they all being algebraic functions of a smaller set of the Casimir operators. According to general principles [115, 116], the Casimir evolution possesses ordinary (KP/Toda) integrability properties, however, the way the generic (non-KP) W-evolution is expressed through it, is unknown. Still, it is this, more general W-evolution, which the knot polynomials are related to. At the same time, there is an important hierarchical parameter $\hbar^{l(\Delta)}$ in (2.50), which measures the algebraic complexity of $\hat{W}(\Delta)$, and this provides an interesting hierarchy of deviations from the KP integrability. It is subject of the remaining paragraphs of this section.

2.4.4 Hurwitz tau-function via Casimir operators

For arbitrary values of w_Δ the Hurwitz tau-function is not an ordinary KP/Toda tau-function in variables $\bar{p}_k$. These latter are generated by the Casimir operators $\hat{C}(k)$:

$$\tau_{KP}\{t|\bar{p}\} = \exp\left(\sum_k t_k \hat{C}(k)\right) \tau_0\{\bar{p}\} \tag{2.70}$$

Like $\hat{W}(\Delta)$, the Casimir operators have the Schur functions $\chi_R\{\bar{p}\}$ as their common eigenfunctions,

$$\hat{C}(k)\,\chi_R = C_R(k)\chi_R \tag{2.71}$$

with the eigenvalues (2.17)

$$C_R(k) = \sum_{i=1}^{l(R)} \Big((R_i - i + 1/2)^k - (-i + 1/2)^k \Big) \tag{2.72}$$

The cut-and-join operators are non-linear combinations of the Casimir operators:

$$\hat{W}(\Delta) = \sum_Q u_Q^\Delta \, \hat{C}(Q) \tag{2.73}$$

where for the Young diagram $Q = \{q_1 \geq q_2 \geq \ldots \geq 0\}$

$$\hat{C}(Q) = \prod_{j=1}^{l(Q)} \hat{C}(q_i), \qquad \frac{\partial}{\partial t_Q} = \prod_{j=1}^{l(Q)} \frac{\partial}{\partial t_{q_i}} \tag{2.74}$$

This means that

$$\tau_H\{w|\bar{p}\} = \exp\left(\sum_{\Delta,Q} w_\Delta u_Q^\Delta \frac{\partial}{\partial t_Q}\right) \tau_{KP}\{t|\bar{p}\}\Bigg|_{t=0} \tag{2.75}$$

The action of this operator, namely, the terms with $l(Q) > 1$, break the KP/Toda integrability. Hopefully, the generic Hurwitz tau-function belongs to the class of generalized tau-functions of [117], but this is yet an open question.

2.4.5 Large N expansion via Casimir operators

Instead of (2.56) one can express the large N expansion through the Casimir operators:

$$Z_{OV}^{\mathcal{K}}\{\bar{p}\} = \exp\left(\sum_\Delta \hbar^{|\Delta|+l(\Delta)-2} \, \tilde{S}_\Delta^{\mathcal{K}}(A|\hbar^2) \, \hat{C}(\Delta)\right) \tau_0^{\mathcal{K}}\{\bar{p}\} \tag{2.76}$$

Here $\tilde{S}_\Delta$ are new combinations of the higher special polynomials S_Δ, and if the shift in (2.17) is chosen to be $1/2$, they are also series in *even* powers of $\hbar$.

2.4.6 Large-R behavior

Since for representations of large sizes $|R|$ the eigenvalues of the Casimir operators grow as

$$C_R(k) \sim \gamma_k |R|^k \tag{2.77}$$

it is clear that the growth of the symmetric group characters is bounded by

$$\varphi_R(\Delta) \lesssim |R|^{|\Delta|} \tag{2.78}$$

This means that in the large R limit eq.(2.76) implies for the HOMFLY polynomial at the generic value of A:

$$\log H_R = |R| \sum_{\Delta:\ \ l(\Delta)=1} (\hbar|R|)^{|\Delta|}\Big(\gamma_{|\Delta|}\sigma_\Delta(A|0) + O(\hbar)\Big) \tag{2.79}$$

which means that in the double scaling limit (used, for example, in the context of the volume conjecture [95])

$$z \longrightarrow 0, \qquad |R| \longrightarrow \infty, \qquad u = \hbar|R| \quad \text{fixed} \tag{2.80}$$

the dominant contribution to $\log H_R$ proportional to $|R|$ is provided by the sum over the symmetric representations Δ with single row Young diagrams, $l(\Delta) = 1$.

Ironically, this does not have *direct* implications for the volume conjecture *per se*: the thing is that it is formulated in the case when N is fixed rather than A, so that also $A = q^N \longrightarrow 1$, and $\sigma_\Delta(A = 1|0) = 0$ for the symmetric representations (this property is implied, for example, by the reduction property [94] $\Lambda_R(q) = \Lambda_\Box(q^{|R|})$ of the Alexander polynomials for the single hook Young diagrams R, see [91]). Therefore, the situation with the volume conjecture in the context of the genus expansion is more tricky [91, 101]. This story has a lot to do with the Mellin-Morton-Rozansky expansion [118] into the inverse Alexander polynomials.

Another obvious observation is that at the same limit (2.80) dominant in (2.76) are the terms with $l(\Delta) = 1$, i.e. *linear* in the Casimir operators. If all other terms were simply thrown away, the Hurwitz tau-function would reduce to a KP one, i.e. we would have a naive KP/Toda integrability for the Ooguri-Vafa partition function. Unfortunately, things are again not so simple: Z_{OV} is defined as a sum over representations R, so that R is not a free parameter, which one can adjust in a desired way.

One can put it differently, formulating the claim in terms of the Plücker relations: as we discussed above in order to a linear combination of the Schur functions χ_R to be a KP tau-function, the coefficients of this combinations have to satisfy the Plücker relations [119]. In the case under consideration this property (the Plücker relations) is satisfied only asymptotically at large $|R|$.

2.4.7 Large N expansion for knot polynomials vs Takasaki-Takebe expansion

Despite an exact reduction to the KP integrability fails, the appearance of the $\hbar$-variable in (2.76) remains very suggestive. In particular, it resembles the famous Takasaki-Takebe description [120] of quasiclassical expansion for the KP tau-functions around their dispersionless approximations. That is, they demonstrate that the quasiclassical limit is described by a nearly-diagonal matrices in the universal Grassmannian, while contributions from every next sub-diagonal is damped by an extra power of $\hbar$. In the free fermion representation of the KP *tau*-functions this is expressed as follows:

$$\tau_{KP}\{u_L(z)|t_k\} = \left\langle \exp\left(\sum_k t_k H_k\right) \exp\left(\frac{1}{\hbar}\sum_L \oint dz u_L(z)\,\bar{\psi}(z)(\hbar\partial_z)^L\psi(z)\right)\right\rangle \tag{2.81}$$

where $u_1(z)$ parameterizes the quasiclassical (dispersionless) tau-functions, and further terms of the loop $\hbar$-expansion are associated with $u_L(z)$, which, in their turn, are associated with the $W^{(L+1)}$ algebra. More exactly, the terms $\oint dz z^{n+L}\,\bar\psi(z)(\hbar\partial_z)^L\psi(z)$ correspond to the action of $W_n^{(L+1)}$-generators on the tau-function.

In (2.81) $H_k = \sum \psi_l\psi^*_{l+k}$ are the Hamiltonians in the free fermion representation giving rise to the KP flows, and the average is defined w.r.t. fermionic vacuum, see [120, 121] for notation and details. Thus, the integral in (2.81) gives a specific parameterization of the group element

$$g =: \exp\left(\oint\oint U(z,z')\bar\psi(z')\psi(z)\right): \tag{2.82}$$

parameterizing the generic KP or Toda-lattice tau-function,

$$\tau_{Toda}\{\bar t, t|g\} = \left\langle \exp\left(\sum_k \bar t_k \bar H_k\right)\ g\ \exp\left(\sum_k t_k H_k\right)\right\rangle \tag{2.83}$$

and consistent with the quasiclassical expansion and formula (2.81) describes how the KP tau-function is formed from the quasiclassical one, as a series in $\hbar$.

Actually, eq.(2.76) describes in a very similar way how the Hurwitz tau-function is formed from the KP one as a series in $\hbar$. Instead of the sub-diagonal terms in the universal Grassmannian, the higher corrections in $\hbar$ are associated with the higher powers of Casimir operators. This analogy made possible by introduction of the auxiliary parameter $\hbar$ in the generic Hurwitz tau-function in the way inspired by the natural problem of the genus expansion of knot polynomials, can shed some light on what is a substitute of the universal Grassmannian [122] as a universal moduli space [123] of the Hurwitz tau-functions. This can also help to develop a substitute of the free fermion representation and embed Hurwitz functions into the general (so far badly studied) world of generalized tau-functions of [117], associated with arbitrary Lie algebras.

Chapter 3

Kontsevich integral

In section 2.3 we introduced HOMFLY polynomials as averaging of the linear group character, but without any specifications and explanation. In this chapter we give details of HOMFLY polynomials and discuss their relations to the Kontsevich integral. We explain relation of HOMFLY polynomials to finite type invariants also known as Vassiliev invariants and discuss their properties.

3.1 Knot invariants from Chern-Simons theory

Recall from Section 2.3 that for a connection A on a principal $SU(N)$-bundle over $\mathbb{R}^3$, where

$$\mathcal{A} = A_i^a(x)\, T^a\, dx^i, \tag{3.1}$$

its holonomy along contour C can be written as:

$$\begin{aligned}\Gamma(C, A) &= P \exp \oint\limits_C A = \\ &= 1 + \oint\limits_C A_i^a(x) T^a dx^i + \oint\limits_C A_{i_1}^{a_1}(x_1) dx_1^i \int\limits_0^{x_1} A_{i_2}^{a_2}(x_2)\, T^{a_1} T^{a_2} dx_2^i + \ldots\end{aligned} \tag{3.2}$$

and the Wilson loop is defined as the trace of the holonomy:

$$W_R(C, \mathcal{A}) = \mathrm{tr}\,_R \Gamma(C, \mathcal{A}). \tag{3.3}$$

Also recall from Definition (2.3.2) that the HOMFLY polynomial is defined as the average of the Wilson loop with Chern-Simons weight:

$$\mathcal{H}_R^{\mathcal{K}} = \langle W_R(\mathcal{K}, \mathcal{A}) \rangle_{\mathrm{CS}}. \tag{3.4}$$

Expanding ordered exponent in a series and taking into account that averaging is going over $A_i^a(x)$ we get the following proposition.

Proposition 3.1.1. *The average $\langle\, W(K)\,\rangle$ is equal to*

$$\langle\, W_R(K)\,\rangle = \sum_{n=0}^{\infty} \int\limits_{o(x_1)<...<o(x_n)} \prod_{k=1}^{n} dx_k^{\mu_k}\,\langle\, A_{\mu_1}^{a_1}(x_1)...A_{\mu_n}^{a_n}(x_n)\,\rangle \mathrm{tr}_R \Big(T^{a_1} T^{a_2} ... T^{a_n} \Big) \tag{3.5}$$

Definition 3.1.2. (Wick theorem) *All correlators with an even number of terms are equal to*

$$\langle\, A_{\mu_1}^{a_1}(x_1)\dots A_{\mu_{2n}}^{a_{2n}}(x_{2n})\,\rangle = \sum_{\Big((i_1,j_1)(i_2,j_2)...(i_n,j_n)\Big)\in P_{2n}} \prod_{k=1}^{n} \langle\, A_{\mu_{i_k}}^{a_{i_k}}(x_{i_k}) A_{\mu_{j_k}}^{a_{j_k}}(x_{i_k})\,\rangle \tag{3.6}$$

where the sum runs over the set of all pairings P_{2n} of $2n$ numbers; and all correlators with an odd number of terms are equal to zero.

Thus, the only thing to define is the pair correlator. It is convenient (see [69]) to make the transformation to Euclidean space $\mathbb{R}\times\mathbb{C}$ as $t = x_0,\ z = x_1 + ix_2,\ \bar{z} = x_1 - ix_2$, introducing $A_z^a = A_1^a - iA_2^a,\ A_{\bar{z}}^a = A_1^a + iA_2^a$.

Definition 3.1.3. *We define pair correlator as follows:*

$$\langle\, A_{\bar{z}}^a(t_1,z_1,\bar{z}_1)\, A_n^b(t_2,z_2,\bar{z}_2)\,\rangle \;=\; 0, \tag{3.7}$$

$$\langle\, A_m^a(t_1,z_1,\bar{z}_1)\, A_n^b(t_2,z_2,\bar{z}_2)\,\rangle \;=\; \epsilon_{mn}\,\delta^{ab}\,\frac{\hbar}{2\pi i}\frac{\delta(t_1-t_2)}{z_1-z_2} \tag{3.8}$$

Then this definition together with the Wick theorem and Proposition 3.1.1 leads to the following proposition.

Proposition 3.1.4. ([70]) *The vacuum expectation value of the Wilson loop $\langle W_R(\mathcal{K},\mathcal{A})\rangle_{\mathrm{CS}}$ is equal to the Kontsevich integral (3.18).*

Thus, Definition 2.3.2 does lead to knot invariants.

Remark. Note that the definition (3.8) together with Wick theorem imply that in series (3.5) at the every order $2n$ group factor is given by $\mathrm{tr}_R\Big(T^{a_{\sigma_p(1)}} T^{a_{\sigma_p(2)}} ... T^{a_{\sigma_p(2n)}}\Big)$, where for a pairing $p = \Big((i_1,j_1)(i_2,j_2)...(i_n,j_n)\Big)$ we define a function σ_p on a set $\{1,2,...,2n\}$ as $\sigma_p(i_k) = i_k,\ \sigma_p(j_k) = i_k$. Then $\mathrm{tr}_R\Big(T^{a_{\sigma_p(1)}} T^{a_{\sigma_p(2)}} ... T^{a_{\sigma_p(2n)}}\Big)$ for any irreducible representation R is a polynomial in N, after that we can continue it for any real N. This extends the LHS and the RHS of the equation (2.35) to any real N. Thus, this remark completes the proof of theorem 2.3.5 for any real N.

Remark. This definition is based on physical considerations via gauge quantum field theory. By this reason we would like to give informal physical explanation of the appearance knot invariants.

3.1.1 Physical interpretation

According to [5] there exists a functional $S_{CS}(A)$ such that the integral averaging of the Wilson loop with the weight $\exp\Big(-\frac{2\pi i}{\hbar}S_{CS}(A)\Big)$ has the following remarkable property:

$$\langle\, W_R(K)\,\rangle = \frac{1}{Z}\int DA\,\exp\Big(-\frac{2\pi i}{\hbar}S_{CS}(A)\Big)\,W_R(C,A) \tag{3.9}$$

where

$$Z = \int DA\,\exp\Big(-\frac{2\pi i}{\hbar}S_{CS}(A)\Big)$$

i.e. the averaging of $W_R(C,A)$ with the weight $\exp\Big(-\frac{2\pi i}{\hbar}S(A)\Big)$ does not depend on the realization C of the knot in $\mathbb{R}^3$, but only on the topological class of equivalence of the knot K (in what follows we will denote the averaging of quantity Q with this weight by $\langle Q\rangle$) and therefore, $\langle\, W(K)\,\rangle$ defines a knot invariant.

The distinguished Chern-Simons action giving the invariant average (3.9) has the following form:

$$S_{CS}(A) = \int\limits_{\mathbb{R}^3} \mathrm{tr}\,(A\wedge dA + \frac{2}{3}A\wedge A\wedge A) \tag{3.10}$$

If we normalize the algebra generators T^a as $\mathrm{tr}\,(T^aT^b) = \delta^{ab}$ and define the structure constants f of algebra g as $[T^a,T^b] = f_{abc}T^c$ then the action takes the form:

$$S_{CS}(A) = \epsilon^{ijk}\int\limits_{\mathbb{R}^3} dx^3\, A^a_i\partial_j A^a_k + \frac{1}{6}f_{abc}A^a_iA^b_jA^c_k$$

Formula (3.9) is precisely the path integral representation of knot invariants.

Holomorphic gauge

It was explained in [70] the Kontsevich integral appears as average value $\langle\, W(K)\,\rangle$ computed in the so called holomorphic gauge. To show it, let us start with decomposition of three-dimensional space $\mathbb{R}^3 = \mathbb{R}\times\mathbb{C}$, i.e. we pass from the coordinates (x_0,x_1,x_2) to $(t,z,\bar{z})$ defined as:

$$t = x_0, \quad z = x_1 + ix_2, \quad \bar{z} = x_1 - ix_2$$

then the for differentials and dual bases we have:

$$dz = dx_1 + idx_2, \quad \partial_z = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2})$$
$$d\bar{z} = dx_1 - idx_2, \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2})$$

The gauge field takes the form:

$$A^a_i dx^i = A^a_0 dx^0 + \frac{1}{2}A^a_z dz + \frac{1}{2}A^a_{\bar{z}} d\bar{z}$$

where:

$$A^a_z = A^a_1 - iA^a_2, \quad A^a_{\bar{z}} = A^a_1 + iA^a_2$$

As usual, the path integral (3.9) is not well defined due to gauge symmetry, to make it convergent we need to fix an appropriate gauge condition. The holomorphic gauge is defined by the following non-covariant condition

$$A_{\bar{z}} = 0 \tag{3.11}$$

The main feature of this gauge is that the initially complex, cubical Chern-Simons action becomes pure quadratic and the resulting path integral becomes Gaussian. This allows one to compute it perturbatively utilizing the Wick theorem. Indeed, in the gauge (3.11) the cubical part of the action (3.10) vanishes:

$$A \wedge A \wedge A|_{A_{\bar{z}}=0} = 0$$

and we end up with the following quadratic action[1]:

$$S(A)|_{A_{\bar{z}}=0} = i \int\limits_{\mathbb{R}^3} dt d\bar{z} dz\, \epsilon^{mn} A^a_m \partial_{\bar{z}} A^a_n \tag{3.12}$$

where ϵ^{mn} is antisymmetric, $m, n \in \{t, z\}$ and $\epsilon^{tz} = 1$. The main ingredient of perturbation theory with action (3.12) is the gauge propagator defined as the inverse of quadratic operator of the action:

$$\langle\, A^a_m(t_1, z_1, \bar{z}_1)\, A^b_n(t_2, z_2, \bar{z}_2)\, \rangle = \Big(\frac{\delta^{ab}}{\hbar}\, \epsilon^{nm} \partial_{\bar{z}}\Big)^{-1} = \epsilon_{mn}\, \delta^{ab}\, \frac{\hbar}{2\pi i} \frac{\delta(t_1 - t_2)}{z_1 - z_2} \tag{3.13}$$

To find it we need the following simple fact about the operator $\partial_{\bar{z}}$:

$$\partial_{\bar{z}}^{-1} = \frac{1}{2\pi i} \frac{1}{z}$$

To prove it, we note that the inverse of Laplace operator $\Delta = \partial_z \partial_{\bar{z}}$ (its Green function) on the complex plane is given by the logarithm function[2]

$$(\partial_z \partial_{\bar{z}})^{-1} = \frac{1}{2\pi i} \log(z\bar{z}), \quad \text{therefore} \quad \partial_{\bar{z}}^{-1} = \frac{1}{2\pi i} \partial_z \log(z\bar{z}) = \frac{1}{2\pi i} \frac{1}{z}$$

Let us show that average value $\langle W(K) \rangle$ in the holomorphic gauge coincides with the Kontsevich integral for the knot. To proceed we introduce an orientation

[1] Of course, the careful gauge fixing involves the Faddeev-Popov procedure which leads to an additional ghost term in the action. Fortunately, in the holomorphic gauge the ghost fields are not coupled to the gauge ones and therefore can be simply integrated away from the path integral. In this way we again arrive to the quadratic action (3.12).

[2] Note that $\partial_{\bar{z}}^{-1}$ is defined up to any holomorphic function, as they are in the kernel of $\partial_{\bar{z}}$:

$$\partial_{\bar{z}}^{-1} = \frac{1}{2\pi i} \frac{1}{z} + f(z)$$

The trick with the Laplacian consists of the following: we restrict the operators to the space of functions with absolute values sufficiently fast decreasing at the infinity. The only holomorpfic function with this property is $f(z) = 0$.

on a knot in $\mathbb{R}^3$. Obviously, if we pick some point p on the knot then $K \setminus p$ is topologically a segment $I_p = (0,1)$. Orientation on I_p naturally defines orientation on $K \setminus p$. We will denote this orientation using symbol o, for example we can compare two points x_1 and x_2 as $o(x_1) < o(x_2)$. The Wilson loop is given by ordered exponent which has the following form:

$$W(K) = \operatorname{tr} P \exp\Big(\oint A^a_\mu(x) T^a dx^\mu \Big) =$$

$$= \sum_{n=0}^{\infty} \int\limits_{o(x_1)<o(x_2)<...<o(x_n)} \prod_{k=1}^{n} dx_k^{\mu_k} \, A^{a_1}_{\mu_1}(x_1) A^{a_2}_{\mu_2}(x_2) ... A^{a_n}_{\mu_n}(x_n) \operatorname{tr} \Big(T^{a_1} T^{a_2} ... T^{a_n} \Big) \tag{3.14}$$

For the average $\langle\, W(K) \,\rangle$ we get:

$$\langle\, W(K) \,\rangle = \sum_{n=0}^{\infty} \int\limits_{o(x_1)<...<o(x_n)} \prod_{k=1}^{n} dx_k^{\mu_k} \,\langle\, A^{a_1}_{\mu_1}(x_1) ... A^{a_n}_{\mu_n}(x_n) \,\rangle \operatorname{tr} \Big(T^{a_1} T^{a_2} ... T^{a_n} \Big)$$

As the action of the theory is quadratic, the average of n fields is not equal to zero only for even n, moreover the Wick theorem gives:

$$\langle\, A^{a_1}_{\mu_1}(x_1) \dots A^{a_{2n}}_{\mu_{2n}}(x_{2n}) \,\rangle = \sum_{\Big((i_1,j_1)(i_2,j_2)...(i_n,j_n)\Big) \in P_{2n}} \prod_{k=1}^{n} \langle\, A^{a_{i_k}}_{\mu_{i_k}}(x_{i_k}) A^{a_{j_k}}_{\mu_{j_k}}(x_{i_k}) \,\rangle \tag{3.15}$$

Here the sum runs over the set of all pairings P_{2n} of $2n$ numbers. An element of this set has the form $p = \Big((i_1,j_1)(i_2,j_2)...(i_n,j_n) \Big)$ where $i_k < j_k$ and the numbers i_k, j_k are all different numbers from the set $\{1,2,...,2n\}$. If $p \in P_{2n}$ is a pairing then we define a function σ_p on a set $\{1,2,...,2n\}$ as follows:

$$\sigma_p(i_k) = i_k, \quad \sigma_p(j_k) = i_k$$

i.e. it returns a minimum number from the pair (i_k, j_k). The Wick theorem (3.15) gives:

$$\langle\, W(K) \,\rangle = \sum_{n=0}^{\infty} \int\limits_{o(x_1)<...<o(x_n)} \sum_{p \in P_{2n}} \prod_{k=1}^{n} dx_{i_k}^{\mu_{i_k}} \, dx_{j_k}^{\mu_{j_k}} \,\langle\, A^{a_{i_k}}_{\mu_{i_k}}(x_{i_k}) A^{a_{j_k}}_{\mu_{j_k}}(x_{i_k}) \,\rangle \,\times$$

$$\times \operatorname{tr} \Big(T^{a_{\sigma_p(1)}} ... T^{a_{\sigma_p(2n)}} \Big)$$

From the propagator (3.13) we have:

$$dx_{i_k}^{\mu_{i_k}} \, dx_{j_k}^{\mu_{j_k}} \,\langle\, A^{a_{i_k}}_{\mu_{i_k}}(x_{i_k}) A^{a_{j_k}}_{\mu_{j_k}}(x_{i_k}) \,\rangle = \frac{\hbar}{2\pi i} \,\delta^{a_{i_k} a_{j_k}} \,(dz_{i_k} dt_{j_k} - dz_{j_k} dt_{i_k}) \,\frac{\delta(t_{i_k} - t_{j_k})}{z_{i_k} - z_{j_k}}$$

therefore,

$$\langle\, W(K) \,\rangle = \sum_{n=0}^{\infty} \frac{\hbar^n}{(2\pi i)^n} \int\limits_{o(z_1)<...<o(z_n)} \sum_{p \in P_{2n}} \prod_{k=1}^{n} \Big((dz_{i_k} dt_{j_k} - dz_{j_k} dt_{i_k}) \,\frac{\delta(t_{i_k} - t_{j_k})}{z_{i_k} - z_{j_k}} \Big) \, G_p$$

where the pairing $p = \Big((i_1, j_1)(i_2, j_2)...(i_n, j_n)\Big)$ and the group element $G_p = \operatorname{tr}\Big(T^{a_{\sigma_p(1)}}T^{a_{\sigma_p(2)}}...T^{a_{\sigma_p(2n)}}\Big)$. To obtain the Kontsevich integral we need to make the last step: integrate out the delta functions $\delta(t_{i_k} - t_{j_k})$ from the integral. To do it, it is convenient to parametrize the knot by the "height" t. Such parametrization of a knot is regular at non-critical points with respect to t direction. In the interval between the critical points we have a well defined function $z(t)$ which parametrizes the knot in $\mathbb{R} \times \mathbb{C}$. Using this parametrization we rewrite the integral in the form:

$$\langle\, W(K)\,\rangle = \sum_{n=0}^{\infty} \frac{\hbar^n}{(2\pi i)^n} \int\limits_{o(z_1)<o(z_2)<...<o(z_n)} \sum_{p\in P_{2n}} (-1)^{p_\downarrow} \prod_{k=1}^{n} dt_{i_k} dt_{j_k} \times$$
$$\times \sum_{p\in P_{2n}} \left(\frac{dz_{i_k}(t_{i_k})}{dt_{i_k}} - \frac{dz_{j_k}(t_{j_k})}{dt_{j_k}} \right) \frac{\delta(t_{i_k} - t_{j_k})}{z_{i_k} - z_{j_k}} G_p \qquad (3.16)$$

Here we should be careful not to forget the sign factor $(-1)^{p_\downarrow}$ where $p_\downarrow$ is the number of down-oriented segments between critical points on the knot entering the integral. On these segments the "height" parameter and the orientation of the knot are opposite, and therefore we should change $dt_i \to -dt_i$ which finally results in factor $(-1)^{p_\downarrow}$. Integrating the t_{j_k} variables we obtain:

$$\langle\, W(K)\,\rangle = \sum_{n=0}^{\infty} \frac{\hbar^n}{(2\pi i)^n} \int\limits_{o(z_1)<...<o(z_n)} \sum_{p\in P_{2n}} (-1)^{p_\downarrow} \times$$
$$\times \prod_{k=1}^{n} dt_{i_k} \left(\frac{dz_{i_k}(t_{i_k})}{dt_{i_k}} - \frac{dz_{j_k}(t_{i_k})}{dt_{j_k}} \right) \frac{1}{z_{i_k} - z_{j_k}} G_p \qquad (3.17)$$

and finally, taking into account that $dz_{j_k}(t_{i_k})/dt_{j_k} = dz_{j_k}(t_{i_k})/dt_{i_k}$ we arrive to the following expression:

$$\boxed{\langle\, W(K)\,\rangle = \sum_{n=0}^{\infty} \frac{\hbar^n}{(2\pi i)^n} \int\limits_{o(z_1)<...<o(z_n)} \sum_{p\in P_{2n}} (-1)^{p_\downarrow} \bigwedge_{k=1}^{n} \frac{dz_{i_k} - dz_{j_k}}{z_{i_k} - z_{j_k}} G_p} \qquad (3.18)$$

The last expression (3.18) is the celebrated Kontsevich integral for knot K.

3.2 Localization of Kontsevich Integral

3.2.1 Multiplicativity and braid representation

The coefficients of Kontsevich integral (3.18) are given in terms of rather sophisticated meromorphic integrals. In this section we describe the localization technique for computing the Kontsevich integral (KI) which gives a simpler combinatorial description of (3.18) [71]. The idea behind the localization is in fact very simple and based on miltiplicativity and factorization of KI and uses braid representation of knots. Let us discuss these properties separately.

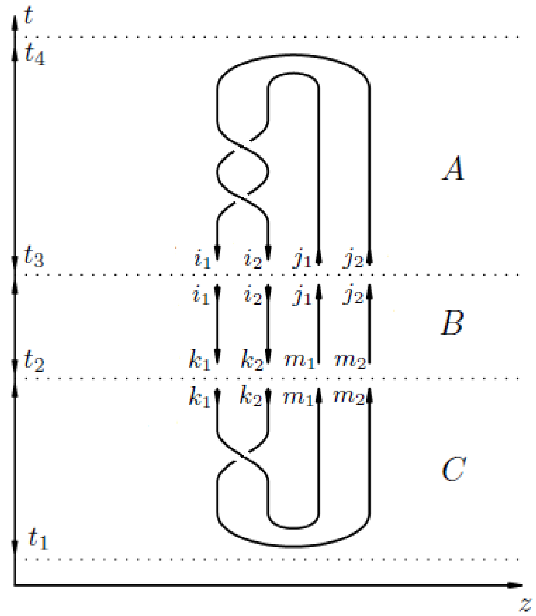


Figure 3.1: Slices

Multiplicative properties

The *multiplicativity* of KI means that we can cut the knot in the finite number of parts and compute KI for these parts separately, then KI for the whole knot can be computed as an appropriate product of these separate integrals. To illustrate the idea, let us consider knot 3_1 (see [80] for knot naming conventions) cut in three parts as in Figure 3.1. The Kontsevich integral computed for separate parts is not a number anymore but an operator of the form $V^{\otimes N_{in}} \to V^{\otimes N_{out}}$ where N_{in} and N_{out} are the numbers of incoming and outgoing lines correspondingly. For example let us cut the knot in three slices (t_4, t_3), (t_3, t_2) and (t_2, t_1). Then the corresponding Kontsevich integrals are given by operators $A^{i_1 i_2}_{j_1 j_2}$, $B^{j_1 j_2 k_1 k_2}_{i_1 i_2 m_1 m_2}$ and $C^{m_1 m_2}_{k_1 k_2}$ where we write their indices explicitly to emphasize that they are finite-dimensional tensors. Lower indices correspond to incoming lines and the upper to the outgoing ones. The value of KI for the whole knot is the product of these tensors:

$$\langle W(K) \rangle = \sum_{\substack{i_1 i_2 j_1 j_2 \\ k_1 k_2 m_1 m_2}} A^{i_1 i_2}_{j_1 j_2} \, B^{j_1 j_2 k_1 k_2}_{i_1 i_2 m_1 m_2} \, C^{m_1 m_2}_{k_1 k_2}$$

For the entire knot KI is a number and not a tensor because the knot is closed and does not have any incoming or outgoing lines. Integral naturally brings us to the main idea of localization: if we are able to represent the knots as a union of finite number of some special "fundamental" parts, then we have to compute KI for these parts only. In order to compute the KI, one needs to choose these special parts in a way such that the KI for them would have the most simple form. Let us consider the middle slice (t_2, t_3) in Figure 3.1. Kontsevich integral B for this part is obviously equal to identity, this is because this piece consists of four lines parallel to t axis and the form $(dz_i - dz_j)/(z_i - z_j)$ in (3.18) vanishes as $dz = 0$ on each vertical line:

$$B = 1^{\otimes 4} \quad \text{or} \quad B^{n_1 n_2 m_1 m_2}_{k_1 k_2 p_1 p_2} = \delta^{n_1}_{k_1}\delta^{n_2}_{k_2}\delta^{m_1}_{p_1}\delta^{m_2}_{p_2}$$

Therefore, it is quite natural to use representations of knots "maximally extended" along the t direction, such that the KI takes the simplest possible form. Such a representation is the well known braid representation of knots:

Proposition 3.2.1. *For any knot K there is a number n (not unique) such that it can be represented as a closure of some element (not unique) from braid group $\mathcal{B}_n$.*

The meaning of this theorem is clear from the examples in Figure 3.2.

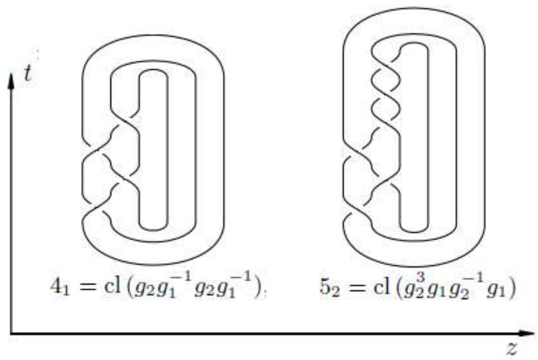


Figure 3.2: Braid representation for knots

Here the knots 4_1 and 5_2 are represented as closures of braids $g_2 g_1^{-1} g_2 g_1^{-1} \in \mathcal{B}_3$ and $g_2^3 g_1 g_2^{-1} g_1 \in \mathcal{B}_3$ correspondingly. The *closure* is the operation that connects the top of the braid with its bottom stringwise. There exists a simple combinatorial algorithm to construct braid representation b for any knot [80].

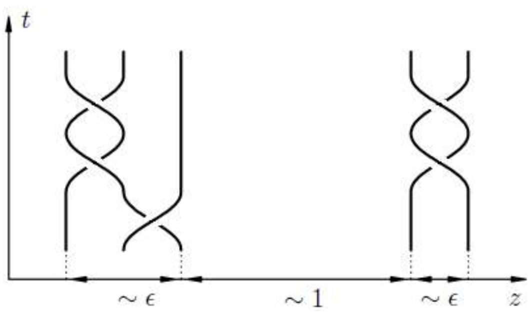


Figure 3.3: Factorization

Factorization

The last thing we need is the *factorization* of Kontsevich integral.

Let us introduce the distance between the strings in the braid. Then the following holds: if one can arrange the strings in the braid in a way that the distance between two groups of them is of order of different powers of some small parameter ϵ, then these two groups give separate contributions to the KI.

More precisely: Consider a braiding $b_n = b_k \otimes b_{n-k} \in \mathcal{B}_n$. Let us assume that the sizes of b_k and b_{n-k} are much less ($O(\epsilon)$) than the distance between them. Then

$$KI(b_n) = KI(b_k) \otimes KI(b_{n-k}) + O(\epsilon)$$

If $b_n = b_k \otimes b_{n-k}$ then the first k strings of the braid do not cross the rest $n-k$ strings. For example in Figure 3.3 the first three strings and the last two form two separate braids. If widths of these braids are ($\sim \epsilon$) much less the distance between them (~ 1), then the factorization theorem implies that the KI is the tensor product of two KI's for each separate braid.

3.2.2 Choice of associators placement

Now, with the help of these properties we are ready to describe the representation of a given knot for which KI takes the simplest form. Let knot K be represented by closure of some braid $b \in \mathcal{B}_n$, then we arrange the strings of the braid such that the distance between k-th and $k+1$-th strings is given by ϵ^{-k} where ϵ is some small formal parameter, Figure 3.4:

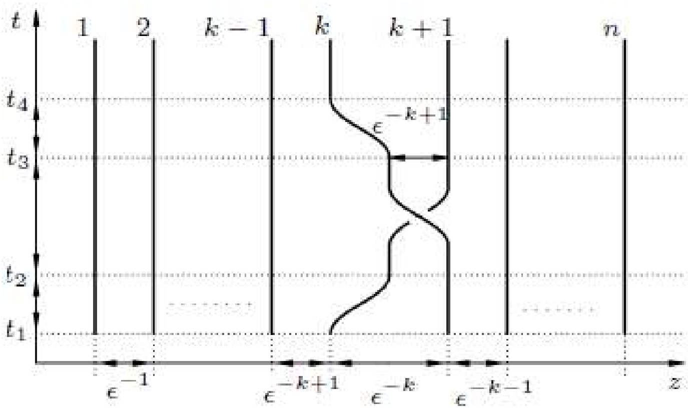


Figure 3.4: R-matrix with associators

i.e. the distance between adjacent strings increases with the number of string. Suppose, that at some slice (t_4, t_1) the braid b has the crossing of k-th and $k+1$-th strings (Figure 3.4). This crossing can be represented as a result of taking three consecutive steps:

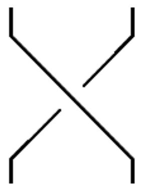


Figure 3.5: R-matrix

1. In slice (t_4, t_3) the k-th string goes closer to the string $k+1$, at the distance ϵ^{-k+1}.
2. In slice (t_3, t_2) two strings are crossed.
3. In slice (t_2, t_1) the k-th string goes to its initial position at the distance ϵ^{-k} from $k+1$ string.

If we arrange the strings in this way, then the distance between k-th and $k+1$-th strings at the interval (t_3, t_2) is much less then the distance to any other string. Therefore, the factorization for KI gives the following contribution for slice (t_3, t_2):

$$KI_{(t_3,t_2)} = R^{\pm 1}_{k,k+1} = 1_1 \otimes ...1_{k-1} \otimes R^{\pm 1} \otimes 1_{k+2} ... \otimes 1_n$$

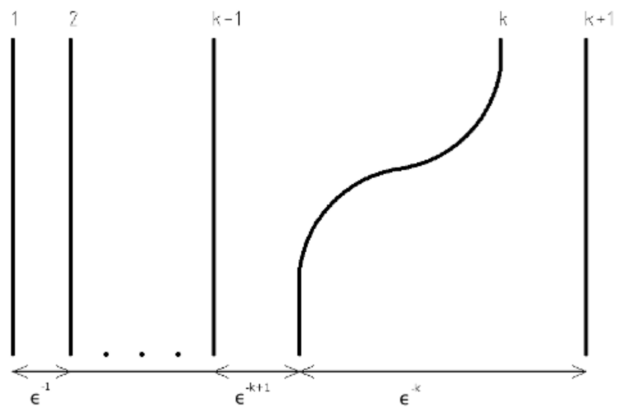


Figure 3.6: Associator

where $R^{\pm 1}$ is the so-called *R-Matrix*, i.e. KI computed for the neighborhood of the crossing point which corresponds to the configuration of strings in Figure 3.5; the sign depends on the orientation of the crossing. Similarly, at slices (t_4, t_3) and (t_2, t_1) the distance between first and k-th string is much less the the distance to the other $n-k$ strings. Therefore, the factorization for the slice (t_4, t_3) gives:

$$KI_{(t_4,t_3)} = \Psi_{1,2,...,k+1} = \Phi_{k+1} \otimes 1^{\otimes n-k-1}, \quad KI_{(t_2,t_1)} = \Psi^{-1}_{1,2,...,k+1} = \Phi^{-1}_k \otimes 1^{\otimes n-k-1}$$

where $\Phi^{\pm 1}_{k+1}$ is the so-called *associator*, given by KI for the configuration of strings represented in Figure 3.6. In summary, we see that the crossing of k-th and $k+1$-th strings in braid representation of the knot gives the contribution to the KI of the form of operator $R^{\pm}_{k,k+1}$ conjugated by operator $\Psi_{1,2,...,k+1}$:

$$\boxed{KI_{(t_4,t_1)} = \Psi_{1,2,...,k+1}\, R_{k,k+1}\, \Psi^{-1}_{1,2,...,k+1}}$$

Denote this tensor by X_k:

$$X_k = \Psi_{1,2,...,k+1}\, R_{k,k+1}\, \Psi^{-1}_{1,2,...,k+1}. \tag{3.19}$$

3.2.3 Formulas for R-Matrices and associators

Explicit expressions for operators R and Ψ can be easily derived from computations of KI for the corresponding pieces of the braid representation of a knot. The KI for neighborhood of a crossing point gives:

$$R = \exp\left(\frac{\hbar}{2\pi i}\Omega\right), \quad \Omega = T^a \otimes T^a \tag{3.20}$$

The calculation of Φ_3 is explained in details in [81], to describe it let us define:

$$\Omega_{12} = \Omega \otimes 1, \quad \Omega_{23} = 1 \otimes \Omega$$

then we have:

$$\Phi_3 = 1^{\otimes 3} + \sum_{k=2}^{\infty} \left(\frac{\hbar}{2\pi i}\right)^k \sum_{m\geq 0} \sum_{\substack{\mathbf{p}>0\, \mathbf{q}>0 \\ |\mathbf{p}|+|\mathbf{q}|=k \\ l(\mathbf{p})=l(\mathbf{q})=m}} (-1)^{|\mathbf{q}|}\, \tau(p_1, q_1, ...p_m, q_m) \times$$

$$\sum_{\substack{l(\mathbf{r})=l(\mathbf{s})=m \\ 0\leq \mathbf{r}\leq \mathbf{p},\, 0\leq \mathbf{s}\leqq \mathbf{q}}} (-1)^{|\mathbf{r}|} \left(\prod_{i=1}^{m} \binom{p_i}{r_i}\binom{q_i}{s_i}\right) \Omega_{23}^{|\mathbf{s}|}\, \Omega_{12}^{p_1-r_1}\Omega_{23}^{q_1-s_1}...\Omega_{12}^{p_m-r_m}\Omega_{23}^{q_m-s_m}\Omega_{12}^{|\mathbf{r}|} \tag{3.21}$$

Where $\mathbf{p} = (p_1, p_2, ..., p_m)$ is a vector with positive integer components. The length of the vector $l(\mathbf{p}) = m$ and $|\mathbf{p}| = \sum p_i$. When we write $\mathbf{p} > \mathbf{q}$, it is understood as $p_i > q_i$ for all i, and $\mathbf{p} > 0$ means that $p_i > 0$ for all i. The coefficients $\tau(p_1, q_1, ...p_m, q_m)$ are expressed through multiple zeta functions as follows:

$$\tau(p_1, q_1, ...p_m, q_m) = \zeta(\underbrace{1, ..., 1}_{p_1-1}, q_1+1, \underbrace{1, ..., 1}_{p_2-1}, q_2+1, ..., q_n+1)$$

such that for example $\tau(1,2) = \zeta(3)$ and $\tau(2,1) = \zeta(1,2)$. The multiple zeta functions are defined as:

$$\zeta(m_1, m_2, ..., m_n) = \sum_{0<k_1<k_2<...<k_n} k_1^{-m_1} k_2^{-m_2} ... k_n^{-m_n}$$

Note, that $\tau(p_1, q_1, ...p_m, q_m) = \tau(q_m, p_m, ...q_1, p_1)$ so that, e.g. $\zeta(1,2) = \zeta(2)$ and $\zeta(1,3) = \zeta(4)$. To define the associators Φ_n for $n > 3$ we need the coproduct operator $\Delta : U(g) \to U^{\otimes 2}(g)$. Its action on the generators of the universal enveloping algebra $U(g)$ is defined as follows:

$$\begin{aligned} &\Delta(T^a) = 1 \otimes T^a + T^a \otimes 1 \\ &\Delta(T^a T^b) = 1 \otimes T^a T^b + T^a \otimes T^b + T^b \otimes T^a + T^a T^b \otimes 1 \\ &.... \end{aligned} \tag{3.22}$$

With the help of this operator the higher associators are expressed through Φ_3 with the help of the following recursive formula:

$$\Phi_{n+1} = \Delta \otimes 1^{\otimes(n-1)}\, \Phi_n \tag{3.23}$$

Formula (3.23) basically means that, e.g. to obtain Φ_4 from Φ_3 one needs to make the following substitutions in the formula for Φ_3:

$$\begin{aligned} 1 \otimes T^a \otimes T^a &\longrightarrow 1 \otimes 1 \otimes T^a \otimes T^a, & (3.24)\\ T^a \otimes 1 \otimes T^a &\longrightarrow 1 \otimes T^a \otimes T^a \otimes T^a + T^a \otimes 1 \otimes T^a \otimes T^a, & (3.25)\\ T^a T^b \otimes T^a \otimes T^b &\longrightarrow 1 \otimes T^a T^b \otimes T^a \otimes T^b + T^a \otimes T^b \otimes T^a \otimes T^b + & (3.26)\\ &+ \quad T^b \otimes T^a \otimes T^a \otimes T^b + T^a T^b \otimes 1 \otimes T^a \otimes T^b & (3.27)\\ &\cdots \end{aligned}$$

and so on. That is, one has to symmetrize the first tensor component of each term of Φ_3 over the first and the second tensor components of Φ_4.

One can derive (3.23) directly from Kontsevich integral consequentially for $n = 3, 4, ...$, but it is simpler to note that in Figure 3.4 the first $k-1$ strings should give equivalent contribution to associator Φ_{k+1} because the distance between first and $k-1$-th strings is much less then the "width of associator" (the distance between the first and the $k+1$-th string). Therefore, operator Δ and the recursive procedure (3.23) have a clear physical meaning of "symmetrization" of contribution of the first $k-1$ strings to associator Φ_{k+1}.

3.2.4 Caps

To complete our consideration we should also find contributions to KI coming from the bottom and the top of the braid's closure. Fortunately, we do not need new operators as these contributions can be represented through already introduced operators Ψ_k.

Indeed, to make a closure of braid $b \in \mathcal{B}_n$ we have to add to the braid n straight strings, such that the total number of strings is $2n$ and then connect $(n-k)$-th string with the $n+k$-th one for all k at the top and the bottom, for example as in Figure 3.7. In what follows we again imply that the distance between our $2n$ strings increases with the number of string, such that the distance between k-th and $k+1$-th ones is of order ϵ^{-k}. Consider the following procedure, first the n-th

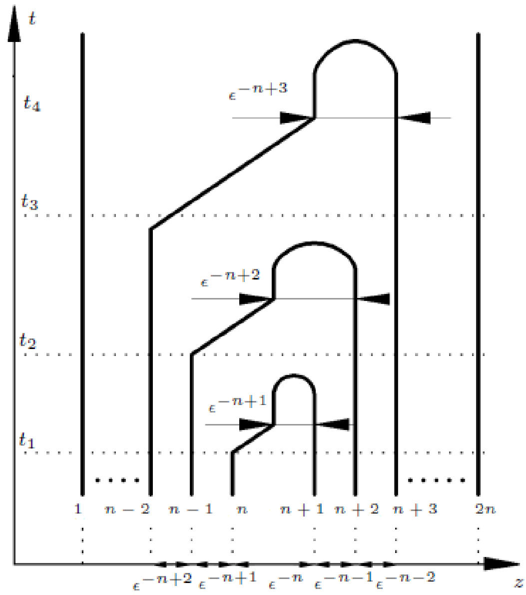


Figure 3.7: Cap

string goes closer to $n+1$ up to the distance ϵ^{-n} (slice (t_1, t_2) in Figure 3.7) then we connect them by a "hat" of width ϵ^{-n}. The factorization theorem gives the contribution $\Psi^{-1}_{1,2\ldots,n+1}$ for this slice (due to the hat contribution being trivial). In the slice (t_2, t_3) we have $2n-2$ strings and we can iterate the procedure one more time which will give the contribution of the form $\Psi^{-1}_{1,2,\ldots n-1,n+2}$. Finally, the whole top closure gives:

$$T_n = \Psi^{-1}_{1,2,2n-1}\Psi^{-1}_{1,2,\ldots n-1,n+2}\ldots\Psi^{-1}_{1,2,\ldots n+1}. \tag{3.28}$$

It is assumed here that the indices corresponding to strings which terminate in a cap are contracted.

Analogously for the bottom of the closure:

$$B_n = \Psi_{1,2,2n-1}\Psi_{1,2,\ldots,n-1,n+2}\ldots\Psi_{1,2,\ldots,n+1} \tag{3.29}$$

Again, one should contract all the indices corresponding to caps.

3.2.5 General combinatorial formula for Kontsevich integral

Now we know the contributions of all parts of the braid closure. In order to write down the answer, let us introduce the symbol $\prod^{\rightarrow}$ representing *the ordered product.*

The answer is then as follows.

Let knot K be represented as the closure of a braid $b \in \mathcal{B}_n$:

$$b = \prod_k^{\rightarrow} g_k$$

then the KI for the knot is given by the following expression[3]*:*

$$\boxed{KI\Big(\overrightarrow{\prod_k} b_{i_k}\Big) = \mathrm{tr}\,\Big(T_n \overrightarrow{\prod_k} X_{i_k} B_n\Big)} \tag{3.30}$$

where the tensors T_n, X_i and B_n are given by formulas (3.28), (3.19) and (3.29) correspondingly.

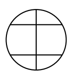
3.2.6 Technique of computation

In order to finally compute the coefficients of Kontsevich integral, one needs to take the following steps.

First of all, note that basic elements of formula (3.30) are contractions $T^a T^a$ where one T^a stands in some tensor component and the other in another one. Note that one can represent it as chord drawn on the knot from the place on the knot corresponding to the first T^a to the place corresponding to the second T^a. Then, in order to compute the given order of (3.30), one has to consider all of the terms in (3.30) of that order, then for every such term draw a chord corresponding to every contracted pair of generators and untie the knot obtaining a *chord diagram.*

More precisely, let n be a given order in $\hbar$ which we want to compute. In this order, Kontsevich integral is a linear combination of group factors $G_p = \mathrm{tr}\,\Big(T^{\sigma_p(1)}T^{\sigma_p(2)}...T^{\sigma_p(2n)}\Big)$ corresponding to pairings $p = \Big((i_1, j_1)(i_2, j_2)...(i_n, j_n)\Big)$.

One may associate with every such group factor a *chord diagram*, which corresponds to the particular pairing in a straightforward way. For example, there is the following correspondance:

$$\mathrm{tr}\,\left(T^a T^b T^c T^a T^c T^b\right) \leftrightarrow$$ 

Thus, Kontsevich integral in order n is a linear combination of chord diagrams with n chords.

In these terms, formal formula (3.30) may be expressed as the following collection of steps.

Let there be associators $\Psi_1, \dots, \Psi_k$ and R-matrices $R_1, \dots, R_l$ assigned to a given knot K. Then, to obtain Kontsevich integral in order n one has to take the sum over all ordered partitions of n into $k + l$ parts:

$$n = \phi_1 + \dots + \phi_k + r_1 + \dots + r_l, \tag{3.31}$$

where all ϕ_i and r_i are nonnegative integers.

The R-matrix part is easier, for every R-matrix R_i we just insert r_i consequential chords (propagators) in the corresponding place on the knot, and take the coefficient $1/r_i!$.

[3]One should understand that the symbol "tr" in (3.30) stands for the contraction of the tensors X_i, B_n and T_n corresponding to their position in the braid.

The associator part is trickier, there we have several summands due to the form of formula (3.21) for the associator, and to the fact that we should include also braids to the left, as it is done in formula (3.23). For every Ψ_i therefore there is a whole sum of ways to insert ϕ_i chords in the knot, with corresponding coefficients. That means that in a given order there are several terms of this order in expression (3.21). This number then multiplies even further when one takes into account (3.23).

Then, for given partition $n = \phi_1 + \cdots + \phi_k + r_1 + \cdots + r_l$, we take the product of all described above parts corresponding to R-matrices and associators. After expanding the brackets, we obtain a sum of several ways to insert n chords in the knot with corresponding coefficients.

The only thing left is to untie the knot into a circle thus obtaining a linear combination of chord diagrams with n chords. Algorithmically, it can be done in a very straightforward way just by assigning an ordinal number to each part of the knot which belongs to an associator or to an R-matrix.

Then we sum over all partitions of n and obtain the coefficients of order n in Kontsevich integral.

3.2.7 KI combinatorially for figure-eight knot

In this paragraph we discuss the combinatorial technique of computing KI coefficients in more detail for the case of the figure-eight knot.

Let K be the figure-eight knot, i.e. knot 4_1. Braid representation for K with associators in correct positions is given in Figure 3.8a. There are 8 associators $\Psi_1, \ldots, \Psi_8$ and 4 R-matrices $R_1, \ldots, R_4$. Corresponding sections of the knot are enclosed in thin dotted and dashed boxes.

Let us describe the computation of KI coefficients of order 2.

Recall that first of all one has to choose an integer partition of $n = 2$ into 12 parts (corresponding to R-matrices and associators). Note that there is no linear in T^aT^a term in (3.21), so all partitions $n = \phi_1 + \cdots + \phi_8 + r_1 + \cdots + r_4$ with any of ϕ_i equal to 1 give vanishing contributions.

Then we will have 4 partitions where both chords are taken from one of the R-matrices. One of the cases is drawn in Figure 3.9d, where the propagators, i.e. chords, are represented by red dashed lines (the thicker ones). Further on, we will have 6 partitions where the chords are taken from two different R-matrices (Figure 3.10e) and 8 partitions where both chords are taken from one of the associators. This latter case is more complicated than the former ones, where we had only one term of a given order, according to (3.20). In the case of associators, however, we will have two complications: first, formula (3.21) has already several terms of a given order and, second, the number of terms is further multiplied in accordance with (3.23) if we have lines to the left, as in the cases of Ψ_2 and Ψ_7. Actually, for the order 2 we have just two relevant terms in (3.21), the one drawn in Figure 3.8b and the one which differs from it by interchanging their endpoints on the central string. Formula (3.23) implies that one should also include the same terms but with the leftmost endpoint shifted to the leftmost string, like in Figure 3.9c.

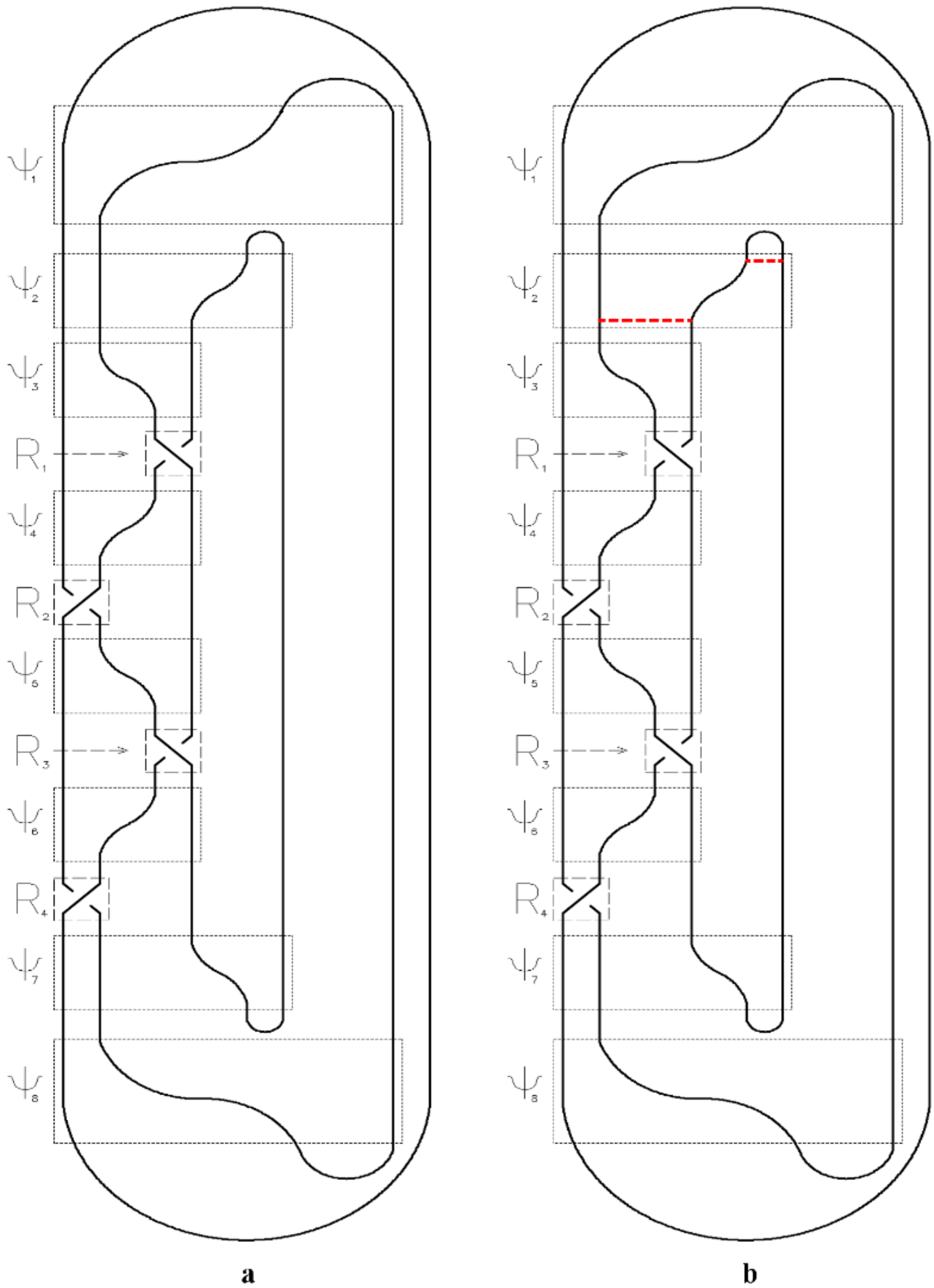


Figure 3.8: Propagators

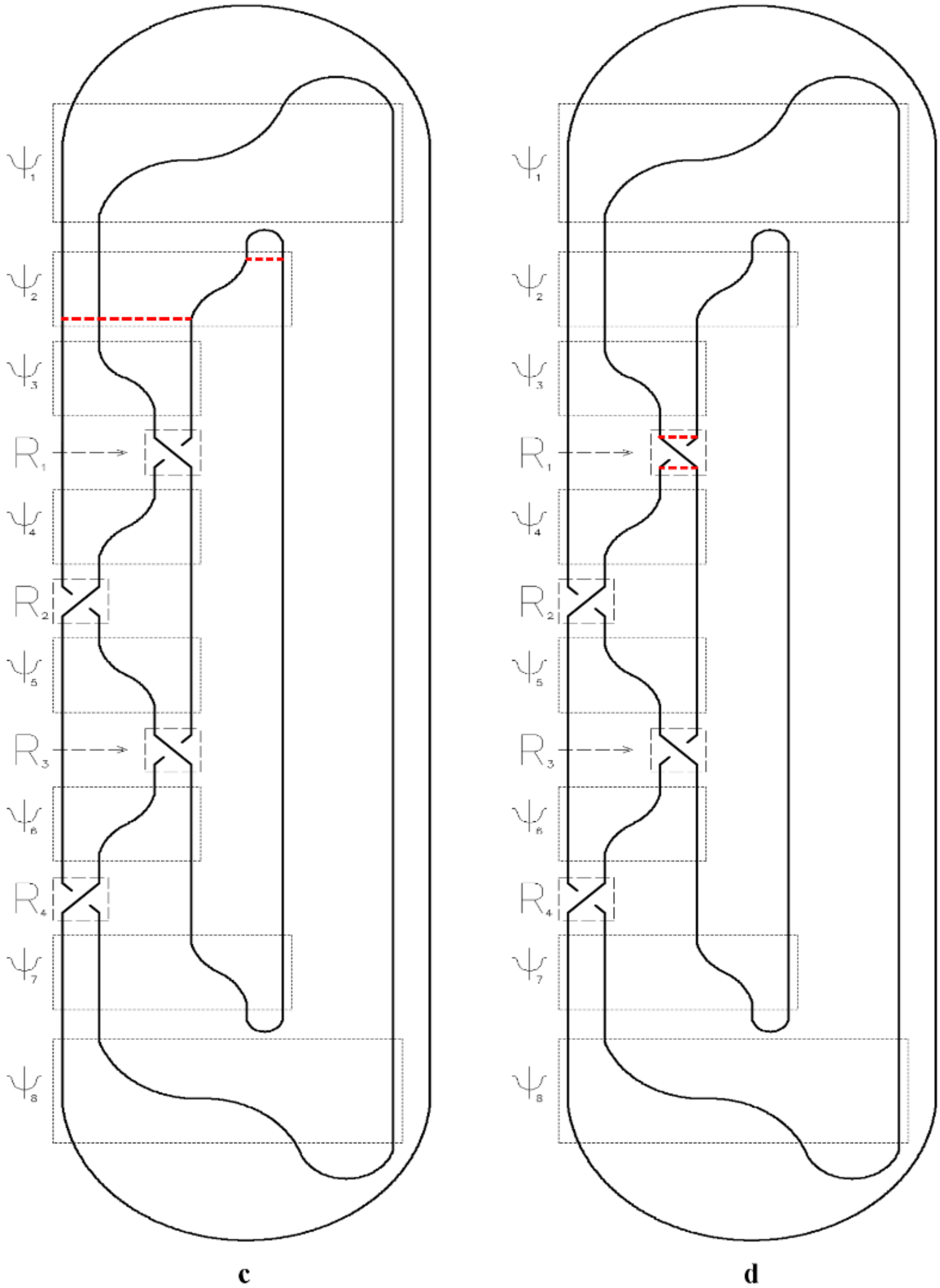


Figure 3.9: Propagators

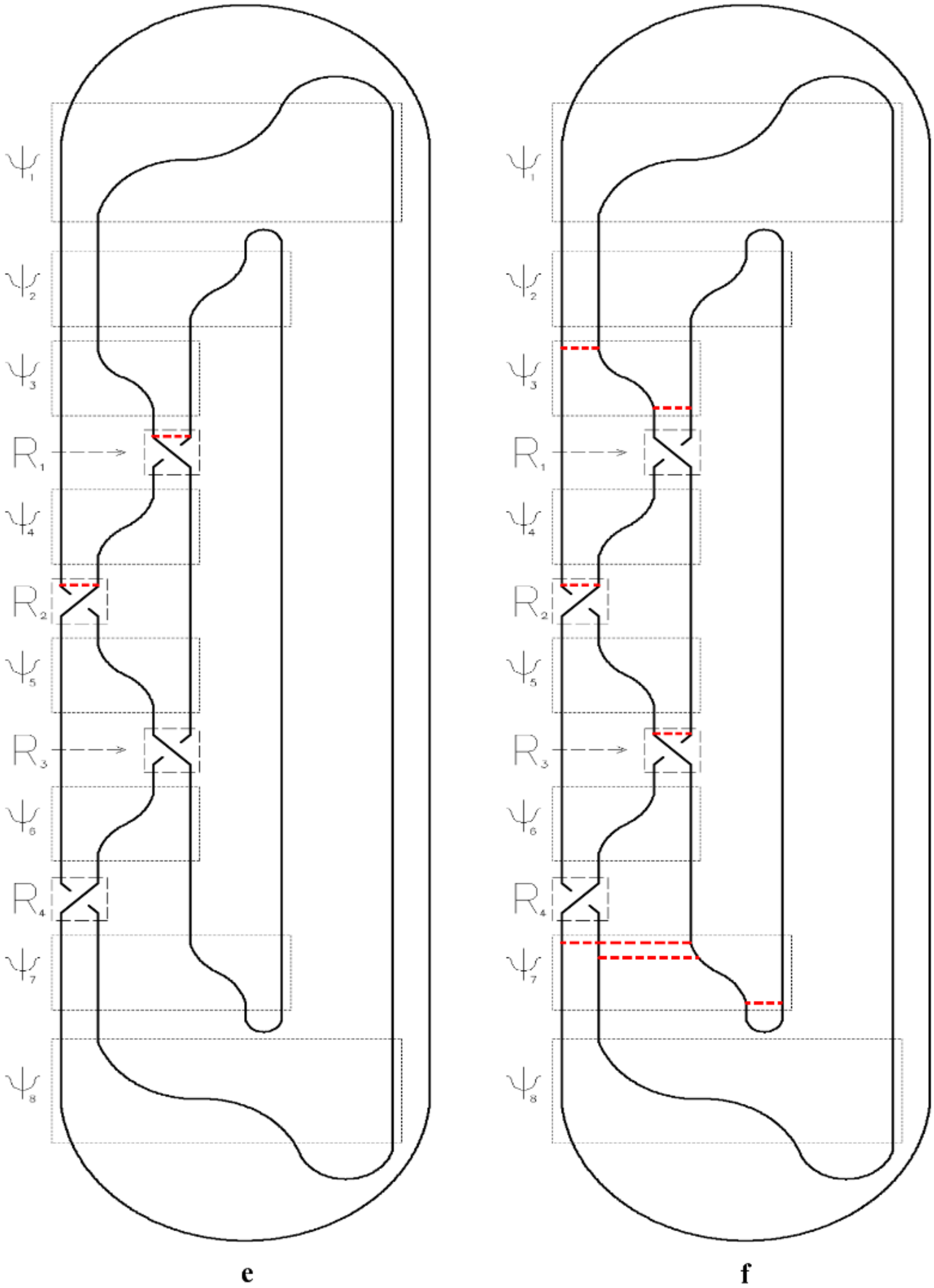


Figure 3.10: Propagators

The terms in KI corresponding to the choice of chords as in Figures 3.8-3.10 are as follows:

1. Figure 3.8b: $\frac{1}{24}\left(T^aT^aT^bT^b\right)$,
2. Figure 3.9c: $\frac{1}{24}\left(T^aT^aT^bT^b\right)$,
3. Figure 3.9d: $\frac{1}{2}\left(T^aT^bT^aT^b\right)$,
4. Figure 3.10e: $\left(T^aT^bT^aT^b\right)$,
5. Figure 3.10f: $i\frac{\zeta(3)}{192\pi^3}\left(T^{a_1}T^{a_2}T^{a_3}T^{a_4}T^{a_5}T^{a_6}T^{a_6}T^{a_7}T^{a_2}T^{a_5}T^{a_1}T^{a_7}T^{a_3}T^{a_4}\right)$,

The final answer in order 2 for knot K is

$$KI_2\left(K_{4_1}\right) = \frac{11}{12}\mathrm{tr}\,\left(T^aT^aT^bT^b\right) - \frac{11}{12}\mathrm{tr}\,\left(T^aT^bT^aT^b\right) \tag{3.32}$$

In orders higher than 2 the mixed terms where some chords are taken from associators and some from R-matrices start to appear. An example of such a chord configuration for order 7 is given in Figure 3.10f.

3.3 From Kontsevich integral to Vassiliev invariants

First of all let us sketchy outline the appearance of Vassiliev invariants in our scheme. Obviously, the mean value $\langle\, W(K)\,\rangle$ has the following structure:

$$\begin{aligned}
\langle\, W_R(C,A)\,\rangle &= \langle\sum_{n=0}^{\infty}\oint dx_1\int dx_2...\int dx_n A^{a_1}(x_1)...A^{a_n}(x_n)\,\mathrm{tr}\,(T^{a_1}...T^{a_n})\,\rangle = \\
&= \sum_{n=0}^{\infty}\oint dx_1\int dx_2...\int dx_n\langle\, A^{a_1}(x_1)...A^{a_n}(x_n)\,\rangle\,\mathrm{tr}\,(T^{a_1}...T^{a_n})= \\
&= \sum_{n=0}^{\infty}\sum_{m=1}^{N_n} V_{n,m}\, G_{n,m}
\end{aligned} \tag{3.33}$$

From this expansion we see that the information about the knot and the gauge group enter $\langle\, W(K)\,\rangle$ separately. The information about the embedding of knot into $\mathbb{R}^3$ is encoded in the integrals of the form:

$$V_{n,m} \sim \oint dx_1\int dx_2...\int dx_n\langle\, A^{a_1}(x_1)A^{a_2}(x_2)...A^{a_n}(x_n)\,\rangle$$

and the information about the gauge group and representation enter the answer as the "group factors":

$$G_{n,m} \sim \mathrm{tr}\,(T^{a_1}T^{a_2}...T^{a_n})$$

$G_{n,m}$ are the group factors called *chord diagrams with n chords*. Chord diagrams with n chords form a vector space of dimension N_n. Despite $\langle W(K)\rangle$ being a knot invariant, the numbers $V_{n,m}$ are not invariants. This is because the group elements $G_{n,m}$ are not independent, and the coefficients $V_{n,m}$ are invariants only up to relations among $G_{n,m}$. In order to pass to the the knot invariants we need to choose in the space of group elements some basis of independent group elements and expand the sum (3.33) in this basis. Finally as it was proven in [79] we arrive to the following infinite product formula:

$$\boxed{\langle\, W_R(K)\,\rangle = \prod_{n=1}^{\infty}\prod_{m=1}^{\mathcal{N}_n} \exp(\hbar^n \mathcal{V}_{n,m}\mathcal{G}_{n,m})} \tag{3.34}$$

where $\mathcal{N}_n$ is the number of independent group elements of degree n and $\mathcal{V}_{n,m}$ are the coefficients of independent group factors $\mathcal{G}_{n,m}$ which are nothing but the Vassiliev invariants of degree n. Note that the number of independent Vassiliev invariants of degree n is given by the number of independent group elements $\mathcal{N}_n$. We list the first several values of $\mathcal{N}_n$:

n	1	2	3	4	5	6
$\mathcal{N}_n$	1	1	1	2	3	5

(3.35)

Formula (3.34) along with table (3.35) means that the expansion of $\langle\, W(K)\,\rangle$ up to order 3 is the following one:

$$\langle W(K)\rangle = 1+\hbar\mathcal{V}_{1,1}\mathcal{G}_{1,1}+\hbar^2\Big(\frac{1}{2!}\mathcal{V}_{1,1}^2\mathcal{G}_{1,1}^2+\mathcal{V}_{2,1}\mathcal{G}_{2,1}\Big)+$$
$$+\hbar^3\Big(\frac{1}{3!}\mathcal{V}_{1,1}^3\mathcal{G}_{1,1}^3+\mathcal{V}_{1,1}\mathcal{V}_{2,1}\mathcal{G}_{1,1}\mathcal{G}_{2,1}+\mathcal{V}_{3,1}\mathcal{G}_{3,1}\Big)+\ldots$$

Note, that here the relations between group factors are taken into account. QFT provides several techniques for computing coefficients of $\hbar$-expansion for $\langle\, W(K)\,\rangle$, each technique leads to some formulae for Vassiliev invariants. The straightforward way is to use the perturbation theory for covariant Lorentz gauge $\partial_i A_i = 0$. Standard quantization technique for Lorentz gauge considered in [63] leads to the following Feynman integral formulae for the first two Vassiliev invariants (integrals are taken along the curve representing the knot):

$$\mathcal{V}_{1,1} = \oint\limits_C dx_1^i \int_0^{x_1} dx_2^j \epsilon_{ijk}\frac{(x_1-x_2)^k}{|x_1-x_2|^3} \tag{3.36}$$

$$\mathcal{V}_{2,1} = \frac{1}{2}\oint\limits_C dx_1^i \int\limits_0^{x_1} dx_2^j \int\limits_0^{x_2} dx_3^k \int\limits_0^{x_3} dx_4^m \epsilon_{pjq}\epsilon_{kis}\frac{(x_4-x_2)^q}{|x_4-x_2|^3}\frac{(x_3-x_1)^s}{|x_3-x_1|^3}+$$
$$-\frac{1}{8}\oint\limits_C dx_1^i \int\limits_0^{x_1} dx_2^j \int\limits_0^{x_2} dx_3^k \int\limits_0^{x_3} dx_4^m \epsilon^{prs}\epsilon_{ipm}\epsilon_{jrn}\epsilon_{kst}\frac{(x_4-x_1)^m}{|x_4-x_1|^3}\frac{(x_4-x_2)^n}{|x_4-x_2|^3}\frac{(x_4-x_3)^t}{|x_4-x_3|^3} \tag{3.37}$$

The first integral here is the so called Gauss integral for self linking number. In general, perturbation theory in Lorentz gauge provides the theory of Vassiliev invariants in terms of rather sophisticated 3d "generalized Gauss integrals".

3.3.1 Chern-Simons definition of Vassiliev invariants

Let us represent the Kontsevich integral (3.18) in the following way:

$$H_R^{\mathcal{K}} = \sum_{n=0}^{\infty} \hbar^n \sum_{m=1}^{\dim(H_n)} V_{n,m}\, G_{n,m}. \tag{3.38}$$

Here, $V_{k,m}$ are some rational coefficients depending on the knot and $G_{k,m}$ are the group factors called the chord diagrams with n chords. The chord diagrams with n chords form the vector space H_n. The coefficients $V_{n,m}$ are not knot invariants because the chord diagrams are not independent, and to express this sum through invariants we have to choose some basis in H_n. The dimensions of H_n are summarized in the table:

n	1	2	3	4	5	6
$dim(H_n)$	1	1	1	3	4	9

(3.39)

In order to pass to Vassiliev invariants we have to choose some basis in the space of chord diagrams. We do it following [68], refer to that paper for details. The so-called trivalent diagrams are introduced in a way represented for orders two and three in Figure 3.11. Group-theoretical rules for graphical representation of chords and trivalent diagrams are presented in Figure 3.12. For the general definition of trivalent diagrams refer to [68], see also [70].

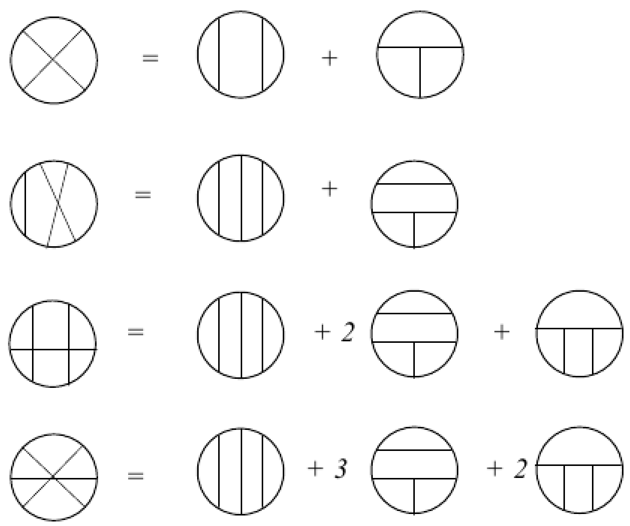


Figure 3.11: Relation between trivalent diagrams and chord diagrams up to order 3

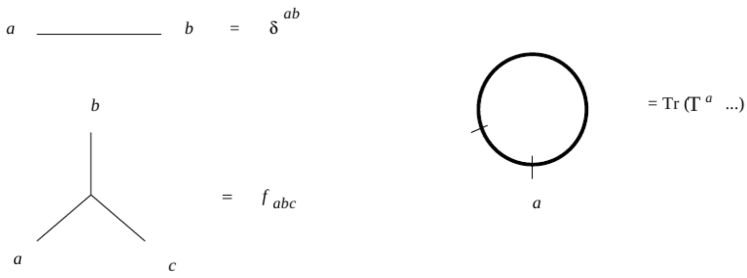
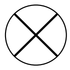
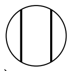
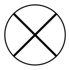
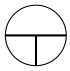


Figure 3.12: Group-theoretical rules

Let us explain the definition of trivalent diagrams on the first relation from Figure 3.11:

$$T^aT^bT^cT^d\delta^{ac}\delta^{bd} = T^aT^bT^aT^b =$$ 

$T^aT^bT^cT^d\delta^{ad}\delta^{bc} = T^aT^bT^bT^a =$  $= T^aT^bT^aT^b - T^aT^bT^aT^b + T^aT^bT^bT^a = T^aT^bT^aT^b - T^aT^b\left(T^aT^b - T^bT^a\right) = T^aT^bT^aT^b - T^aT^b\left[T^aT^b\right] = T^aT^bT^aT^b - f^{abc}T^aT^bT^c =$  $-$ 

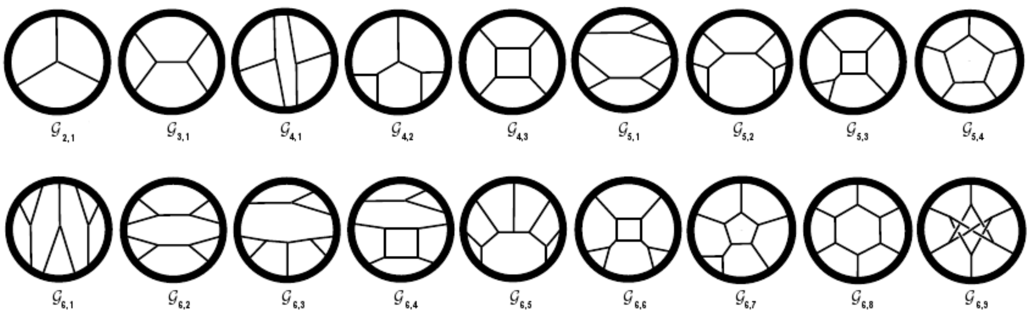


Figure 3.13: Trivalent diagrams

In Figure 3.13 one can find a collection of trivalent diagrams that form the so-called *canonical basis* $\{\mathcal{G}_{ij}\}$ of H_n up to order six. In the fundamental representation their explicit expressions are given in the following table:

$$
\begin{array}{cc}
\mathcal{G}_{2,1} = -\frac{1}{4}N^2 + \frac{1}{4} & \mathcal{G}_{6,1} = -\frac{1}{64}N^6 + \frac{3}{64}N^4 - \frac{3}{64}N^2 + \frac{1}{64} \\
\mathcal{G}_{3,1} = -\frac{1}{8}N^3 + \frac{1}{8}N & \mathcal{G}_{6,2} = \frac{1}{64}N^6 - \frac{1}{32}N^4 + \frac{1}{64}N^2 \\
\mathcal{G}_{4,1} = \frac{1}{16}N^4 - \frac{1}{8}N^2 + \frac{1}{16} & \mathcal{G}_{6,3} = \frac{1}{64}N^6 - \frac{1}{32}N^4 + \frac{1}{64}N^2 \\
\mathcal{G}_{4,2} = -\frac{1}{16}N^4 + \frac{1}{16}N^2 & \mathcal{G}_{6,4} = -\frac{1}{64}N^6 + \frac{3}{64}N^2 - \frac{1}{32} \\
\mathcal{G}_{4,3} = \frac{1}{16}N^4 + \frac{1}{16}N^2 - \frac{1}{8} & \mathcal{G}_{6,5} = -\frac{1}{64}N^6 + \frac{1}{64}N^4 \\
\mathcal{G}_{5,1} = \frac{1}{32}N^5 - \frac{1}{16}N^3 + \frac{1}{32}N & \mathcal{G}_{6,6} = \frac{1}{64}N^6 + \frac{1}{64}N^4 - \frac{1}{32}N^2 \\
\mathcal{G}_{5,2} = -\frac{1}{32}N^5 + \frac{1}{32}N^3 & \mathcal{G}_{6,7} = \frac{1}{64}N^6 - \frac{1}{64}N^2 \\
\mathcal{G}_{5,3} = \frac{1}{32}N^5 + \frac{1}{32}N^3 - \frac{1}{16}N & \mathcal{G}_{6,8} = \frac{1}{64}N^6 + \frac{1}{64}N^2 - \frac{1}{32} \\
\mathcal{G}_{5,4} = \frac{1}{32}N^5 - \frac{1}{32}N & \mathcal{G}_{6,9} = \frac{3}{64}N^4 - \frac{5}{64}N^2 + \frac{1}{32}
\end{array}
\tag{3.40}
$$

Using this basis we rewrite (3.38) through invariants:

$$
H_R^{\mathcal{K}} = \sum_{n=0}^{\infty} \hbar^n \sum_{m=1}^{\dim(H_n)} \mathcal{V}_{n,m}\,\mathcal{G}_{n,m} \tag{3.41}
$$

Here $\mathcal{V}_{ij}$ are the so called finite-type or Vassiliev invariants of knots. They depend only on the knot under consideration but not on the group and its representation.

Now let us introduce the primitive Vassiliev invariants. It is a well known fact that the expansion of logarithm of any correlator in any QFT contains only connected Feynman diagrams (for more details about this situation in the Chern-Simons perturbation theory see [79]). This fact immediately leads to the following summation of

$$
H_R^{\mathcal{K}} = \prod_{n=0}^{\infty} \prod_{m=1}^{\mathcal{N}_n} \exp\left(\hbar^n \mathcal{V}^c_{n,m}\,\mathcal{G}^c_{n,m}\right), \tag{3.42}
$$

where $\mathcal{G}^c$ are connected diagrams, $\mathcal{V}^c$ are primitive Vassiliev invariants. Here $\mathcal{N}_n$ is dimension of the space of connected chord diagrams (or equivalently the space of primitive Vassiliev invariants). The dimensions of these spaces up to order 6 are given in the following table:

n	1	2	3	4	5	6
$\mathcal{N}_n$	1	1	1	2	3	5

(3.43)

The meaning of the expression (3.42) is that $\mathcal{V}_{i,j}$ in (3.41) are not independent. In fact only those coefficients $\mathcal{V}_{ij}$ are independent, for which the corresponding diagram $\mathcal{G}_{ij}$ is connected. Comparing $\hbar$ expansion of (3.42) with (3.41) we, for example, find:

$$\begin{aligned}
\mathcal{V}_{4,1} &= \tfrac{1}{2}\,\mathcal{V}_{2,1}^2 \\
\mathcal{V}_{5,1} &= \mathcal{V}_{2,1}\mathcal{V}_{3,1}, \\
\mathcal{V}_{6,1} &= \tfrac{1}{6}\mathcal{V}_{2,1}^3, \\
\mathcal{V}_{6,2} &= \tfrac{1}{2}\,\mathcal{V}_{3,1}^2, \\
\mathcal{V}_{6,3} &= \mathcal{V}_{2,1}\mathcal{V}_{4,2}, \\
\mathcal{V}_{6,4} &= \mathcal{V}_{2,1}\mathcal{V}_{4,3}.
\end{aligned} \tag{3.44}$$

And finally: *The Vassiliev invariants form a graded ring freely generated by primitive invariants.*

3.4 Loop expansion of knot polynomials

Now we study relation between large N expansion

$$\begin{aligned}
H_R^{\mathcal{K}}(q,A) &= \exp\left(\sum_{\Delta} \hbar^{|\Delta|+l(\Delta)-2}\; S_{\Delta}^{\mathcal{K}}\big(A^2,\hbar^2\big)\;\varphi_R(\Delta)\right), \\
S_{\Delta}^{\mathcal{K}}(A^2,\hbar^2) &= \sum_g \sigma_{\Delta}^{\mathcal{K}}(g;A)\,\hbar^{2g},
\end{aligned} \tag{3.45}$$

and loop expansion

$$H_R^{\mathcal{K}} = \prod_{n=0}^{\infty}\prod_{m=1}^{\mathcal{N}_n} \exp\left(\hbar^n \mathcal{V}_{n,m}^c\,\mathcal{G}_{n,m}^c\right). \tag{3.46}$$

Also we get explicit expression for higher special polynomials $\sigma_{\Delta}^{\mathcal{K}}(g;A)$ through Vassiliev invariants.

Let us reduce the latter series to the form of the former series. For this purpose one needs to introduce variable $\alpha = N\hbar$ and separate it in the expansion (3.46). Variable $\hbar$ is given explicitly there, while variable N is included in $\mathcal{G}_{n,m}^c$ implicitly. Actually, $\mathcal{G}_{n,m}^c$ is a polynomial in N of order not higher than n, i.e. we have

$$\begin{aligned}
\mathcal{G}_{n,m}^c &= \sum_{k=0}^{n} c_{n,m,k}^{(R)} N^k, \\
c_{n,m,k}^{(R)} &= \sum_{|\Delta|+l(\Delta)-2\le n-k} c_{n,m,k}(\Delta)\varphi_R(\Delta),
\end{aligned} \tag{3.47}$$

where $c_{n,m,k}(\Delta) \in \mathbb{Q}$. Indeed, let us consider coefficient of the expansion (3.46) in front of $\hbar^n$: $\sum_{m=1}^{\mathcal{N}_n} \mathcal{V}_{n,m}^c\,\mathcal{G}_{n,m}^c$. Factors $\mathcal{G}_{n,m}^c$ do not depend on a knot $\mathcal{K}$ and $\mathcal{V}_{n,m}^c$ are linearly independent. Let us take $\mathcal{N}_n$ different knots such that the following matrix is not degenerate[4]:

$$\left(\mathcal{V}_{n,i}^{\mathcal{K}_j}\right)_{i,j=1,\dots,\mathcal{N}_n}. \tag{3.48}$$

[4]Index c in $\mathcal{V}^c$ we omit below in this paragraph for brevity.

Then $\sum\limits_j \mathcal{V}_{n,j}^{\mathcal{K}_i}\,\mathcal{G}_{n,j}^c$ is equal to the corresponding coefficient of the expansion (3.45) in front of $\hbar^n$, which is a linear combination of $\varphi_R(\Delta)$, denoted by $f_R(n)^{\mathcal{K}}$:

$$\sum_j \mathcal{V}_{n,j}^{\mathcal{K}_i}\,\mathcal{G}_{n,j}^c = f_R(n)^{\mathcal{K}_i}. \tag{3.49}$$

Since the matrix $\left(\mathcal{V}_{n,j}^{\mathcal{K}_i}\right)$ is not degenerate, then we can inverse it and obtain expression for $\mathcal{G}_{n,j}^c$ in terms of $\varphi_R(\Delta)$ what is given in formula (3.47). Particular Young diagrams using in the summation (3.47) are determined by formula (3.45) (Conjecture 2.4.3) of the explicit form of large N expansion.

Therefore, HOMFLY polynomial takes the form

$$\begin{aligned} H_R^{\mathcal{K}} &= \exp\left(\sum_{n=0}^{\infty}\sum_{m=1}^{\mathcal{N}_n}\hbar^n\mathcal{V}_{n,m}^c\,\mathcal{G}_{n,m}^c\right) = \\ &= \exp\left(\sum_{\Delta}\varphi_R(\Delta)\sum_{n=0}^{\infty}\hbar^u\sum_{\substack{k\geq 2-u\\ k\geq 0}}\alpha^k\sum_{m=1}^{\mathcal{N}_{u+k}}c_{u+k,m,k}(\Delta)\mathcal{V}_{u+k,m}^c\right) \qquad (3.50)\\ u &:= 2n+|\Delta|+l(\Delta)-2 \qquad (3.51)\end{aligned}$$

Remark. Note that $2-u>0$ only for $u=0 \Leftrightarrow n=0, \Delta=[1]$ and $u=1 \Leftrightarrow n=0, \Delta=[2]$. For all other n and Δ value $2-u\leq 0$, hence, we can omit condition $k\geq 2-u$ keeping in mind the above note.

So, we have

$$\begin{aligned} H_R^{\mathcal{K}} &= \exp\left(\sum_{\Delta}\varphi_R(\Delta)\sum_{g=0}^{\infty}\hbar^u\sigma_{\Delta}^{\mathcal{K}}(g;A)\right) = \qquad (3.52)\\ &= \exp\left(\sum_{\Delta}\varphi_R(\Delta)\sum_{n=0}^{\infty}\hbar^u\sum_{k\geq 0}\alpha^k\sum_{m=1}^{\mathcal{N}_{u+k}}c_{u+k,m,k}(\Delta)\mathcal{V}_{u+k,m}^c\right) \qquad (3.53)\end{aligned}$$

Comparison of two last formulas (large N expansion and Vassiliev expansion) leads to the folowing theorem modulo conjecture 2.4.3.

Theorem 3.4.1. *(Conditional theorem) Higher special polynomials are related to Vassiliev invarants as follows:*

$$\sigma_{\Delta}^{\mathcal{K}}(g;A) = \sum_{k\geq 0}\alpha^k\sum_{m=1}^{\mathcal{N}_{u+k}}c_{u+k,m,k}(\Delta)\mathcal{V}_{u+k,m}^c \tag{3.54}$$

Note that the above remark from this point of view just means that $\sigma_{[1]}^{\mathcal{K}}(0)\sim\alpha^2+O(\alpha^3)$ and $\sigma_{[2]}^{\mathcal{K}}(0)\sim\alpha+O(\alpha^2)$.

3.4.1 Polynomial relations for Vassiliev invariants

Let us carefully investigate formula (3.54). First, in the left hand side there is a polynomial in variable $A = \exp(\alpha)$, and in the right hand side there is a power series in α. In order to satisfy this equality (”polynomial” = ”power series”), there should be relations on the coefficients of the power series, i.e. on the Vassiliev invariants. Here we consider only the case $\Delta = \Box$, $n = 0$. It corresponds to HOMFLY polynomial for $q = 1$.

$$H_{\Box}^{\mathcal{K}}(q=1, A) = \exp\left(\sigma_{\Box}^{\mathcal{K}}(0; A)\right) \tag{3.55}$$

Then taking into account that for this case $u = 2n + |\Delta| + l(\Delta) - 2 = 0$ from equation (3.54) we have the following:

$$\begin{aligned} H_{\Box}^{\mathcal{K}}(q=1, A) &= \exp\left(\sigma_{\Box}^{\mathcal{K}}(0; A)\right) = \exp\left(\sum_{k\geq 0} \alpha^k \sum_{m=1}^{\mathcal{N}_k} c_{k,m,k}(\Box)\mathcal{V}_{k,m}^c\right) \\ &= \exp\left(\sum_{k\geq 0} \alpha^k v_k\right) = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} V_k, \end{aligned} \tag{3.56}$$

where v_k and V_k are defined by this formula. It is clear that V_k are some linear combinations of *non-primitive* Vassiliev invariants of order k.

Second, we arrange all knots by the number of strands r in the braid $\mathcal{B}_{\mathcal{K}}$ corresponding to the minimum braid representation. Then with the help of MFW inequality bounding the braid index by the A-breadth of the HOMFLY polynomial [2] we have the following:

$$H_{\Box}^{\mathcal{K}_r}(q, A) = \sum_{i=0}^{r-1} x_i^{\mathcal{K}}(q) A^{y(\mathcal{K})+2i} = A^{y(\mathcal{K})} \sum_{i=0}^{r-1} x_i^{\mathcal{K}}(q) A^{2i}, \tag{3.57}$$

where $x_i^{\mathcal{K}}(q)$ are polynomials in q depending on the knot $\mathcal{K}$, $y(\mathcal{K})$ is a numerical function depending on the knot, which is nothing but the writhe. In other words, HOMFLY has r terms (not more) as a polynomial in A. It is clear that the same is true for special polynomial $\sigma_{\Box}^{\mathcal{K}}(A)$, when $q = 1$. Considering this we can write condition (3.56) as follows:

$$x_1 A^{y_1} + x_2 A^{y_2} + \dots + x_r A^{y_r} = V_0 + V_1 \alpha + V_2 \frac{\alpha^2}{2!} + V_3 \frac{\alpha^3}{3!} + \dots \tag{3.58}$$

Let us substitute $A = \exp(\alpha)$, expand the LHS in α and introduce the following vectors

$$\vec{e}_k = \left(1,\, y_k,\, y_k^2,\, y_k^3,\, \dots\right) \tag{3.59}$$

$$\vec{V} = (V_0,\, V_1,\, V_2,\, V_3,\, \dots) \tag{3.60}$$

Then condition (3.58) means that vector $\vec{V}$ lies in the linear span of the vectors $\vec{e}_1, \dots, \vec{e}_r$:

$$\vec{V} \in < \vec{e}_1, \dots, \vec{e}_r > \tag{3.61}$$

Let us introduce shifted vector $\vec{V}{<}i{>}$ as

$$\vec{V}{<}i{>} = (V_i, V_{i+1}, V_{i+2}, V_{i+3}, \dots) \tag{3.62}$$

Any shifted vector $\vec{V}{<}i{>}$ also lies in the linear span of the vectors $\vec{e}_1, \dots, \vec{e}_r$.

Let us consider $r+1$ shifted vectors $\{\vec{V}{<}i_1{>}, \vec{V}{<}i_2{>}, \dots, \vec{V}{<}i_r{>}, \vec{V}{<}i_{r+1}{>}\}$ and write them in the matrix form:

$$M = \begin{pmatrix} V_{i_1} & V_{i_1+1} & V_{i_1+2} & V_{i_1+3} & \dots \\ V_{i_2} & V_{i_2+1} & V_{i_2+2} & V_{i_2+3} & \dots \\ \dots & & & & \\ V_{i_r} & V_{i_r+1} & V_{i_r+2} & V_{i_r+3} & \dots \\ V_{i_{r+1}} & V_{i_{r+1}+1} & V_{i_{r+1}+2} & V_{i_{r+1}+3} & \dots \end{pmatrix} \tag{3.63}$$

Since $\vec{V}{<}i_1{>}, \vec{V}{<}i_2{>}, \dots, \vec{V}{<}i_r{>}, \vec{V}{<}i_{r+1}{>} \in < \vec{e}_1, \dots, \vec{e}_r >$, then any new matrix, consisted from any $r+1$ columns of the matrix (3.63), is a degenerate. Thus, we have proved the following theorem.

Theorem 3.4.2. *Vassiliev invariants of the knot $\mathcal{K}$ with r strands in the minimum braid representation satisfy the following relation*

$$\det M_{j_1 \dots j_{r+1}} = 0, \tag{3.64}$$

where M is defined by (3.63) and a set of numbers $\{j_1, \dots, j_{r+1}\}$ denotes column numbers, every number j_k takes value from 1 to ∞.

Let us give example of such particular matrix. Let $\{i_1 = 0, i_2 = 1, i_3 = 2, \dots, i_{r+1} = r\}$ and $\{j_1 = 1, j_2 = 2, j_3 = 3, \dots, j_{r+1} = r+1\}$, then from (3.63) and (3.64) we get the following relation

$$\boxed{\det_{1 \le i,j \le r+1} V_{r+j-i} = 0} \tag{3.65}$$

3.4.2 Numerical results for invariants up to order 6 and families of knots

Now one can use computational technique from the previous section 3.1.1 and results from subsection 3.3.1 to calculate the Kontsevich integral. In this way we calculate Vassiliev invariants up to level 6 inclusive for knots with number of self-intersections up to 14 inclusive. These results are available in [80]. Here we list just a few examples:

	$\mathcal{V}_{2,1}$	$\mathcal{V}_{3,1}$	$\mathcal{V}_{4,1}$	$\mathcal{V}_{4,2}$	$\mathcal{V}_{4,3}$	$\mathcal{V}_{5,1}$	$\mathcal{V}_{5,2}$	$\mathcal{V}_{5,3}$	$\mathcal{V}_{5,4}$
3_1	4	-8	8	$\frac{62}{3}$	$\frac{10}{3}$	-32	$-\frac{176}{3}$	$-\frac{32}{3}$	-8
4_1	-4	0	8	$\frac{34}{3}$	$\frac{14}{3}$	0	0	0	0
5_1	12	-40	72	174	26	-480	$-\frac{2512}{3}$	$-\frac{448}{3}$	-104
5_2	8	-24	32	$\frac{268}{3}$	$\frac{44}{3}$	-192	-368	-64	-56
6_1	-8	8	32	$\frac{116}{3}$	$\frac{52}{3}$	-64	$-\frac{304}{3}$	$-\frac{64}{3}$	-24
6_2	-4	8	8	$\frac{34}{3}$	$\frac{38}{3}$	-32	$-\frac{208}{3}$	$-\frac{64}{3}$	-24
6_3	4	0	8	$\frac{14}{3}$	$-\frac{14}{3}$	0	0	0	0
7_1	24	-112	288	684	100	-2688	$-\frac{13888}{3}$	$-\frac{2464}{3}$	-560
7_2	12	-48	72	222	34	-576	-1152	-192	-176
7_3	20	88	200	$\frac{1510}{3}$	$\frac{242}{3}$	1760	$\frac{9520}{3}$	$\frac{1696}{3}$	440
7_4	16	64	128	$\frac{1016}{3}$	$\frac{184}{3}$	1024	$\frac{5824}{3}$	$\frac{1024}{3}$	320
7_5	16	-64	128	$\frac{968}{3}$	$\frac{136}{3}$	-1024	$-\frac{5440}{3}$	$-\frac{928}{3}$	-224
7_6	4	-16	8	$\frac{158}{3}$	$\frac{34}{3}$	-64	$-\frac{544}{3}$	$-\frac{64}{3}$	-48
7_7	-4	-8	8	$-\frac{14}{3}$	$-\frac{10}{3}$	32	$\frac{112}{3}$	$\frac{64}{3}$	-8

(3.66)

	$\mathcal{V}_{6,1}$	$\mathcal{V}_{6,2}$	$\mathcal{V}_{6,3}$	$\mathcal{V}_{6,4}$	$\mathcal{V}_{6,5}$	$\mathcal{V}_{6,6}$	$\mathcal{V}_{6,7}$	$\mathcal{V}_{6,8}$	$\mathcal{V}_{6,9}$
3_1	$\frac{32}{3}$	32	$\frac{248}{3}$	$\frac{40}{3}$	$\frac{5071}{30}$	$\frac{58}{15}$	$\frac{3062}{45}$	$\frac{17}{18}$	$\frac{271}{30}$
4_1	$-\frac{32}{3}$	0	$-\frac{136}{3}$	$-\frac{56}{3}$	$-\frac{1231}{30}$	$\frac{142}{15}$	$-\frac{1742}{45}$	$\frac{79}{18}$	$-\frac{271}{30}$
5_1	288	800	2088	312	$\frac{41151}{10}$	$\frac{2494}{15}$	$\frac{7634}{5}$	$\frac{43}{2}$	$\frac{1951}{10}$
5_2	$\frac{256}{3}$	288	$\frac{2144}{3}$	$\frac{352}{3}$	$\frac{22951}{15}$	$-\frac{28}{5}$	$\frac{29764}{45}$	$\frac{137}{9}$	$\frac{1351}{15}$
6_1	$-\frac{256}{3}$	32	$-\frac{928}{3}$	$-\frac{416}{3}$	$-\frac{2791}{15}$	$\frac{884}{15}$	$-\frac{10084}{45}$	$\frac{343}{9}$	$-\frac{871}{15}$
6_2	$-\frac{32}{3}$	32	$-\frac{136}{3}$	$-\frac{152}{3}$	$\frac{2129}{30}$	$\frac{662}{15}$	$-\frac{1862}{45}$	$\frac{463}{18}$	$-\frac{751}{30}$
6_3	$\frac{32}{3}$	0	$\frac{56}{3}$	$-\frac{56}{3}$	$\frac{511}{30}$	$\frac{418}{15}$	$-\frac{1858}{45}$	$\frac{65}{18}$	$-\frac{449}{30}$
7_1	2304	6272	16416	2400	$\frac{160231}{5}$	$\frac{21548}{15}$	$\frac{58148}{5}$	163	$\frac{7351}{5}$
7_2	288	1152	2664	408	$\frac{60431}{10}$	$-\frac{826}{15}$	$\frac{38942}{15}$	$\frac{497}{6}$	$\frac{3471}{10}$
7_3	$\frac{4000}{3}$	3872	$\frac{30200}{3}$	$\frac{4840}{3}$	$\frac{121855}{6}$	382	$\frac{73862}{9}$	$\frac{2437}{18}$	$\frac{6559}{6}$
7_4	$\frac{2048}{3}$	2048	$\frac{16256}{3}$	$\frac{2944}{3}$	$\frac{168062}{15}$	$-\frac{1176}{5}$	$\frac{233288}{45}$	$\frac{898}{9}$	$\frac{11102}{15}$
7_5	$\frac{2048}{3}$	2048	$\frac{15488}{3}$	$\frac{2176}{3}$	$\frac{156422}{15}$	$\frac{5912}{15}$	$\frac{170888}{45}$	$\frac{730}{9}$	$\frac{7142}{15}$
7_6	$\frac{32}{3}$	128	$\frac{632}{3}$	$\frac{136}{3}$	$\frac{19471}{30}$	$-\frac{474}{5}$	$\frac{17342}{45}$	$\frac{401}{18}$	$\frac{1711}{30}$
7_7	$-\frac{32}{3}$	32	$\frac{56}{3}$	$\frac{40}{3}$	$\frac{2849}{30}$	$-\frac{218}{15}$	$\frac{3418}{45}$	$-\frac{161}{18}$	$\frac{449}{30}$

(3.67)

There is also one more technique to compute these invariants. It was suggested by Alvarez and Labastida in [82]. We use their technique to check our results.

Willerton's fish and families of knots

On 18 Vassiliev invariant up to order six

$$\mathcal{V}_{2,1}, \mathcal{V}_{3,1}, \mathcal{V}_{4,1}, \mathcal{V}_{4,2}, \mathcal{V}_{4,3}, \mathcal{V}_{5,1}, \mathcal{V}_{5,2}, \mathcal{V}_{5,3}, \mathcal{V}_{5,4}, \mathcal{V}_{6,1}, \mathcal{V}_{6,2}, \mathcal{V}_{6,3}, \mathcal{V}_{6,4}, \mathcal{V}_{6,5}, \\ \mathcal{V}_{6,6}, \mathcal{V}_{6,7}, \mathcal{V}_{6,8}, \mathcal{V}_{6,9}$$

there are 6 relations

$$\mathcal{V}_{4,1} = 1/2\, \mathcal{V}_{2,1}^2, \tag{3.68}$$

$$\mathcal{V}_{5,1} = \mathcal{V}_{2,1}\mathcal{V}_{3,1}, \tag{3.69}$$

$$\mathcal{V}_{6,1} = \frac{1}{6}\mathcal{V}_{2,1}^3, \tag{3.70}$$

$$\mathcal{V}_{6,2} = 1/2\, \mathcal{V}_{3,1}^2, \tag{3.71}$$

$$\mathcal{V}_{6,3} = \mathcal{V}_{2,1}\mathcal{V}_{4,2}, \tag{3.72}$$

$$\mathcal{V}_{6,4} = \mathcal{V}_{2,1}\mathcal{V}_{4,3}. \tag{3.73}$$

Remaining 12 Vassiliev invariants are generally thought to be independent.

However, if one plots $\mathcal{V}_{3,1}$ against $\mathcal{V}_{2,1}$ for all knots of given number of crossings, one obtains filled region with form resembling fish, which was discovered by Willerton in [83] and is hence called Willerton's fish, see Figure 3.14. Note, that for prime knots, which are actually plotted in Figure 3.14 coordinates are "quantized" – they are integer multiples of $\mathcal{V}'s$ for the trefoil. For higher Vassiliev invariants in our chosen basis it is not the case. See Appendix 5.1 where we present some new even more peculiar figures representing relations between Vassiliev invariants.

The boundaries of the fish can be fitted with cubic polynomial. This suggests, that there are some additional relations on Vassiliev invarants.

Conjecture 3.4.3. (Willerton's conjecture [83]) *There is a condition for any knot $\mathcal{K}$ on Vassiliev invariants of the following form:*
cubic in $\mathcal{V}_{2,1}(\mathcal{K}) \leq (\mathcal{V}_{3,1}(\mathcal{K}))^2 \leq$ another cubic in $\mathcal{V}_{2,1}(\mathcal{K})$

We suggest that inequalities appear, because Willerton did not consider Vassiliev invariants of order 6, they were unknown at that time. Thus, now it is natural to search for relations on Vassiliev invariants up to 6 order, i.e. to search for vanishing polynomials in $\mathcal{V}$ of order up to 6. If such polynomials can be found it would mean that Vassiliev invariants are not independent. However, it turned out, that *there are no such vanishing expressions.*

Nevertheless, the following expression vanishes for all knots up to 6 crossings and for quite big number of knots with more crossings, for example, for 92 out of 165 knots with 10 crossings:

$$F_1 = \frac{16\mathcal{V}_{2,1}}{15} + 2\mathcal{V}_{2,1}^2 + \mathcal{V}_{2,1}^3 - 6\mathcal{V}_{3,1} - 6\mathcal{V}_{2,1}\mathcal{V}_{3,1} - 3\mathcal{V}_{3,1}^2 - 4\mathcal{V}_{4,2} - 6\mathcal{V}_{2,1}\mathcal{V}_{4,2} + \\ + 4\mathcal{V}_{4,3} + 6\mathcal{V}_{2,1}\mathcal{V}_{4,3} + 6\mathcal{V}_{5,2} - 6\mathcal{V}_{5,3} - 6\mathcal{V}_{5,4} + 6\mathcal{V}_{6,5} - 6\mathcal{V}_{6,6} - 6\mathcal{V}_{6,7} - 6\mathcal{V}_{6,8}. \tag{3.74}$$

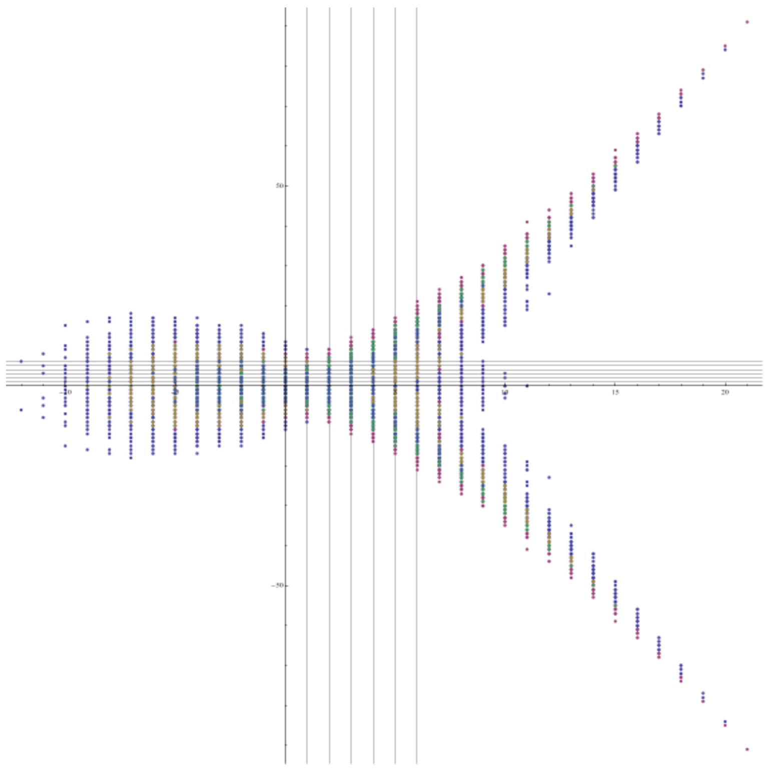


Figure 3.14: Willerton's fish, i.e. $\mathcal{V}_{3,1}$ plotted against $\mathcal{V}_{2,1}$ for knots with up to 14 crossings

This is very untrivial, because the general polynomial of order 6 has 17 free coefficients, while polynomial (3.74) vanishes on 92 knots among solely the knots with 10 crossings!

Another numerical experimental fact is that the value of (3.74) is generally close to the value of *writhe number* (see section 3.5 for the definition), and tends to coincide with it, but not for all knots.

3.5 Temporal gauge

In section 3.1 we declare that from Chern-Simons theory in Lorentz gauge one can get integral representations for Vassiliev invariants (3.36, 3.37). Despite beautiful the integral representation for Vassiliev invariants is too complicated to compute and it is difficult to use them for investigations. From the other side, the knots are simple combinatorial objects and it is not surprisingly that simpler combinatorial formulas for the invariants should exist. In this section we are going to define combinatorial objects related to the knots called *writhe numbers*. It will be shown that writhe numbers can be derived from the Wilson loop operator in temporal gauge. However it turns out that it is impossible to express Vassiliev invariants through such writhe numbers. Because of this reason we define *colored writhe numbers* in a combinatorial way by analogy with the combinatorial definition of writhe numbers, because temporal gauge consideration does not lead us to them.

Then it is possible to express Vassiliev invariants through colored writhe number up to order 4 at least (this fact was found in [65, 66]).

3.5.1 Writhe numbers from temporal gauge

Consider Chern-Simons action in the temporal gauge $A_0 = 0$. Then the propagator takes the following form:

$$\langle A_0^a(x), A_\mu^b(y)\rangle = 0, \quad \mu = 0,1,2, \tag{3.75}$$

$$\langle A_\mu^a(x), A_\nu^b(y)\rangle = \frac{1}{2}\varepsilon^{\mu\nu}\delta^{ab}\delta(x_1-y_1)\delta(x_2-y_2)\mathrm{sign}(\mathrm{x}_0-\mathrm{y}_0), \quad \mu = 0,1,2. \tag{3.76}$$

Now let us consider the vacuum expectation value of the Wilson loop operator (3.33):

$$\langle W(K)\rangle = \sum_{n=0}^{\infty}\oint dx_1\int dx_2 ... \int dx_n \langle A^{a_1}(x_1)A^{a_2}(x_2)...A^{a_3}(x_n)\rangle \mathrm{tr}\,(T^{a_1}T^{a_2}...T^{a_n}). \tag{3.77}$$

Taking into account the propagators (3.75), (3.76) consider term of vev with $n = 2$ in details:

$$\int dx_\mu \int dy_\nu \langle A(x_\mu)A(x_\nu))\rangle = \frac{1}{2}\int\int dx_\mu dy_\nu \varepsilon^{\mu\nu}\delta^{ab}\delta(x_1-y_1)\delta(x_2-y_2)\mathrm{sign}(\mathrm{x}_0-\mathrm{y}_0). \tag{3.78}$$

Let us parametrize the knot by a parameter t running from 0 to 1, then we can rewrite the last integral in the following form:

$$\int_0^1\int_0^1 dt_1 dt_2\left(\frac{dx_1}{dt_1}\frac{dy_2}{dt_2}-\frac{dx_2}{dt_1}\frac{dy_1}{dt_2}\right)\delta\left(x_1(t_1)-y_1(t_2)\right)\delta\left(x_2(t_1)-y_2(t_2)\right)\cdot \tag{3.79}$$

$$\cdot\mathrm{sign}\left(\mathrm{x}_0(\mathrm{t}_1)-\mathrm{y}_0(\mathrm{t}_2)\right) \tag{3.80}$$

To perform the integration we need to solve the following equations:

$$\begin{cases} x_1(t_1)-y_1(t_2) = 0 \\ \\ x_2(t_1)-y_2(t_2) = 0 \end{cases} \tag{3.81}$$

The solutions of these equations are the self-intersection points of two-dimensional curve $(x_1(t), x_2(t))$ which is the projection of the knot c on the plane (x_1, x_2). Let us denote by $t_1^k < t_2^k$ the values of the parameter t in the intersection points, then the two-dimensional delta-function in the integral can be represented in the following form:

$$\delta(x_1(t_1)-y_1(t_2))\,\delta(x_2(t_1)-y_2(t_2)) = \sum_k \frac{\left(\delta(t_1-t_1^k)\delta(t_2-t_2^k)+\delta(t_1-t_2^k)\delta(t_2-t_1^k)\right)}{|\frac{dx_1}{dt_1}\frac{dy_2}{dt_2}-\frac{dx_2}{dt_1}\frac{dy_1}{dt_2}|}$$

Substituting this expression into (3.81) and integrating over t_1 and t_2 we arrive to the following simple expression:

$$\int dx_\mu \int dy_\nu \langle A(x_\mu)A(x_\nu))\rangle = \sum_k \epsilon_k, \tag{3.82}$$

where the quantities ϵ_k are the "sings" of the intersection points. They can take values ± 1 and are defined in the following way:

$$\epsilon_k = \frac{\frac{dx_1}{dt_1}(t_1^k)\,\frac{dy_2}{dt_2}(t_2^k) - \frac{dx_2}{dt_1}(t_1^k)\,\frac{dy_1}{dt_2}(t_2^k)}{|\frac{dx_1}{dt_1}(t_1^k)\,\frac{dy_2}{dt_2}(t_2^k) - \frac{dx_2}{dt_1}(t_1^k)\,\frac{dy_1}{dt_2}(t_2^k)|}\,\mathrm{sign}(x_0(t_1^k) - y_0(t_2^k)) \tag{3.83}$$

Thus, Chern-Simons theory in temporal gauge naturally leads us to the following notion.

Definition 3.5.1. *Writhe number for oriented knot $w(\mathcal{K})$ is defined as the sum of the signs of all the crossings:*

$$w_1(\mathcal{K}) := \sum_p \varepsilon(p), \tag{3.84}$$

where the crossing signs are $+1$ or -1 as indicated in Figure 3.15.

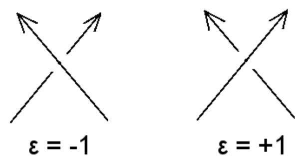


Figure 3.15: Crossing signs

Writhe numbers can be represented graphically, if one gives another equivalent definition. Let us choose the origin on the knot and the orientation. When we are going along the knot, we meet every crossing point twice. We enumerate all crossing points by increasing sequence of the natural numbers. Then every crossing point is defined by the pair of numbers (i_1, i_2), $i_1 \neq i_2$ (see, for example, Figure 3.16). So, we get the following different definition of the writhe number.

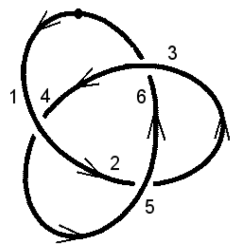


Figure 3.16: Trefoil

Definition 3.5.2. *Writhe number for oriented knot $w(\mathcal{K})$ is defined as the sum of the crossing signs:*

$$w_1(\mathcal{K}) := \sum_{i_1<i_2} \varepsilon_{i_1 i_2}. \tag{3.85}$$

Now we can depict the writhe number by a chord diagram, which is called the Gauss diagram:

$$w_1(\mathcal{K}) = \sum_{i_1<i_2} \varepsilon_{i_1 i_2} \equiv \tag{3.86}$$

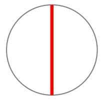


Diagram language is more comfortable and illustrative for higher writhe numbers, definitions of which we give below.

It turns out that the second Vassiliev invariant cannot be expressed through the just defined writhe number. Hence, we modify the definition of the writhe number. First, we introduce two types of the crossing signs ε.

Definition 3.5.3.

$$\varepsilon^{ou}_{i_1 i_2} = \begin{cases} \varepsilon_{i_1 i_2}, & \textit{if the strand which led to the point } (i_1 i_2) \textit{ is } \textbf{over} \textit{ another strand} \\ 0, & \textit{otherwise} \end{cases} \tag{3.87}$$

$$\varepsilon^{uo}_{i_1 i_2} = \begin{cases} \varepsilon_{i_1 i_2}, & \textit{if the strand which led to the point } (i_1 i_2) \textit{ is } \textbf{under} \textit{ another strand} \\ 0, & \textit{otherwise} \end{cases} \tag{3.88}$$

Thus based on the last definition we can define two types of colored writhe numbers.

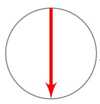
Definition 3.5.4. *Colored writhe numbers for oriented knot are defined as the following sums*

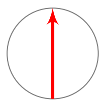
$$w_1^{ou} := \sum_{i_1<i_2} \varepsilon^{ou}_{i_1 i_2} \tag{3.89}$$

$$w_1^{uo} := \sum_{i_1<i_2} \varepsilon^{uo}_{i_1 i_2} \tag{3.90}$$

Colored writhe numbers can be denoted by the following diagrams:

$$w_1^{ou} \equiv \qquad , \qquad w_1^{uo} \equiv \tag{3.91}$$





Remark 3.5.5. *We always read chord diagrams clockwise from the top. It is equivalent to introducing an origin point O slightly to the right from the top.*

3.5.2 Higher writhe numbers

Any definition from previous subsection can be generalized in a straightforward way. However because the colored writhe numbers are more general objects than the ordinary ones, we generalize only the last definition (3.5.4).

Definition 3.5.6. *Higher colored writhe number of n-th order for oriented knot $w(\mathcal{K})$ is the sum*

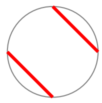
$$w_n^{\sigma_1..\sigma_n}(\mathcal{K}) := \sum_{i_1<\cdots<i_{2n}} \varepsilon^{\sigma_1}_{i_{m_1}i_{m_2}} \dots \varepsilon^{\sigma_n}_{i_{m_{2n-1}}i_{m_{2n}}}, \tag{3.92}$$

where $\sigma_1, \dots, \sigma_n$ take values ou *or* uo.

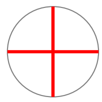
In graphical approach indices $m_1, ..., m_{2n}$ correspond to the points on the circle, $\varepsilon_{i_{m_k}i_{m_l}}$ corresponds to a line between points m_k and m_l (see the following example).

Example 3.5.7. *For order 2 we have only two diagrams:*

$$\sum_{i_1<i_2<i_3<i_4} \varepsilon_{i_1i_2}\varepsilon_{i_3i_4} = \tag{3.93}$$



$$\sum_{i_1<i_2<i_3<i_4} \varepsilon_{i_1i_3}\varepsilon_{i_2i_4} = \tag{3.94}$$

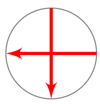


3.5.3 Relations between higher writhe numbers

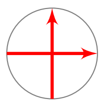
Higher colored writhe numbers are not independent. There are linear relations between them. These relations complicate the study of Vassiliev invariants because often it is difficult to reduce answers and understand their combinatorial context. For illustrative purpose we list here relations for orders 2, 3 and 4.

Order 2

There is the only relation:



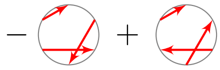
$=$

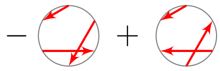


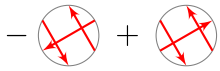
(3.95)

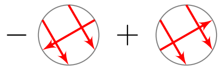
Order 3

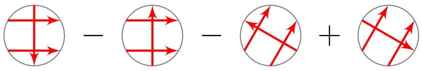
There are already 22 relations:

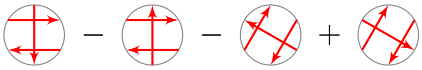
 $= 0$ (3.96)

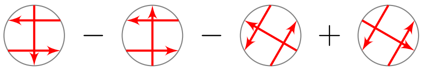
 $= 0$ (3.97)

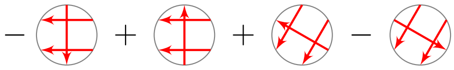
 $= 0$ (3.98)

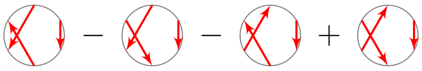
 $= 0$ (3.99)

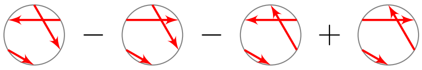
 $= 0$ (3.100)

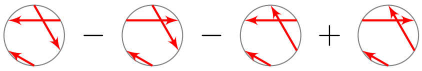
 $= 0$ (3.101)

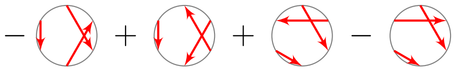
 $= 0$ (3.102)

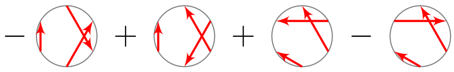
 $= 0$ (3.103)

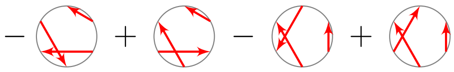
 $= 0$ (3.104)

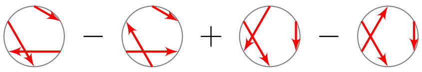
 $= 0$ (3.105)

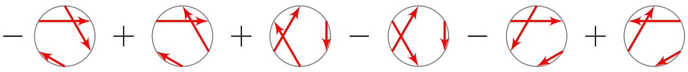
 $= 0$ (3.106)

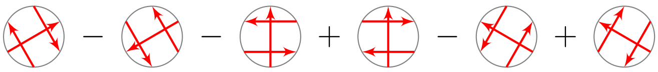
 $= 0$ (3.107)

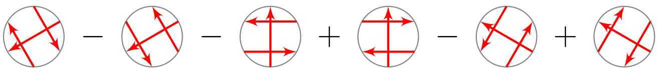
 $= 0$ (3.108)

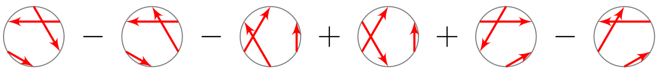
 $= 0$ (3.109)

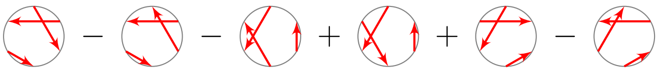
 $= 0$ (3.110)

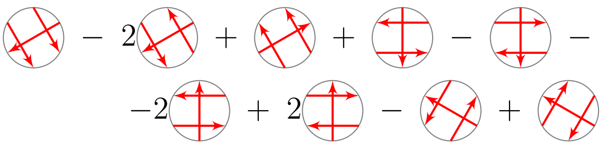
 $= 0$ (3.111)

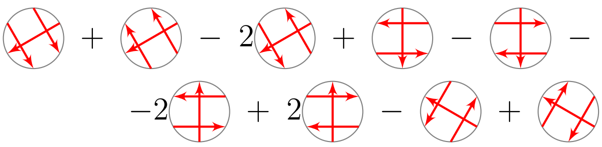
 $= 0$ (3.112)

 $= 0$ (3.113)

 $= 0$ (3.114)

 $= 0$ (3.115)

 $= 0$ (3.116)

 $= 0$ (3.117)

Let us note that all of the relations do not mix diagrams with different topology.

Order 4

There are 195 relations and we do not list all of them. We just note that in order 4 relations do not mix connected and disconnected diagrams and diagrams with different topology.

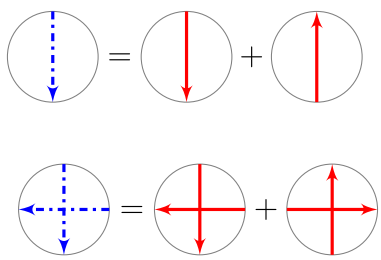
3.5.4 Vassiliev invariants via higher writhe numbers

With the help of higher colored writhe numbers one can give very easy and elegant combinatorial formulas for Vassiliev invariants.

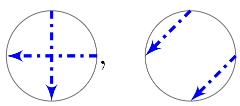
Numerical experimental data

From our numerical results (mentioned in section 3.4.2) we find some useful properties of Vassiliev invariants. All of the results in this subsection are purely experimental ones.

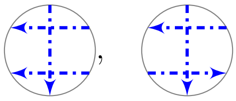
- Vassiliev invariants are can always be represented by sums of colored writhes with symmetric chord diagrams with respect to change of all over-crossings to under-crossings and under-crossings to over-crossings. It follows from the fact that Vassiliev invariants do not change under reflection. We call such combinations *reflective*. Therefore it is convenient to introduce reflective combinations of writhes as follows:



- Vassiliev invariants can always be represented by sums of writhes with connected chord diagrams. For example, the first diagram in the figure below is connected while the second is not:



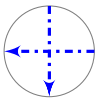
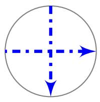
- Vassiliev invariants are always equal to the sum of writhes whose chord diagrams are *irreducible*, which means that we can not stick together any two chords in the diagram. Here we implicitly assume that to the left of the top point of the diagram we have the basepoint. Chords separated by the basepoint cannot be stuck together. For example, the first diagram in the figure below is reducible and the second one is irreducible:

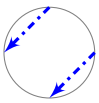
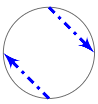
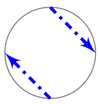
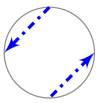
  

  This property actually means that Vassiliev invariants of order k do not mix with invariants of order less than k.

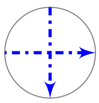
Order 2

One confusing fact related to writhes is their dependence on the choice of the base point on the knot projection. To construct invariants we should use only invariant combination of writhes that do not depend on such a choice. Invariant combination of writhes with two chords are the following ones:

$C_1 =$  $C_2 =$ 

$C_3 =$  $+$  $C_4 =$  $+$ 

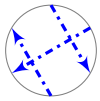
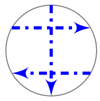
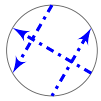
The second Vassiliev invariant should be some linear combination of these four expressions. We find:

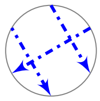
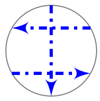
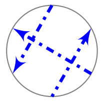
$\mathcal{V}_{2,1} =$  (3.118)

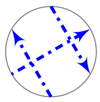
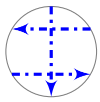
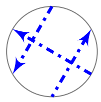
Note that due to the presence of the basepoint this diagram is irreducible.

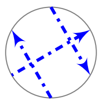
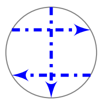
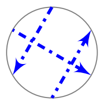
Order 3

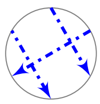
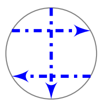
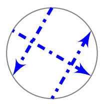
Invariant combinations of irreducible and connected writhes are the following ones:

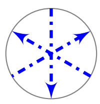
$C_1 =$  $+$  $+$ 

$C_2 =$  $+$  $+$ 

$C_3 =$  $+$  $+$ 

$C_4 =$  $+$  $+$ 

$C_5 =$  $+$  $+$ 

$C_6 =$ 

We found the following interesting property of the first five combinations:

$$C_1 = C_2 = C_3 = C_4 = C_5 \tag{3.119}$$

For the third Vassiliev invariant we get:

$$\mathcal{V}_{3,1} = \frac{1}{2} C_1 + C_6 \tag{3.120}$$

Chapter 4

Structures of superpolynomials

4.1 General idea

It was discovered that different knot homology theories related to polynomial invariants. The basic idea is to define a doubly graded homology theory $Hom_{i,j}(\mathcal{K})$ and construct the polynomial invariant of a knot $\mathcal{K}$ as its graded Euler characteristic with respect to one of the gradings. Such theory is called a categorification of the knot polynomial invariant.

In a such approach the Jones polynomial J is the graded Euler characteristic of the doubly graded Khovanov homology $Hom^{Kh}_{i,j}(\mathcal{K})$;

$$J(q) = \sum_{i,j} (-1)^j q^i \dim Hom^{Kh}_{i,j}(\mathcal{K}), \tag{4.1}$$

where the grading i is called the Jones grading, and j is called the homological grading. Originally $H^{Kh}_{i,j}$ were constructed combinatorially in terms of skein theory [32].

There is a generalization of Khovanov's theory [33] to categorified $sl(N)$ polynomial invariant $\mathcal{H}_N(q)$, which is given by graded Euler characteristic of their homology $\mathrm{HKR}^N_{i,j}(\mathcal{K})$:

$$\mathcal{H}_N(q) = \sum_{i,j} (-1)^j q^i \dim \mathrm{HKR}^N_{i,j}(\mathcal{K}). \tag{4.2}$$

This theory is known as Khovanov-Rozansky homology. For $N = 2$ this theory is equivalent to the Khovanov homology.

Also there is a categorification of the Alexander polynomial. It is the knot Floer homology $\mathrm{HFK}_j(\mathcal{K}; i)$, introduced in [34, 35]:

$$\Delta(q) = \sum_{i,j} (-1)^j q^i \dim \mathrm{HFK}_j(\mathcal{K}; i). \tag{4.3}$$

All polynomials above are related, because they can be derived from a single polynomial invariant known as the HOMFLY polynomial. Although all knot homology theories above categorify polynomial invariants of knots in the same class, their constructions are very different. However in [24] there was formulated the goal to unify all doubly graded homology theories (the Khovanov-Rozansky $sl(N)$ homology, knot Floer homology and their various deformations) into a single theory.

Unfortunately, such unified theory is not constructed yet. Instead, there is explicit the description of its properties, which is very detailed, powerful and instructive and give many non-trivial predictions, which can be verified.

4.2 DAHA-superpolynomials

As we discussed above at the present moment there is no definition of superpolynomials for arbitrary knots and representations. However there are well-defined polynomials for torus knots, which regards as superpolynomials. There are some different approaches to construct them: via homology and differentials [24], via q,t-Catalan numbers [28], via refined Chern-Simons theory [78], via generalized Rosso-Jones formula [77] and via double affine Hecke algebra (DAHA) of type A_N [26, 27]. All of them give same answers. However for definitions we use approach via DAHA following [27]. Actually therein superpolynomials are defined by the whole non-trivial procedure and in the rest of this section we briefly introduce it.

First, we have to recall the definition of DAHA and its polynomial representation. Second, with the help of $SL_2(\mathbb{Z})$ action and evaluation map we define superpolynomials.

4.2.1 DAHA

Let us consider formal parameters q and t, and define field of fractions

$$\mathbb{K} = \mathbb{C}(q,t) \tag{4.4}$$

of the ring of constants $\mathbb{C}[q^{\pm 1}, t^{\pm 1}]$. Also we consider the algebra of polynomials

$$V_N = \mathbb{K}[x_1, \dots, x_N]. \tag{4.5}$$

Denoting $n_i(R)$ the multiplicity of the part i, let us define the following scalar product for power sums $p_k = \sum_i x_i^k$

$$\langle p_R | p_{R'} \rangle = \delta_{RR'} \prod_k n_k! k^{n_k} \prod_i \frac{1-q^{R_i}}{1-t^{R_i}}, \tag{4.6}$$

which can be explicitly realized by

$$\langle f(p_k) | g(p_k) \rangle = f\left(k \frac{1-q^k}{1-t^k} \frac{\partial}{\partial p_k} \right) g(p_k) \bigg|_{p_k=0} \tag{4.7}$$

Introduce the monomial symmetric functions $m_R = \mathrm{Sym}(x_1^{R_1}, x_2^{R_2}, \dots)$. Then the Macdonald polynomials are the polynomials given by the following expansion:

$$M_R = m_R + \sum_{R'<R} c_{RR'} m_{R'} \tag{4.8}$$

with coefficients c_{RR} defined by the orthogonality condition

$$\langle M_R | M_{R'} \rangle = 0 \qquad \text{if } R \neq R' \tag{4.9}$$

Now we define DAHA of type A_N, introduced in [84], following [27].

Definition 4.2.1. *The algebra $\mathbf{H}_N$ is defined over $\mathbb{K}$ by generators $T_i^{\pm 1}$ for $i \in \{1, ..., N-1\}$, and ${X_j}^{\pm 1}, {Y_j}^{\pm 1}$ for $j \in \{1, ..., N\}$, under the following relations:*

$$\begin{array}{rcl} (T_i + t)(T_i - t^{-1}) & = & 0, \\ T_i T_{i+1} T_i & = & T_{i+1} T_i T_{i+1}, \\ [T_i, T_k] & = & 0 \textit{ for } |i-k| > 1, \\ T_i X_i T_i & = & X_{i+1}, \\ {T_i}^{-1} Y_i {T_i}^{-1} & = & Y_{i+1}, \\ [T_i, X_k] & = & 0, \\ [T_i, Y_k] & = & 0 \textit{ for } |i-k| > 1, \\ [X_j, X_k] & = & 0, \\ [Y_j, Y_k] & = & 0, \\ Y_1 X_1 \ldots X_N & = & q X_1 \ldots X_N Y_1, \\ {X_1}^{-1} Y_2 & = & Y_2 {X_1}^{-1} {T_1}^{-2}. \end{array} \tag{4.10}$$

4.2.2 Polynomial representation

Now we consider the polynomial representation for $\mathbf{H}_N$:

$$\mathbf{H}_N \longrightarrow \mathrm{End}(V_N). \tag{4.11}$$

Then T_i are given by the Demazure-Lusztig operators:

$$T_i = t^{-1} s_i + (t^{-1} - t) \frac{s_i - 1}{x_i / x_{i+1} - 1} \tag{4.12}$$

where $s_i = (i, i+1)$ are the simple reflections, and elements X_i act as multiplication by x_i.

Let us introduce the operators ∂_i and γ on V_N as follows:

$$\partial_i(f) = f(x_1, \ldots, x_{i-1}, q^2 x_i, x_{i+1}, \ldots x_N) \tag{4.13}$$

$$\gamma = s_{N-1} \cdots s_1 \partial_1. \tag{4.14}$$

Then setting

$$Y_i = T_i \cdots T_{N-1} \gamma T_1^{-1} \cdots T_i^{-1}, \tag{4.15}$$

we have that the operators T_i, X_j, Y_j on V_N satisfy the relations of the DAHA.

4.2.3 DAHA-superpolynomials

Let $w = s_{i_1} \cdots s_{i_l}$ is a reduced decomposition of $w \in S_N$ and $T_w = T_{i_1} \cdots T_{i_l}$. If we define

$$e = \frac{1}{[N]_t^-!} \sum_{w \in S_N} t^{-l(w)} \cdot T_w, \tag{4.16}$$

where $[N]_t^-! = [1]_t^- \cdots [N]_t^-$, $[N]_t^- = \dfrac{1-t^{-2N}}{1-t^{-2}}$, then we get $e^2 = e$. Thus, we constructed the complete idempotent $e \in \mathbf{H}_N$ ([86]). With its help we define spherical DAHA $\mathbf{SH}_N$ as the subalgebra of $\mathbf{H}_N$:

$$\mathbf{SH}_N = e \cdot \mathbf{H}_N \cdot e. \tag{4.17}$$

Let

$$\tau_+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \tau_- = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \tag{4.18}$$

denote the generators of $SL_2(\mathbb{Z})$. These operators acts on DAHA generators as follows[1] (see [85, 86]):

$$\begin{array}{rcl} \tau_+(X_i) &=& X_i, \\ \tau_+(T_i) &=& T_i, \\ \tau_+(Y_i) &=& Y_i X_i (T_{i-1}^{-1} \cdots T_i^{-1})(T_i^{-1} \cdots T_{i-1}^{-1}) \\ \tau_-(Y_i) &=& Y_i, \\ \tau_-(T_i) &=& T_i, \\ \tau_-(X_i) &=& X_i Y_i (T_{i-1} \cdots T_i)(T_i \cdots T_{i-1}) \end{array} \tag{4.19}$$

They extend to automorphisms of $\mathbf{H}_N$ and preserve $\mathbf{SH}_N$.

For any pair of coprime integers (n, m), let us choose a matrix:

$$\gamma^{n,m} = \begin{pmatrix} x & n \\ y & m \end{pmatrix} \in SL_2(\mathbb{Z}) \tag{4.20}$$

such that $\gamma^{n,m}(0,1) = (n, m)$.

Let us introduce the elements

$$M_{R,N}^{n,m} := \gamma^{n,m}\Big(e \cdot M_R(Y_1, ..., Y_N) \cdot e\Big). \tag{4.21}$$

Then we have the following proposition.

Proposition 4.2.2. *(see [85, 86]) The elements $M_{R,N}^{n,m}$ do not depend on the choice of x and y in $\gamma_{n,m}$ (4.20).*

[1]Under notation $T_i \cdots T_{i-1}$ we imply cyclic order $T_i \cdots T_{N-1} \cdot T_1 \cdots T_{i-1}$.

The last ingridient we needed is the evaluation map $\varepsilon_N : V_N \longrightarrow \mathbb{K}$, defined as follows:

$$\varepsilon_N(f) \;=\; f(t^{N-1}, t^{N-2}, \dots, 1). \tag{4.22}$$

Now, following [26], we define the *DAHA-superpolynomials*:

$$\mathcal{P}^{n,m}_{R,N}(q,t) = \varepsilon_N(M^{n,m}_{R,N} \cdot 1) \in \mathbb{K}. \tag{4.23}$$

Here we apply the element $M^{n,m}_{R,N}$ to $1 \in V_N$ in the polynomial representation.

The following two propositions were conjectured in [26] and proved in [27].

Proposition 4.2.3. (Stabilization). *There exists a polynomial $\mathcal{P}^{T[n,m]}_R(A,q,t)$, which we refer to as non-normalized superpolynomial, such that:*

$$\mathcal{P}^{n,m}_{R,N}(q,t) = \mathcal{P}^{T[n,m]}_R(A = t^N, q, t) \tag{4.24}$$

Definition 4.2.4. *Normalized DAHA-superpolynomials for torus knots $T[n,m]$ are defined as follows:*

$$P^{T[n,m]}_R(A,q,t) := \frac{\mathcal{P}^{T[n,m]}_R(A,q,t)}{\mathcal{P}^{T[1,0]}_R(A,q,t)} \tag{4.25}$$

Then the following proposition takes place.

Proposition 4.2.5. (Duality). *The normalized superpolynomials P for transposed diagrams are related by the equation*

$$P^{T[n,m]}_{R^t}(A,q,t) = (-q)^{(1-n)|R|} P^{T[n,m]}_R(A, t^{-1}, q^{-1}). \tag{4.26}$$

4.3 Superpolynomials and their symmetric properties

In section (2.3) we considered symmetric properties of HOMFLY polynomials and proved that they belong to the algebra of shifted symmetric functions Ł. In this section we address same question to the superpolynomials.

4.3.1 Symmetric properties

Recall that in order to prove that the HOMFLY polynomials are shifted symmetric functions we used three facts among other things (see section 2.3.1):

1. the HOMFLY polynomials are averages of the characters $GL(N)$ (Schur polynomials):

$$\mathcal{H}^{\mathcal{K}}_R = \langle\, \chi_R\Big(p(U)\Big)\, \rangle; \tag{4.27}$$

2. expansion of Schur polynomials in terms of symmetric group characters

$$\langle\, \chi_R\Big(p(U)\Big)\,\rangle = \langle \sum_{|\Delta|=|R|} d_R\, \varphi_R(\Delta)\, p_\Delta(U)\,\rangle; \tag{4.28}$$

3. and the averaging is over U

$$\langle \sum_{|\Delta|=|R|} d_R\, \varphi_R(\Delta)\, p_\Delta\,\rangle = \sum_{|\Delta|=|R|} d_R\, \varphi_R(\Delta) \langle\, p_\Delta\,\rangle. \tag{4.29}$$

Thus, from the fact that Schur polynomial is antisymmetric function in variables $r_i = R_i - i$ it follows that $\mathcal{H}_R^{\mathcal{K}}$ is antisymmetric too. Therefore, normalized HOMFLY polynomial, which is the ratio of $\mathcal{H}_R^{\mathcal{K}}$ and $\chi_R(p^*)$, is the symmetric function in r_i.

In the case of superpolynomials we deal with the Macdonald polynomials instead of the Schur polynomials as we discussed in the previous section. Actually, the matrix $\gamma^{n,m}$ in (4.21) acts only on power sums:

$$p_k^N := Y_1^k + ... + Y_N^k. \tag{4.30}$$

Since the Macdonald polynomial $M_{R,N}$ is polynomial in the power sums p_k^N, then the $M_{R,N}^{n,m}$ are polynomials in $\gamma^{n,m}(p_k^N)$.

Futhermore, to compute $\varepsilon_N(f)$ for any symmetric function f one can expand it in terms of the p_k and apply the next formula from [42]:

$$\varepsilon_N(p_k) = \frac{1-t^{kN}}{1-t^k} = \frac{1-A^k}{1-t^k}, \tag{4.31}$$

where we capture the large N-dependence in the new variable $A = t^N$.

According to (4.21) and (4.23) the non-normalized superpolynomial $\mathcal{P}_R^{T[n,m]}(A,q,t)$ is defined as

$$\mathcal{P}_R^{T[n,m]}(A,q,t) = \varepsilon_N\left(\gamma^{n,m}\left(e \cdot M_R\left(Y_1,...,Y_N\right)\cdot e\right)\cdot 1\right), \tag{4.32}$$

and we can rewrite it as

$$\mathcal{P}_R^{T[n,m]}(A,q,t) = \langle\, M_R \cdot 1\,\rangle_{n,m} \tag{4.33}$$

Then introducing $M_R = \sum\limits_\Delta K_R(\Delta)\, p_\Delta,\ K_R(\Delta) \in \mathbb{K}$ we get

$$\mathcal{P}_R^{T[n,m]}(A,q,t) = \sum_\Delta K_R(\Delta) \langle\, p_\Delta \cdot 1\,\rangle_{n,m}. \tag{4.34}$$

Thus, symmetric properties of superpolynomials are determined by symmetric properties of Macdonald polynomials, namely by $K_R(\Delta)$.

Let us note that $\mathcal{P}_R^{T[1,0]}(A,q,t)$ is equal to the Macdonald polynomial at the special points $p_k^* = \frac{A^k - A^{-k}}{t^k - t^{-k}}$, which explicitly given as follows [78]:

$$M_R^* = \prod_{k=0}^{\beta-1} \prod_{1\le j<i\le N} \frac{q^{R_j - R_i} t^{i-j} q^k - q^{R_i - R_j} t^{j-i} q^{-k}}{t^{i-j} q^k - t^{j-i} q^{-k}} \tag{4.35}$$

If we substitute $t = q^\beta$, where β is a new independent variable, then from this formula it obviously follows that they are antisymmetric in $\{\mu_i = R_i - \beta i\}$ rather than $\{r_i = R_i - i\}$. However at the present moment we do not manage to prove that the Macdonald polynomial at the general points $\bar{p}_k$ are antisymmetric functions in μ_i. If we had formulas for Macdonald polynomials similar to determinant formulas for Schur polynomials, which explicitly demonstrate antisymmetric property in r_i, then we could determine Macdonald's symmetry properties. However we do not have such formulas, the closest analogy is given in [87, 88]. Anyway, we hope that points $p_k^* = \frac{A^k - A^{-k}}{t^k - t^{-k}}$ are general enough and that Macdonald polynomials in general points $\bar{p}_k$ are antisymmetric functions in μ_i. Then it would imply a normalized superpolynomial is symmetric function in μ_i that in turn would be meaningful argument that normalized superpolynomial belongs to the algebra of β-shifted symmetric functions, i.e. algebra of polynomials symmetric in variables $\mu_i = R_i - \beta i$. Let us give the second evidence to support this conjecture.

The Macdonald polynomials are uniquely defined as the common system of eigenfunctions of the commuting set of operators generalizing the Casimir operators, which are nothing but the Ruijsenaars Hamiltonians [57]:

$$\hat{H}_k = \sum_{i_1 < \ldots < i_k} \frac{1}{\Delta(x)} \hat{P}_{i_1} \ldots \hat{P}_{i_k} \Delta(x) \, \hat{Q}_{i_1} \ldots \hat{Q}_{i_k}, \qquad [\hat{H}_k, \hat{H}_m] = 0 \tag{4.36}$$

where the Vandermonde determinant $\Delta(x) = \det_{ij} x_i^{N-j} = \prod_{i<j}^{N} (x_i - x_j)$ and the shift operators are defined as:

$$\hat{P}_k = q^{\beta x_k \partial_{x_k}}, \quad \hat{Q}_k = q^{(1-\beta) x_k \partial_{x_k}} \tag{4.37}$$

The spectrum of (4.36) can be defined from the eigenvalues of spectral operator:

$$\left(\sum_{k=0}^{n} z^k \hat{H}_k \right) M_R(x_1, \ldots, x_n) = \prod_{i=1}^{\infty} (1 + z \, q^{R_i + \beta(n-i)}) M_R(x_1, \ldots, x_n) \tag{4.38}$$

Thus, the eigenvalues are symmetric in $\mu_i = R_i - \beta i$. This appeals to construct a full set of symmetric polynomials in $\{\mu_i = R_i - \beta i\}$. Unfortunately, in this case there is no a counterpart of the Schur-Weyl duality known, and the Macdonald polynomials are not simple characters. Hence, there is no a distinguished set of symmetric functions of $\mu_i = R_i - \beta i$, and we construct the full sets of symmetric polynomials in $\{\mu_i\}$ in two ways in order to present the superpolynomial as a deformed Hurwitz exponential.

Thus, based on the above arguments we formulate the following conjecture.

Conjecture 4.3.1. *Normalized DAHA-superpolynomials belong to the algebra of functions symmetric in variables $\mu_i = R_i - \beta i$.*

This conjecture is also supported by explicit calculations, when we consider genus expansion and loop expansion below. However, first, we give two multiplicative bases for that algebra.

4.3.2 Beta-deformation of Casimir operators

As the first basis one can consider corresponding *power sums*, which can be treated as a naive β-deformation of the Casimirs eigenvalues:

$$C_R^\beta(k) = \sum_i \left(R_i - \beta i + \frac{1}{2}\right)^k - \left(-\beta i + \frac{1}{2}\right)^k. \tag{4.39}$$

It is clear that for $\beta = 1$ they are equal to (2.17). They are also clearly symmetric in $R_i - \beta i$. Hence, one can use them to construct a full set of symmetric polynomials in analogy with formula (2.21):

$$\hat{C}^\beta(\Delta) \quad = \quad \prod_{j=1}^{l(\Delta)} \hat{C}^\beta(\delta_i). \tag{4.40}$$

4.3.3 Operators $\hat{T}_k$

Since we are interested in a basis symmetric in $\mu_i = R_i - \beta i$, it is allowed not to depend on q at all. Indeed, there are "intermediate" symmetric functions, which generalize the Schur functions but still solve the (trigonometric) Calogero-Moser system. These are the *Jack polynomials* J_R [42]. They are obtained from the Macdonald polynomials just in the limit of $q \to 1$. Since the Jack polynomials are defined to be eigenvalues of the trigonometric Calogero-Moser Hamiltonians $\hat{H}_i$, one can consider their generating function:

$$\left(u^{|R|} \sum_{i=0}^{\infty} u^{-i}\, \hat{H}_i\right) J_R = \mathcal{T}(u) J_R \tag{4.41}$$

the generating function for the eigenvalues being

$$\mathcal{T}(u) = \prod_{(i,j)\in\lambda} \Big(u - (i-1)\beta + (j-1)\Big) \tag{4.42}$$

Then, it is natural to define *the operators* $\hat{T}_k^\beta$ such that

$$\begin{aligned} \hat{T}_k^\beta J_R &= T_k^\beta(R) J_R, \\ T_k^\beta(R) &= \sum_{i,j} \Big((j-1) - \beta(i-1)\Big)^{k-1}, \end{aligned} \tag{4.43}$$

These operators play an important role in the theory of Jack polynomials, the latter being somewhat mysteriously connected [58] with the AGT relations [59]. It is possible to express $C_R^\beta(k)$ through linear combinations of $T_k^\beta(R)$.

The full set of functions is then given in complete analogy with (2.21):

$$T_\Delta^\beta(R) := \prod_{i=1}^{l(\Delta)} T_{\Delta_i}^\beta(R). \tag{4.44}$$

For explicit calculations we used exactly this basis. Thus, one can write the most general form of the β-deformation of the Hurwitz exponential for the superpolynomial.

Conjecture 4.3.2. *The dependence on the irreducible representation R of the DAHA-superpolynomials is given as follows:*

$$P_R^{\mathcal{K}} = \exp\Big\{\sum_{\Delta} \omega_\Delta^{\mathcal{K}} \cdot T_\Delta^\beta(R)\Big\} \tag{4.45}$$

Particular values of constants ω_Δ can be dealt with by particular perturbative expansions, similarly to the HOMFLY case. We again consider the two particular examples: the genus expansion (4.46) and the loop expansion (4.50,4.61).

4.3.4 Large N expansion for superpolynomials

The large N expansion of the superpolynomial is given by $\hbar \to 0$, $N \to \infty$, $\hbar N =$ const (i.e. $A = e^{N\hbar/2}$ is an arbitrary variable), β is arbitrary and it is a straightforward generalization of the Hurwitz exponential at $\beta = 1$, that is, $\varphi_R(\Delta) \to T_R^\beta(\Delta)$ and $S(\hbar^2, A) \to \mathcal{S}(\hbar^2, \beta, A)$.

Conjecture 4.3.3. *Large N expansion for DAHA-superpolynomials is equal to the following*

$$P_R^{\mathcal{K}}(q,t,A) \;=\; \exp\Big\{\sum_{\Delta} \hbar^{|\Delta|+l(\Delta)-2} \cdot \mathcal{S}_\Delta^{\mathcal{K}}(\hbar^2,\beta,A) \cdot T_\Delta^\beta(R)\Big\}, \tag{4.46}$$

where

$$\mathcal{S}_\Delta^{\mathcal{K}}(\hbar^2,\beta,A) = \sum_{n=0}^{\infty} \hbar^{2n} s_\Delta^{\mathcal{K}}(n) \tag{4.47}$$

This conjecture is supported by the following three reasons. First, making computational experiments with particular superpolynomials (see Appendix 5.3), we see that their expansions are very similar to the expansions of HOMFLY polynomials (2.44). By this reason we suggest that the definition of superpolynomial expansion has same structure. Second, we think that the dependence from representation R should be described by some nice understandable functions. Actually, they should be β-deformed counterparts of symmetric group characters, but at the present moment we do not know such deformation. In the previous subsections, based on symmetric properties of Macdonald polynomials, we gave our arguments why we consider exactly $T_\Delta^\beta(R)$ as a full basis. Third, we think that in the order n of the expansion contributes only diagrams satisfing condition

$$|\Delta| + l(\Delta) \le n+2.$$

At zero and first orders our definition is agreed with the expansion from [29]:

$$P_R(A,q,t)_{q=e^{\hbar/2},\ t=e^{\beta\hbar/2}} = P_\square^{|R|} + \hbar\left(\nu_{R^t} - \beta\nu_R\right)\sigma_\square^{|R|-2}\sigma_2 + \ldots, \tag{4.48}$$

where $\sigma_\square$ is a special polynomial, σ_2 is a higher special polynomial of the first order (see [6] for details), $\nu_R = \sum_i R_i(i-1)$.

4.3.5 Loop expansion for superpolynomials

Considering particular examples of knots we have discovered that there are two essentially different cases: *thin* knots and *thick* knots [24].

Definition 4.3.4. *A knot $\mathcal{K}$ is thin if all terms of the superpolynomial have the same degree. Otherwise, the knot is thick.*

Remark. In this definition it is implied that a superpolynomial is considered in variables $(\mathbf{a}, \mathbf{q}, \mathbf{t})$, which are related with ours as follows:

$$\mathbf{q} = t, \qquad \mathbf{t} = -q/t, \qquad A \ = \ \mathbf{a}\sqrt{-\mathbf{t}}.$$

For thin knots the $\mathbf{t}$-grading is determined by the $\mathbf{a}$- and $\mathbf{q}$-gradings. Thus, the superpolynomial is determined by the HOMFLY polynomial completely. For thick knots the situation is different. At the present moment it is unclear how to determine their terms in a general way. First example of thick knot is the torus knot $8_{19} = T[3,4]$, second example is non-torus 9_{42}.

Thin knots

The loop expansion for the superpolynomial $P_R^{\mathcal{K}}(A|q|t)$ is provided with $\hbar \to 0$, N and β are fixed in the variables:

$$q = e^{\hbar/2}, \quad A = e^{N\hbar/2}, \quad t = e^{\beta\hbar/2}. \tag{4.49}$$

Since when $t = q$ the superpolynomial reduces to the HOMFLY polynomial, we have (3.41) for $\beta = 1$.

Based on computational experiments with particular superpolynomials, we guess the conjecture about general form of loop expansion.

Conjecture 4.3.5. *In the case of thin knots the loop expansion of superpolynomials has the form*

$$P_R^{\mathcal{K}}(A, q, t) = \sum_{i=0}^{\infty} \hbar^i \sum_{j=1}^{\mathcal{N}_i^{\beta}} D_{i,j}^{(R)} \mathcal{V}_{i,j}^{\mathcal{K}}, \tag{4.50}$$

where $D_{i,j}^{(R)}$ are beta-deformations of trivalent diagrams, $\mathcal{V}_{i,j}^{\mathcal{K}}$ **are the same Vassiliev invariants** *as in (3.41).*

Let us emphasize that $\mathcal{V}_{i,j}^{\mathcal{K}}$ are exactly same Vassiliev invariants as in HOMFLY case. Therefore, the superpolynomials of the thin knot do not contain any new information about the knot as compared with the HOMFLY case. However, the structure of group factors is different: $\mathcal{G}_{i,j} \to D_{i,j}$. We have calculated few examples explicitly.

Remark. Let us emphasize that we also checked this conjecture for non-torus superpolynomials, which we found in [24].

Group factors in HOMFLY case (trivalent diagrams, Figure 3.13)

$$\begin{aligned}
\mathcal{G}_{2,1}^{(R)} &= \frac{1}{4}\left(-|R|\cdot N^2 - 2\varphi_R([2])\cdot N + |R|^2\right) &(4.51)\\
\mathcal{G}_{3,1}^{(R)} &= \frac{1}{8}N\left(|R|\cdot N^2 + 2\varphi_R([2])\cdot N - |R|^2\right) &(4.52)\\
\mathcal{G}_{4,1}^{(R)} &= \left(\mathcal{G}_{2,1}^{(R)}\right)^2 &(4.53)\\
\mathcal{G}_{4,2}^{(R)} &= \frac{1}{16}N^2\left(-|R|\cdot N^2 - 2\varphi_R([2])\cdot N + |R|^2\right) &(4.54)\\
\mathcal{G}_{4,3}^{(R)} &= \frac{1}{16}\Big(|R|\cdot N^4 + 6\varphi_R([2])\cdot N^3 + \\
&+ 16\left(\frac{3}{4}\varphi_R([3]) + \frac{7}{8}\varphi_R([1,1]) + \frac{1}{16}\varphi_R([1])\right)\cdot N^2 - \\
&- 16\left(\frac{1}{2}\varphi_R([4]) + \varphi_R([2,1]) + \frac{1}{2}\varphi_R([2])\right)\cdot N - &(4.55)\\
&- \Big(2\varphi_R([1]) + 28\varphi_R([1,1]) + 72\varphi_R([1,1,1]) + 24\varphi_R([3,1]) - 48\varphi_R([2,2])\Big)\Big)
\end{aligned}$$

Beta-deformed group factors are[2]

$$\begin{aligned}
D_{2,1}^{(R)} &= \frac{1}{4}\left(-|R|\cdot N^2 - 2\left(T_2^\beta - \frac{1}{2}(\beta-1)T_1\right)\cdot N + \beta|R|^2\right) &(4.56)\\
D_{3,1}^{(R)} &= \frac{1}{4}\left(-2N + 1 - \beta\right)D_{2,1}^{(R)} &(4.57)\\
D_{4,1}^{(R)} &= \left(D_{2,1}^{(R)}\right)^2 &(4.58)\\
D_{4,2}^{(R)} &= \left(\frac{1}{4}\left(-2N + 1 - \beta\right)\right)^2 D_{2,1}^{(R)} &(4.59)\\
D_{4,3}^{(R)} &= \frac{1}{16}\Big(|R|\cdot N^4 + 6\left(T_2^\beta - \frac{1}{2}(\beta-1)T_1\right)\cdot N^3 + \\
&+ 16\left(\frac{3}{4}T_3^\beta - \frac{5}{8}(\beta-1)T_2^\beta + \frac{7}{64}(\beta-1)^2T_1 + \frac{1}{16}\beta T_1^2\right)\cdot N^2 - \\
&- 16\Big(\frac{1}{2}T_4^\beta - \frac{15}{16}(\beta-1)T_3^\beta + \frac{7}{32}(\beta-1)^2T_2^\beta + \frac{1}{16}(\beta-1)\left(T_2^\beta\right)^2 - \\
&- \frac{1}{64}(\beta-1)(3\beta^2 - 2\beta + 3)T_1\Big)\cdot N + f\Big) &(4.60)
\end{aligned}$$

Thick knots

For the thick knots, the loop expansion in $\hbar$ is different than for thin knots. Also based on computational experiments with particular superpolynomials for thick knots, we guess the conjecture about general form of loop expansion.

Conjecture 4.3.6. *In the case of thick knots the loop expansion of superpolyno-*

[2]The last term f of $D_{4,3}^{(R)}$ is still unknown.

mials has the form

$$P_R^{\mathcal{K}}(A,q,t) = \sum_{i=0}^{\infty} \hbar^i \sum_{j=1}^{\mathcal{N}_i^\beta} D_{i,j}^{(R)} \mathcal{V}_{i,j}^{\mathcal{K}} \quad + \quad (\beta - 1) \cdot \sum_{i=0}^{\infty} \hbar^i \sum_{j=1}^{\mathcal{M}_i^\beta} \Xi_{i,j}^{(R)} \rho_{i,j}^{\mathcal{K}}, \tag{4.61}$$

where the first sum is the same as for the thin knots, while the second sum is different: $\Xi_{i,j}^{(R)}$ *are new group structure factors and* $\rho_{i,j}^{\mathcal{K}}$ *are some numbers different from the Vassiliev invariants of the first sum.*

We think that the second sum is **crucially new**. One can ask if $\rho_{i,j}^{\mathcal{K}}$ could be also related with the Vassiliev invariants, maybe, they are some linear combinations of $\mathcal{V}_{i,j}^{\mathcal{K}}$. In order to answer this question, we recall a definition of invariants of the finite type (we follow the text-book [1]).

Definition 4.3.7. *Any knot invariant can be extended to knots with double points by means of the Vassiliev skein relation, depicted on the Figure 4.1.*

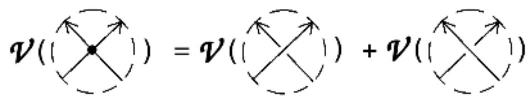


Figure 4.1: Vassiliev skein relation

Using the Vassiliev skein relation recursively, one can extend any knot invariant to knots with an arbitrary number of double points. There are many ways to do this, since one can choose to resolve double points in an arbitrary order. However, the result is independent of the choice.

Definition 4.3.8. *A knot invariant is said to be a Vassiliev invariant (or a finite type invariant) of order* $\leq n$ *if its extension vanishes on all singular knots with more than* n *double points. A Vassiliev invariant is said to be of order* n *if it is of order* $\leq n$ *but not of order* $\leq n-1$.

Proposition 4.3.9. $\rho_{i,j}$ *are not Vassiliev invariants of order* $\leq i$.

Applying the skein relation recursively to the simplest thick knot $T[3,4] = 8_{19}$ with 4 double points, we have explicitly checked that $\rho_{2,1}$ and $\rho_{3,1}$ are not the Vassiliev invariants of order 3 at least. It is possible that they are invariants of higher order, e.g. of 26 or 42. However, this would look quite unusual, since $\rho_{i,j}$ have natural graduation by powers of $\hbar$ as well as the Vassiliev invariants $v_{i,j}$. This question clearly deserves a further detailed analysis.

Examples of $\rho^{\mathcal{K}}_{\mathrm{i,j}}$ for thick torus knots

$\mathcal{K}$	$v_{3,1}$	$\rho_{3,1}$
$T[3,3k+1]$	$4k(3k+1)(3k+2)$	$-\dfrac{k^2(k+1)}{4}$
$T[3,3k+2]$	$4(k+1)(3k+1)(3k+2)$	$-\dfrac{k(k+1)^2}{4}$
$T[4,4k+1]$	$\dfrac{80k(2k+1)(4k+1)}{3}$	$-\dfrac{k^2(4k+3)}{2}$
$T[4,4k+3]$	$\dfrac{80(k+1)(2k+1)(4k+3)}{3}$	$-\dfrac{(k+1)^2(4k+1)}{2}$
$T[5,5k+1]$	$\dfrac{100k(5k+1)(5k+2)}{3}$	$-\dfrac{7k^2(5k+3)}{4}$
$T[5,5k+2]$	$\dfrac{20(5k+1)(5k+2)(5k+3)}{3}$	$-\dfrac{k(35k^2+42k+11)}{4}$
$T[5,5k+3]$	$\dfrac{20(5k+2)(5k+3)(5k+4)}{3}$	$-\dfrac{(k+1)(35k^2+28k+4)}{4}$
$T[5,5k+4]$	$\dfrac{100(k+1)(5k+3)(5k+4)}{3}$	$-\dfrac{7(k+1)^2(5k+2)}{4}$

We computed this table for chosen superpolynomials of torus knots for k from 1 to 5.

Chapter 5

Appendix

5.1 Appendix A. More pictures on Vassiliev invariants

Here we list several more pictures of the type of Figure 3.14, representing "relations" beween Vassiliev invariants. Very interesting is the form of the plot of $\mathcal{V}_{6,5}$ against $\mathcal{V}_{5,2}$, which is depicted in Figure 5.1**a**.

The following expression characterizes with good accuracy the long "horns" part of the figure by vanishing on most of the corresponding knots:

$$F_3 = -10752+\frac{48616\mathcal{V}_{2,1}}{15}-178\mathcal{V}_{2,1}^2+\mathcal{V}_{2,1}^3-1254\mathcal{V}_{3,1}+66\mathcal{V}_{2,1}\mathcal{V}_{3,1}+356\mathcal{V}_{4,2}-6\mathcal{V}_{2,1}\mathcal{V}_{4,2}$$
$$-356\mathcal{V}_{4,3}-3\mathcal{V}_{3,1}^2+6\mathcal{V}_{2,1}\mathcal{V}_{4,3}-66\mathcal{V}_{5,2}+66\mathcal{V}_{5,3}+66\mathcal{V}_{5,4}+6\mathcal{V}_{6,5}-6\mathcal{V}_{6,6}-6\mathcal{V}_{6,7}-6\mathcal{V}_{6,8}, \tag{5.1}$$

see Figure 5.1**b**, where only points on which the expression vanishes are left.

The plot of $\mathcal{V}_{5,2}$ against $\mathcal{V}_{4,2}$ has vaguely the same form, and moreover the same expression (5.1) vanishes on most of the points in the "horns" part, see Figures 5.2**a**, 5.2**b**.

Plots of $5\mathcal{V}_{4,2} - 31\mathcal{V}_{4,3}$ against $\mathcal{V}_{3,1}$ and $\mathcal{V}_{5,3}$ are also rather interesting, see Figures 5.3**a**, 5.3**b**.

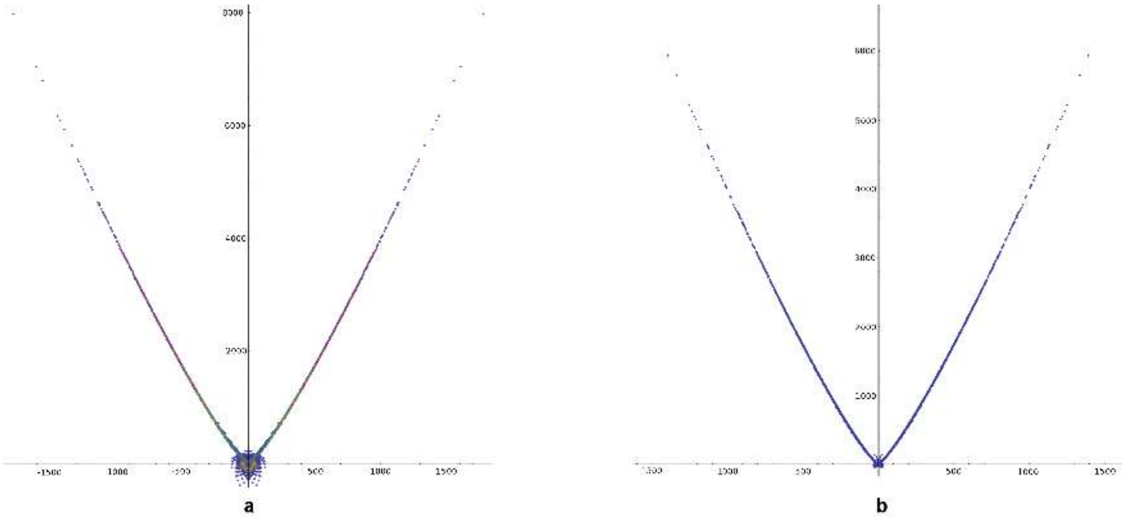


Figure 5.1: **a**: horns, i.e. $\mathcal{V}_{6,5}$ plotted against $\mathcal{V}_{5,2}$ for knots with up to 14 crossings; **b**: same with constraint (5.1)

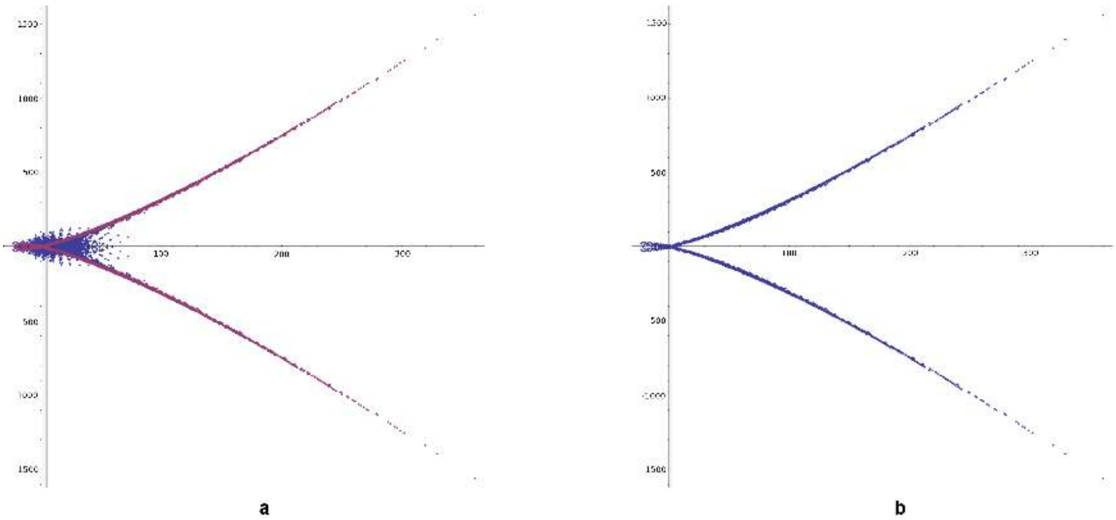


Figure 5.2: **a**: horns, i.e. $\mathcal{V}_{5,2}$ plotted against $\mathcal{V}_{4,2}$ for knots with up to 14 crossings; **b**: same with constraint (5.1)

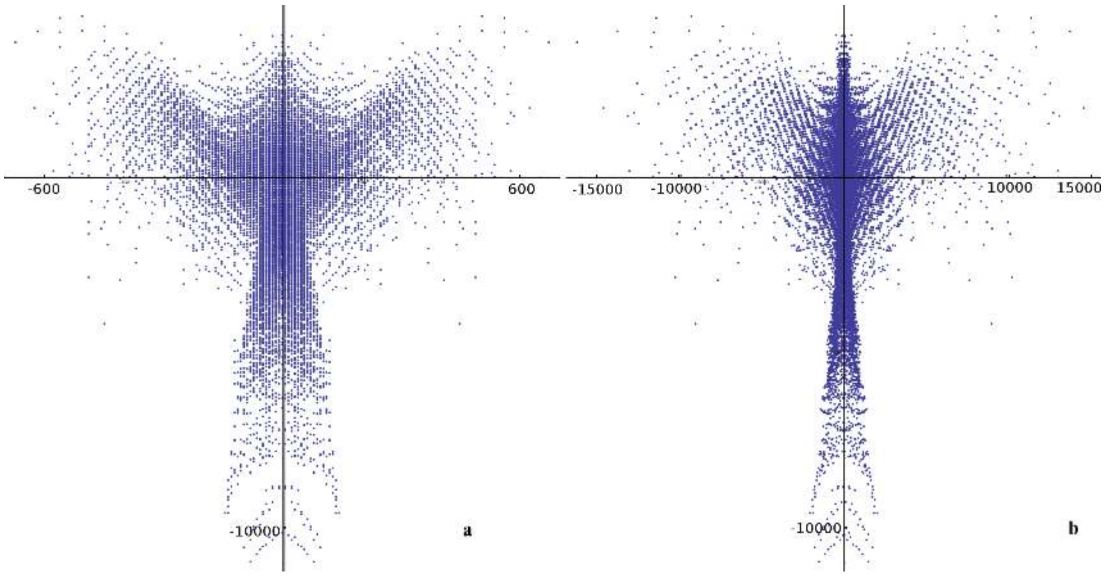


Figure 5.3: Birds, i.e. $5\mathcal{V}_{4,2} - 31\mathcal{V}_{4,3}$ plotted against $\mathcal{V}_{3,1}$ (Figure **a**) and $5\mathcal{V}_{4,2} - 31\mathcal{V}_{4,3}$ plotted against $\mathcal{V}_{5,3}$ (Figure **b**), for knots with up to 14 crossings

5.2 Appendix B. Special polynomials

In this section we list examples of higher special polynomials $\sigma_{\Delta}^{\mathcal{K}}(g)$ for some knots. Recall that they appear in large N expansion of HOMFLY polynomials in the following way (2.44):

$$\begin{aligned} H_R^{\mathcal{K}}(q,A) &= \exp\left(\sum_{\Delta} \hat{\hbar}^{|\Delta|+l(\Delta)-2}\, S_{\Delta}^{\mathcal{K}}\big(A^2,\hat{\hbar}^2\big)\, \varphi_R(\Delta)\right), \qquad (5.2) \\ S_{\Delta}^{\mathcal{K}}(A^2,\hbar^2) &= \sum_g \sigma_{\Delta}^{\mathcal{K}}(g)\, \hat{\hbar}^{2g}, \end{aligned}$$

where for our convenience we use $\hat{\hbar} = \frac{\hbar}{\sigma_{\square}^2}$ and $\sigma_{\square} = H_{\square}^{\mathcal{K}}(q{=}1, A)$.

5.2.1 Trefoil

$$
\begin{aligned}
\sigma_{[1]}(0) &= \log\frac{2-A^2}{A} \\
\sigma_{[2]}(0) &= \frac{(A-1)(A+1)(3A^2-5)}{A^2} \\
\sigma_{[1,1]}(0) &= \frac{1}{2}\frac{(2-A^2)^2(-22A^2+17+9A^4)}{A^4} \\
\sigma_{[3]}(0) &= \frac{(A^2-2)(27A^2-44)(A-1)^2(A+1)^2}{2A^4} \\
\sigma_{[2,1]}(0) &= \frac{4}{3}\frac{(A^2-2)^3(A-1)(A+1)(27A^4-65A^2+43)}{A^6} \\
\sigma_{[1,1,1]}(0) &= \frac{(A^2-2)^5(243A^6-781A^4+905A^2-376)}{3A^8} \\
\sigma_{[4]}(0) &= \frac{(A^2-2)^2(A-1)(A+1)(432A^6-1568A^4+1829A^2-683)}{6A^6} \\
\sigma_{[3,1]}(0) &= \frac{(A^2-2)^4(-11390A^2+15643A^4-9540A^6+2187A^8+3060)}{8A^8} \\
\sigma_{[2,2]}(0) &= \frac{(A^2-2)^4(6089-21872A^2+30112A^4-18412A^6+4203A^8)}{12A^8} \\
\sigma_{[5]}(0) &= \frac{(A^2-2)^3(10125A^6-36872A^4+42814A^2-15576)(A-1)^2(A+1)^2}{24A^8}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{[1]}(1) &= \frac{(2-A^2)^3}{A^4} \\
\sigma_{[2]}(1) &= \frac{5}{8}\frac{(A^2-2)^4(A-1)(A+1)(7A^2-17)}{A^6} \\
\sigma_{[1,1]}(1) &= \frac{1}{2}\frac{(2-A^2)^2(-22A^2+17+9A^4)}{A^4} \\
\sigma_{[3]}(1) &= \frac{(A^2-2)(27A^2-44)(A-1)^2(A+1)^2}{2A^4}
\end{aligned}
$$

$$
\sigma_{[1]}(2) = 0
$$

5.2.2 Knot $T[2,5]$

$$
\begin{aligned}
\sigma_{[1]}(0) &= \log\frac{3-2A^2}{A} \\
\sigma_{[2]}(0) &= \frac{(A-1)(A+1)(17A^2-23)}{A^2} \\
\sigma_{[1,1]}(0) &= \frac{(2A^2-3)^2(83A^4-194A^2+123)}{2A^4} \\
\sigma_{[3]}(0) &= \frac{(2A^2-3)(440A^2-587)(A-1)^2(A+1)^2}{2A^4} \\
\sigma_{[2,1]}(0) &= \frac{2(2A^2-3)^3(A-1)(A+1)(1427A^4-3257A^2+1926)}{3A^6} \\
\sigma_{[1,1,1]}(0) &= \frac{(-3+2A^2)^5(-14191+37290A^2-33663A^4+10429A^6)}{3A^8} \\
\sigma_{[4]}(0) &= \frac{(2A^2-3)^2(A-1)(A+1)(20303A^6-67601A^4+74088A^2-26712)}{6A^6} \\
\sigma_{[3,1]}(0) &= \frac{(-3+2A^2)^4(206277-781148A^2+1106241A^4-696058A^6+164376A^8)}{8A^8} \\
\sigma_{[2,2]}(0) &= \frac{(-3+2A^2)^4(-1532420A^2+2165076A^4-1360752A^6+320633A^8+408399)}{12A^8} \\
\sigma_{[5]}(0) &= \frac{(-3+2A^2)^3(1373736A^6-4574446A^4+4997116A^2-1785093)(A^2-1)^2}{24A^8}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{[1]}(1) &= \frac{(2A^2-3)^3(A-2)(A+2)}{A^4} \\
\sigma_{[2]}(1) &= \frac{235(2A^2-3)^4(A-1)(A+1)(3A^2-5)}{8A^6} \\
\sigma_{[1,1]}(1) &= \frac{(-3+2A^2)^6(263-379A^2+133A^4)}{A^8} \\
\sigma_{[3]}(1) &= \frac{2(-3+2A^2)^5(6320A^2-8063)(A-1)^2(A+1)^2}{3A^8}
\end{aligned}
$$

$$
\sigma_{[1]}(2) = -\frac{(-3+2A^2)^7}{A^8}
$$

5.2.3 Knot $T[3,4]$

$$\begin{aligned}
\sigma_{[1]}(0) &= \log\frac{A^4-5A^2+5}{A^2}\\
\sigma_{[2]}(0) &= \frac{(A-1)(A+1)(8A^6-59A^4+139A^2-98)}{A^4}\\
\sigma_{[1,1]}(0) &= \frac{(A^4-5A^2+5)^2(-477A^6+1312A^4-1529A^2+650+64A^8)}{2A^8}\\
\sigma_{[3]}(0) &= \frac{(A^4-5A^2+5)(192A^8-1907A^6+6989A^4-11049A^2+6160)(A-1)^2(A+1)^2}{2A^8}\\
\sigma_{[2,1]}(0) &= \frac{(A^4-5A^2+5)^3(A-1)(A+1)}{3A^{12}}\Big(2048A^{10}-20610A^8+81573A^6-\\
&- 157361A^4+145965A^2-52195\Big)\\
\sigma_{[1,1,1]}(0) &= \frac{(A^4-5A^2+5)^5}{6A^{16}}\Big(-252572A^{10}+1073187A^8-2398926A^6+\\
&+ 2954196A^4-1899956A^2+500665+24576A^{12}\Big)\\
\sigma_{[4]}(0) &= \frac{(A^4-5A^2+5)^2(A-1)(A+1)}{6A^{12}}\Big(8192A^{14}-118975A^{12}+723388A^{10}-\\
&- 2374550A^8+4517500A^6-4952762A^4+2889917A^2-692540\Big)\\
\sigma_{[3,1]}(0) &= \frac{(A^4-5A^2+5)^4}{8A^{16}}\Big(7056904+82299068A^4-36938869A^2-101809662A^6+\\
&+ 76452748A^8+10172398A^{12}-35724884A^{10}+110592A^{16}-1618975A^{14}\Big)\\
\sigma_{[2,2]}(0) &= \frac{(A^4-5A^2+5)^4}{12A^{16}}\Big(13937464+161289145A^4-72632094A^2-198933448A^6+\\
&+ 148949118A^8+19703861A^{12}-69398084A^{10}+212928A^{16}-3126850A^{14}\Big)\\
\sigma_{[5]}(0) &= \frac{(A^4-5A^2+5)^3(A-1)^2(A+1)^2}{24A^{16}}\Big(512000A^{16}-8768755A^{14}+\\
&+ 64624025A^{12}-266794450A^{10}+672123335A^8-1053157134A^6+\\
&+ 997632397A^4-520542457A^2+114331124\Big)
\end{aligned}$$

$$\begin{aligned}
\sigma_{[1]}(1) &= \frac{-5(A^4-5A^2+5)^3(A^2-2)}{A^8}\\
\sigma_{[2]}(1) &= \frac{(A^4-5A^2+5)^4(A-1)(A+1)(680A^6-4579A^4+11027A^2-8498)}{8A^{12}}\\
\sigma_{[1,1]}(1) &= \frac{(A^4-5A^2+5)^6(-2719A^6+8334A^4-10751A^2+4950+336A^8)}{2A^{16}}\\
\sigma_{[3]}(1) &= \frac{(A^4-5A^2+5)^5(A-1)^2(A+1)^2}{6A^{16}}\Big(27600A^8-236483A^6+772316A^4-\\
&- 1129785A^2+601045\Big)
\end{aligned}$$

$$\sigma_{[1]}(2) = -\frac{-(A^4-5A^2+5)^7(A^2-6)}{A^{16}}$$

5.2.4 Eight-figure knot

$$
\begin{aligned}
\sigma_{[1]}(0) &= \log\frac{A^4-A^2+1}{A^2}\\
\sigma_{[2]}(0) &= \frac{(A-1)(A+1)(A^2+1)(2A^4-3A^2+2)}{A^4}\\
\sigma_{[1,1]}(0) &= \frac{(A^4-A^2+1)^2(-9A^6+6A^4-9A^2+4A^8+4)}{2A^8}\\
\sigma_{[3]}(0) &= \frac{(A^4-A^2+1)(12A^8-A^6-3A^4-A^2+12)(A-1)^2(A+1)^2}{2A^8}\\
\sigma_{[2,1]}(0) &= \frac{(A^4-A^2+1)^3(A^4-1)(32A^8-83A^6+94A^4-83A^2+32)}{3A^{12}}\\
\sigma_{[1,1,1]}(0) &= \frac{(A^4-A^2+1)^5(-283A^{10}+308A^8-260A^6+96A^{12}+308A^4-283A^2+96)}{6A^{16}}\\
\\
\sigma_{[1]}(1) &= \frac{-(A^4-A^2+1)^3}{A^6}\\
\sigma_{[2]}(1) &= \frac{(A^4-A^2+1)^4(A^4-1)(10A^4-43A^2+10)}{8A^{12}}\\
\sigma_{[1,1]}(1) &= \frac{(A^4-A^2+1)^6(-7A^6+10A^4-7A^2+A^8+1)}{2A^{16}}\\
\sigma_{[3]}(1) &= \frac{(A^4-A^2+1)^5(105A^8-109A^6-123A^4-109A^2+105)(A^2-1)^2}{6A^{16}}\\
\\
\sigma_{[1]}(2) &= 0
\end{aligned}
\tag{5.3}
$$

5.3 Examples of genus expansion for superpolynomials

5.3.1 Trefoil

Let us write explicitly few terms of genus expansion for the torus knot $T[2,3]$:

$$P_R(q=e^{\hbar/2},t=e^{\beta\hbar/2},A) \;=\; \tilde{s}_R(1)\cdot\exp\Big\{\,\hbar\cdot\tilde{s}_R(2)+\hbar^2\cdot\tilde{s}_R(3)+\ldots\Big\} \tag{5.4}$$

$$\tilde{s}_R(1)=(2-A^2)^{|R|} \tag{5.5}$$

$$\tilde{s}_R(2)=\tfrac{(A^2-1)(3A^2-5)}{(-2+A^2)^2}T_2^\beta(R)-\tfrac{(A^2-1)(\beta-1)}{-2+A^2}T_1^\beta(R) \tag{5.6}$$

$$
\begin{aligned}
\tilde{s}_R(3)=&-\tfrac{1}{2}\tfrac{(8A^2-13)(A^2-1)^2}{(-2+A^2)^4}T_3^\beta(R)+\tfrac{(A^2-1)(2A^4-8A^2+7)(\beta-1)}{(-2+A^2)^4}T_2^\beta(R)-\\
&-\tfrac{1}{2}\tfrac{(A^2-1)\big((1-\beta+\beta^2)A^4+(-4\beta^2-4+6\beta)A^2+4\beta^2-7\beta+4\big)}{(-2+A^2)^4}T_1^\beta(R)-\\
&-\tfrac{1}{2}\tfrac{(-9A^4+21A^2+A^6-15)\beta}{(-2+A^2)^4}\Big(T_1^\beta(R)\Big)^2
\end{aligned}
\tag{5.7}
$$

Summary

Knot theory is an area of low-dimensional topology, which studies topological properties of knots. A knot is a closed curve without self-intersections in the 3-space. We do not distinguish between a knot and any continuous deformations of this knot which can be performed without self-intersections. In other words, we think about a knot as if it is made from easily deformable rubber, which we cannot cut and glue. One of the main questions in the knot theory is how to distinguish knots. To answer this question we want to find such properties of a knot which depend only on the equivalence class of the knot. This idea gives rise to the theory of knot invariants, a major part of the knot theory. A knot (or link) invariant is a function from the set of knots to some other with values depending only on equivalence classes of knots. Any representative from the class can be chosen to calculate the invariant. There is no restriction on the kind of objects in the target space. For example, they could be integers, polynomials, matrices or groups. In 1980's the HOMFLY polynomial was invented, defined through skein relations. Then it was generalized to a whole family of quatum invariants labeled by Young diagrams R. Such colored HOMFLY polynomials, defined via Chern-Simons theory or quantum groups, are central objects of the present theory of knot invarants. The point is that they involve a lot of structures, very often hidden and implicit, which reveal connections of knots with many subjects of mathematics and mathematical physics.

So, in particular, we discuss a connection of HOMFLY polynomials with Hurwitz covers and represent a generating function for the HOMFLY polynomial of a given knot in all representations as Hurwitz partition function, i.e. the dependence of the HOMFLY polynomials on representation R is naturally captured by symmetric group characters (cut-and-join eigenvalues). The genus expansion and the loop expansion through Vassiliev invariants explicitly demonstrate this phenomenon. We study the genus expansion (also known as the large N expansion) and discuss its properties. Then we also consider the loop expansion in details. In particular, we give an algorithm to calculate Vassiliev invariants, give some examples and discuss relations among Vassiliev invariants.

In the last chapter we consider superpolynomials for torus knots defined via double affine Hecke algebra. We claim that the superpolynomials are not functions of Hurwitz type: symmetric group characters do not provide an adequate linear basis for their expansions. Deformation to superpolynomials is, however, straightforward in the multiplicative basis: the Casimir operators are β-deformed to Hamiltonians of the Calogero-Moser-Sutherland system. Applying this trick to the genus and Vassiliev expansions, we observe that the deformation is fully

straightforward only for the thin knots. Beyond the family of thin knots additional algebraically independent terms appear in the Vassiliev expansions. This can suggest that the superpolynomials do in fact contain more information about knots than the colored HOMFLY and Kauffman polynomials.

Samenvatting

Knopentheorie is een deelgebied van de laag-dimensionale topologie, die topologische eigenschappen van knopen bestudeert. Een knoop is een gesloten kromme in de drie-dimensionale ruimte, die geen snijpunten met zichzelf heeft. We maken geen verschil tussen een knoop en een willekeurige continue vervorming van de knoop, die gemaakt kan worden zo dat de knoop zichzelf niet snijdt.

Een van de belangrijkste vragen van de knopentheorie is hoe we het verschil tussen twee knopen kunnen zien. Om deze vraag te kunnen beantwoorden, willen we de eigenschappen ven knopen vinden die alleen afhankelijk zijn van de equivalentieklasse van de knoop. Vanuit dit idee is de theorie van knopeninvarianten ontwikkeld, die tegenwoordig het grootste deel van knopentheorie is.

Een knopeninvariant is een functie gedefinieerd op de verzameling van knopen met waarden die uitsluitend afhankelijk zijn van de equivalentieklasse van de knoop. Om de waarde van de invariant op een equivalentieklasse te berekenen kunnen we in dit geval een willekeurig knoop van deze klasse gebruiken. Er is geen beperking op de mogelijke ruimte van de waarden van een knoopinvariant. Dat kunnen gehele getallen, of polynomen, of matrices, of elementen van een bepaalde groep zijn.

Een van de belangrijkste knopeninvarianten is het zogenoemde HOMFLY-polynoom. Het is ontdekt in de jaren 1980 met een definitie die gebruik maakt van de zogenoemde skein-relaties. Vervolgens werd het gegeneraliseerd tot een volledige verzameling van kwantum invarianten die afhankelijk zijn van een keuze van een Young-diagram. Deze gekleurde HOMFLY-polynomen kunnen door middel van Chern-Simons theorie of kwantumgroepen gedefinieerd worden. Ze staan in de belangstelling van de moderne theorie van knopeninvarianten, want ze bevatten veel structuren, die heel vaak verborgen en niet-expliciet zijn, en deze structuren verbinden de knopentheorie met veel andere gebieden in wiskunde en mathematische fysica.

In het proefschrift bespreken we een verbinding tussen de knopentheorie en Hurwitztheorie. We geven een formule voor de HOMFLY-polynoom in alle representaties als een partitiefunctie van Hurwitz. Dat betekent dat de afhankelijkheid van de HOMFLY-polynomen van de representatie uitgedrukt kan worden in termen van de karakters van de symmetrische groep. Verschillende ontwikkelingen van de HOMFLY-polynomen geven daarmee meer inzicht, in bijzonder de ontwikkeling in termen van de Vassiliev-invarianten. Het proefschrift bevat veel nieuwe resultaten over de Vassiliev-invarianten. In het bijzonder, geven we een algoritme voor de berekening van deze invarianten, en bespreken hun onderlinge relaties.

In het laatste deel van het proefschrift bespreken we de zogenoemde superpolynomen van de knopen gedefinieerd door de double affine Hecke algebras. De superpolynomen zijn niet de functies van het Hurwitz-type, en de karakters van de symmetrische groep geven geen goede basis voor de ontwikkeling van deze invarianten. Toch kunnen we veel structuur achter deze invarianten opmerken, en, gebaseerd op onze berekeningen stellen we een aantal vermoedens over de superpolynomen voor.

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