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HIGGS LINE BUNDLES, GREEN-LAZARSFELD SETS, AND MAPS OF KÄHLER MANIFOLDS TO CURVES

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ABSTRACT. Let X be a compact Kähler manifold. The set $\mathrm{char}(X)$ of one-dimensional complex valued characters of the fundamental group of X forms an algebraic group. Consider the subset of $\mathrm{char}(X)$ consisting of those characters for which the corresponding local system has nontrivial cohomology in a given degree d. This set is shown to be a union of finitely many components that are translates of algebraic subgroups of $\mathrm{char}(X)$. When the degree d equals 1, it is shown that some of these components are pullbacks of the character varieties of curves under holomorphic maps. As a corollary, it is shown that the number of equivalence classes (under a natural equivalence relation) of holomorphic maps, with connected fibers, of X onto smooth curves of a fixed genus > 1 is a topological invariant of X. In fact it depends only on the fundamental group of X.

Let X denote a compact Kähler manifold. Call two holomorphic maps $f: X \to C$ and $f': X \to C'$, where C and C' are curves, equivalent if there is an isomorphism $\sigma: C \to C'$ such that $f' = \sigma \circ f$. Fix an integer $g > 1$, and consider the set of equivalence classes of surjective holomorphic maps, with connected fibers, of X onto smooth curves of genus g. We will see that this set is finite and that its cardinality $N_g(X)$ depends only on the fundamental group of X.

This result is deduced from a structure theorem for certain homologically defined sets of characters. A character of X is a homomorphism of $\pi_1(X)$ into $\mathbb{C}^*$; it is unitary if the image of $\pi_1(X)$ lies in the unit circle $U(1)$. The set $\mathrm{char}(X)$ of characters forms an affine algebraic group. For every character $\varrho \in \mathrm{char}(X)$, we let $\mathbb{C}_\varrho$ denote the local system or locally constant sheaf on X whose monodromy representation is given by ϱ. For each pair of integers i and m, we define the subset $\Sigma^i_m(X)$ of $\mathrm{char}(X)$ to consist of those characters ϱ for which $\dim H^i(X, \mathbb{C}_\varrho) \geq m$. We will denote $\Sigma^i_1(X)$ by $\Sigma^i(X)$, and we will suppress the dependence on X when there is no danger of confusion. We will call a subset S of $\mathrm{char}(X)$ a unitary translate of an affine subtorus if there exists a unitary character $\varrho \in \mathrm{char}(X)$ such that ϱS is a connected algebraic subgroup.

Theorem 1. *For X, i, and m as above, the set Σ^i_m is a union of finitely many unitary translates of affine subtori.*

By a component of Σ^i_m, we will mean a unitary translate of an affine subtorus $T \subseteq \Sigma^i_m$ that is maximal with respect to inclusion. Using results of Beauville [B1], [B2], we can explicitly describe the positive dimensional components of Σ^1.

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Theorem 2. *Any positive-dimensional component of* Σ^1 *is a translate of an affine subtorus by a torsion element in* $\mathrm{char}(X)$. *If* $T \subseteq \Sigma^1$ *is a positive-dimensional component containing the trivial character, then there exists a surjective holomorphic map with connected fibers* $f: X \to C$ *onto a smooth curve of genus at least two such that* $T = f^* \mathrm{char}(C)$

Corollary. *If* $g \geq 2$ *then* $N_g(X)$ *is finite and it depends only* $\pi_1(X)$. *In other words, if* X' *is another compact Kähler manifold with* $\pi_1(X') \cong \pi_1(X)$ *then* $N_g(X') = N_g(X)$.

Sketch of proof. Using the theorem, we see that $N_g(X)$ counts the number of $2g$-dimensional components of $\Sigma^1(X)$ containing the trivial character. Σ^1 has a purely group theoretic description: $\varrho \in \Sigma^1(X)$ if and only if $H^1(\pi_1(X), \mathbb{C}_\varrho) \neq 0$. Therefore, an isomorphism $\varphi: \pi_1(X) \cong \pi_1(X')$ induces a bijection $\varphi^*: \mathrm{char}(X') \to \mathrm{char}(X)$ such that $\varphi^*(\Sigma^1(X')) = \Sigma^1(X)$. □

Using Hodge theory, we can give a different, more analytic description of $\mathrm{char}(X)$. By a Higgs line bundle, we mean a pair (L, θ) consisting of a holomorphic line bundle L whose first Chern class $c_1(L)$ lies in the torsion subgroup $H^2(X, \mathbb{Z})_{\mathrm{tors}}$, together with a holomorphic 1-form θ. The set of Higgs lines bundles $\mathrm{Higgs}(X)$ can be endowed with the structure of a complex Lie group by identifying it with the product of the Picard torus $\mathrm{Pic}^0(X)$, $H^2(X, \mathbb{Z})_{\mathrm{tors}}$ and the vector space of holomorphic 1-forms. We define a map $\psi: \mathrm{char}(X) \to \mathrm{Higgs}(X)$ as follows: $\psi(\varrho) = (L_\varrho, \theta_\varrho)$, where L_ϱ is the holomorphic bundle whose sheaf of sections is $\mathbb{C}_\varrho \otimes_{\mathbb{C}} O_X$ and θ_ϱ is the $(1, 0)$ part of $\log \|\varrho\|$ viewed as a cohomology class under the isomorphism $H^1(X, \mathbb{R}) \cong \mathrm{Hom}(\pi_1(X), \mathbb{R})$. Then ψ is an isomorphism of topological groups (but not of complex Lie groups). Simpson [S] introduced the concept of a Higgs bundle of arbitrary rank on a Kähler manifold; however, the notion of Higgs line bundle also occurs implicitly in the work of Green and Lazarsfeld [GL1], [GL2] and Beauville.

Before describing the image of Σ^i_m under ψ, we need to define the cohomology group of a Higgs line bundle (L, θ)

$$H^{pq}(L, \theta) = \frac{\ker(H^q(X, \Omega^p_X \otimes L) \overset{\wedge\theta}{\to} H^q(X, \Omega^{p+1}_X \otimes L))}{\mathrm{im}(H^q(X, \Omega^{p-1}_X \otimes L) \overset{\wedge\theta}{\to} H^q(X, \Omega^p_X \otimes L))}.$$

The next theorem follows by combining the results of Green and Lazarsfeld [GL1, 3.7] with those of Simpson [S, 3.2].

Theorem 3. *For each* i *there is an isomorphism*

$$H^i(X, \mathbb{C}_\varrho) \cong \bigoplus_{p+q=i} H^{pq}(\psi(\varrho)).$$

We define the sets

$$\sigma^{pq}_m = \{(L, \theta) \in \mathrm{Higgs}(X) | \dim H^{pq}(L, \theta) \geq m\},$$
$$S^{pq}_m = \{L \in \mathrm{Pic}^0(X) | \dim H^q(X, \Omega^p_X \otimes L) \geq m\}.$$

The set S^{pq}_m was defined by Green and Lazarsfeld; it equals the intersection of σ^{pq}_m with $\mathrm{Pic}^0(X) \times \{0\}$.

Corollary. *$\psi(\Sigma^i_m) = \bigcup_\mu \bigcap_{0\le k\le i} \sigma^{k,i-k}_{\mu(k)}$, where μ runs over all partitions of m, i.e., functions $\mu\colon \{0\cdots i\} \to \{0, 1, 2, \dots\}$ such that $\Sigma\mu(k) = m$.*

Let $\mathbb{R}^+$ denote the set of positive real numbers viewed as a group under multiplication. A number $t \in \mathbb{R}^+$ acts on a Higgs line bundle by the rule $t * (L, \theta) = (L, t\theta)$. We can transfer this action to $\mathrm{char}(X)$ via ψ, namely, $t * \varrho = \psi^{-1}(t * \psi(\varrho))$. After choosing generators for $\pi_1(X)$, we can identify the connected components of $\mathrm{char}(X)$ with a product of $\mathbb{C}^*$'s. Under this identification the $\mathbb{R}^+$ action is described by

$$t * (r_1 e^{i\lambda_1}, r_2 e^{i\lambda_2}, \dots) = (r_1 e^{it\lambda_1}, r_2 e^{it\lambda_2}, \dots)$$

where $r_1, r_2, \dots \lambda_1, \dots \in \mathbb{R}$.

We can now indicate the idea of the proof of the first theorem. Using a Cech complex, it is possible to write down equations for Σ^i_m, so we conclude that this is an algebraic subset of $\mathrm{char}(X)$. The corollary to Theorem 3 shows that this set is stable under the $\mathbb{R}^+$ action. The theorem now follows from

Proposition. *If $V \subseteq (\mathbb{C}^*)^n$ is a closed irreducible subvariety stable under the above $\mathbb{R}^+$ action, then V is a unitary translate of an affine subtorus.*

Sketch of proof. The Zariski closure of any orbit $\mathbb{R}^+ * v$, with $v \in (\mathbb{C}^*)^n$, can be shown to be a unitary translate of an affine subtorus. One then checks that for a sufficiently general point $v \in V$, the orbit $\mathbb{R}^+ * v$ is Zariski dense in V. □

As a corollary to Theorem 1, we obtain a new proof of a theorem of Green and Lazarsfeld [GL2] about the structure of S^{pq}_m. We say that a subset T of the Picard group $\mathrm{Pic}(X)$ is a translate of a complex subtorus if there is an element $\tau \in \mathrm{Pic}(X)$ such that $\tau + T$ is a connected complex Lie subgroup.

Corollary. *There exist a finite number of translates of complex subtori T_i of $\mathrm{Pic}(X)$ and subspaces V_i of the space of holomorphic 1-forms on X with $\dim T_i = \dim V_i$, such that σ^{pq}_m is a union of $T_i \times V_i$. In particular S^{pq}_m is the union of those T_i contained in $\mathrm{Pic}^0(X)$.*

Sketch of proof. σ^{pq}_m is an analytic subvariety of $\mathrm{Higgs}(X)$. Choose an irreducible component U of this set. Let $i = p + q$ and for $k \in \{0, \dots i\}$ define

$$\mu(k) = \max\{n \mid U \subseteq \sigma^{k,i-k}_n\}.$$

Then U is an irreducible component of $\bigcap_i \sigma^{k,i-k}_{\mu(k)}$ that is not contained in $\bigcap_i \sigma^{k,i-k}_{\mu'(k)}$ for any other partition μ' of $M = \sum_j \mu(j)$. Thus U is an irreducible component of $\psi(\Sigma^i_M)$. By the theorem, it can be shown that any irreducible component of $\psi(\Sigma^i_M)$ is the image under ψ of a unitary translate of an affine subtorus; such a set is of the form $T \times V$, where T is a translate of a complex subtorus of $\mathrm{Pic}(X)$ and V is a subspace of 1-forms of the same dimension. □

We will call an unramified cover of X with abelian Galois group an abelian cover. The maximal abelian cover X^{ab} is obtained as the quotient of the universal cover by the commutator subgroup $\pi_1(X)'$. The Galois group of X^{ab} over X is precisely $H_1(X, \mathbb{Z})$. The homology groups $H_i(X^{\mathrm{ab}}, \mathbb{Z})$ are finitely generated as $\mathbb{Z}[H_1(X, \mathbb{Z})]$-modules although not necessarily as abelian groups. Our next theorem give partial support to some conjectures of Beauville [B2] and Catanese [C] on the structure of Green-Lazarsfeld sets.

Theorem 4. *Fix an integer N. Suppose that $H^i(X^{\mathrm{ab}}, \mathbb{Z})$ is a finitely generated abelian group for all $i < N$. Then*

(a) $\Sigma^i(X)$ *consists of a finite set of torsion points of* $\mathrm{char}(X)$ *whenever* $i < N$.

(a′) $S_1^{pq}(X)$ *consists of a finite set of torsion points in* $\mathrm{Pic}^0(X)$ *whenever* $p + q < N$.

(b) *There is a finite sheeted abelian cover $X' \to X$ such that $\Sigma^i(X') = \{1\}$ where 1 is the trivial character whenever $i < N$.*

(b′) $S_1^{pq}(X') = \{O_X\}$ *whenever* $p + q < N$.

(c) $\Sigma^N(X)$ *has a positive-dimensional component if and only if* $H^N(X^{\mathrm{ab}}, \mathbb{Q})$ *is infinite-dimensional.*

(c′) $S_1^{pq}(X)$ *has a positive-dimensional component for some p and q, with $p + q = N$, if and only if $H^N(X^{\mathrm{ab}}, \mathbb{Q})$ is infinite-dimensional.*

Sketch of proof of (a). Let V be a finite-dimensional $\mathbb{C}$-vector space upon which $A = H_1(X, \mathbb{Z})$ acts. A character ϱ will be called a weight of V if there is a nonzero $v \in V$ such that for all $a \in A$, $av = \varrho(a)v$. We prove a vanishing/nonvanishing theorem: $H^0(A, V \otimes_{\mathbb{C}} \mathbb{C}_\varrho) = 0$ if ϱ^{-1} is a weight of V, otherwise $H^p(A, V \otimes_{\mathbb{C}} \mathbb{C}_\varrho) = 0$ for all p. Let W be the union of the set of weights of $H^i(X^{\mathrm{ab}}, \mathbb{C}) = H^i(X^{\mathrm{ab}}, \mathbb{Z}) \otimes \mathbb{C}$ with $i < N$, and let W^{-1} be the set of inverses of these weights. Associated to the cover X^{ab} there is a spectral sequence

$$E_2^{pq} = H^p(A, H^q(X^{\mathrm{ab}}, \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}_\varrho) \Rightarrow H^{p+q}(X, \mathbb{C}_\varrho).$$

This together with the vanishing/nonvanishing theorem implies that $\bigcup_{i<N} \Sigma^i(X) = W^{-1}$. Therefore the sets $\Sigma^i(X)$ are finite when $i < N$, and so by Theorem 1 they must consist of unitary characters.

Let K be the number field obtained by adjoining to $\mathbb{Q}$ all the eigenvalues of generators of A acting on $H^i(X^{\mathrm{ab}}, \mathbb{Z})$ with $i < N$. Then W is defined over the ring of integers O_K of K. In other words there is a subset $W' \subset \mathrm{Hom}(\pi_1(X), O_K^*)$ such that $W = \bigcup_{i:K\to\mathbb{C}} i(W')$. Since we have shown that the characters in W are also unitary, it follows by a theorem of Kronecker that they must have finite order. □

Corollary. *The following are equivalent.*

(a) $H_1(\pi_1(X)', \mathbb{Q})$ *is infinite-dimensional.*

(b) *There is a finite sheeted abelian cover of X that maps onto a curve of genus at least two.*

Sketch of proof of (a) ⇒ (b). If $H_1(\pi_1(X)', \mathbb{Q}) \cong H_1(X^{\mathrm{ab}}, \mathbb{Q}) \cong H^1(X^{\mathrm{ab}}, \mathbb{Q})$ is infinite-dimensional then $\Sigma^1(X)$ has a positive-dimensional component. By theorem 2, this component is a translate of an affine subtorus by a torsion element. Therefore there is a finite abelian cover X' of X such that the pull back of this component, which lies in $\Sigma^1(X')$, contains the trivial character. Then Theorem 2 shows that X' maps onto a curve of genus at least 2. □

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