High-Energy Factorization and Small-X Deep Inelastic Scattering Beyond Leading Order — Source link [↗]

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High-energy factorization and small-x deep inelastic scattering beyond leading order [*]

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Abstract

High-energy factorization in QCD is investigated beyond leading order and its relationship to the factorization theorem of mass singularities is established to any collinear accuracy. Flavour non-singlet observables are shown to be regular at small x order by order in perturbation theory. In the singlet sector, we derive the relevant master equations for the space-like evolution of gluons and quarks. Their solution enables us to sum next-to-leading corrections to the small-x behaviour of quark anomalous dimensions and deep inelastic scattering coefficient functions. We present results in both $\overline{\rm MS}$ and DIS factorization schemes.

1. Introduction

Hadronic processes at large transferred momentum p_t are accurately investigated by using perturbative QCD. The comparison between theoretical predictions and jet physics data from high-energy colliders has enhanced our confidence in perturbative QCD up to the 10% accuracy level [1]. The reason for this success is that not only the non-perturbative (higher-twist) contributions vanish as powers of Λ/p_t in the hard-scattering regime $p_t \gg \Lambda$ (Λ being the QCD scale), but mainly the fact that logarithmic corrections to the naïve parton model (i.e. lowest-order perturbative QCD) are systematically computable and, in most cases, known as a power series expansion in the 'small' (due to asymptotic freedom) running coupling $\alpha_S(p_t^2) \sim (\beta_0 \ln p_t^2/\Lambda^2)^{-1}$.

Precise quantitative tests of QCD and searches for new physics at present and future hadron colliders are nonetheless carried out at an increasingly large centre-of-mass energy $\sqrt{S}$, thus opening up a new kinematic region characterized by small values of the ratio $x = p_t^2/S$ ($S \gg p_t^2$). In this small-x regime, our capability to make perturbative QCD predictions decreases. The reason is twofold. First, the parton densities $f(x, p_t^2)$ of the incoming hadrons (which are necessary as inputs in any perturbative calculation) are poorly known at very small values of x. Second, the perturbative expansion is slowly (or badly) convergent because of the presence of logarithmic corrections of the type $\alpha_S^n(p_t^2) \ln^m x$: higher powers of $\alpha_S(p_t^2)$ associated with multiple hard-jet production can indeed be compensated by large enhancing factors $\ln(S/p_t^2)$. Note that this second point also affects the first one. In the case that a reliable set of small-x parton densities can be extracted from a single experiment, we should be able to predict accurately their perturbative evolution with p_t^2.

The discussion above makes clear that our understanding of QCD in the small-x regime has still to be improved. In particular, on the theoretical side, it is highly desirable to keep under control and to evaluate reliably the QCD component which can be computed perturbatively. This can be done by combining the customary perturbative approach to hard-scattering processes with an improved perturbative expansion which systematically sums classes (leading, next-to-leading, and so on) of small-x logarithmic contributions to all orders in α_S.

At present, the QCD multiparton matrix elements have been computed to double logarithmic accuracy in the small-x region [2]–[6]. They can be used to study the structure of hadronic final states in small-x processes, thus predicting some new distinctive features [3]–[5] such as the increase of particle (jet) multiplicity and the suppression of large rapidity gaps. Only some phenomenological investigations have been carried out [7], and more detailed analyses are needed. In this paper we do not consider the issue of the structure of the final states but rather concentrate on the evaluation of higher-order corrections for total cross sections.

The leading high-energy contributions to total cross sections are single-logarithmic terms $(\alpha_S \ln x)^n$ (higher powers of $\ln x$ cancel in this case) due to multiple gluon exchanges in the t-channel. In the case of hard processes which are directly coupled to gluons in the naïve parton model, these leading logarithmic contributions can be resummed to all orders in perturbation theory by using the high-energy or $k_\perp$-factorization theorem [8]–[10]. The basic idea [8] is to replace the collinear (or parton pole) factorization by gluon Regge pole factorization.

Considering, for instance, the simplest small-x process initiated by hard-gluon interactions at the Born level, namely the heavy-flavour photoproduction process

$$\gamma(p_1) + h(p) \rightarrow Q(p_3) + \bar{Q}(p_4) + X, \tag{1.1}$$

one can write the total cross section in the following factorized form [8]

$$4M^2 \, \sigma(x, M^2) = \int d^2\mathbf{k} \int_x^1 \frac{dz}{z} \, \hat{\sigma}(x/z, \mathbf{k}^2/M^2, \alpha_S(M^2)) \, \mathcal{F}(z, \mathbf{k}) \, . \tag{1.2}$$

Here the heavy flavour mass M $(M \gg \Lambda)$ defines the hard scale of the process and $x \equiv 4M^2/S$ $(S \simeq 2p_1 \cdot p \gg M^2)$.

In Eq. (1.2) $\hat{\sigma}$ is the basic high-energy hard cross section for the subprocess $\gamma + g(k) \rightarrow Q\bar{Q}$, computed to the lowest order in α_S as a function of the transverse momentum $\mathbf{k}$ of the incoming off-shell (essentially transverse $k \simeq zp + k_\perp$, $k^2 \simeq -\mathbf{k}^2$) gluon $g(k)$. On the other hand, $\mathcal{F}(z, \mathbf{k})$ is the unintegrated gluon density of the incoming hadron $h(p)$ and is related to the customary gluon density $f_g(x, M^2)$ via $k_\perp$-integration

$$x f_g(x, M^2) \sim \int_0^{M^2} d^2\mathbf{k} \; \mathcal{F}(x, \mathbf{k}) \; . \tag{1.3}$$

Therefore the $k_\perp$-dependent factorization in Eq. (1.2) reduces to the leading-order collinear factorization [11] for $S \gg M^2 \gg k_\perp^2$. However it holds also for $S \gg k_\perp^2 \sim M^2$, thus controlling all the logarithmically-enhanced terms $(\alpha_S \ln x)^n$ associated with hard-gluon radiation in the final state.

The resummation of the leading $\ln x$ contributions follows from noticing that the hard cross section $\hat{\sigma}$ is well-behaved at high energy (i.e. $\hat{\sigma}(x, \mathbf{k}^2/M^2, \alpha_S) \sim x$ modulo $\ln x$-terms, for $x \rightarrow 0$). Therefore the large perturbative corrections $(\alpha_S \ln x)^n$ in the cross section (1.2) are generated precisely by the $k_\perp$-integration from the ones in the gluon density $\mathcal{F}(x, \mathbf{k})$, as given by the Balitskii-Fadin-Kuraev-Lipatov (BFKL) equation [12]

$$\mathcal{F}(x, \mathbf{k}) \sim \frac{1}{2\pi} e^{-\lambda \ln x} (\mathbf{k}^2)^{-\frac{1}{2}} \; , \tag{1.4}$$

$$\lambda = 4 C_A \frac{\alpha_S}{\pi} \ln 2 \; . \tag{1.5}$$

Inserting Eq. (1.4) into Eq. (1.2), one obtains the following perturbative result for the total cross section at very high energy [8]

$$4M^2 \sigma(x, M^2) \sim x^{-\lambda} (M^2)^{\frac{1}{2}} \; h(1/2) \; , \tag{1.6}$$

$$h(1/2) \equiv \frac{1}{2} \int_0^\infty \frac{d\mathbf{k}^2}{\mathbf{k}^2} \left(\frac{\mathbf{k}^2}{M^2} \right)^{\frac{1}{2}} \int_0^1 \frac{dx}{x} \; \hat{\sigma}(x, \mathbf{k}^2/M^2, \alpha_S) \; . \tag{1.7}$$

The main features of Eqs. (1.6), (1.7), derived from the $k_\perp$-factorization formula (1.2), are the following. The total cross section increases at high energy with a universal (process-independent) power behaviour S^λ, λ being the intercept of the perturbative QCD pomeron in Eq. (1.5). This result is the consequence of the very steep behaviour (1.4) of the gluon density[1] at small x and large $k_\perp$. The normalization of the total cross section instead depends on the process and the process-dependent factor $h(1/2)$ derives from the detailed and calculable transverse-momentum dynamics of the hard subprocess.

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[1] Note, however, that if the gluon density has a non-perturbative component steeper than the perturbative one, the former dominates over the latter at high energy.

The resummation of the leading-order contributions at high energy is a crude (although mandatory) approximation. This is true from both the theoretical (the perturbative QCD pomeron violates unitarity to leading order) and phenomenological (terms of relative order α_S are systematically neglected) sides. The evaluation of subleading contributions is therefore relevant (i) to include corrections necessary to restore unitarity at asymptotic energies and (ii) to estimate the accuracy and set the limits of applicability of the leading-order formalism.

Unitarization effects have been extensively studied in Refs. [13]–[16]. Although a systematic calculational approach based on first principles is still missing, the likely conclusion emerging from these studies is that the full restoration of unitarity can be achieved only after the inclusion of higher-twist corrections.

As regards the leading-twist contributions to hard-scattering processes, a calculational program of high-energy logarithms based on Regge behaviour is being pursued by Fadin and Lipatov [17], and the evaluation of the two-loop correction to the BFKL kernel now seems feasible.

In this paper we follow a different approach towards the computation of subleading corrections at high energy. We show how the $k_\perp$-factorization theorem can be extended beyond leading order in a consistent way with all-order (leading-twist) collinear factorization. This allows us to set up a systematic logarithmic expansion both for hard coefficient functions and parton anomalous dimensions. Moreover, once these quantities have been computed to a certain logarithmic accuracy, they can unambiguously be supplemented with non-logarithmic (finite-x) contributions exactly calculable to any fixed order in perturbation theory. Note, also, that a further advantage of this approach is that the effects of the running coupling can be included exactly (at least in principle), thus avoiding the infrared instabilities encountered in phenomenological attempts [13,18,19] to extend the BFKL equation beyond leading order. Obviously, with the present attitude, we abandon any demand to predict the *absolute* behaviour of the cross sections at very high energies (very small x) because higher-twist corrections are systematically neglected. However, since logarithmic scaling violations are systematically under control, we think that such an approach can be useful in making quantitative phenomenological predictions at high (but finite) energies and large transferred momenta [20].

The outline of the paper is as follows. In Section 2 we first recall the general framework of leading-twist collinear factorization. Then, on the basis of power counting arguments, we show how the high-energy factorization of Ref. [8] can be extended beyond leading order and consistently matched with all-order collinear factorization, in terms of resummed anomalous dimensions and coefficient functions. In Section 3 we start our calculational program in dimensional regularization by deriving the master equation for the high-energy behaviour of the gluon forward scattering amplitudes and computing the ensuing anomalous dimensions and normalization factor in the (modified) minimal subtraction scheme. The analogous calculation for the quark channel is performed in Section 4. Here, we obtain an algebraic equation for the quark anomalous dimensions to next-to-leading logarithmic accuracy $\alpha_S(\alpha_S \ln x)^n$, and present its explicit solution up to the six-loop order. In Section 5 we turn our attention to the calculation of the coefficient functions for deep inelastic lepton-hadron scattering. We compute both the longitudinal and transverse coefficient functions by resumming the logarithmic

contributions $\alpha_S(\alpha_S \ln x)^n$. We also consider the all-order generalization of the DIS factorization scheme and obtain explicit resummed expressions for the corresponding quark anomalous dimensions with next-to-leading logarithmic accuracy. Section 6 is devoted to summarizing our main results, and may also serve as a guide for the reader mostly interested in extracting perturbative QCD results for phenomenological applications. The definition of singlet and non-singlet parton densities and a few mathematical details are left to Appendices A to C.

Some of the results derived in this paper have already been presented in Refs. [21,22].

2. QCD factorizations

This section is devoted to setting up the formal basis of the high-energy factorization. We start in Subsection 2.1 by introducing the factorization theorem of collinear singularities and defining the parton densities and coefficient functions. Then in Subsection 2.2 we recall the proof of this factorization theorem to leading-twist order in the context of dimensional regularization. This formal apparatus is used in Subsection 2.3 to develop a simple power counting at high energy and to show how high-energy factorization can be carried out consistently with all-order collinear factorization. The high-energy factorization formulae are $k_\perp$-dependent and can be considered as the generalization of collinear factorization in terms of *unintegrated* parton densities and *off-shell* coefficient functions. Alternatively, high-energy factorization can be compared with Regge factorization [23]. The gluon Green functions and the two-gluon irreducible kernels entering as building blocks in the high-energy factorization discussed in Subsection 2.3 are the analogue of the reggeon trajectory and its residue. However, this analogy has to be taken with caution since it does not properly account for the issue of factorization of collinear singularities. Finally, in Subsection 2.4, we show how the high-energy factorization leads to resummed anomalous dimensions and coefficient functions.

2.1. Hard processes and parton densities

The factorization theorem of collinear (mass) singularities [24,25,11] states that, in a general hard collision (i.e. a scattering process involving a large transferred momentum $< p_t^2 > = Q^2 \gg \Lambda^2$) of incoming hadrons, all long-distance (non-perturbative) effects can be factorized into universal (process-independent) parton densities thus leading to a perturbatively calculable dependence on the hard scattering scale Q^2.

Considering, for the sake of simplicity, the case of a single incoming hadron (like the heavy-flavour photoproduction process (1.1) or deep inelastic lepton-hadron scattering), one can write the dimensionless cross section $F(x,Q^2) \sim Q^2\sigma(x,Q^2)$ as follows ($x \equiv Q^2/S$)

$$F(x,Q^2) = \sum_a \int_x^1 dz\, C_a(x/z; \alpha_S(\mu_F^2), Q^2/\mu_F^2)\, f_a(z,\mu_F^2) \,. \tag{2.1}$$

Here C_a are the process-dependent coefficient functions, f_a are the parton densities ($a = q_i, \bar{q}_i, g,\ i = 1, \ldots, N_f,\ N_f$ being the number of flavours) of the incoming hadron and μ_F^2 is an arbitrary factorization scale such that $\mu_F^2 \gg \Lambda^2$. The observable F is independent of μ_F^2. Correspondingly the μ_F^2-dependence of C_a on the r.h.s. of Eq. (2.1) is exactly cancelled by that of f_a. Moreover, both the α_S-dependence of the coefficient functions and the scale dependence of the parton densities are computable as a power series expansion in α_S. In particular, the parton densities fulfil the renormalization group evolution equations

$$\frac{d\, f_a(x, \mu^2)}{d \ln \mu^2} = \sum_b \int_x^1 \frac{dz}{z} P_{ab}(\alpha_S(\mu^2), z)\, f_b(x/z, \mu^2)\,, \qquad (2.2)$$

where $P_{ab}(\alpha_S, z)$ are generalized Altarelli-Parisi splitting functions which are computable in QCD perturbation theory:

$$P_{ab}(\alpha_S, x) = \sum_{n=1}^{\infty} \left(\frac{\alpha_S}{2\pi}\right)^n P_{ab}^{(n-1)}(x)\,. \qquad (2.3)$$

The corresponding expansion for the coefficient function is

$$C_a(x; \alpha_S, Q^2/\mu^2) = (\alpha_S)^p \left[C_a^{(0)}(x) + \sum_{n=1}^{\infty} \left(\frac{\alpha_S}{2\pi}\right)^n C_a^{(n)}(x; Q^2/\mu^2) \right]\,, \qquad (2.4)$$

where the integer power p depends on the process.

The x-dependence of the factorization formula (2.1) and of the evolution equations (2.2) can be diagonalized by introducing the N-moments of the cross section

$$F_N(Q^2) = \int_0^1 dx\, x^{N-1} F(x, Q^2) \qquad (2.5)$$

and the analogous moments of any other function of x. It is also convenient to define the rescaled parton densities $\tilde{f}_a$

$$\tilde{f}_a(x, \mu^2) \equiv x f_a(x, \mu^2)\,, \qquad (2.6)$$

and the anomalous dimension matrix $\gamma_{ab,N}$ [1]

$$\gamma_{ab,N}(\alpha_S) \equiv \int_0^1 dx\, x^N P_{ab}(\alpha_S, x) = P_{ab,N+1}(\alpha_S)\,. \qquad (2.7)$$

In the N-moment space, Eqs. (2.1), (2.2) respectively read as follows

$$F_N(Q^2) = \sum_a C_{a,N}(\alpha_S(\mu_F^2), Q^2/\mu_F^2)\, \tilde{f}_{a,N}(\mu_F^2)\,, \qquad (2.8)$$

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[1] Our definition differs from the standard one in which $\gamma_N = P_N$.

$$\frac{d\,\tilde{f}_{a,N}(\mu^2)}{d\ln\mu^2} = \sum_b \gamma_{ab,N}(\alpha_S(\mu^2))\,\tilde{f}_{b,N}(\mu^2)\ . \tag{2.9}$$

Eq. (2.1) is correct apart from terms vanishing as inverse powers of Q^2 at high Q^2 (higher-twist corrections). In this regime, a consistent perturbative use of the collinear factorization formula (2.1) requires the knowledge of $\{P_{ab}^{(0)}, C_a^{(0)}\}$ (leading order), $\{P_{ab}^{(1)}, C_a^{(1)}\}$ (next-to-leading order) and so on. Therefore, QCD predictions for the cross section are usually obtained by computing in fixed-order perturbation theory both the splitting functions and the coefficient functions. The splitting functions up to two-loop accuracy have been known for a long time [25]–[28]. Next-to-leading order coefficient functions have been computed for most processes [29], and, in the cases of deep inelastic lepton-hadron scattering (DIS) [30] and Drell-Yan process [31], the next-to-next-to-leading terms $C_a^{(2)}$ are also known.

As discussed in Section 1, higher-order contributions to the cross section and, hence, to splitting and coefficient functions are logarithmically enhanced at small x. More precisely, in the small-x limit we have[2]

$$P_{ab}^{(n-1)}(x) \sim \frac{1}{x}\left[\ln^{n-1}x + \mathcal{O}(\ln^{n-2}x)\right]\ , \tag{2.10}$$

$$C_a^{(n)}(x) \sim \ln^{n-1}x + \mathcal{O}(\ln^{n-2}x)\ . \tag{2.11}$$

This small-x behaviour corresponds, in N-moment space, to singularities for $N \to 0$ in the form

$$\gamma_{ab,N}(\alpha_S) = \sum_{k=1}^{\infty}\left[\left(\frac{\alpha_S}{N}\right)^k A_{ab}^{(k)} + \alpha_S\left(\frac{\alpha_S}{N}\right)^k B_{ab}^{(k)} + \ldots\right]\ , \tag{2.12}$$

$$C_{a,N}(\alpha_S, Q^2/\mu^2) = \alpha_S^p\, C_{a,N=0}^{(0)}\left[1 + \sum_{k=1}^{\infty}\left(\frac{\alpha_S}{N}\right)^k D_a^{(k)}(Q^2/\mu^2)\right.$$
$$\left. + \alpha_S\left(\frac{\alpha_S}{N}\right)^k E_a^{(k)}(Q^2/\mu^2) + \ldots\right]\ . \tag{2.13}$$

These singularities may spoil the convergence of the perturbative expansions (2.3) and (2.4) at small x (high energy). Nonetheless one can consider an improved perturbative expansion obtained by resumming the *leading* $(A_{ab}^{(k)}, D_a^{(k)})$, *next-to-leading* $(B_{ab}^{(k)}, E_a^{(k)})$, etc., coefficients in the high-energy regime. Once these coefficients are known, they can be combined [32] with Eqs. (2.3) and (2.4) (after subtracting the resummed logarithmic terms in order to avoid double counting), to obtain a prediction throughout the region of x where $\alpha_S\ln(1/x) \lesssim 1$ (or $\alpha_S/N \lesssim 1$), which is much larger than the domain $\alpha_S\ln(1/x) \ll 1$ where the α_S-perturbative expansions (2.3), (2.4) are applicable.

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[2] According to our definition, the lowest-order coefficient function $C^{(0)}$ is normalized in such a way that $C^{(0)}(x) \sim x$, modulo $\ln x$ corrections, at small x.

As outlined in Section 1 and discussed in detail in Ref. [8] (see also Section 5), the $k_\perp$-factorization formula (1.2) allows one to resum the leading high-energy contributions $(\alpha_S \ln x)^n$ $((\alpha_S/N)^n)$ both for the anomalous dimensions and for the coefficient function. By comparing Eqs. (1.2) and (2.1) and considering the relation (1.3) between unintegrated and full parton densities, we see that the $k_\perp$-factorization is, in a sense, more general than the collinear factorization in Eq. (2.1). The latter is recovered after the transverse momentum integration in the former. This relationship can be made explicit in a simple way to leading order [8]. However, beyond leading order, a careful analysis of the low-$k_\perp$ integration region in (1.2) is needed in order to disentangle non-perturbative and higher-twist effects from the perturbative ones.

This issue can be investigated from a formal viewpoint by considering cross sections at parton level instead of hadronic cross sections. In the partonic cross section, the non-perturbative contributions show up in terms of collinear divergences. Once these divergences are properly regularized, they can be factorized and subtracted by a procedure of renormalization of the bare parton densities. This procedure can be carried out to any collinear accuracy, although the ensuing anomalous dimensions and coefficient functions are no longer separately regularization/factorization scheme independent beyond one-loop level.

The point we want to address in the following is that the $k_\perp$-factorization can be consistently implemented beyond leading order without spoiling the all-order factorization of collinear singularities. In particular, the resummation of next-to-leading contributions at high energy can be performed by properly taking care of the factorization scheme dependence of the splitting and coefficient functions. To this end, let us first recall the formal basis of collinear factorization.

2.2. Collinear factorization

In order to present a formal derivation of the factorization theorem of collinear singularities, we follow the technique developed by Curci, Furmanski and Petronzio [25]. The dimensionless cross section $F(x, Q^2)$ in Eq. (2.1) is first expressed in terms of partonic cross sections and parton distributions in the form[3] (Fig. 1)

$$F = F^{(0)}(\ldots, p) \; \tilde{f}^{(0)}(p, \ldots) + [F_4^{(0)}(\ldots; p_1, p_2) \; \tilde{f}_4^{(0)}(p_1, p_2; \ldots) + \ldots] \,, \tag{2.14}$$

where, in the limiting case of on-shell partons $(p_i^2 = 0)$, the first term on the r.h.s. picks out the leading-twist contribution we are interested in. Nonetheless, the partonic cross section $F^{(0)}$ is collinear divergent in the on-shell limit, so that both $F^{(0)}$ and $\tilde{f}^{(0)}$ have to be regarded as properly regularized 'bare' quantities.

We use the standard procedure of dimensional regularization, in which the bare cross section $F^{(0)}$ is evaluated in $n = 4 + 2\varepsilon$ space-time dimensions, considering $(n - 2)$ helicity states for gluons and 2 helicity states for quarks. The corresponding dimensional-

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[3] Here and in the following we use a symbolic notation in which the product $C = AB$ of two kernels A and B understands the integration over the intermediate momenta (or the corresponding product in N-moment space) and the sum over the intermediate parton species and their spin and colour indices.
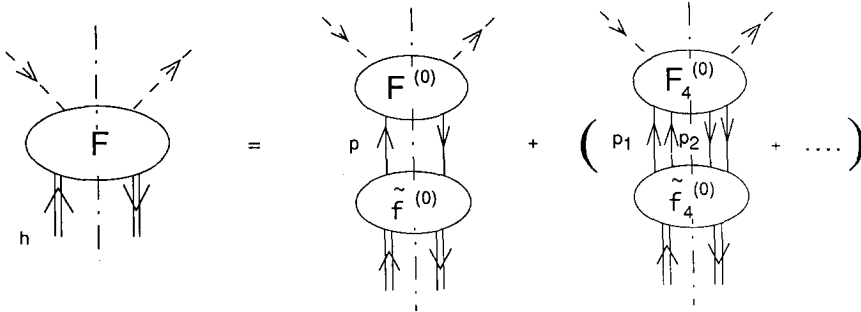
Fig. 1. Factorization of the physical cross section F in terms of partonic cross sections $(F^{(0)}, F_4^{(0)}, \ldots)$ and parton distributions $(\tilde{f}^{(0)}, \tilde{f}_4^{(0)}, \ldots)$. In the case of on-shell partons $(p_i^2 = 0)$, the first term on the r.h.s. and those in parenthesis represent respectively the leading- and higher-twist contributions.
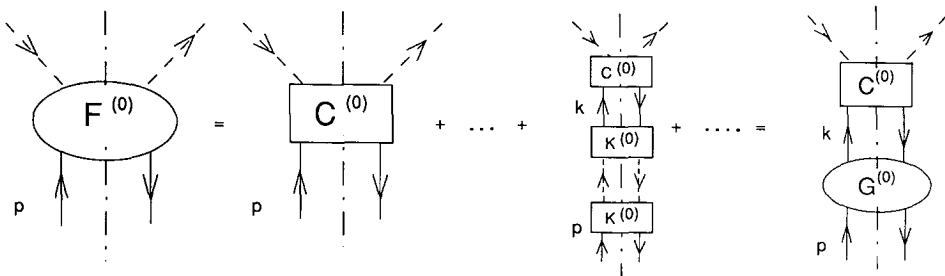

Fig. 2. Expansion of the partonic cross section $F^{(0)}$ in two-particle irreducible (2PI) kernels.

regularization scale is denoted by μ. Once $F^{(0)}$ has been renormalized (using, for instance, the $\overline{\text{MS}}$ renormalization scheme), collinear singularities are automatically regularized and show up as single poles in $1/\varepsilon$. The factorization theorem allows one to subtract these poles from $F^{(0)}$ and factorize them (to all orders in α_S) into process-independent transition functions Γ, according to

$$F^{(0)} = C \, \Gamma \,, \tag{2.15}$$

where the coefficient function C is finite for $\varepsilon \to 0$. Using the transition functions Γ to define the 'physical' parton densities $\tilde{f}$

$$\tilde{f} = \Gamma \tilde{f}^{(0)} \,, \tag{2.16}$$

one then recovers the factorization formula (2.8) by performing the limit $\varepsilon \to 0$.

The factorization procedure leading to Eq. (2.15) is simplified if we evaluate the gauge-invariant partonic cross section $F^{(0)}$ in a physical gauge. Denoting by $p^\mu = P(1,0,1)$ the incoming parton momentum (Fig. 2), we introduce the following Sudakov parametrization for any other momentum k

$$k^\mu = zp^\mu + k_\perp^\mu + \frac{k^2 + \mathbf{k}^2}{z}\frac{\bar{p}^\mu}{2p\cdot\bar{p}} \,, \quad k_\perp^\mu = (0,\mathbf{k},0) \,, \quad \bar{p}^\mu = P(1,0,-1) \,, \tag{2.17}$$

and we choose the axial gauge $\bar{p} \cdot A = 0$, where the sum over the gluon helicities is given by the polarization tensor

$$d^{\mu\nu}(k) = -g^{\mu\nu} + \frac{k^{\mu}\bar{p}^{\nu} + \bar{p}^{\mu}k^{\nu}}{\bar{p} \cdot k} \,. \tag{2.18}$$

Then we consider the expansion of $F^{(0)}$ in terms of kernels $C^{(0)}$ and $K^{(0)}$ which are two-particle irreducible (2PI) in the t-channel:

$$F^{(0)} = C^{(0)}(1 + K^{(0)} + K^{(0)}K^{(0)} + \ldots) \equiv C^{(0)} \, \mathcal{G}^{(0)} \,, \tag{2.19}$$

$$\mathcal{G}^{(0)} = 1 + K^{(0)} + K^{(0)}K^{(0)} + \ldots = \frac{1}{1 - K^{(0)}} \,. \tag{2.20}$$

In the axial gauge in which we are working, the 2PI amplitudes are free from mass singularities [33]. Therefore all the collinear divergences originate from the integrations over the momenta coming out from the kernels $K^{(0)}$ and connecting them to each other in the (process-independent) bare Green function $\mathcal{G}^{(0)}$. The factorization formula (2.15) can now be realized by introducing a suitable projection operator $\mathcal{P}_C$ which decouples $C^{(0)}$ and $\mathcal{G}^{(0)}$ in the spin indices and extracts the singular part of the $d^n k$ integrals (i.e. poles in ε) thus decoupling $C^{(0)}$ and $\mathcal{G}^{(0)}$ also in momentum space.

For each kernel, one can write the decomposition $K^{(0)} = (1 - \mathcal{P}_C)K^{(0)} + \mathcal{P}_C K^{(0)}$, where all the singularities are due to the second term on the r.h.s. Applying this procedure in an iterative way, one obtains

$$\mathcal{G}^{(0)} = \mathcal{G} \, \Gamma \tag{2.21}$$

where all the ε-poles have been subtracted from the 'renormalized' Green function $\mathcal{G}$:

$$\mathcal{G} = \frac{1}{1 - (1 - \mathcal{P}_C)K^{(0)}} = 1 + (1 - \mathcal{P}_C)K^{(0)} + (1 - \mathcal{P}_C)[K^{(0)}(1 - \mathcal{P}_C)K^{(0)}] + \ldots \,, \tag{2.22}$$

and associated with the transition function Γ

$$\Gamma = \frac{1}{1 - (\mathcal{P}_C K)} = 1 + (\mathcal{P}_C K) + (\mathcal{P}_C K)(\mathcal{P}_C K) + \ldots \,, \tag{2.23}$$

$$K \equiv K^{(0)} \, \mathcal{G} \,. \tag{2.24}$$

The coefficient function C in Eq. (2.15) is thus identified with

$$C = C^{(0)} \, \mathcal{G} \,. \tag{2.25}$$

In order to show the consistency of collinear and $k_{\perp}$-factorization (see Section 2.3), we have to recall the form of the projection operator $\mathcal{P}_C$ [25]. Let us denote its action on helicity and momentum space respectively by $P_C^{(s)}$ and $P_C^{(\varepsilon)}$, so that $\mathcal{P}_C = P_C^{(\varepsilon)} \otimes P_C^{(s)}$. If A and B are two kernels connected by a parton of momentum k (Fig. 3), the action
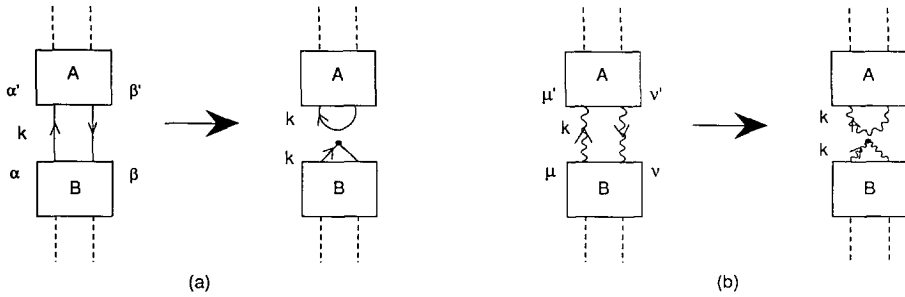
Fig. 3. Action of the collinear projector $\mathcal{P}_C$ on the spin indices of two kernels A and B in the cases of an intermediate (a) quark and (b) gluon state.

on the helicity space is

$$A \, P_C^{(s)} \, B = A(\ldots, k)_{\ldots \alpha' \beta'} \left(\frac{1}{2} \slashed{k} \right)_{\alpha' \beta'} \left(\frac{\slashed{\bar{p}}}{2 \bar{p} \cdot k} \right)_{\alpha \beta} B_{\alpha \beta \ldots}(k, \ldots) \,, \tag{2.26}$$

when the connecting parton is a quark and

$$A \, P_C^{(s)} \, B = A(\ldots, k)_{\ldots \mu' \nu'} \frac{d^{\mu' \nu'}(k)}{n - 2} (-g^{\mu \nu}) B_{\mu \nu \ldots}(k, \ldots) \,, \tag{2.27}$$

when the connecting parton is a gluon. The operator $P_C^{(\varepsilon)}$ sets $k^\mu = z p^\mu$ on the left-hand side (A), performs the $dk^2 d^{n-2}\mathbf{k}$ integration up to the factorization scale μ_F^2 on the right-hand side (B), and extracts the ensuing poles in ε.

Note from Eqs. (2.26), (2.27) that $P_C^{(s)}$ acts on the left by performing the average over the parton helicities in *n-dimensions*. Note also that $P_C^{(\varepsilon)}$ is not (at least in principle) unambiguously defined. The factorization of Eqs. (2.15), (2.21) in terms of collinear-finite and divergent (for $\varepsilon \to 0$) contributions can still be achieved if $P_C^{(\varepsilon)}$ extracts not only the ε-poles but also any finite contribution for $\varepsilon \to 0$. This leads to the factorization-scheme dependence of both anomalous dimensions and coefficient functions. The factorization scheme is completely specified once $P_C^{(\varepsilon)}$ has been uniquely defined. Equivalently, one can specify the explicit (and finite) ε-dependence of the transition functions Γ. One of the most commonly used scheme is the *modified minimal subtraction* ($\overline{\text{MS}}$) scheme, in which the transition functions have the form [25]

$$\Gamma_{ab,N}(\alpha_S(\mu_F^2/\mu^2)^\varepsilon, \varepsilon) = \left[P_\alpha \exp \left(\frac{1}{\varepsilon} \int_0^{\alpha_S(\mu_F^2/\mu^2)^\varepsilon S_\varepsilon} \frac{d\alpha}{\alpha} \gamma_N(\alpha) \right) \right]_{ab} \,. \tag{2.28}$$

In this equation we have reintroduced the explicit dependence on the parton labels and N-moment indices. The symbol P_α denotes, as usual, the path-ordered integration of the anomalous dimension matrix $\gamma_{ab,N}$. Note the presence of the ε-finite factor $S_\varepsilon = \exp\{-\varepsilon[\psi(1) + \ln 4\pi]\}$ ($\psi(z)$ is the Euler ψ-function), which characterizes the $\overline{\text{MS}}$-scheme. The only formal simplification we have used in Eq. (2.28) is that we have
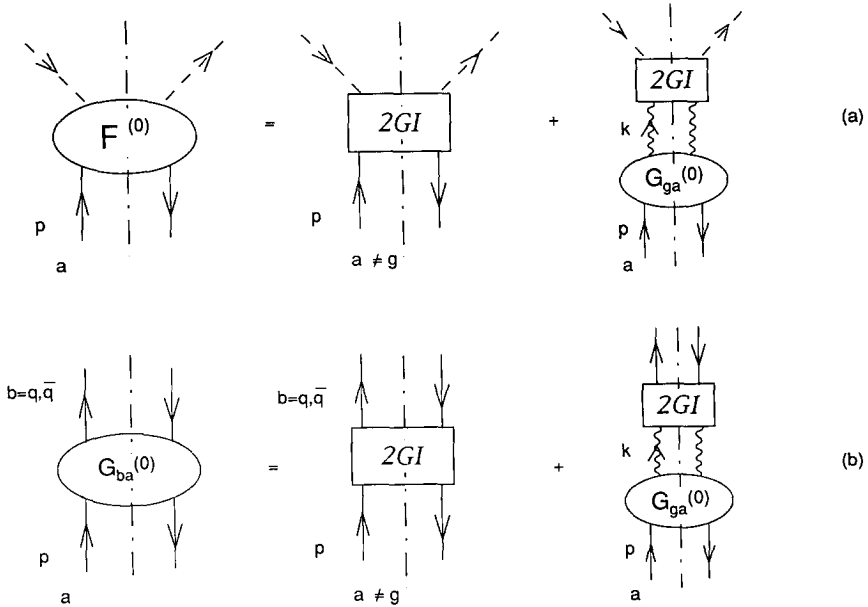
Fig. 4. Expansion in two-gluon irreducible (2GI) kernels at high energy for (a) the partonic cross section $F^{(0)}$ and (b) the (singlet) quark Green functions $\mathcal{G}_{qa}$, $\mathcal{G}_{\bar{q}a}$.

considered the case of a fixed coupling constant α_S. As shown in the following, this simplification is sufficient for the purposes of the present paper.

2.3. Power counting and factorization at high energy

The expansion in 2PI kernels introduced in the previous subsection is particularly useful to discuss the high-energy behaviour. High-energy (or small-x) logarithmic contributions are indeed generated by multiple gluon exchanges in the t-channel. Therefore we are led to consider kernels which are two-gluon irreducible (2GI).

For the parton cross section $F_a^{(0)}$ ($a = q_i, \bar{q}_i, g$), we single out the part which is 2GI by selecting the first (starting from above in Fig. 4a) two-gluon intermediate state. Considering the small-N limit in N-moment space, the 2GI kernel behaves as $\alpha_S^p (1 + \alpha_S + \ldots)$, where the first term corresponds to the tree-approximation and the remaining terms stand for corrections which are subleading at high energy. The large perturbative contributions $(\alpha_S/N)^k$ are thus generated precisely by k-integration from the ones in the gluon Green functions $\mathcal{G}_{ga}^{(0)}$ ($a = q_i, \bar{q}_i, g$). In particular, since flavour non-singlet parton cross sections (Appendix A) get no contribution from pure-gluon intermediate states, we can immediately conclude that *non-singlet* anomalous dimensions and coefficient functions are *regular* for $N \to 0$ order by order in α_S.

A decomposition similar to that for $F_a^{(0)}$ can be performed also for the (flavour singlet) quark Green function (Fig. 4b). Since the 2GI kernel behaves in this case as $\alpha_S (1 + \alpha_S + \ldots)$, we see that the quark anomalous dimensions contribute to *next-to-leading* terms $\alpha_S(\alpha_S/N)^k$ in the high-energy limit.

Note that the expansion in 2GI kernels is more general than that in 2PI kernels. The 2GI kernels in Fig. 4a and Fig. 4b can still be expanded respectively as $(C_a^{(0)} + \sum_{b \neq g} C_b^{(0)} K_{ba}^{(0)} + \ldots)$ and $(K_{ba}^{(0)} + \sum_{c \neq g} K_{bc}^{(0)} K_{ca}^{(0)} + \ldots)$. However, only the tree-level approximations for $C_a^{(0)}$ and $K_{ba}^{(0)}$ ($b \neq g$) contribute to leading order in α_S/N. A similar simplification does not occur for the gluon Green function $\mathcal{G}_{ga}^{(0)}$ because the gluon kernels $K_{ga}^{(0)}$ contain terms of the type $(\alpha_S/N)^k$ to any order k (modulo dynamical cancellations) in α_S.

The expansion in 2GI kernels described so far allows one a simple power counting at high energy. The next step towards high-energy factorization consists of decoupling the 2GI kernels and the gluon Green functions with α_S/N fixed. This factorization is conceptually different from the collinear one, where, roughly speaking, one expands in α_S (or ε) with α_S/ε fixed. In particular, we cannot perform the collinear limit $k^2 \rightarrow 0$ to any fixed order in α_S, because small-x contributions $(\alpha_S/N)^k$ are associated with any value of k [8]. On the other side, we do not want to spoil the collinear factorization, so that the high-energy factorization has to be valid for any value of ε (i.e., in any number of space-time dimensions). We are going to show that the $k_\perp$-factorization procedure introduced in Ref. [8] can be carried out consistently with all-order collinear factorization.

The high-energy limit of the product $A_g \mathcal{G}_{ga}^{(0)}$, A_g being the 2GI kernel involved in the decomposition of Fig. 4, was discussed in detail in [8]. Considering the decomposition of A_g in Lorentz-invariant amplitudes, it was shown that the leading high-energy behavior can be extracted via $k_\perp$-factorization, i.e. by inserting into the two-gluon intermediate state a $k_\perp$-dependent projection operator $\mathcal{P}_H$ as follows

$$A_g \, \mathcal{G}_{ga}^{(0)} = A_g \, \mathcal{P}_H \, \mathcal{G}_{ga}^{(0)} + \ldots . \tag{2.29}$$

As in the case of the collinear projector $\mathcal{P}_C$ in Section 2.2, we introduce the notation $\mathcal{P}_H = P_H^{(\varepsilon)} \otimes P_H^{(s)}$, where $P_H^{(s)}$ and $P_H^{(\varepsilon)}$ denote respectively the action of the high-energy projector $\mathcal{P}_H$ on helicity and momentum space. $P_H^{(s)}$ acts as follows

$$A_g \, P_H^{(s)} \, \mathcal{G}_{ga}^{(0)} = A(\ldots, k)_g^{\cdots \mu'\nu'} \frac{k_{\perp\mu'} k_{\perp\nu'}}{\mathbf{k}^2} (-g_{\mu\nu}) \mathcal{G}_{ga}^{(0)\mu\nu}(k, p) , \tag{2.30}$$

whilst $P_H^{(\varepsilon)}$ sets $k^\mu = zp^\mu + k_\perp^\mu$ on the left-hand side (A_g) and integrates the right-hand side ($\mathcal{G}_{ga}^{(0)}$) over the invariant mass k^2 at fixed $k_\perp$. Note that the $k_\perp$-dependence is left unaffected by $\mathcal{P}_H$ and, in particular, the 2GI kernel A_g has to be evaluated with an incoming off-shell (essentially transverse $k^2 = -\mathbf{k}^2$) gluon.

Equation (2.29) generalizes the $k_\perp$-factorization formula (1.2) to $n = 4 + 2\varepsilon$ space-time dimensions. The key point, however, is not just the formal resemblance between Eqs. (2.29), (2.30) and (1.2), but rather the fact that $\mathcal{P}_H$ selects the correct high-energy behaviour in any number of dimensions. We mean that, for instance, in performing the approximation (2.29), we are not neglecting any contribution of order $(\alpha_S/N)^k \cdot \varepsilon$ with respect to the four-dimensional case. This statement is a consequence of the fact that $\mathcal{P}_H$ is a 'true' projection operator:

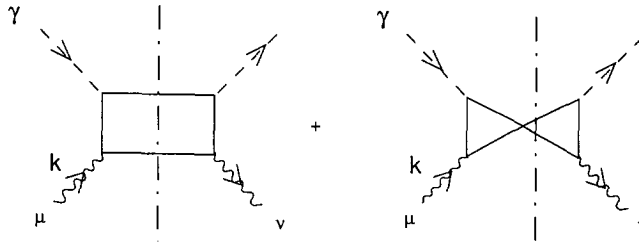$$\mathcal{P}_H^2 = \mathcal{P}_H , \tag{2.31}$$

Fig. 5. Quark-antiquark contribution to the lowest-order absorptive part $A_{\mu\nu}$ of the scattering amplitude $\gamma g \rightarrow \gamma g$.

and fulfils the property

$$\mathcal{P}_H \supseteq \mathcal{P}_C \quad (\mathcal{P}_H = \mathcal{P}_C \text{ iff } k_\perp = 0) . \tag{2.32}$$

Eqs. (2.31) and (2.32) are self-evident for the momentum space components $P_H^{(\varepsilon)}, P_C^{(\varepsilon)}$ and follows from the simple relations ($<>_\phi$ denotes the average over the $n-3$ azimuthal angles in the transverse momentum space)

$$(-g_{\mu\nu}) \frac{k_\perp^\mu k_\perp^\nu}{\mathbf{k}^2} = 1 , \quad < \frac{k_\perp^\mu k_\perp^\nu}{\mathbf{k}^2} >_\phi \stackrel{k_\perp \rightarrow 0}{=} \frac{d^{\mu\nu}(k = zp)}{n-2} , \tag{2.33}$$

for the spin components $P_H^{(s)}, P_C^{(s)}$.

Eq. (2.32) guarantees the consistency between high-energy factorization and collinear factorization. Due to Eq. (2.32) we can first perform the high-energy approximation in Eq. (2.29) and then proceed to the all-order factorization of collinear singularities by applying iteratively the collinear projector $\mathcal{P}_C$, as described in Section 2.2.

2.4. Resummation at high energy

In order to describe how in practice this procedure works and leads to the high-energy resummation, let us consider the case involving a 2GI kernel which is collinear safe. In particular, we refer to the heavy-flavour photoproduction process (1.1) already introduced in Section 1. The corresponding 2GI kernel A_g to lowest order in α_S is given in Fig. 5: the dashed line denotes the incoming on-shell photon and the full lines correspond to the heavy-flavour pair produced in the final state. Considering the case of an incoming-gluon partonic state and using the high-energy approximation in Eq. (2.29), we immediately obtain the factorized formula (1.2) (in $n = 4 + 2\varepsilon$ dimensions), which we can rewrite in the N-moment space as follows

$$4M^2 \sigma_N(M^2) = \int d^{2+2\varepsilon}\mathbf{k} \ \hat{\sigma}_N(\mathbf{k}^2/M^2, \alpha_S(M^2/\mu^2)^\varepsilon; \varepsilon)$$

$$\cdot \mathcal{F}_N^{(0)}(\mathbf{k}; \alpha_S, \mu, \varepsilon) \ \tilde{f}_{g,N}^{(0)}(\mu, \varepsilon) . \tag{2.34}$$

Here $\hat{\sigma} \sim A_{\mu\nu}(k) \ k_\perp^\mu k_\perp^\nu/\mathbf{k}^2$, $\tilde{f}_g^{(0)}$ is the bare gluon distribution and, according to the action of the high-energy projector $\mathcal{P}_H$, we have introduced the $k_\perp$-dependent gluon Green function

$$\mathcal{F}^{(0)}(z, \mathbf{k}; \alpha_S, \mu, \varepsilon) = \int \frac{dk^2}{2(2\pi)^{4+2\varepsilon}} \left(-g_{\mu\nu} \mathcal{G}_{gg}^{(0)\mu\nu}(k, p) \right) . \tag{2.35}$$

Eq. (2.34) has to be regarded as the bare (and collinear regularized) version of the factorization formula (1.2), in the sense that it still contains collinear poles in ε which, according to Eq. (2.21), can be factorized in the form

$$\mathcal{F}_N^{(0)}(\mathbf{k}; \alpha_S, \mu, \varepsilon) = \frac{\gamma_{gg,N}(\alpha_S)}{\pi \mathbf{k}^2} \, \tilde{R}_N(\mathbf{k}, \mu_F, \alpha_S; \mu, \varepsilon) \, \Gamma_{gg,N}(\alpha_S(\mu_F^2/\mu^2)^\varepsilon, \varepsilon) , \tag{2.36}$$

where the $\overline{\text{MS}}$-scheme gluon transition function is given by[4]

$$\Gamma_{gg,N}(\alpha_S, \varepsilon) = \exp \left\{ \frac{1}{\varepsilon} \int\limits_0^{\alpha_S S_\varepsilon} \frac{d\alpha}{\alpha} \, \gamma_{gg,N}(\alpha) \right\} . \tag{2.37}$$

The function $\tilde{R}_N$ in Eq. (2.36) has no ε-poles order by order in α_S, and cannot depend on the dimensional regularization scale μ in the limit $\varepsilon \to 0$. Moreover, since the l.h.s. of Eq. (2.36) is independent of the factorization scale μ_F, the only $\mathbf{k}$-dependence of $\tilde{R}_N$ allowed by dimensional arguments, for $\varepsilon = 0$, is the following

$$\tilde{R}_N(\mathbf{k}, \mu_F, \alpha_S; \mu, \varepsilon = 0) = R_N(\alpha_S) \, (\mathbf{k}^2/\mu_F^2)^{\gamma_{gg,N}(\alpha_S)} . \tag{2.38}$$

The reduced cross section $\hat{\sigma}$ in Eq. (2.34) is collinear safe because it corresponds to a 2PI kernel of the type $C^{(0)}$ in Eq. (2.19). Therefore, after using the transition function $\Gamma_{gg,N}$ in (2.36) to 'renormalize' the bare gluon density $\tilde{f}^{(0)}$ as in Eq. (2.16), we can safely perform the $\varepsilon \to 0$ limit in Eq. (2.34) and obtain

$$4M^2 \, \sigma_N(M^2) = C_N(\alpha_S, M^2/\mu_F^2) \, \tilde{f}_{g,N}(\mu_F^2) , \tag{2.39}$$

$$C_N(\alpha_S, M^2/\mu_F^2) = h_N(\gamma_{gg,N}(\alpha_S)) \, R_N(\alpha_S) \, (M^2/\mu_F^2)^{\gamma_{gg,N}(\alpha_S)} , \tag{2.40}$$

where the process-dependent part h_N of the coefficient function C_N is given by the following $k_\perp$-transform of the hard cross section $\hat{\sigma}$

$$h_N(\gamma) \equiv \gamma \int\limits_0^\infty \frac{d\mathbf{k}^2}{\mathbf{k}^2} \left(\frac{\mathbf{k}^2}{M^2} \right)^\gamma \, \hat{\sigma}_N(\mathbf{k}^2/M^2, \alpha_S; \varepsilon = 0) . \tag{2.41}$$

The result in Eq. (2.40) gives the resummed expression (including the dependence on the factorization scale μ_F^2) for the coefficient function C_N to the leading order $(\alpha_S/N)^k$, provided the gluon anomalous dimensions $\gamma_{gg,N}$ and the process-independent function R_N are known to the same accuracy. A similar result was first derived in Ref. [8]. The only difference with respect to [8] is that Eq. (2.40) takes into account the explicit dependence on the process-independent but *factorization-scheme dependent* function R_N.

---

[4] We limit ourselves to the case of a fixed coupling constant α_S. In fact, the running coupling effects lead to subleading contributions $\alpha_S(\alpha_S/N)^k$ and can thus be neglected in the present leading-logarithmic analysis.

This point is essential for precise phenomenological predictions at finite energies, when one has to combine the resummed coefficient function (2.40) with fixed-order non-logarithmic contributions computed in a well defined factorization scheme of collinear singularities. Moreover this issue is relevant to extending the high-energy resummation to subleading orders, where a corresponding scheme dependence of the anomalous dimensions comes into play (see, for instance, Section 5).

Note, however, that the presence of the process independent factor R_N in Eq. (2.40) is no longer relevant (at least, to leading order) if one limits oneself to considering only ratios of cross sections. In this case R_N and the factorization scale dependence cancel in the ratio, thus leading to an absolute prediction in terms of hard scales and $h_N(\gamma)$ functions of the type in Eq. (2.41) [8].

The high-energy contributions $(\alpha_S/N)^k$ are embodied in Eqs. (2.29), (2.34) through the $k_\perp$-integration of the gluon Green function $\mathcal{G}_{ga}^{(0)}$. As discussed in [8] and recalled in Section 1, in the case of four space-time dimensions the resummation of the leading terms $(\alpha_S/N)^k$ in the gluon density (anomalous dimensions) is accomplished by the BFKL equation [12]. The analogous master equation in $n = 4 + 2\varepsilon$ dimensions, which is necessary to compute both the anomalous dimensions and the function R_N, is derived and discussed in the following section.

3. The gluon Green functions

The leading high-energy behaviour of the gluon Green functions $\mathcal{G}_{gg}^{(0)}(k,p)$ and $\mathcal{G}_{gq}^{(0)}(k,p)$ can be easily derived by generalizing the soft-gluon insertion technique in Refs. [3,5,34] to the case of n dimensions. In the present paper we do not repeat all the detailed calculations described in [3,34], but we simply sketch the main steps and properties which are necessary for the n-dimensional generalization.

The starting observation is that the high-energy contributions $(\alpha_S/N)^k$ to the gluon Green function are produced by radiation of soft gluons, that is real and virtual gluons carrying a very small fraction x of the longitudinal momentum p of the incoming parton. Therefore, in the soft-gluon approach, the matrix element $M^{(k+1)}(k,p)$, contributing to $\mathcal{G}^{(0)}$ to the $(k+1)$−loop order, is obtained from $M^{(k)}$ by the insertion of an additional (real or virtual) soft gluon with momentum q. Using the soft approximation for vertices and propagators, this insertion can in turn be factorized, leading to the recurrence relation

$$|M^{(k+1)}(k,p)|^2 = g_s^2 \left\{ [M^{(k)}(k+q,p)]^\dagger [J_{\text{soft}}^{(R)}(q)]^2 M^{(k)}(k+q,p) \right.$$
$$\left. - [M^{(k)}(k,p)]^\dagger [J_{\text{soft}}^{(V)}(q)]^2 M^{(k)}(k,p) \right\} . \tag{3.1}$$

The explicit expressions for the real and virtual soft-gluon currents $J_{\text{soft}}^{(R)}$ and $J_{\text{soft}}^{(V)}$ can be found in Refs. [3,34]. The point we want to emphasize here is that they do not explicitly depend on the number n of space-time dimensions in which the soft-gluon factorization is carried out. As a matter of fact, the n-dependence of the matrix element $M^{(k+1)}$ can only be due to the spin structure of the vertices (the scalar propagators $i/(q^2 + i\epsilon)$ are the same in any number of dimensions). It turns out that a *single*

(essentially *eikonal*) helicity flow dominates to leading-logarithmic accuracy in the high-energy limit. Thus, the relevant QCD matrix elements are the same as for $n = 4$ and the only difference comes from the n-dimensional phase space over which Eq. (3.1) has to be integrated.

Using the recurrence relation (3.1), performing the sum over the number k of loops and following exactly the same steps as in Refs. [3,34], one obtains an integral equation for the gluon Green function $\mathcal{G}^{(0)}$. More precisely, considering the gluon density in Eq. (2.35), one finds ($\bar{\alpha}_S \equiv C_A \alpha_S/\pi$, $C_A = N_c$ being the number of colours)

$$
\mathcal{F}_N^{(0)}(\mathbf{k}; \alpha_S, \mu, \varepsilon) = \delta^{(2+2\varepsilon)}(\mathbf{k}) + \frac{\bar{\alpha}_S}{N} \int \frac{d^{2+2\varepsilon}\mathbf{q}}{(2\pi\mu)^{2\varepsilon}} \frac{1}{\pi\mathbf{q}^2} \left\{ \mathcal{F}_N^{(0)}(\mathbf{k} - \mathbf{q}; \alpha_S, \mu, \varepsilon) \right.
$$
$$
\left. - \frac{\mathbf{k} \cdot (\mathbf{k} - \mathbf{q})}{(\mathbf{k} - \mathbf{q})^2} \mathcal{F}_N^{(0)}(\mathbf{k}; \alpha_S, \mu, \varepsilon) \right\} . \tag{3.2}
$$

Some comments are in order. In the case of $n = 4$ dimensions ($\varepsilon = 0$), after azimuthal average over $\mathbf{q}$, Eq. (3.2) reproduces the BFKL equation [12,35]. Moreover, the integrand (not the phase space) of the homogeneous term in Eq. (3.2) is exactly the same as for the BFKL equation. As discussed above, this is a straightforward consequence of the dynamical dominance of a single gluon polarization to the present accuracy. Note, however, an essential difference with respect to the BFKL equation: the full kernel of Eq. (3.2) is not scale invariant. Indeed, scale invariance is broken by the dimensional regularization procedure and the eigenfunctions of the kernel are no longer simple powers $(\mathbf{k}^2)^\gamma$. From a physical viewpoint this means that whilst small and large transverse momentum regions contribute equally to the BFKL equation, they are now weighted asymmetrically. The breaking of scale invariance in Eq. (3.2) allows one to regularize the collinear singularities in terms of ε-poles. Once these poles have been factorized and subtracted, as in Eqs. (2.36), (2.38), one can perform the $\varepsilon \to 0$ limit and recover scale invariance at the expenses of an additional dependence on the factorization scale as dictated by the leading-twist behaviour of the parton densities (cross sections).

A further consequence of the lack of scale invariance in the kernel of the gluon Green function is that no simple technique is available to diagonalize the integral equation (3.2). However, it is possible (see Appendix B) to solve it as a formal power series in α_S with ε-dependent coefficients. The result reads as follows

$$
\mathcal{F}_N^{(0)}(\mathbf{k}; \mu, \alpha_S, \varepsilon) = \delta^{(2+2\varepsilon)}(\mathbf{k})
$$
$$
+ \frac{\Gamma(1+\varepsilon)}{(\pi\mathbf{k}^2)^{1+\varepsilon}} \sum_{k=1}^{\infty} \left[\frac{\bar{\alpha}_S}{N} S_\varepsilon \frac{e^{\varepsilon\psi(1)}}{\Gamma(1+\varepsilon)} \left(\frac{\mathbf{k}^2}{\mu^2} \right)^\varepsilon \right]^k c_k(\varepsilon) , \tag{3.3}
$$

where we have explicitly introduced the $\overline{\text{MS}}$-scheme factor $S_\varepsilon = \exp[-\varepsilon(\psi(1) + \ln 4\pi)]$, and the coefficients c_k are defined by the recurrence relation

$$
c_1(\varepsilon) = 1 , \qquad c_{k+1}(\varepsilon) = c_k(\varepsilon) I_k(\varepsilon) \quad (k \geq 1) , \tag{3.4}
$$

with ($\Gamma(z)$ is the Euler Γ-function)

$$I_k(\varepsilon) = \frac{1}{\varepsilon} \frac{\Gamma^2(1+\varepsilon)}{\Gamma(1+2\varepsilon)} \left[\frac{\Gamma(1+2\varepsilon)\,\Gamma(k\,\varepsilon)\,\Gamma(1-k\varepsilon)}{\Gamma((1+k)\,\varepsilon)\,\Gamma(1+(1-k)\,\varepsilon)} - \Gamma(1+\varepsilon)\,\Gamma(1-\varepsilon) \right] .$$

$$(3.5)$$

The gluon Green function (3.3) is formally a distribution acting on the transverse-momentum space in $n-2 = 2+2\varepsilon$ dimensions. Following the procedure of factorization of collinear singularities described in Section 2.2, one can extract from $\mathcal{F}_N^{(0)}$ the transition function $\Gamma_{gg,N}$ as in Eq. (2.36), thus leading to a 'renormalized' gluon density which, for finite ε, is a very cumbersome $k_\perp$-distribution. Therefore, it is more convenient to introduce the gluon Green function $G^{(0)}$ integrated up to the factorization scale $\mu_F^2 = Q^2$:

$$G_{gg,N}^{(0)}(\alpha_S\,(Q^2/\mu^2)^\varepsilon, \varepsilon) \equiv \int d^{2+2\varepsilon}\mathbf{k}\ \mathcal{F}_N^{(0)}(\mathbf{k}; \alpha_S, \mu, \varepsilon)\ \Theta(Q^2 - \mathbf{k}^2) .$$

$$(3.6)$$

Note that $G^{(0)}$ does not depend on α_S e Q^2/μ^2 independently, but only on the product $\alpha_S(Q^2/\mu^2)^\varepsilon$. Performing the $k_\perp$-integration of Eq. (3.3) we find

$$G_{gg,N}^{(0)}(\alpha_S\,(Q^2/\mu^2)^\varepsilon, \varepsilon) = 1 + \sum_{k=1}^{\infty} \left[\frac{\bar{\alpha}_S}{N}\,S_\varepsilon\,\frac{e^{\varepsilon\psi(1)}}{\Gamma(1+\varepsilon)} \left(\frac{Q^2}{\mu^2}\right)^\varepsilon \right]^k \frac{1}{k\,\varepsilon}\,c_k(\varepsilon) .$$

$$(3.7)$$

Introducing explicitly the parton indices, the collinear factorization in Eq. (2.21) reads $G_{gg,N}^{(0)} = \sum_a G_{ga,N}\Gamma_{ag,N}$. However, from the power counting in Section 2.3, we know that the quark anomalous dimensions $\gamma_{qa,N}$ (and hence the corresponding transition functions $\Gamma_{qa,N}$) are subleading at high energy. Therefore, to leading order in $(\alpha_S/N)^k$, we can write

$$G_{gg,N}^{(0)}(\alpha_S(Q^2/\mu^2)^\varepsilon, \varepsilon) = G_{gg,N}(\alpha_S(Q^2/\mu^2)^\varepsilon, \varepsilon)\ \Gamma_{gg,N}(\alpha_S(Q^2/\mu^2)^\varepsilon, \varepsilon) ,$$

$$(3.8)$$

where $\Gamma_{gg,N}$ is given by Eq. (2.37), in terms of the gluon anomalous dimension $\gamma_{gg,N}$. By comparing Eqs. (3.7) and (3.8), one can compute $\gamma_{gg,N}$ and the 'renormalized' Green function $G_{gg,N}$ for any value of ε. Moreover, from Eqs. (3.6) e (2.36) it follows that the function R_N in Eq. (2.38) is related to the $\varepsilon \to 0$ limit of $G_{gg,N}$

$$R_N(\alpha_S) = G_{gg,N}(\alpha_S(Q^2/\mu^2)^\varepsilon, \varepsilon)|_{\varepsilon=0} .$$

$$(3.9)$$

The recurrence factor $I_n(\varepsilon)$ behaves like $1/\varepsilon$ for $\varepsilon \to 0$. Therefore, in agreement with Eqs. (2.37) and (3.8), the power series expansion (3.7) has at most a single ε-pole for any power of α_S. More precisely, for $\varepsilon \to 0$ we have

$$I_k(\varepsilon) \simeq \frac{1}{k\,\varepsilon}\left(1 + \mathcal{O}(\varepsilon^3)\right) \qquad (\varepsilon \to 0) ,$$

$$(3.10)$$

and correspondingly

$$G^{(0)}_{gg,N}(\alpha_S(Q^2/\mu^2)^\varepsilon,\varepsilon) = 1 + \sum_{k=1}^{\infty} \left[\frac{1}{\varepsilon} \frac{\bar{\alpha}_S}{N} S_\varepsilon \frac{e^{\varepsilon\psi(1)}}{\Gamma(1+\varepsilon)} \left(\frac{Q^2}{\mu^2}\right)^\varepsilon \right]^k \frac{1}{k!} \left(1 + \mathcal{O}(\varepsilon^3)\right)$$

$$\simeq \exp\left[\frac{1}{\varepsilon} \frac{\bar{\alpha}_S}{N} S_\varepsilon \frac{e^{\varepsilon\psi(1)}}{\Gamma(1+\varepsilon)} \left(\frac{Q^2}{\mu^2}\right)^\varepsilon \right]$$

$$\cdot \left(1 + \mathcal{O}\left(\left(\frac{\bar{\alpha}_S}{N}\right)^3\right) + \mathcal{O}\left(\frac{1}{\varepsilon}\left(\frac{\bar{\alpha}_S}{N}\right)^4\right)\right). \tag{3.11}$$

Comparing Eqs. (3.8) and (3.11), we see that the slow departure of $I_k(\varepsilon)$ from its leading-pole approximation $I_k(\varepsilon) \simeq 1/k\varepsilon$ gives rise to subleading collinear corrections, for both $\gamma_{gg,N}$ and R_N, which are of relative order α_S^3 in the high-energy regime:

$$\gamma_{gg,N} = \frac{\bar{\alpha}_S}{N}\left[1 + \mathcal{O}\left(\left(\frac{\bar{\alpha}_S}{N}\right)^3\right)\right], \quad R_N = 1 + \mathcal{O}\left(\left(\frac{\bar{\alpha}_S}{N}\right)^3\right). \tag{3.12}$$

Terms of relative order α_S and α_S^2 are present in the various Feynman diagrams contributing to the gluon Green function but they cancel in the sum. This cancellation, which is automatically embodied in the master equation (3.2), does no longer occur in higher perturbative orders. The resummed expressions for $\gamma_{gg,N}$ and R_N to the leading accuracy $(\alpha_S/N)^n$ are derived in Appendix B. The results are the following.

The dominant contribution $\gamma_N(\alpha_S)$ to the gluon anomalous dimensions

$$\gamma_{gg,N} = \gamma_N(\alpha_S) + \mathcal{O}(\alpha_S(\alpha_S/N)^k) \tag{3.13}$$

is obtained by solving the implicit equation (Fig. 6)

$$1 = \frac{\bar{\alpha}_S}{N} \chi(\gamma_N(\alpha_S)), \tag{3.14}$$

where the characteristic function $\chi(\gamma)$ is expressed in terms of the Euler ψ-function

$$\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma) = \frac{1}{\gamma}\left[1 + \sum_{k=1}^{\infty} 2\,\zeta(2k+1)\,\gamma^{2k+1}\right] \tag{3.15}$$

and $\zeta(n)$ is the Riemann ζ-function. The solution of Eq. (3.14) in power series of the coupling constant gives [1] ($\zeta(3) \simeq 1.202$, $\zeta(5) \simeq 1.037$):

$$\gamma_N(\alpha_S) = \sum_{n=1}^{\infty} g_n \left(\frac{\bar{\alpha}_S}{N}\right)^n = \frac{\bar{\alpha}_S}{N} + 2\zeta(3)\left(\frac{\bar{\alpha}_S}{N}\right)^4 + 2\zeta(5)\left(\frac{\bar{\alpha}_S}{N}\right)^6 + \mathcal{O}\left(\left(\frac{\bar{\alpha}_S}{N}\right)^7\right). \tag{3.16}$$

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[1] The explicit values of the coefficients g_n, up to $n = 14$, can be found in Ref. [5].
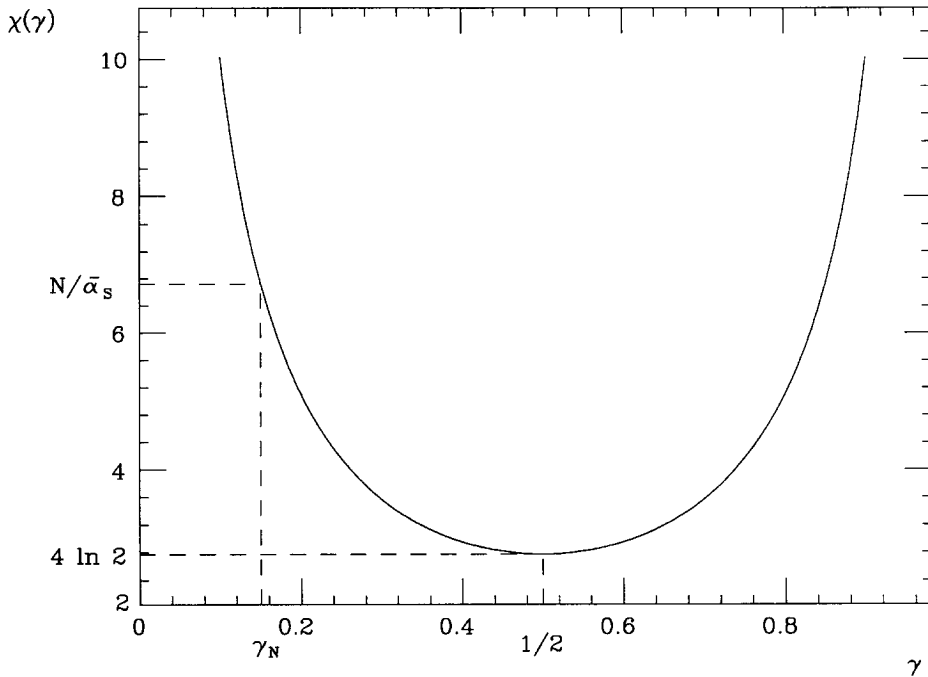
Fig. 6. The BFKL characteristic function $\chi(\gamma)$ for $0 < \gamma < 1$. γ_N is the BFKL anomalous dimension.

The function $R_N(\alpha_S)$ is given by

$$R_N(\alpha_S) = \left\{ \frac{\Gamma(1 - \gamma_N)\,\chi(\gamma_N)}{\Gamma(1 + \gamma_N)\,[-\gamma_N\,\chi'(\gamma_N)]} \right\}^{\frac{1}{2}}$$

$$\cdot \exp\left\{ \gamma_N\,\psi(1) + \int\limits_0^{\gamma_N} d\gamma\, \frac{\psi'(1) - \psi'(1 - \gamma)}{\chi(\gamma)} \right\} , \qquad (3.17)$$

where χ and χ' are the characteristic function in Eq. (3.15) and its first derivative, respectively. The α_S-dependence of the r.h.s. in Eq. (3.17) is implicit in that of the gluon anomalous dimension $\gamma_N = \gamma_N(\alpha_S)$. The first perturbative terms are ($\zeta(4) = 2\zeta^2(2)/5 \simeq 1.082$):

$$R_N(\alpha_S) = 1 + \frac{8}{3}\zeta(3)\left(\frac{\bar{\alpha}_S}{N}\right)^3 - \frac{3}{4}\zeta(4)\left(\frac{\bar{\alpha}_S}{N}\right)^4 + \frac{22}{5}\zeta(5)\left(\frac{\bar{\alpha}_S}{N}\right)^5 + \mathcal{O}\left(\left(\frac{\bar{\alpha}_S}{N}\right)^6\right) .$$

$$(3.18)$$

Some comments are in order.

The result in Eqs. (3.13), (3.14) for the gluon anomalous dimension is exactly the celebrated BFKL anomalous dimension [12]. In the present paper this result has been derived by consistently carrying out the procedure of factorization of the collinear singularities in dimensional regularization. However it is worth noting that the same expression for the anomalous dimension can be obtained by using alternative and less

sophisticated regularization prescriptions of the collinear singularities (for instance, considering Eq. (3.2) directly in $n = 4$ dimensions and regularizing it by keeping the incoming gluon slightly off-shell, via the replacement $\delta^{(2)}(\mathbf{k}) \rightarrow \delta(\mathbf{k}^2 - \mu^2)/\pi$ in the inhomogeneous term [34,35]). The reason for this has to be traced back to the property of the kernel in the master equation (3.2) of being collinear regular for $\varepsilon \rightarrow 0$: the collinear divergences in the solution of Eq. (3.2) originate only from the fact that the inhomogeneous term is not sufficiently smooth for $\mathbf{k} \rightarrow 0$. As a result, in the small-N limit, the gluon anomalous dimensions to leading accuracy $(\alpha_S/N)^k$ are *regularization/factorization scheme independent* (and related via Eq. (3.14) to the eigenvalues $\chi(\gamma)$ of the BFKL kernel) within a wide class of schemes. This class includes all the schemes which do not introduce pathologically singular terms of the type α_S^k/N^{k+p} $(p \geq 1)$ in the perturbative calculation at high energy [2], or, more precisely, which do not violate the high-energy power counting discussed in Section 2.3.

The BFKL anomalous dimension in Eqs. (3.14), (3.16) departs rather slowly from its one-loop contribution. However for very small values of x, corresponding to sizeable values of $\bar{\alpha}_S/N \sim \mathcal{O}(1)$, γ_N increases quite fast and for $\bar{\alpha}_S/N = (4 \ln 2)^{-1}$ reaches the saturation value $\gamma = 1/2$ at which $\chi(\gamma)$ has a minimum (Fig. 6). For still larger values of $\bar{\alpha}_S/N$ there are two complex conjugate branches of γ_N, coming from the pinching with the symmetrical solution of Eq. (3.14) at $\gamma = 1 - \gamma_N$. Therefore the resummation of the singular terms $(\alpha_S/N)^k$ builds up a stronger singularity at $N = \lambda \equiv 4 \bar{\alpha}_S \ln 2 \simeq 2.65 \alpha_S$ [12]. As discussed in Section 1, this branch point singularity (known as the *perturbative QCD pomeron*) is responsible for the steep behaviour $x^{-\lambda}$ of the gluon density $\tilde{f}_g(x, \mu^2)$, generated by the perturbative QCD evolution at high scale μ^2.

The function $R_N(\alpha_S)$ in Eq. (3.17), on the other hand, depends on the factorization scheme more than the gluon anomalous dimensions. For instance, going from the $\overline{\text{MS}}$-scheme result (3.17) to the MS scheme (in this scheme the transition function Γ_{gg} is obtained by setting $S_\varepsilon = 1$ on the r.h.s. of Eq. (2.37)), $R_N(\alpha_S)$ has to be multiplied by the factor $\exp[-\gamma_N(\alpha_S)(\psi(1) + \ln 4\pi)]$. This scheme dependence of $R_N(\alpha_S)$ has to be compensated in physical observables by subleading contributions of order $\alpha_S(\alpha_S/N)^k$ in the anomalous dimensions.

Since $R_N(\alpha_S)$ is related to the $\varepsilon \rightarrow 0$ limit of the renormalized gluon Green function $G_{gg,N}$ (see Eq. (3.9)), it can be regarded as the normalization factor of the perturbative QCD pomeron. Note, in particular, that $R_N(\alpha_S)$ is singular at the saturation value $\gamma = 1/2$ of the BFKL anomalous dimension $\gamma_N(\alpha_S)$:

$$R_N(\alpha_S) \simeq \text{const.} \left(\frac{1}{1 - 2\gamma_N(\alpha_S)}\right)^{\frac{1}{2}} \qquad (\gamma_N \rightarrow 1/2) . \qquad (3.19)$$

This behaviour is related to the branch point singularity of the BFKL anomalous dimension at $N = 4 \bar{\alpha}_S \ln 2$ and signals the ultimate failure of the leading-twist approach to the mass singularity factorization and the onset of the multi-Regge factorization at extreme energies (see Ref. [8] for a more detailed discussion of this issue).

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[2] This feature of the gluon anomalous dimensions was first pointed out by T. Jaroszewicz [36].

We have so far considered the leading high-energy behaviour of the gluon Green function $\mathcal{G}_{gg}^{(0)}$. In order to complete our discussion on the gluon channel, we conclude this section by examining the Green function $\mathcal{G}_{gq}^{(0)}$ with an incoming quark or antiquark.

Let us denote by $\mathcal{F}_q^{(0)}$ the $k_\perp$-dependent Green function defined by replacing $\mathcal{G}_{gg}^{(0)}$ by $\mathcal{G}_{gq}^{(0)}$ on the r.h.s. of Eq. (2.35). The corresponding master equation to leading accuracy $(\alpha_S/N)^k$ is the following

$$\mathcal{F}_{q,N}^{(0)}(\mathbf{k}; \alpha_S, \mu, \varepsilon) = \frac{C_F}{C_A} \frac{\bar{\alpha}_S}{N} \frac{1}{(2\pi\mu)^{2\varepsilon}} \frac{1}{\pi\mathbf{k}^2} + \frac{\bar{\alpha}_S}{N} \int \frac{d^{2+2\varepsilon}\mathbf{q}}{(2\pi\mu)^{2\varepsilon}} \frac{1}{\pi\mathbf{q}^2}$$
$$\cdot \left\{ \mathcal{F}_{q,N}^{(0)}(\mathbf{k} - \mathbf{q}; \alpha_S, \mu, \varepsilon) \right.$$
$$\left. - \frac{\mathbf{k} \cdot (\mathbf{k} - \mathbf{q})}{(\mathbf{k} - \mathbf{q})^2} \mathcal{F}_{q,N}^{(0)}(\mathbf{k}; \alpha_S, \mu, \varepsilon) \right\}. \tag{3.20}$$

Comparing Eqs. (3.20) and (3.2), we see that $\mathcal{F}_{q,N}^{(0)}$ and $\mathcal{F}_N^{(0)}$ fulfil the same integral equation, apart from a different inhomogeneous term. In particular, it is straightforward to check that the solution of Eq. (3.20) can be expressed in terms of the pure-gluon Green function $\mathcal{F}_N^{(0)}$ as follows

$$\mathcal{F}_{q,N}^{(0)}(\mathbf{k}; \alpha_S, \mu, \varepsilon) = \frac{C_F}{C_A} \left[\mathcal{F}_N^{(0)}(\mathbf{k}; \alpha_S, \mu, \varepsilon) - \delta^{(2+2\varepsilon)}(\mathbf{k}) \right]. \tag{3.21}$$

The $k_\perp$-integrated Green function $G_{gq}^{(0)}$ is thus given by

$$G_{gq,N}^{(0)}(\alpha_S(Q^2/\mu^2)^\varepsilon, \varepsilon) \equiv \int d^{2+2\varepsilon}\mathbf{k}\, \mathcal{F}_{q,N}^{(0)}(\mathbf{k}; \alpha_S, \mu, \varepsilon)\, \Theta(Q^2 - \mathbf{k}^2)$$
$$= \frac{C_F}{C_A} \left[G_{gg,N}^{(0)}(\alpha_S(Q^2/\mu^2)^\varepsilon, \varepsilon) - 1 \right]. \tag{3.22}$$

The result (3.22) allows us to compute the anomalous dimensions $\gamma_{gq,N}$ by performing simple algebraic manipulations. We have first to factorize from $G_{gq,N}^{(0)}$ the collinear singularities according to Eq. (2.21). Introducing explicitly the parton indices, $G_{gq,N}^{(0)} = \sum_a G_{ga,N}\Gamma_{aq,N}$, and using the fact that the quark transition functions Γ_{qb} are subleading at high energy, we obtain:

$$G_{gq,N}^{(0)}(\alpha_S(Q^2/\mu^2)^\varepsilon, \varepsilon) = G_{gg,N}(\alpha_S(Q^2/\mu^2)^\varepsilon, \varepsilon)\, \Gamma_{gq,N}(\alpha_S(Q^2/\mu^2)^\varepsilon, \varepsilon)$$
$$+ G_{gq,N}(\alpha_S(Q^2/\mu^2)^\varepsilon, \varepsilon). \tag{3.23}$$

Inserting Eqs. (3.23) and (3.8) into Eq. (3.22) we get the identity

$$C_A \left[G_{gg,N}(\alpha_S, \varepsilon)\, \Gamma_{gq,N}(\alpha_S, \varepsilon) + G_{gq,N}(\alpha_S, \varepsilon) \right]$$
$$= C_F \left[G_{gg,N}(\alpha_S, \varepsilon)\, \Gamma_{gg,N}(\alpha_S, \varepsilon) - 1 \right], \tag{3.24}$$

which we can rewrite in the following form

$$[C_A \, G_{gq,N}(\alpha_S,\varepsilon) - C_F \, (G_{gg,N}(\alpha_S,\varepsilon) - 1)] \, \frac{1}{G_{gg,N}(\alpha_S,\varepsilon)}$$

$$= C_F \, [\Gamma_{gg,N}(\alpha_S,\varepsilon) - 1] - C_A \, \Gamma_{gq,N}(\alpha_S,\varepsilon) \, . \tag{3.25}$$

We now notice that, order by order in perturbation theory, the renormalized Green functions G_{ab} are regular for $\varepsilon \to 0$ whilst the transition functions Γ_{ab} are series of poles. Therefore the only solution to Eq. (3.25) is

$$G_{gq,N}(\alpha_S,\varepsilon) = \frac{C_F}{C_A} \, [G_{gg,N}(\alpha_S,\varepsilon) - 1] \, , \tag{3.26}$$

$$\Gamma_{gq,N}(\alpha_S,\varepsilon) = \frac{C_F}{C_A} \, [\Gamma_{gg,N}(\alpha_S,\varepsilon) - 1] \, . \tag{3.27}$$

In particular, since to leading accuracy $(\alpha_S/N)^k$ we have

$$\Gamma_{gq,N}(\alpha_S,\varepsilon) = \frac{1}{\varepsilon} \int_0^{\alpha_S \, S_\varepsilon} \frac{d\alpha}{\alpha} \, [\gamma_{gq,N}(\alpha) + \gamma_{gg,N}(\alpha) \, \Gamma_{gq,N}(\alpha,\varepsilon)] \, , \tag{3.28}$$

$$\Gamma_{gg,N}(\alpha_S,\varepsilon) = 1 + \frac{1}{\varepsilon} \int_0^{\alpha_S} \frac{d\alpha}{\alpha} \, \gamma_{gg,N}(\alpha) \, \Gamma_{gg,N}(\alpha,\varepsilon) \, , \tag{3.29}$$

from Eq. (3.27) we obtain:

$$\gamma_{gq,N}(\alpha_S) = \frac{C_F}{C_A} \, \gamma_{gg,N}(\alpha_S) + \mathcal{O}(\alpha_S(\alpha_S/N)^k)$$

$$= \frac{C_F}{C_A} \, \gamma_N(\alpha_S) + \mathcal{O}(\alpha_S(\alpha_S/N)^k) \, , \tag{3.30}$$

$\gamma_N(\alpha_S)$ being the BFKL anomalous dimension. We see that the coefficients of the leading terms $(\alpha_S/N)^k$ in $\gamma_{ga,N}$ $(a = q,g)$ are equal, apart from an overall factor given by the ratio of the colour charges of the initial-state parton a.

4. The quark Green functions

From the high energy power counting discussed in Section 2.3, we know that the gluon channel dominates to leading logarithmic accuracy in the small-x regime. Beyond leading order, however, the quark sector has to be considered on an equal footing with the gluon sector. Actually, from a phenomenological viewpoint, the knowledge of the next-to-leading quark anomalous dimensions may be more relevant than that of the corresponding corrections to the gluon anomalous dimensions. The reason for this is that the most accurate information on small-x parton densities is coming out from HERA data [37] on deep inelastic structure functions, which couple directly to quarks (and not gluons).
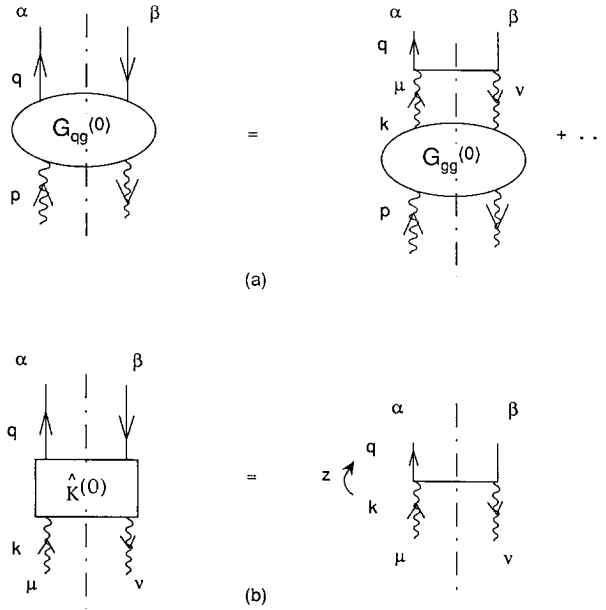
Fig. 7. (a) High-energy factorization of the gluon → quark Green function $\mathcal{G}_{qg}$ and (b) the corresponding off-shell kernel $\hat{K}_{qg}$.

In order to compute the (flavour-singlet) quark anomalous dimensions we consider the Green function $\mathcal{G}_{qa}^{(0)}$:

$$\mathcal{G}_{qa}^{(0)} = \sum_b K_{qb}^{(0)} \, \mathcal{G}_{ba}^{(0)} \,, \tag{4.1}$$

or, more precisely, its expansion in 2GI kernels (Fig. 4b), and we apply the high-energy factorization formula (2.29). We thus arrive at the analogue of the $k_\perp$-factorization formula (2.34):

$$\mathcal{G}_{qg}^{(0)\,\alpha\beta}(q,p) = \int d^{2+2\varepsilon}\mathbf{k} \int_0^1 \frac{dy}{y} \left(\hat{K}_{\mu\nu}^{(0)\,\alpha\beta}(q,k) \frac{k_\perp^\mu \, k_\perp^\nu}{\mathbf{k}^2} \right)_{|\,k=yp+k_\perp}$$

$$\cdot \, \mathcal{F}^{(0)}(y,\mathbf{k};\alpha_S,\mu,\varepsilon) \,, \tag{4.2}$$

where $\hat{K}^{(0)}$ (Fig. 7) is the 2GI kernel $K_{qg}^{(0)}$ to the lowest order in α_S (μ, ν and α, β are respectively the spin indices of the incoming gluon and outgoing quark). Moreover, since we are interested in the anomalous dimensions rather than the coefficient function, it is also convenient to apply the collinear projector $\mathcal{P}_C$ and consider (as in the gluon sector) the Green function $G_{qg}^{(0)}$ integrated over $q_\perp$ and the invariant mass q^2 up to the factorization scale $\mu_F^2 = Q^2$ (we use the Sudakov parametrization $q^\mu = x\,p^\mu + q_\perp^\mu + (q^2 + \mathbf{q}^2)\,\bar{p}^\mu/2\,x\,p\cdot\bar{p}$)

$$\mathcal{P}_C \, \mathcal{G}_{qg}^{(0)} \equiv G_{qg}^{(0)}(x, \alpha_S(Q^2/\mu^2)^\varepsilon, \varepsilon)$$

$$= \int \frac{dq^2 \, d^{2+2\varepsilon}\mathbf{q}}{2(2\pi)^{4+2\varepsilon}} \left(\frac{\bar{\slashed{p}}}{2\bar{p} \cdot q} \right)_{\alpha\beta} \mathcal{G}_{qg}^{(0) \, \alpha\beta}(q, p) \; \Theta\left(Q^2 - |q^2|\right) . \tag{4.3}$$

Inserting Eq. (4.2) into Eq. (4.3) and taking the N-moments we get

$$G_{qg,N}^{(0)}(\alpha_S(Q^2/\mu^2)^\varepsilon, \varepsilon)$$

$$= \int d^{2+2\varepsilon}\mathbf{k} \; \hat{K}_{qg,N}(\mathbf{k}^2/Q^2, \alpha_S(Q^2/\mu^2)^\varepsilon; \varepsilon) \; \mathcal{F}_N^{(0)}(\mathbf{k}; \alpha_S, \mu, \varepsilon) . \tag{4.4}$$

The factorization formula (4.4) relates the quark Green function to the $k_\perp$-distribution $\mathcal{F}^{(0)}$ in the same way as Eq. (2.34) relates the heavy-flavour cross section to the gluon density. The *off-shell* kernel $\hat{K}_{qg,N}$, obtained from $\hat{K}^{(0)}(q, k)$ after integration over q and explicitly calculated in Appendix C, is indeed regular for $N \to 0$ so that the resummation of the singular terms $(\alpha_S/N)^k$ in $G_{qg,N}^{(0)}$ is achieved by $k_\perp$-integration of the corresponding terms in $\mathcal{F}^{(0)}$. Note, however, an essential physical difference with respect to Eq. (2.34). Unlike the hard cross section $\hat{\sigma}_N$, the off-shell kernel $\hat{K}_{qg,N}$ is *not* collinear safe. The collinear singularity arises from the integration of the splitting process $g \to q\bar{q}$ in Fig. 7b and shows up as an ε-pole in the n-dimensional on-shell case $\mathbf{k}^2 = 0$:

$$\hat{K}_{qg}(z, \mathbf{k}^2 = 0, \alpha_S(Q^2/\mu^2)^\varepsilon; \varepsilon) = z \, \frac{\alpha_S}{2\pi} \, S_\varepsilon \, \frac{e^{\varepsilon\psi(1)}}{\Gamma(1+\varepsilon)} \left(\frac{(1-z)Q^2}{\mu^2} \right)^\varepsilon$$

$$\cdot \frac{1}{\varepsilon} T_R \, \frac{(1-z)^2 + z^2 + \varepsilon}{1+\varepsilon} \quad , \tag{4.5}$$

or, alternatively, as a logarithmic divergence in the on-shell limit $\mathbf{k}^2 \to 0$ for the four dimensional case:

$$\hat{K}_{qg}\left(z, \mathbf{k}^2/Q^2, \alpha_S, \varepsilon = 0\right) = z \, \frac{\alpha_S}{2\pi} \, \Theta\left(Q^2 - z\,\mathbf{k}^2\right) \cdot T_R\left[\left((1-z)^2 + z^2\right) \ln \frac{Q^2}{z\,\mathbf{k}^2} \right.$$

$$\left. - (1 - 6z(1-z)) \left(1 - \frac{z\,\mathbf{k}^2}{Q^2}\right) \right] . \tag{4.6}$$

Note that the coefficients of both the ε-pole and $\ln(Q^2/\mathbf{k}^2)$-term are correctly proportional to the n-dimensional Altarelli-Parisi splitting function $P_{qg}^{(0)}$ in one-loop order:

$$P_{qg}^{(0)}(z; \varepsilon) = T_R \, \frac{(1-z)^2 + z^2 + \varepsilon}{1+\varepsilon} . \tag{4.7}$$

Actually, the full off-shell kernel $\hat{K}_{qg,N}$ in Eq. (4.4) can consistently be interpreted as the integral of a generalized *off-shell* (and positive definite!) splitting function $\hat{P}_{qg}^{(0)}$. The explicit calculation in Appendix C gives

$$\hat{K}_{qg}(z, \mathbf{k}^2/Q^2, \alpha_S(Q^2/\mu^2)^\varepsilon; \varepsilon) = z \, \Theta \left(Q^2 - z \, \mathbf{k}^2\right) \int\limits_0^{(1-z)(Q^2 - z\mathbf{k}^2)} \frac{d\tilde{\mathbf{q}}^2}{\tilde{\mathbf{q}}^2} \left(\frac{\tilde{\mathbf{q}}^2}{\mu^2}\right)^\varepsilon$$

$$\cdot \frac{\alpha_S}{2\pi} S_\varepsilon \frac{e^{\varepsilon \psi(1)}}{\Gamma(1+\varepsilon)} \hat{P}_{qg}^{(0)}(z, \mathbf{k}^2/\tilde{\mathbf{q}}^2; \varepsilon) , \qquad (4.8)$$

where the splitting function is

$$\hat{P}_{qg}^{(0)}(z, \mathbf{k}^2/\tilde{\mathbf{q}}^2; \varepsilon) = T_R \left(\frac{\tilde{\mathbf{q}}^2}{\tilde{\mathbf{q}}^2 + z(1-z)\mathbf{k}^2}\right)^2$$

$$\cdot \left[\frac{(1-z)^2 + z^2 + \varepsilon}{1+\varepsilon} + 4z^2(1-z)^2 \frac{\mathbf{k}^2}{\tilde{\mathbf{q}}^2}\right] , \qquad (4.9)$$

and $\tilde{\mathbf{q}} = \mathbf{q} - z\,\mathbf{k}$ denotes the boost-invariant (along the k-direction) transverse momentum transferred in the splitting process $g \to q\bar{q}$ of Fig. 7b.

The fact that the kernel $\hat{K}_{qg}$ is not collinear safe prevents us from taking the $\varepsilon \to 0$ limit in Eq. (4.4) even after having factorized the transition function $\Gamma_{gg,N}$ as in Eqs. (2.34), (2.36). The non-polynomial ε-dependence of $\hat{K}_{qg}$ (due to both $\hat{P}_{qg}^{(0)}$ and the phase space integration in Eq. (4.8)), as much as the off-shell dependence of $\hat{P}_{qg}^{(0)}$, are thus responsible for non-trivial (and factorization scheme dependent) quark anomalous dimensions $\gamma_{qg,N}$ in higher perturbative orders.

Using the power series solution (3.3) for the gluon distribution $\mathcal{F}_N^{(0)}$ and performing the $k_\perp$-integration in Eq. (4.4), we can determine the corresponding power series expansion for the quark Green function $G_{qg,N}^{(0)}$. Considering the $N \to 0$ limit for $\hat{K}_{qg,N}$ (i.e., neglecting subdominant corrections at high energy) we find (see Appendix C)

$$G_{qg,N}^{(0)}(\alpha_S(Q^2/\mu^2)^\varepsilon, \varepsilon) = \frac{\alpha_S}{2\pi} T_R \, S_\varepsilon \frac{e^{\varepsilon \psi(1)}}{\Gamma(1+\varepsilon)} \left(\frac{Q^2}{\mu^2}\right)^\varepsilon \frac{1}{\varepsilon} \frac{4+\varepsilon}{(3+\varepsilon)(2+\varepsilon)}$$

$$\cdot \left\{1 + \sum_{k=1}^\infty \left[\frac{\bar{\alpha}_S}{N} S_\varepsilon \frac{e^{\varepsilon \psi(1)}}{\Gamma(1+\varepsilon)} \left(\frac{Q^2}{\mu^2}\right)^\varepsilon\right]^k \frac{1}{k\varepsilon} d_k(\varepsilon)\right\}$$

$$+ \mathcal{O}\left(\alpha_S^2(\alpha_S/N)^k\right) , \qquad (4.10)$$

where the coefficients $d_k(\varepsilon)$ are expressed in terms of $c_k(\varepsilon)$ in Eq. (3.4) as follows

$$d_k(\varepsilon) = \frac{1}{k+1} \frac{4+(1-3k)\varepsilon}{4+\varepsilon}$$

$$\cdot \frac{\Gamma(1+k\varepsilon)\,\Gamma(1-k\varepsilon)\,\Gamma(1+\varepsilon)\,\Gamma(4+\varepsilon)}{\Gamma(1+(1+k)\varepsilon)\,\Gamma(4+(1-k)\varepsilon)} c_k(\varepsilon) . \qquad (4.11)$$

On the other side, the procedure of factorization of the collinear singularities in Eq. (2.21) gives $G_{qg,N}^{(0)} = \sum_a G_{qa,N} \Gamma_{ag,N}$, which, neglecting terms of order $\alpha_S^2(\alpha_S/N)^k$, can be written as

$$G_{qg,N}^{(0)}(\alpha_S(Q^2/\mu^2)^\varepsilon,\varepsilon) = G_{qg,N}(\alpha_S(Q^2/\mu^2)^\varepsilon,\varepsilon)\,\Gamma_{gg,N}(\alpha_S(Q^2/\mu^2)^\varepsilon,\varepsilon)$$

$$+\,\Gamma_{qg,N}(\alpha_S(Q^2/\mu^2)^\varepsilon,\varepsilon)\;, \tag{4.12}$$

$$\Gamma_{qg,N}(\alpha_S,\varepsilon) = \frac{1}{\varepsilon}\int\limits_0^{\alpha_S S_\varepsilon}\frac{d\alpha}{\alpha}\,\gamma_{qg,N}(\alpha)\,\Gamma_{gg,N}(\alpha,\varepsilon)\;. \tag{4.13}$$

Remember that $G_{qg,N}$ is finite for $\varepsilon \to 0$ order by order in α_S. The structure of the ε-poles on the r.h.s. of Eq. (4.12) thus defines $G_{qg,N}$ and the transition function $\Gamma_{qg,N}$ uniquely. It follows that, comparing the expansion (4.10) with Eqs. (4.12), (4.13) and using the known result for $\Gamma_{gg,N}$ (i.e. the leading-order gluon anomalous dimension (3.13)), we can compute the anomalous dimension $\gamma_{qg,N}$ (and the renormalized Green function $G_{qg,N}$) order by order in perturbation theory.

We have explicitly performed this calculation up to the sixth order and the result for the quark anomalous dimensions in the $\overline{\text{MS}}$ scheme reads

$$\gamma_{qg,N}(\alpha_S) = \frac{\alpha_S}{2\pi}T_R\frac{2}{3}\left\{1 + \frac{5}{3}\frac{\bar\alpha_S}{N} + \frac{14}{9}\left(\frac{\bar\alpha_S}{N}\right)^2 + \left[\frac{82}{81} + 2\zeta(3)\right]\left(\frac{\bar\alpha_S}{N}\right)^3 + \left[\frac{122}{243}\right.\right.$$

$$+\,\frac{25}{6}\zeta(3)\right]\left(\frac{\bar\alpha_S}{N}\right)^4 + \left[\frac{146}{729} + \frac{14}{3}\zeta(3) + 2\zeta(5)\right]\left(\frac{\bar\alpha_S}{N}\right)^5$$

$$+\,\mathcal{O}\left(\left(\frac{\bar\alpha_S}{N}\right)^6\right)\right\}$$

$$\simeq \frac{\alpha_S}{2\pi}T_R\frac{2}{3}\left\{1 + 1.67\frac{\bar\alpha_S}{N} + 1.56\left(\frac{\bar\alpha_S}{N}\right)^2 + 3.42\left(\frac{\bar\alpha_S}{N}\right)^3\right.$$

$$+\,5.51\left(\frac{\bar\alpha_S}{N}\right)^4 + 7.88\left(\frac{\bar\alpha_S}{N}\right)^5 + \mathcal{O}\left(\left(\frac{\bar\alpha_S}{N}\right)^6\right)\right\}\;. \tag{4.14}$$

The calculation is straightforward but already very cumbersome to this order in perturbation theory[1], and we are not able to provide an explicit resummed formula for γ_{qg} to all orders (some all-order features of γ_{qg} are presented in Appendix C). However we emphasize that this is just an (open) algebraic problem related to the use of the $\overline{\text{MS}}$ factorization scheme, that is the highly non-trivial ε-dependence of the coefficients $d_k(\varepsilon)$ in Eq. (4.11) and the non-local structure of the ε-poles in Eq. (4.13). The series (4.10) for the quark Green function indeed contains all the necessary information on the anomalous dimensions $\gamma_{qg,N}$ to any perturbative order $\alpha_S(\alpha_S/N)^k$. In the next section we show that, choosing a different factorization scheme of mass singularities, we can explicitly resum all the next-to-leading terms $\alpha_S(\alpha_S/N)^k$ in $\gamma_{qg,N}$.

The coefficients of the first two terms in the curly bracket of Eq. (4.14) agree with the known one- and two-loop anomalous dimensions in the $\overline{\text{MS}}$ scheme [27,28]. The higher-order contributions can be used to estimate the effect of these small-x

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[1] Note that, because of the cancellations involved in the gluon sector (see Eqs. (3.10), (3.12)), the first three coefficients in the perturbative expansion (4.14) are entirely related to the ε-dependence of the off-shell kernel $\hat{K}_{qg,N}$.

corrections at intermediate values of x. In particular, the $\mathcal{O}(\alpha_S^3)$ term in Eq. (4.14) can be combined with the existing $\mathcal{O}(\alpha_S^2)$ calculations of the coefficient functions for the DIS [30] and Drell-Yan [31] processes in order to check the stability of the fixed-order perturbative expansion in the x-range accessible at present. Note also that, unlike the case of the gluon anomalous dimensions (3.13), (3.16), all the perturbative coefficients in Eq. (4.14) are non-vanishing (and positive). Therefore in the quark sector one may expect [22] (and finds [20]) an earlier departure from the fixed-order perturbative behaviour.

The flavour-singlet anomalous dimension $\gamma_{qq,N}^S$ (Appendix A) starts in order α_S^2 and its resummed expression at high energy is related in a simple way to that for $\gamma_{qg,N}$, namely

$$\gamma_{qq,N}^S(\alpha_S) = \frac{C_F}{C_A}\left[\gamma_{qg,N}(\alpha_S) - \frac{\alpha_S}{2\pi}T_R\frac{2}{3}\right] + \mathcal{O}(\alpha_S^2\,(\alpha_S/N)^k)\ . \tag{4.15}$$

The derivation of Eq. (4.15) is similar to that of the analogous Eq. (3.30) for the gluon sector. One has to apply the high-energy factorization procedure of Section 2.3 to the Green function $G_{qq}^{S\,(0)}$ (here, the superscript S denotes the fact that the incoming and outgoing quarks carry a different flavour), thus obtaining the following $k_\perp$-factorization formula

$$G_{qq,N}^{S\,(0)}(\alpha_S(Q^2/\mu^2)^\varepsilon, \varepsilon)$$
$$= \int d^{2+2\varepsilon}\mathbf{k}\ \hat{K}_{qg,N}(\mathbf{k}^2/Q^2, \alpha_S(Q^2/\mu^2)^\varepsilon; \varepsilon)\ \mathcal{F}_{q,N}^{(0)}(\mathbf{k}; \alpha_S, \mu, \varepsilon)\ . \tag{4.16}$$

This equation differs from Eq. (4.4) only by the replacement of the pure-gluon $k_\perp$-distribution $\mathcal{F}_N^{(0)}$ by $\mathcal{F}_{q,N}^{(0)}$ (see Eq. (3.20)). Therefore, using Eq. (3.21), one can relate the two Green functions $G_{qq}^{S\,(0)}$ and $G_{qg}^{(0)}$:

$$G_{qq,N}^{S\,(0)}\left(\alpha_S\left(\frac{Q^2}{\mu^2}\right)^\varepsilon, \varepsilon\right) = \frac{C_F}{C_A}\left[G_{qg,N}^{(0)}\left(\alpha_S\left(\frac{Q^2}{\mu^2}\right)^\varepsilon, \varepsilon\right)\right.$$
$$\left. - \hat{K}_{qg,N}\left(\mathbf{k}=0, \alpha_S\left(\frac{Q^2}{\mu^2}\right)^\varepsilon, \varepsilon\right)\right]\ . \tag{4.17}$$

The factorization of collinear singularities in $G_{qq,N}^{S\,(0)}$ is achieved by the following high-energy relations

$$G_{qq,N}^{S\,(0)}(\alpha_S, \varepsilon) = \Gamma_{qq,N}^S(\alpha_S, \varepsilon) + G_{qq,N}^S(\alpha_S, \varepsilon) + G_{qg,N}(\alpha_S, \varepsilon)\,\Gamma_{gq,N}(\alpha_S, \varepsilon)\ , \tag{4.18}$$

$$\Gamma_{qq,N}^S(\alpha_S, \varepsilon) = \frac{1}{\varepsilon}\int\limits_0^{\alpha_S\,S_\varepsilon}\frac{d\alpha}{\alpha}\left[\gamma_{qq,N}^S(\alpha) + \gamma_{qg,N}(\alpha)\,\Gamma_{gq,N}(\alpha, \varepsilon)\right]\ . \tag{4.19}$$

Using Eqs. (4.18), (4.19), (3.27) one can express $G_{qq}^{S\,(0)}$ as a function of G_{qq}^S, γ_{qq}^S, G_{qg}, γ_{qg}, Γ_{gg}. Analogously, by means of Eqs. (4.12), (4.13) one can express $G_{qg}^{(0)}$

as a function of G_{qg}, γ_{qg}, Γ_{gg}. Inserting the expressions obtained in this way into Eq. (4.17), one ends up with the equation

$$G_{qq,N}^{S}(\alpha_S, \varepsilon) + \frac{1}{\varepsilon} \int\limits_{0}^{\alpha_S S_\varepsilon} \frac{d\alpha}{\alpha} \gamma_{qq,N}^{S}(\alpha)$$

$$= \frac{C_F}{C_A} \left[G_{qg,N}(\alpha_S, \varepsilon) + \frac{1}{\varepsilon} \int\limits_{0}^{\alpha_S S_\varepsilon} \frac{d\alpha}{\alpha} \gamma_{qg,N}(\alpha) - \hat{K}_{qg,N}(\mathbf{k}=0, \alpha_S; \varepsilon) \right], \qquad (4.20)$$

from which the result (4.15) follows.

It is worth noting that the all-order relation (4.15) between $\gamma_{qq,N}^{S}$ and $\gamma_{qg,N}$, as well as its leading-order analogue (3.30), have been derived using only algebraic identities at high energy, with no reference to the details of the dimensional-regularization prescription. In particular they remain true using dimensional reduction [38], a dimensional regularization scheme which explicitly respects supersymmetric Ward identities. In this sense, Eqs. (3.30) and (4.15) can be considered as high-energy limits of the supersymmetry identity $\gamma_{qq} + \gamma_{gq} = \gamma_{qg} + \gamma_{gg}$ valid in $N = 1$ supersymmetric Yang-Mills theory, i.e. for $C_F = T_R = C_A$. It follows that in the supersymmetric case the gluon anomalous dimensions γ_{gg} and γ_{gq} coincide also at the next-to-leading level $\alpha_S(\alpha_S/N)^k$. This property may be useful as a technical tool to check and simplify the calculation of the (still unknown) next-to-leading corrections in the gluon sector.

5. Deep inelastic scattering at small x

5.1. Structure functions and parton densities

The cross section for deep inelastic lepton-hadron scattering is given in terms of the customary structure functions $F_i(x, Q^2)$ ($i = 1, 2, 3$). Here Q^2 denotes the square of the momentum transferred by the scattered lepton and x is the Bjorken variable. In the following we only consider the scattering process occurring through the exchange of a single photon. As far as the hadronic component is concerned, this approximation simply amounts to neglecting F_3, which is a non-singlet structure function and, hence, non-singular at small x (see Section 2.3). We also present our results in terms of the structure functions F_2 and F_L, $F_L(x, Q^2) = F_2(x, Q^2) - 2xF_1(x, Q^2)$ being the longitudinal structure function.

In the naïve parton model the DIS structure functions are related to the parton densities of the incoming hadron as follows (e_i are the quark charges)

$$F_2(x, Q^2) = \sum_{i=1}^{N_f} e_i^2 \left[\tilde{f}_{q_i}(x, Q^2) + \tilde{f}_{\bar{q}_i}(x, Q^2) \right] \quad , \qquad (5.1)$$

$$F_L(x, Q^2) = 0 \quad . \qquad (5.2)$$

Taking into account perturbative QCD corrections in leading-twist order and using the factorization theorem of mass singularities, Eqs. (5.1), (5.2) become

$$F_i(x, Q^2) = \frac{1}{N_f} \left(\sum_{j=1}^{N_f} e_j^2 \right) F_i^S(x, Q^2) + F_i^{NS}(x, Q^2) \quad (i = 2, L) , \tag{5.3}$$

where the singlet and non-singlet components F^S and F^{NS} are given by

$$
\begin{aligned}
F_i^S(x, Q^2) = \int_x^1 \frac{dz}{z} \; & [C_i^S(z; \alpha_S(\mu_F^2), Q^2/\mu_F^2) \; \tilde{f}_S(x/z, \mu_F^2) \\
& + C_i^g(z; \alpha_S(\mu_F^2), Q^2/\mu_F^2) \; \tilde{f}_g(x/z, \mu_F^2)] ,
\end{aligned}
\tag{5.4}
$$

$$F_i^{NS}(x, Q^2) = \int_x^1 \frac{dz}{z} \; C_i^{NS}(z; \alpha_S(\mu_F^2), Q^2/\mu_F^2) \sum_{j=1}^{N_f} e_j^2 \; \tilde{f}_{q_j}^{(+)}(x/z, \mu_F^2) . \tag{5.5}$$

Here $\tilde{f}_S$ and $\tilde{f}_{q_j}^{(+)}$ are respectively the singlet and non-singlet quark densities defined in Appendix A, and C_i^A ($A = S, g, NS$) are the coefficient functions [1] computable as power series in $\alpha_S(\mu_F^2)$. Note that, because of the naïve parton model relation (5.1), we have $C_2^A(z) = \delta(1 - z) + \mathcal{O}(\alpha_S)$ for $A = S, NS$, and $C_2^g(z) = \mathcal{O}(\alpha_S)$. Therefore also the gluon density $\tilde{f}_g$ contributes to F_2 beyond the lowest order. Moreover, since $C_L^A = \mathcal{O}(\alpha_S)$ is not vanishing, the Callan-Gross relation (5.2) is violated.

At present, the coefficient functions C_i^A are completely known up to $\mathcal{O}(\alpha_S^2)$ [30,39]. The non-singlet coefficients C_i^{NS} are not enhanced by $\ln x$ terms in higher orders. The all-order resummation of the logarithmic contributions $\alpha_S^{n+2}\ln^n x$ (or $\alpha_S(\alpha_S/N)^{n+1}$ in the N-moment space) to C_i^g and C_i^S is performed in the next subsection.

5.2. Coefficient functions

The coefficients functions C_i^A are evaluated starting from the expression $F_i = \sum_a F_{ia}^{(0)} \tilde{f}_a^{(0)}$ of the hadronic structure functions F_i in terms of the partonic structure functions $F_{ia}^{(0)}$, and then performing the collinear factorization as in Eq. (2.15). Considering first the gluon structure functions $F_{ig}^{(0)}$, we have

$$F_{ig}^{(0)} = \frac{1}{N_f} \left(\sum_{j=1}^{N_f} e_j^2 \right) [C_i^g \Gamma_{gg} + C_i^S 2 N_f \Gamma_{qg}] . \tag{5.6}$$

On the other side, from the high-energy factorization in Section 2.3 (see Fig. 4a), we obtain

---

[1] The coefficient functions in Eqs. (5.4), (5.5) are normalized according to our notation in Section 2.1 (see, in particular, Eq. (2.11)). This normalization differs from that often used in the literature [30].

$$F_{ig}^{(0)}(x; \alpha_S(Q^2/\mu^2)^\varepsilon, \varepsilon)$$

$$= \frac{1}{N_f} \left(\sum_{j=1}^{N_f} e_j^2 \right) \int d^{2+2\varepsilon}\mathbf{k}$$

$$\cdot \int_x^1 \frac{dz}{z} \, \hat{\sigma}_i^g(z, \mathbf{k}^2/Q^2, \alpha_S(Q^2/\mu^2)^\varepsilon; \varepsilon) \, \mathcal{F}_N^{(0)}(x/z, \mathbf{k}; \alpha_S, \mu, \varepsilon) \,, \tag{5.7}$$

where $\mathcal{F}^{(0)}$ is the $k_\perp$-dependent gluon Green function in Eq. (2.35), and $\hat{\sigma}_i^g$ are obtained by applying the high-energy projector $\mathcal{P}_H$ to the lowest-order $q\bar{q}$ contribution to the $\gamma^*g \to \gamma^*g$ absorptive part $A_{\mu\nu}$ (Fig. 5), as in Eq. (2.30). The off-shell cross sections $\hat{\sigma}_i^g$ have been explicitly evaluated in Refs. [8] ($i = 2$) and [32] ($i = L$) for the case of massive quarks and $n = 4$ dimensions. The generalization to the massless case and $n = 4 + 2\varepsilon$ dimensions is straightforward.

Eq. (5.7) is the dimensionally-regularized version of the $k_\perp$-factorization formula (1.2). We can thus properly address the issue of collinear-singularity factorization. The main point to be noticed is that the off-shell cross sections $\hat{\sigma}_L^g$ and $\hat{\sigma}_2^g$ have a different collinear behaviour. As a consequence of the Callan-Gross relation (5.2) (i.e., the fact that on-shell partons do not couple directly to a longitudinally polarized photon), $\hat{\sigma}_L^g$ is collinear safe. Its on-shell and $\varepsilon = 0$ limit is indeed finite:

$$\hat{\sigma}_L^g(z, \mathbf{k} = 0, \alpha_S(Q^2/\mu^2)^\varepsilon; \varepsilon)$$

$$= \frac{\alpha_S}{2\pi} S_\varepsilon \left(\frac{Q^2}{\mu^2} \right)^\varepsilon N_f T_R \left[8 z^2 (1 - z) + \mathcal{O}(\varepsilon) \right] \,. \tag{5.8}$$

On the contrary, $\hat{\sigma}_2^g$ is not collinear safe and its on-shell limit has an ε-pole proportional to the lowest-order Altarelli-Parisi splitting function $P_{qg}^{(0)}$ in Eq. (4.7):

$$\hat{\sigma}_2^g(z, \mathbf{k} = 0, \alpha_S(Q^2/\mu^2)^\varepsilon; \varepsilon)$$

$$= \frac{\alpha_S}{2\pi} S_\varepsilon \left(\frac{Q^2}{\mu^2} \right)^\varepsilon N_f \frac{1}{\varepsilon} \left[z P_{qg}^{(0)}(z; \varepsilon) + \mathcal{O}(\varepsilon) \right] \,. \tag{5.9}$$

Therefore, the factorization structures (5.7) for $F_{Lg}^{(0)}$ and $F_{2g}^{(0)}$ are respectively analogous to Eq. (2.34) for the heavy-flavour case and Eq. (4.4) for the quark Green function. The factorization of collinear singularities has to be carried out accordingly.

We start considering the longitudinal structure function. In the high-energy limit, both the coefficient function C_L^S and the transition function Γ_{qg} are of order $\alpha_S(\alpha_S/N)^k$ ($k \geq 0$). Therefore from Eq. (5.6) we have

$$F_{Lg}^{(0)} = \frac{1}{N_f} \left(\sum_{j=1}^{N_f} e_j^2 \right) \left[C_L^g \Gamma_{gg} + \mathcal{O}\left(\alpha_S^2 (\alpha_S/N)^k \right) \right] \,, \tag{5.10}$$

and we see that, in order to compute C_L^g, we have to factorize Γ_{gg} on the r.h.s. of Eq. (5.7). Using the factorization formula (2.36) and proceeding as in Section 2.4,

we arrive at the following expression for the N-moments of the longitudinal coefficient function

$$C_{L,N}^g(\alpha_S, Q^2/\mu_F^2) = h_{L,N}(\gamma_N(\alpha_S))\ R_N(\alpha_S)\ (Q^2/\mu_F^2)^{\gamma_N(\alpha_S)} + \mathcal{O}(\alpha_S^2(\alpha_S/N)^k)\,, \tag{5.11}$$

where the function $h_{L,N}(\gamma)$ is given in terms of the off-shell cross section $\hat{\sigma}_L^g$:

$$h_{L,N}(\gamma) = \gamma \int\limits_0^\infty \frac{d\mathbf{k}^2}{\mathbf{k}^2} \left(\frac{\mathbf{k}^2}{Q^2}\right)^\gamma \hat{\sigma}_{L,N}^g(\mathbf{k}^2/Q^2, \alpha_S; \varepsilon = 0)\,. \tag{5.12}$$

The explicit evaluation of the function $h_{L,N}(\gamma)$ requires the knowledge of the off-shell cross section $\hat{\sigma}_L^g$ in $n = 4$ dimensions. The analogous calculation for massive quarks has been performed in great detail in Refs. [8,32]. Therefore in this paper we limit ourselves to presenting only the final result. In particular, since the off-shell cross section $\hat{\sigma}_L^g$ in Eq. (5.7) vanishes uniformly in $k_\perp$ in the high-energy limit $z \to 0$ (as discussed in Section 2.3, this property follows from the fact that only 2GI kernels contribute to $\hat{\sigma}_L^g$), the function $h_{L,N}(\gamma)$ is weakly N-dependent:

$$h_{L,N}(\gamma) = h_L(\gamma)\,(1 + \mathcal{O}(N)) \quad (N \to 0)\,. \tag{5.13}$$

In order to evaluate the dominant terms $\alpha_S(\alpha_S/N)^k$ in Eq. (5.11), we can thus set $N = 0$ in Eqs. (5.12), (5.13). Computing explicitly the corresponding function $h_L(\gamma)$, we find

$$h_L(\gamma) = \frac{\alpha_S}{2\pi}\, N_f\, T_R\, \frac{4\,(1-\gamma)}{3-2\gamma}\, \frac{\Gamma^3(1-\gamma)\,\Gamma^3(1+\gamma)}{\Gamma(2-2\gamma)\,\Gamma(2+2\gamma)}\,. \tag{5.14}$$

The results in Eqs. (5.11), (5.14) give the resummed expression for the coefficient function $C_{L,N}^g$ in the $\overline{\text{MS}}$ scheme to the logarithmic accuracy $\alpha_S(\alpha_S/N)^k$, including the corresponding dependence on the factorization scale μ_F. The resummation effect is incorporated in (5.11) through the (α_S/N)-dependence of the BFKL anomalous dimension (3.14) and the γ-dependence of R_N and $h_{L,N}$ as given by Eqs. (3.17) and (5.12).

Let us now consider the structure function F_2. As noted above, the pattern of collinear singularities in Eq. (5.7) is similar to that in the corresponding Eq. (4.4) for the quark Green function. Eq. (5.7) thus contains all the relevant information on both the DIS coefficient function C_2^g and the quark anomalous dimensions γ_{qg}. As a matter of fact, since in the small-N limit $C_{2,N}^S = 1 + \mathcal{O}(\alpha_S(\alpha_S/N)^k)$, from Eq. (5.6) we obtain the analogue of Eq. (4.12):

$$F_{2g,N}^{(0)}(\alpha_S(Q^2/\mu^2)^\varepsilon, \varepsilon) = \frac{1}{N_f} \left(\sum_{j=1}^{N_f} e_j^2\right) \left[C_{2,N}^g(\alpha_S(\mu_F^2/\mu^2)^\varepsilon, Q^2/\mu_F^2; \varepsilon)\right.$$
$$\cdot\ \Gamma_{gg,N}(\alpha_S(\mu_F^2/\mu^2)^\varepsilon, \varepsilon) + 2\,N_f\,\Gamma_{qg,N}(\alpha_S(\mu_F^2/\mu^2)^\varepsilon, \varepsilon)$$
$$\left. +\ \mathcal{O}(\alpha_S^2(\alpha_S/N)^k)\right]\,. \tag{5.15}$$

Starting from Eq. (5.7) and factorizing the collinear singularities according to Eq. (5.15), one can compute $C_{2,N}^g$ and $\gamma_{qg,N}$ order by order in perturbation theory.

Obviously, this algebraic problem is by no means simpler than that encountered in Section 4 for the evaluation of $\gamma_{qg,N}$ and we are not able to provide an explicit resummed formula for $C_{2,N}^g$. Nonetheless, Eq. (5.7) can be used in a very simple way for deriving an all-order relation between $C_{2,N}^g$ and $\gamma_{qg,N}$.

The main observation is that, setting $\mu_F^2 = Q^2$ on the r.h.s. of Eq. (5.15) and performing the derivative with respect to Q^2, we obtain a factorized structure similar to Eq. (5.10):

$$\frac{\partial}{\partial \ln Q^2} F_{2g,N}^{(0)}(\alpha_S(Q^2/\mu^2)^\varepsilon, \varepsilon)$$

$$= \frac{1}{N_f} \left(\sum_{j=1}^{N_f} e_j^2 \right) \left[\gamma_{gg,N}(\alpha_S(Q^2/\mu^2)^\varepsilon) \, C_{2,N}^g(\alpha_S(Q^2/\mu^2)^\varepsilon, 1; \varepsilon) \right.$$

$$\left. + 2N_f \, \gamma_{qg,N}(\alpha_S(Q^2/\mu^2)^\varepsilon, \varepsilon) + \varepsilon \, \alpha_S \frac{\partial}{\partial \alpha_S} C_{2,N}^g(\alpha_S(Q^2/\mu^2)^\varepsilon, 1; \varepsilon) \right]$$

$$\cdot \Gamma_{gg,N}(\alpha_S(Q^2/\mu^2)^\varepsilon, \varepsilon) + \mathcal{O}(\alpha_S^2(\alpha_S/N)^k) \,. \tag{5.16}$$

Here, all the collinear singularities are factorized into the gluon transition function Γ_{gg} whereas the term in the square bracket, much like C_L^g in Eq. (5.10), is finite as $\varepsilon \to 0$. Stated differently, the off-shell kernel $\partial \hat{\sigma}_2^g / \partial \ln Q^2$ in Eq. (5.7) (although not $\hat{\sigma}_2^g$ itself) is collinear safe, and its $\varepsilon = 0$ limit is related to the linear combination $\gamma_{gg} C_2^g + 2N_f \gamma_{qg}$. Therefore, proceeding as in the case of the longitudinal structure function, we obtain

$$\gamma_N(\alpha_S) \, C_{2,N}^g(\alpha_S, Q^2/\mu_F^2 = 1) + 2N_f \, \gamma_{qg,N}(\alpha_S) = h_{2,N}(\gamma_N(\alpha_S)) \, R_N(\alpha_S)$$
$$+ \mathcal{O}(\alpha_S^2(\alpha_S/N)^k) \,, \tag{5.17}$$

where R_N is given in Eq. (3.17), $\gamma_N(\alpha_S)$ is the BFKL anomalous dimension and the function $h_{2,N}(\gamma)$ is

$$h_{2,N}(\gamma) = \gamma \int\limits_0^\infty \frac{d\mathbf{k}^2}{\mathbf{k}^2} \left(\frac{\mathbf{k}^2}{Q^2} \right)^\gamma \frac{\partial}{\partial \ln Q^2} \hat{\sigma}_{2,N}^g(\mathbf{k}^2/Q^2, \alpha_S; \varepsilon = 0) \,. \tag{5.18}$$

The off-shell kernel $\partial \hat{\sigma}_{2,N}^g / \partial \ln Q^2$ in $n = 4$ dimensions has been computed in Ref. [8] for the case of massive quarks. Performing the massless limit, we find

$$h_{2,N}(\gamma) = h_2(\gamma) \, (1 + \mathcal{O}(N)) \quad (N \to 0) \,, \tag{5.19}$$

$$h_2(\gamma) = \frac{\alpha_S}{2\pi} N_f T_R \frac{2 \, (2 + 3\gamma - 3\gamma^2)}{3 - 2\gamma} \frac{\Gamma^3(1-\gamma) \, \Gamma^3(1+\gamma)}{\Gamma(2-2\gamma) \, \Gamma(2+2\gamma)} \,. \tag{5.20}$$

The results in Eqs. (5.17), (5.19), (5.20) provide the explicit resummation of the contributions $\alpha_S (\alpha_S/N)^k$ to the coefficient function $C_{2,N}^g$ in terms of $h_2(\gamma_N)$ and the quark anomalous dimensions $\gamma_{qg,N}$. In particular, using the six-loop expression (4.14)

for $\gamma_{qg,N}$, one can compute $C^g_{2,N}$ up to the five-loop order. The dependence of $C^g_{2,N}$ on the factorization scale μ^2_F is given by

$$
\begin{aligned}
C^g_{2,N}(\alpha_S, Q^2/\mu^2_F) &= C^g_{2,N}(\alpha_S, Q^2/\mu^2_F = 1)\left(\frac{Q^2}{\mu^2_F}\right)^{\gamma_N(\alpha_S)} \\
&\quad + 2N_f\frac{\gamma_{qg,N}(\alpha_S)}{\gamma_N(\alpha_S)}\left[\left(\frac{Q^2}{\mu^2_F}\right)^{\gamma_N(\alpha_S)} - 1\right] \\
&\quad + \mathcal{O}(\alpha^2_S(\alpha_S/N)^k) .
\end{aligned}
$$
(5.21)

The singlet coefficient functions C^S_L, C^S_2 in Eq. (5.4) can be evaluated starting from parton structure functions $F^{(0)}_{iq}$ with an incoming quark. These structure functions fulfil a $k_\perp$-factorization formula similar to Eq. (5.7) with the replacement $\mathcal{F}^{(0)} \to \mathcal{F}^{(0)}_q$, $\mathcal{F}^{(0)}_q$ being the $k_\perp$-distribution with an incoming quark in Eq. (3.20). One can apply the algebraic manipulations analogous to those used in Sections 3 and 4 for evaluating γ_{gq} and γ^S_{qq}, thus obtaining the following colour charge relations

$$
C^{PS}_{L,N}\left(\alpha_S, \frac{Q^2}{\mu^2_F}\right) = \frac{C_F}{C_A}\left[C^g_{L,N}\left(\alpha_S, \frac{Q^2}{\mu^2_F}\right) - \frac{\alpha_S}{2\pi}N_f T_R\frac{4}{3}\right] + \mathcal{O}\left(\alpha^2_S(\alpha_S/N)^k\right) ,
$$
(5.22)

$$
\begin{aligned}
C^{PS}_{2,N}\left(\alpha_S, \frac{Q^2}{\mu^2_F}\right) &= \frac{C_F}{C_A}\left[C^g_{2,N}\left(\alpha_S, \frac{Q^2}{\mu^2_F}\right) - \frac{\alpha_S}{2\pi}N_f T_R\frac{2}{3}(1 + 2\ln\frac{Q^2}{\mu^2_F})\right] \\
&\quad + \mathcal{O}\left(\alpha^2_S(\alpha_S/N)^k\right) .
\end{aligned}
$$
(5.23)

Here, for the sake of convenience, we have introduced the pure-singlet coefficient functions $C^{PS}_i = C^S_i - C^{NS}_i$. They have the same singular small-x behaviour as the singlet functions C^S_i and start in $\mathcal{O}(\alpha^2_S)$ in perturbation theory.

The perturbative expansions of the resummed results derived in this section read:

$$
\begin{aligned}
C^g_{L,N}(\alpha_S, Q^2/\mu^2_F = 1) &= \frac{\alpha_S}{2\pi}T_R N_f\frac{4}{3}\Bigg\{1 - \frac{1}{3}\frac{\bar{\alpha}_S}{N} \\
&\quad + \left[\frac{34}{9} - \zeta(2)\right]\left(\frac{\bar{\alpha}_S}{N}\right)^2 + \left[-\frac{40}{27} + \frac{1}{3}\zeta(2) + \frac{8}{3}\zeta(3)\right]\left(\frac{\bar{\alpha}_S}{N}\right)^3 \\
&\quad + \left[\frac{1216}{81} - \frac{34}{9}\zeta(2) - \frac{14}{9}\zeta(3) - 6\zeta(4)\right]\left(\frac{\bar{\alpha}_S}{N}\right)^4 + \mathcal{O}\left(\left(\frac{\bar{\alpha}_S}{N}\right)^5\right)\Bigg\} \\
&\simeq \frac{\alpha_S}{2\pi}T_R N_f\frac{4}{3}\Bigg\{1 - 0.33\frac{\bar{\alpha}_S}{N} + 2.13\left(\frac{\bar{\alpha}_S}{N}\right)^2 + 2.27\left(\frac{\bar{\alpha}_S}{N}\right)^3 + 0.43\left(\frac{\bar{\alpha}_S}{N}\right)^4 \\
&\quad + \mathcal{O}\left(\left(\frac{\bar{\alpha}_S}{N}\right)^5\right)\Bigg\} ,
\end{aligned}
$$
(5.24)

$$C_{2,N}^g(\alpha_S, Q^2/\mu_F^2 = 1) = \frac{\alpha_S}{2\pi} T_R N_f \frac{2}{3} \left\{ 1 + \left[\frac{43}{9} - 2\zeta(2) \right] \frac{\bar{\alpha}_S}{N} + \left[\frac{1234}{81} - \frac{13}{3}\zeta(2) \right. \right.$$

$$\left. + \frac{4}{3}\zeta(3) \right] \left(\frac{\bar{\alpha}_S}{N} \right)^2 + \left[\frac{7412}{243} - \frac{71}{9}\zeta(2) + \frac{89}{9}\zeta(3) - 12\zeta(4) \right] \left(\frac{\bar{\alpha}_S}{N} \right)^3$$

$$+ \left[\frac{50012}{729} - \frac{466}{27}\zeta(2) + \frac{910}{27}\zeta(3) - \frac{28}{3}\zeta(2)\,\zeta(3) - 26\zeta(4) \right.$$

$$\left. + \frac{24}{5}\zeta(5) \right] \left(\frac{\bar{\alpha}_S}{N} \right)^4 + \mathcal{O}\left(\left(\frac{\bar{\alpha}_S}{N} \right)^5 \right) \right\}$$

$$\simeq \frac{\alpha_S}{2\pi} T_R N_f \frac{2}{3} \left\{ 1 + 1.49\frac{\bar{\alpha}_S}{N} + 9.71 \left(\frac{\bar{\alpha}_S}{N} \right)^2 + 16.43 \left(\frac{\bar{\alpha}_S}{N} \right)^3 + 39.11 \left(\frac{\bar{\alpha}_S}{N} \right)^4 \right.$$

$$\left. \mathcal{O}\left(\left(\frac{\bar{\alpha}_S}{N} \right)^5 \right) \right\}. \tag{5.25}$$

The first two coefficients in Eqs. (5.24), (5.25) agree with those recently computed in Refs. [30,40]. We regard this agreement as a non-trivial check of our results. Note also that the three- and four-loop coefficients (and the five-loop coefficient in Eq. (5.25)!) are substantially larger than the two-loop ones. We thus argue that the higher-order contributions computed in this paper may have a phenomenological relevance already at the values of x accessible at the HERA ep-collider [37].

Concluding this subsection, we point out that $k_\perp$-factorization formulae similar to Eq. (5.7) have recently been used [19] with the phenomenological aim of relating the original BFKL equation [12] to the DIS structure functions.

5.3. The DIS factorization scheme

In the previous sections we have repeatedly noted that the parton densities are not physical observables. Indeed they depend on the regularization/factorization scheme used for removing the parton level collinear singularities. This freedom in defining the parton densities means that, starting from the $\overline{\text{MS}}$ densities $\tilde{f}_a$, one can introduce a new set $\tilde{f}'_a$ of parton densities via an invertible transformation

$$\tilde{f}'_{a,N}(\mu^2) = \sum_b U_{ab,N}(\alpha_S(\mu^2))\, \tilde{f}_{b,N}(\mu^2)\,. \tag{5.26}$$

Obviously a similar transformation applies to the coefficient functions, in order to leave the physical cross section unchanged. The evolution of the new parton densities with μ^2 is controlled by the new anomalous dimension matrix

$$\gamma'_{ab,N} = \left[\beta(\alpha_S) \left(\alpha_S \frac{\partial}{\partial \alpha_S} U \right) U^{-1} + U\gamma U^{-1} \right]_{ab,N}\,. \tag{5.27}$$

The transformation matrix U_{ab} has a power series expansion in α_S such that $U_{ab,N}(\alpha_S) = \delta_{ab} + \mathcal{O}(\alpha_S)$, and has to fulfil the following physical constraints:

i) flavour and charge conjugation invariance

$$U_{q_i g} = U_{\bar{q}_i g} \equiv U_{qg} , \qquad U_{g q_i} = U_{g \bar{q}_i} \equiv U_{gq}$$

$$U_{q_i q_j} = U_{\bar{q}_i \bar{q}_j} \equiv \left(\delta_{ij} - \frac{1}{N_f} \right) U_{qq}^{NS} + \frac{1}{2 N_f} \left(U^{(V)} + U_{SS} \right) ,$$

$$U_{q_i \bar{q}_j} = U_{\bar{q}_i q_j} \equiv \left(\delta_{ij} - \frac{1}{N_f} \right) U_{q\bar{q}}^{NS} - \frac{1}{2 N_f} \left(U^{(V)} - U_{SS} \right) ; \tag{5.28}$$

ii) fermion number conservation

$$U_{q_i q_j, N=0} - U_{q_i \bar{q}_j, N=0} = \delta_{ij} \tag{5.29}$$

or, equivalently, $U_{qq, N=0}^{NS} = U_{q\bar{q}, N=0}^{NS} = U_{N=0}^{(V)}/2$;
iii) longitudinal momentum conservation

$$\sum_a U_{ab, N=1} = 1 . \tag{5.30}$$

Eq. (5.28) is the analogue of Eq. (A.5) in Appendix A for the anomalous dimensions γ_{ab}. In particular, it guarantees that the flavour singlet and non-singlet sectors are decoupled in any regularization/factorization scheme. The matrix components U_{qq}^{NS}, $U_{q\bar{q}}^{NS}$, $U^{(V)}$ introduced in Eq. (5.28) act on the flavour non-singlet parton densities, whilst U_{SS}, U_{qg}, U_{gq}, U_{gg} control the transformation on the singlet sector.

Higher-order QCD calculations for hadron collisions are usually performed in two different factorization schemes of collinear singularities, the $\overline{MS}$ scheme, used so far in this paper, and the DIS scheme [41]. After having regularized the collinear singularities in the parton matrix elements, the DIS-scheme parton densities $\tilde{f}_a^{(DIS)}$ are defined by enforcing the constraint that the DIS structure function $F_2(x, Q^2)$ has the same expression as in the naïve parton model. In particular, in the one-photon approximation to deep inelastic lepton-hadron scattering, the relation (5.1) is true to all orders in perturbation theory:

$$F_2(x, Q^2) = \sum_{i=1}^{N_f} e_i^2 \left[\tilde{f}_{q_i}^{(DIS)}(x, Q^2) + \tilde{f}_{\bar{q}_i}^{(DIS)}(x, Q^2) \right] . \tag{5.31}$$

Equivalently, one can say that in the DIS scheme the DIS coefficient functions are [2]

$$C_2^{NS \ (DIS)}(z; \alpha_S(Q^2), Q^2/\mu_F^2 = 1) = C_2^{S \ (DIS)}(z; \alpha_S(Q^2), Q^2/\mu_F^2 = 1) = \delta(1 - z) ,$$
$$C_2^{g \ (DIS)}(z; \alpha_S(Q^2), Q^2/\mu_F^2 = 1) = 0 . \tag{5.32}$$

In order to evaluate higher-order contributions in the small-x regime, the DIS scheme offers some computational and phenomenological advantages [22]. The former amounts to the fact that in the DIS scheme one can explicitly resum the corrections $\alpha_S (\alpha_S/N)^k$

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[2] Note that Eqs. (5.32) hold true only for a factorization scale $\mu_F^2 = Q^2$.

for the quark anomalous dimensions to all orders in α_S, as we shall illustrate below. As to the latter, we notice that the next-to-leading contributions $\alpha_S (\alpha_S/N)^k$ to the gluon anomalous dimensions are still unknown. Therefore, the knowledge of the quark anomalous dimensions in the DIS scheme may facilitate phenomenological investigations of the small-x behaviour of the structure function $F_2(x, Q^2)$.

The comment above applies once the DIS scheme has been defined to all orders in perturbation theory. The point is that Eq. (5.31) (or, equivalently, (5.32)) fixes only the quark densities unambiguously. The relation between the singlet quark density in the DIS scheme and the $\overline{\text{MS}}$-scheme parton densities is

$$\tilde{f}_{S,N}^{(\text{DIS})}(\mu^2) = C_{2,N}^S(\alpha_S(\mu^2), 1) \, \tilde{f}_{S,N}(\mu^2) + C_{2,N}^g(\alpha_S(\mu^2), 1) \, \tilde{f}_{g,N}(\mu^2) \qquad (5.33)$$

or, in terms of the matrix U_{ab} in Eq. (5.26),

$$U_{SS,N}(\alpha_S) = C_{2,N}^S(\alpha_S, 1) \, , \quad 2 N_f U_{qg,N}(\alpha_S) = C_{2,N}^g(\alpha_S, 1) \, . \qquad (5.34)$$

The DIS-scheme gluon density, instead, still remains ambiguous and is given by an arbitrary combination of gluon and singlet quark densities in the $\overline{\text{MS}}$ scheme

$$\tilde{f}_{g,N}^{(\text{DIS})}(\mu^2) = U_{gq,N}(\alpha_S(\mu^2)) \, \tilde{f}_{S,N}(\mu^2) + U_{gg,N}(\alpha_S(\mu^2)) \, \tilde{f}_{g,N}(\mu^2) \, , \qquad (5.35)$$

with the only constraint (5.30), which reads

$$U_{gg,N=1}(\alpha_S) = 1 - C_{2,N=1}^g(\alpha_S, 1) \, , \quad U_{gq,N=1}(\alpha_S) = 1 - C_{2,N}^S(\alpha_S, 1) \, . \qquad (5.36)$$

The convention introduced in Ref. [41] for defining $\tilde{f}_g^{(\text{DIS})}$ up to $\mathcal{O}(\alpha_S)$ amounts to extending Eq. (5.36) to any value of N in order α_S. A natural generalization of this convention is to require Eq. (5.36) to be valid for any N and to *all orders* in α_S. Doing that, the DIS-scheme gluon density is completely defined.

Note, however, that for the purposes of our all-order calculation, it is not necessary to specify the actual form of the two matrix elements $U_{gq,N}$ and $U_{gg,N}$ in Eq. (5.35). We just assume that they are chosen not to be extremely singular at high energies, i.e. they should not contain leading-order contributions of the type $(\alpha_S/N)^k$ for $N \to 0$. This is sufficient to ensure that most of the $\overline{\text{MS}}$-scheme results obtained in the previous sections remain valid in the DIS scheme. In particular the gluon anomalous dimensions $\gamma_{ga}^{(\text{DIS})}$ and the longitudinal coefficient functions $C_L^{(\text{DIS})}$ are

$$\gamma_{ga,N}^{(\text{DIS})}(\alpha_S) = \gamma_{ga,N}(\alpha_S) + \mathcal{O}(\alpha_S^2(\alpha_S/N)^k) \quad (a = g, q) \, , \qquad (5.37)$$

$$C_{L,N}^{A\,(\text{DIS})}\left(\alpha_S(\mu_F^2), \frac{Q^2}{\mu_F^2}\right) = C_{L,N}^A\left(\alpha_S(\mu_F^2), \frac{Q^2}{\mu_F^2}\right) + \mathcal{O}\left(\alpha_S^2(\frac{\alpha_S}{N})^k\right) \quad (A = g, S) \, ,$$

where the resummed expressions for the $\overline{\text{MS}}$-scheme anomalous dimensions γ_{ga} and coefficient functions C_L^A are given in Eqs. (3.14), (3.30), (5.11), (5.22).

The quark anomalous dimensions, instead, do not coincide any longer (to this logarithmic accuracy) with the corresponding anomalous dimensions in the $\overline{\text{MS}}$ scheme. Using Eqs. (5.27) and (5.33) we obtain

$$\gamma_{qg,N}^{(\mathrm{DIS})}(\alpha_S) = \gamma_{qg,N}(\alpha_S) + \frac{1}{2 N_f} C_{2,N}^g(\alpha_S, 1)\, \gamma_{gg,N}(\alpha_S) + \mathcal{O}(\alpha_S^2(\alpha_S/N)^k)\ .$$

$$(5.38)$$

On the other hand, the expression on the r.h.s. has been computed with logarithmic accuracy $\alpha_S\,(\alpha_S/N)^k$ in Section 5.2. Inserting Eqs. (5.17), (5.20) into (5.38), we find the following resummed expression for the quark anomalous dimension in the DIS scheme:

$$\gamma_{qg,N}^{(\mathrm{DIS})}(\alpha_S) = \frac{\alpha_S}{2\pi} T_R \frac{2 + 3\gamma_N - 3\gamma_N^2}{3 - 2\gamma_N} \frac{\Gamma^3(1 - \gamma_N)\,\Gamma^3(1 + \gamma_N)}{\Gamma(2 + 2\gamma_N)\,\Gamma(2 - 2\gamma_N)} R_N(\alpha_S)$$
$$+ \mathcal{O}\left(\alpha_S^2(\alpha_S/N)^k\right)\ .$$

$$(5.39)$$

The colour charge relation (4.15) is still true in the DIS scheme:

$$\gamma_{qq,N}^{S\,(\mathrm{DIS})}(\alpha_S) = \frac{C_F}{C_A}\left[\gamma_{qg,N}^{(\mathrm{DIS})}(\alpha_S) - \frac{\alpha_S}{2\pi} T_R \frac{2}{3}\right] + \mathcal{O}(\alpha_S^2\left(\alpha_S/N\right)^k)\ .$$

$$(5.40)$$

Some comments are in order. The DIS-scheme result (5.39) for the quark anomalous dimensions has to be contrasted with the results discussed in Section 4 for the $\overline{\mathrm{MS}}$ scheme. The algebraic complications of the $\overline{\mathrm{MS}}$ scheme prevented us from obtaining resummed expressions in closed form for γ_{qg} and C_2^g separately. We were able to explicitly resum only the combination in Eq. (5.17), which turns out to be equivalent to the anomalous dimensions in the DIS scheme. This computational simplification has a more physical origin. The singlet sector of the deep inelastic lepton-hadron scattering is characterized by four physical observables, which can be studied in QCD perturbation theory: the structure functions F_2^S, F_L^S and their first derivatives with respect to Q^2. The $\overline{\mathrm{MS}}$ scheme describes these observables in terms of eight different quantities: four coefficient functions C_i^A ($i = 2, L$, $A = g, S$) and the four matrix elements of the singlet anomalous dimensions. Obviously, only some linear combinations of them have to be regarded as physical observables. The DIS scheme, reducing to two the non-trivial coefficient functions ($C_2^S = 1$, $C_2^g = 0$), limits the number of arbitrary unphysical quantities necessary to describe the scattering process. The ensuing anomalous dimensions are more easily computable to all orders in α_S because they are more directly related to observable scaling violations.

Using the expansion (3.16) for the BFKL anomalous dimension γ_N and the expression (3.17) for $R_N(\alpha_S)$, the first perturbative terms of the quark anomalous dimension (5.39) can be readily computed:

$$\gamma_{qg,N}^{(\mathrm{DIS})} = \frac{\alpha_S}{2\pi} T_R \frac{2}{3}\left\{1 + \frac{13}{6}\frac{\bar\alpha_S}{N} + \left(\frac{71}{18} - \zeta(2)\right)\left(\frac{\bar\alpha_S}{N}\right)^2\right.$$
$$+ \left[\frac{233}{27} - \frac{13}{6}\zeta(2) + \frac{8}{3}\zeta(3)\right]\left(\frac{\bar\alpha_S}{N}\right)^3$$
$$+ \left[\frac{1276}{81} - \frac{71}{18}\zeta(2) + \frac{91}{9}\zeta(3) - 6\zeta(4)\right]\left(\frac{\bar\alpha_S}{N}\right)^4 + \left[\frac{8384}{243} - \frac{233}{27}\zeta(2)\right.$$

$$+ \frac{710}{27}\zeta(3) - \frac{20}{3}\zeta(2)\,\zeta(3) - 13\,\zeta(4) + \frac{22}{5}\zeta(5)\Big]\left(\frac{\bar{\alpha}_S}{N}\right)^5$$

$$+ \mathcal{O}\left(\left(\frac{\bar{\alpha}_S}{N}\right)^6\right)\Bigg\}$$

$$\simeq \frac{\alpha_S}{2\pi}T_R\frac{2}{3}\left\{1 + 2.17\frac{\bar{\alpha}_S}{N} + 2.30\left(\frac{\bar{\alpha}_S}{N}\right)^2 + 8.27\left(\frac{\bar{\alpha}_S}{N}\right)^3 + \right.$$

$$\left. + 14.92\left(\frac{\bar{\alpha}_S}{N}\right)^4 + 29.23\left(\frac{\bar{\alpha}_S}{N}\right)^5 + \mathcal{O}\left(\left(\frac{\bar{\alpha}_S}{N}\right)^6\right)\right\}. \tag{5.41}$$

The coefficients of the first two terms in the curly bracket agree with the one- and two-loop calculations in the DIS scheme [27,28,30]. Moreover, all the coefficients in Eq. (5.41) are systematically larger than the corresponding $\overline{\text{MS}}$-scheme coefficients in Eq. (4.14). This behaviour is due to the additional contribution of the coefficient function C_2^g in Eq. (5.38), and it is likely to persist in higher orders.

Note also that the all-order expression (5.39) is analytic for $0 \leq \gamma_N < 1/2$. Thus, independenly of the value of α_S, the leading trajectory in N-moment space is still given by the BFKL pomeron. As the BFKL anomalous dimension γ_N increases towards its saturation value at $\gamma_N = 1/2$, the quark anomalous dimension quickly increases, approaching a singularity due to the pomeron normalization factor $R_N(\alpha_S)$ (see Eq. (3.19)). This increase of $\gamma_{qg}^{(\text{DIS})}$ leads to strong scaling violations, although the singularity at $\gamma_N = 1/2$ is cancelled in physical observables by analogous contributions to the resummed coefficient functions in Eq. (5.37).

6. Summary

In the present paper we have shown how the high-energy factorization theorem [8] can be extended beyond the leading logarithmic accuracy in a manner which is consistent with the all-order factorization of collinear singularities. Much effort has been devoted to investigating the issue of the dependence on the factorization scheme of parton densities and coefficient functions. This analysis has led to the (off-shell) $k_\perp$-factorization in dimensional regularization represented schematically by Eq. (2.29) (see also Eqs. (4.2) and (5.7)). Eq. (2.29) has then been used to study the high-energy (or small-x) behaviour of deep inelastic scattering processes.

A first general consequence of Eq. (2.29) is that flavour non-singlet observables are regular at small x order by order in perturbation theory.

As regards the singlet sector, $k_\perp$-factorization allows one to sum classes of logarithmic corrections to all orders in α_S. To this end, one has to evaluate 2GI kernels (see Eq. (2.29)) in fixed-order perturbation theory and use the master equations (3.2), (3.20) for the gluon Green functions.

Eqs. (3.2) and (3.20) are the generalization of the BFKL equation [12] to the case of $n = 4 + 2\varepsilon$ space-time dimensions. Their solution is discussed in Section 3. In particular, the calculation of the gluon anomalous dimensions $\gamma_{gg,N}(\alpha_S)$, $\gamma_{gq,N}(\alpha_S)$ (see Eqs. (3.13), (3.14), (3.30)) has been carried out to the leading logarithmic

accuracy $(\alpha_S/N)^k$ in the context of dimensional regularization and the BFKL results on the pomeron trajectory have been re-derived. Besides this, we have been able to compute (to the same accuracy) the normalization factor $R_N(\alpha_S)$ (see Eq. (3.17)) of the perturbative QCD pomeron in the $\overline{\text{MS}}$ factorization scheme.

The quark sector enters the QCD evolution equations to next-to-leading logarithmic order $\alpha_S (\alpha_S \ln x)^k$. Using high-energy factorization, in Section 4 we have evaluated the corresponding quark Green functions. We have shown that the result in Eq. (4.10) originates from an integral equation whose kernel is related to a generalized (off-shell) Altarelli-Parisi splitting function (see Eq. (4.9)). We have also discussed how Eq. (4.10) can be used for evaluating the small-N limit of the quark anomalous dimensions $\gamma_{qg,N}(\alpha_S)$ and $\gamma^S_{qq,N}(\alpha_S)$. The result of our explicit calculation up to six-loop order is given in Eq. (4.14).

As an example of application of high-energy factorization to a specific hard process, in Section 5 we have considered deep inelastic lepton-hadron scattering (for the case of heavy-flavour production see Refs. [8]–[10] and Section 2.4). Resummed expressions to next-to-leading accuracy $\alpha_S (\alpha_S/N)^k$ for the DIS coefficient functions C_2 and C_L are presented in Eqs. (5.11), (5.17), (5.21), (5.22), (5.23). These results are given in the $\overline{\text{MS}}$ factorization scheme. In Section 5.3, we have also introduced an all-order generalization of the DIS factorization scheme first proposed in Ref. [41]. Within this scheme, where most of the DIS coefficient functions are trivial (see Eqs. (5.32) and (5.37)), we have obtained the next-to-leading resummed expressions (5.39), (5.40) for the quark anomalous dimensions.

Quantifying precisely the phenomenological consequences of the results presented here is a matter of detailed numerical investigations. However, the size of the next-to-leading-order coefficients in the perturbative expansions (4.14), (5.24), (5.25) and (5.41) suggests that these contributions may have phenomenological relevance in accurate analyses of scaling violations, already at the values of x $(x \sim 10^{-3} \div 10^{-4})$ accessible at present hadron colliders. In particular, since the first leading-order coefficients of the gluon anomalous dimensions are vanishing (see Eqs. (3.16)), the next-to-leading corrections in the quark sector computed in this paper may be quite important for the study of the proton structure functions being measured at HERA. A fully consistent analysis to next-to-leading logarithmic order obviously requires also the computation of the still unknown (to this accuracy) gluon anomalous dimensions. We hope to report progress on this subject in the near future.

Acknowledgments

Appendix A

The rescaled parton densities $\tilde{f}_a(x, \mu^2) = x f_a(x, \mu^2)$ introduced in Eq. (2.6) fulfil the evolution equations

$$\frac{d \tilde{f}_a(x, \mu^2)}{d \ln \mu^2} = \sum_b \int_0^1 dz \, P_{ab}(\alpha_S(\mu^2), z) \, \tilde{f}_b(x/z, \mu^2) \,, \tag{A.1}$$

where $P_{ab}(\alpha_S, z)$ are the generalized Altarelli-Parisi splitting functions in Eq. (2.3). According to our normalization, the splitting functions have the following one-loop expressions

$$P_{gg}^{(0)}(z) = 2 C_A \left[\left(\frac{1}{1-z} \right)_+ - 1 + \frac{1-z}{z} + z(1-z) \right]$$
$$+ \left(\frac{11}{6} C_A - \frac{2}{3} T_R N_f \right) \delta(1-z) \,,$$
$$P_{gq_i}^{(0)}(z) = P_{g\bar{q}_i}^{(0)}(z) = C_F \frac{1 + (1-z)^2}{z} \,,$$
$$P_{q_i g}^{(0)}(z) = P_{\bar{q}_i g}^{(0)}(z) = T_R \left[z^2 + (1-z)^2 \right] \,, \tag{A.2}$$

$$P_{q_i q_j}^{(0)}(z) = P_{\bar{q}_i \bar{q}_j}^{(0)}(z) = C_F \left(\frac{1+z^2}{1-z} \right)_+ \delta_{ij} \,, \quad P_{q_i \bar{q}_j}^{(0)}(z) = P_{\bar{q}_i q_j}^{(0)}(z) = 0 \,, \tag{A.3}$$

in terms of the $SU(N_c)$ colour factors ($N_c = 3$ is the number of colours)

$$C_A = N_c \,, \quad C_F = \frac{N_c^2 - 1}{2 N_c} \,, \quad \text{Tr} \, (t^a t^b) = \delta_{ab} T_R = \frac{1}{2} \delta_{ab} \,. \tag{A.4}$$

The leading-order splitting functions $P_{ab}^{(0)}(z)$ (which have been known for a long time [26]) are factorization theorem invariants, i.e. they do not depend on the explicit procedure to factorize collinear singularities. The physical reason for this is that they are directly related to observable scaling violations in deep inelastic scattering processes. On the contrary, splitting functions and anomalous dimensions beyond one-loop order do depend on the regularization and factorization schemes of collinear singularities. Nonetheless, due to charge conjugation invariance and $SU(N_f)$ flavour symmetry of QCD, they satisfy the following scheme-independent properties

$$\gamma_{q_i g} = \gamma_{\bar{q}_i g} \equiv \gamma_{qg} \,, \quad \gamma_{g q_i} = \gamma_{g \bar{q}_i} \equiv \gamma_{gq} \,,$$
$$\gamma_{q_i q_j} = \gamma_{\bar{q}_i \bar{q}_j} \equiv \gamma_{qq}^{NS} \delta_{ij} + \gamma_{qq}^S \,, \quad \gamma_{q_i \bar{q}_j} = \gamma_{\bar{q}_i q_j} \equiv \gamma_{q\bar{q}}^{NS} \delta_{ij} + \gamma_{q\bar{q}}^S \,. \tag{A.5}$$

The symmetry properties (A.5) imply that the anomalous dimensions matrix γ_{ab} has only seven independent components. Correspondingly, three flavour non-singlet ($\tilde{f}^{(V)}$, $\tilde{f}_{q_i}^{(-)}$, $\tilde{f}_{q_i}^{(+)}$) and two flavour singlet ($\tilde{f}_S$, $\tilde{f}_g$) parton densities can be introduced so that the evolution equations (A.1)) (or (2.9)) are completely diagonalized (in the partonic

space) for the non-singlet sector. One explicitly finds (we drop the overall dependence on N, μ and α_S)

$$\frac{d \ln \tilde{f}^{(V)}}{d \ln \mu^2} = \gamma^{(V)} , \qquad \frac{d \ln \tilde{f}_{q_i}^{(-)}}{d \ln \mu^2} = \gamma^{(-)} , \qquad \frac{d \ln \tilde{f}_{q_i}^{(+)}}{d \ln \mu^2} = \gamma^{(+)} , \qquad (A.6)$$

where

$$\tilde{f}^{(V)} \equiv \sum_{j=1}^{N_f} (\tilde{f}_{q_j} - \tilde{f}_{\bar{q}_j}) , \qquad \tilde{f}_{q_i}^{(\pm)} \equiv \tilde{f}_{q_i} \pm \tilde{f}_{\bar{q}_i} - \frac{1}{N_f} \sum_{j=1}^{N_f} (\tilde{f}_{q_j} \pm \tilde{f}_{\bar{q}_j}) , \qquad (A.7)$$

and the non-singlet anomalous dimensions are given by

$$\gamma^{(V)} = \gamma_{qq}^{NS} - \gamma_{q\bar{q}}^{NS} + N_f (\gamma_{qq}^{S} - \gamma_{q\bar{q}}^{S}) , \qquad \gamma^{(\pm)} = \gamma_{qq}^{NS} \pm \gamma_{q\bar{q}}^{NS} . \qquad (A.8)$$

The evolution equations are instead still coupled in the singlet sector:

$$\frac{d \tilde{f}_S}{d \ln \mu^2} = [\gamma_{qq}^{NS} + \gamma_{q\bar{q}}^{NS} + N_f (\gamma_{qq}^{S} + \gamma_{q\bar{q}}^{S})] \tilde{f}_S + 2 N_f \gamma_{qg} \tilde{f}_g ,$$

$$\frac{d \tilde{f}_g}{d \ln \mu^2} = \gamma_{gq} \tilde{f}_S + \gamma_{gg} \tilde{f}_g , \qquad (A.9)$$

where the quark singlet density is defined by $\tilde{f}_S = \sum_{i=1}^{N_f} (\tilde{f}_{q_i} + \tilde{f}_{\bar{q}_i})$.

From Eq. (A.2) we see that all the three non-singlet anomalous dimensions are degenerate in one-loop order. The anomalous dimensions in two-loop order were computed in Refs. [25,27,28]. In this order the degeneracy mentioned above is partially removed because $\gamma_{q\bar{q}}^{NS} \neq 0$. However we still have $\gamma^{(V)}(\alpha_S) = \gamma^{(-)}(\alpha_S) + O(\alpha_S^3)$ since γ_{qq}^{S} and $\gamma_{q\bar{q}}^{S}$ coincide in $O(\alpha_S^2)$. The equality between γ_{qq}^{S} and $\gamma_{q\bar{q}}^{S}$ is expected to be violated starting from $O(\alpha_S^3)$.

The high-energy power counting in Section 2.3 implies that the non-singlet anomalous dimensions in Eq. (A.8) (and hence $\gamma_{qq,N}^{NS}$, $\gamma_{q\bar{q},N}^{NS}$, $\gamma_{qq,N}^{S} - \gamma_{q\bar{q},N}^{S}$) are regular for $N \to 0$ order by order in α_S. In other words, the corresponding n-loop splitting functions $P^{(n-1)}$ in Eq. (2.3) are less singular than $1/x$ for $x \to 0$. All the high-energy contributions α_S^n / N^k ($n \geq k \geq 1$) are thus associated with the gluon anomalous dimensions $\gamma_{gg,N}$, $\gamma_{gq,N}$ and with the quark anomalous dimensions $\gamma_{qg,N}$, $\gamma_{qq,N}^{S} \simeq \gamma_{q\bar{q},N}^{S}$.

Appendix B

In order to obtain the solution of the master equation (3.2) for the $k_\perp$-distribution $\mathcal{F}^{(0)}$, it is convenient to introduce the dimensionless distribution $\bar{\mathcal{F}}$, as follows

$$\mathcal{F}_N^{(0)}(\mathbf{k}; \alpha_S, \mu, \varepsilon) \equiv \delta^{(2+2\varepsilon)}(\mathbf{k}) + \frac{\Gamma(1+\varepsilon)}{(\pi \mathbf{k}^2)^{1+\varepsilon}} \bar{\mathcal{F}}_N(\alpha_S(\mathbf{k}^2/\mu^2)^\varepsilon, \varepsilon) . \qquad (B.1)$$

Note that $\bar{\mathcal{F}}$ does not depend on ε, α_S and $\mathbf{k}$ independently. In fact, rewriting Eq. (3.2) in terms of $\bar{\mathcal{F}}$, we see that it only depends on ε and the combination $\alpha_S(\mathbf{k}^2/\mu^2)^\varepsilon$:

$$\bar{\mathcal{F}}_N(\alpha_S(\mathbf{k}^2/\mu^2)^\varepsilon, \varepsilon) = \frac{\bar{\alpha}_S}{N} S_\varepsilon \frac{e^{\varepsilon \psi(1)}}{\Gamma(1+\varepsilon)} \left(\frac{\mathbf{k}^2}{\mu^2}\right)^\varepsilon$$

$$+ \frac{\bar{\alpha}_S}{N} \left(\frac{\mathbf{k}^2}{\mu^2}\right)^\varepsilon \int \frac{d^{2+2\varepsilon}\mathbf{q}}{(2\pi)^{2\varepsilon}} \frac{1}{\pi \mathbf{q}^2 (\mathbf{k} - \mathbf{q})^2}$$

$$\cdot \left[\frac{\mathbf{k}^2}{((\mathbf{k} - \mathbf{q})^2)^\varepsilon} \bar{\mathcal{F}}_N(\alpha_S((\mathbf{k} - \mathbf{q})^2/\mu^2)^\varepsilon, \varepsilon) \right.$$

$$\left. - \frac{\mathbf{k} \cdot (\mathbf{k} - \mathbf{q})}{(\mathbf{k}^2)^\varepsilon} \bar{\mathcal{F}}_N(\alpha_S(\mathbf{k}^2/\mu^2)^\varepsilon, \varepsilon) \right] . \tag{B.2}$$

Then, we perform the shift $\mathbf{q} \to \mathbf{q}' = \mathbf{q} - \mathbf{k}$ of the integration variable in Eq. (B.2), and introduce the ratio $\tau = |\mathbf{q}'|/|\mathbf{k}|$ and the angle θ between $\mathbf{q}'$ and $\mathbf{k}$. The integral equation (B.2) now reads

$$\bar{\mathcal{F}}_N(\alpha_S(\mathbf{k}^2/\mu^2)^\varepsilon, \varepsilon) = \frac{\bar{\alpha}_S}{N} S_\varepsilon \frac{e^{\varepsilon \psi(1)}}{\Gamma(1+\varepsilon)} \left(\frac{\mathbf{k}^2}{\mu^2}\right)^\varepsilon \tag{B.3}$$

$$\cdot \left\{ 1 + \frac{2\Gamma(1+\varepsilon)}{\sqrt{\pi}\,\Gamma(\frac{1}{2}+\varepsilon)} \int\limits_0^\infty \frac{d\tau}{\tau} \int\limits_0^\pi d\theta \frac{(\sin\theta)^{2\varepsilon}}{1 + 2\tau\cos\theta + \tau^2} \right.$$

$$\cdot \left[\bar{\mathcal{F}}_N(\alpha_S(\tau^2\mathbf{k}^2/\mu^2)^\varepsilon, \varepsilon) \right.$$

$$\left. \left. + \tau^{1+2\varepsilon} \cos\theta\, \bar{\mathcal{F}}_N(\alpha_S(\mathbf{k}^2/\mu^2)^\varepsilon, \varepsilon) \right] \right\} .$$

The angular integration in Eq. (B.3) can be performed in terms of associated Legendre functions. However we did not find the ensuing representation of any convenience.

The perturbative solution of Eq. (B.3) has the following expression

$$\bar{\mathcal{F}}_N(\alpha_S(\mathbf{k}^2/\mu^2)^\varepsilon, \varepsilon) = \sum_{k=1}^\infty \left[\frac{\bar{\alpha}_S}{N} S_\varepsilon \frac{e^{\varepsilon \psi(1)}}{\Gamma(1+\varepsilon)} \left(\frac{\mathbf{k}^2}{\mu^2}\right)^\varepsilon \right]^k c_k(\varepsilon) . \tag{B.4}$$

Inserting equation (B.4) into Eq. (B.3), we obtain the recurrence relation (3.4) for the perturbative coefficients $c_k(\varepsilon)$. The recurrence factor $I_k(\varepsilon)$ is given by

$$I_k(\varepsilon) = I(\gamma = k\varepsilon; \varepsilon) , \tag{B.5}$$

where $I(\gamma; \varepsilon)$ is the following integral

$$I(\gamma; \varepsilon) = \frac{2\Gamma(1+\varepsilon)}{\sqrt{\pi}\,\Gamma(\frac{1}{2}+\varepsilon)} \int\limits_0^\infty \frac{d\tau}{\tau} \int\limits_0^\pi d\theta \frac{(\sin\theta)^{2\varepsilon}}{1 + 2\tau\cos\theta + \tau^2} (\tau^{2\gamma} + \tau^{1+2\varepsilon}\cos\theta)$$

$$= \frac{2\pi\Gamma(1+\varepsilon)}{\sqrt{\pi}\,\Gamma(\frac{1}{2}+\varepsilon)} \int\limits_0^\pi d\theta\, (\sin\theta)^{2\varepsilon-1} \left[\frac{\sin((1-2\gamma)\theta)}{\sin(2\pi\gamma)} + \frac{\cos\theta\,\sin(2\varepsilon\theta)}{\sin(2\pi\varepsilon)} \right]$$

$$= \frac{1}{\varepsilon} \frac{\Gamma^2(1+\varepsilon)}{\Gamma(1+2\varepsilon)} \left[\frac{\Gamma(1+2\varepsilon)\,\Gamma(\gamma)\,\Gamma(1-\gamma)}{\Gamma(\varepsilon+\gamma)\,\Gamma(1+\varepsilon-\gamma)} - \Gamma(1+\varepsilon)\,\Gamma(1-\varepsilon) \right] \tag{B.6}$$

Using Eqs. (B.5) and (B.6) we obtain Eq. (3.5).

The integral $I(\gamma;\varepsilon)$ in Eq. (B.6) represents the action of the kernel of the master equation (B.2) (or, equivalently, (3.2)) on a test function $\mathcal{F}(\mathbf{k})$ behaving like $(\mathbf{k}^2)^\gamma$. Therefore the fact that $I(\gamma;\varepsilon)$ is finite for $\varepsilon \to 0$ and γ fixed is a consequence of the collinear regularity of the kernel of the BFKL equation in four dimensions. Note also that this regularity is achieved through the cancellation of collinear singularities which are present separately in the real and virtual contributions, i.e. in the first and second terms in the square bracket of Eq. (B.2) or (B.6).

As discussed in Section 3, the lack of scale invariance in the master equation (B.2) does not allow us to find a general solution for $\bar{\mathcal{F}}$ for any value of ε. This means that we are not able to resum the formal power series expansion in Eq. (B.4). Nevertheless, we can obtain an explicit all-order solution in the relevant limit of small ε values. This limit is sufficient to compute the transition function Γ_{gg} (i.e. the gluon anomalous dimensions) and the associated normalization factor R_N in Eqs. (3.8), (3.9).

To do this, let us perform the derivative of Eq. (B.3) with respect to $\ln\alpha_S$ and then divide both sides by $\bar{\mathcal{F}}$. We thus obtain

$$
\varepsilon \frac{\partial}{\partial \ln \alpha_S} \ln \bar{\mathcal{F}}_N(\alpha_S, \varepsilon) = \varepsilon + \frac{\bar{\alpha}_S}{N} S_\varepsilon \frac{e^{\varepsilon \psi(1)}}{\Gamma(1+\varepsilon)} \frac{2\Gamma(1+\varepsilon)}{\sqrt{\pi}\,\Gamma(\frac{1}{2}+\varepsilon)}
$$
$$
\cdot \int_0^\infty \frac{d\tau}{\tau} \int_0^\pi d\theta \frac{(\sin\theta)^{2\varepsilon}}{1+2\tau\cos\theta+\tau^2}
$$
$$
\cdot \left[\frac{\bar{\mathcal{F}}_N(\alpha_S \tau^{2\varepsilon}, \varepsilon)}{\bar{\mathcal{F}}_N(\alpha_S, \varepsilon)} \varepsilon \frac{\partial}{\partial \ln \alpha_S} \ln \bar{\mathcal{F}}_N(\alpha_S \tau^{2\varepsilon}, \varepsilon) \right.
$$
$$
\left. + \tau^{1+2\varepsilon} \cos\theta \,\varepsilon \frac{\partial}{\partial \ln \alpha_S} \ln \bar{\mathcal{F}}_N(\alpha_S, \varepsilon) \right] . \tag{B.7}
$$

Unlike Eq. (B.3), Eq. (B.7) is homogenous with respect to $\bar{\mathcal{F}}$. Therefore we can easily factorize the singular transition function $\Gamma_{gg,N}(\alpha_S, \varepsilon)$.

More precisely, let us notice that the distribution $\bar{\mathcal{F}}$ introduced in Eq. (B.1) is related to the integrated gluon Green function (3.6) as follows

$$
\bar{\mathcal{F}}_N(\alpha_S (Q^2/\mu^2)^\varepsilon, \varepsilon) = Q^2 \frac{\partial}{\partial Q^2} G_{gg,N}^{(0)}(\alpha_S(Q^2/\mu^2)^\varepsilon, \varepsilon)
$$
$$
= \varepsilon \frac{\partial}{\partial \ln \alpha_S} G_{gg,N}^{(0)}(\alpha_S(Q^2/\mu^2)^\varepsilon, \varepsilon) . \tag{B.8}
$$

Using the factorization formula (3.8),

$$
G_{gg,N}^{(0)}(\alpha_S, \varepsilon) = G_{gg,N}(\alpha_S, \varepsilon) \Gamma_{gg,N}(\alpha_S, \varepsilon)
$$
$$
= G_{gg,N}(\alpha_S, \varepsilon) \exp\left(\frac{1}{\varepsilon} \int_0^{\alpha_S S_\varepsilon} \frac{d\alpha}{\alpha} \gamma_N(\alpha) \right) , \tag{B.9}
$$

we thus have

$$\bar{\mathcal{F}}_N(\alpha_S, \varepsilon) = \left[\gamma_N(\alpha_S S_\varepsilon) + \varepsilon \frac{\partial}{\partial \ln \alpha_S} \ln G_{gg,N}(\alpha_S, \varepsilon) \right] G_{gg,N}(\alpha_S, \varepsilon) \, \Gamma_{gg,N}(\alpha_S, \varepsilon) \, .$$

$$(B.10)$$

Since the anomalous dimensions $\gamma_N(\alpha_S S_\varepsilon)$ and the renormalized Green function $G_{gg,N}(\alpha_S, \varepsilon)$ are regular for $\varepsilon \to 0$, it follows from Eq. (B.10) that the relevant functions involved in Eq. (B.7) and Eq. (B.7) itself can be expanded in ε around $\varepsilon = 0$. A straightforward calculation gives

$$\varepsilon \frac{\partial}{\partial \ln \alpha_S} \ln \bar{\mathcal{F}}_N(\alpha_S \tau^{2\varepsilon}, \varepsilon) = \left[\gamma_N(\alpha_S S_\varepsilon) + \varepsilon \left(\frac{\partial \ln \gamma_N(\alpha_S)}{\partial \ln \alpha_S} + \frac{\partial \ln R_N(\alpha_S)}{\partial \ln \alpha_S} \right) \right]$$
$$\cdot \left[1 + 2\varepsilon \ln \tau \frac{\partial \ln \gamma_N(\alpha_S)}{\partial \ln \alpha_S} + \mathcal{O}(\varepsilon^2) \right] \, ,$$

$$\frac{\bar{\mathcal{F}}_N(\alpha_S \tau^{2\varepsilon}, \varepsilon)}{\bar{\mathcal{F}}_N(\alpha_S, \varepsilon)} = \left[1 + 2\varepsilon \ln \tau \left(\frac{\partial \ln \gamma_N(\alpha_S)}{\partial \ln \alpha_S} + \frac{\partial \ln R_N(\alpha_S)}{\partial \ln \alpha_S} \right) \right. \quad (B.11)$$
$$\left. + 2\varepsilon \ln^2 \tau \frac{\partial \gamma_N(\alpha_S)}{\partial \ln \alpha_S} + \mathcal{O}(\varepsilon^2) \right] \exp\{2\gamma_N(\alpha_S S_\varepsilon) \ln \tau\} \, ,$$

where $R_N(\alpha_S) = G_{gg,N}(\alpha_S, \varepsilon = 0)$. Correspondingly, Eq. (B.7) reads as follows (we drop the overall explicit dependence on α_S)

$$\gamma_N + \varepsilon \left(\frac{\partial \ln \gamma_N}{\partial \ln \alpha_S} + \frac{\partial \ln R_N}{\partial \ln \alpha_S} \right) + \mathcal{O}(\varepsilon^2) = \varepsilon + \left[\gamma_N + \varepsilon \left(\frac{\partial \ln \gamma_N}{\partial \ln \alpha_S} + \frac{\partial \ln R_N}{\partial \ln \alpha_S} \right) \right]$$
$$\cdot \frac{\bar{\alpha}_S}{N} \Omega(\gamma_N; \varepsilon) \, I(\gamma_N; \varepsilon) \, , \quad (B.12)$$

where we have introduced the (second-order) differential operator Ω

$$\Omega(\gamma_N; \varepsilon) = 1 + \varepsilon \left[\left(2 \frac{\partial \ln \gamma_N}{\partial \ln \alpha_S} + \frac{\partial \ln R_N}{\partial \ln \alpha_S} \right) \frac{\partial}{\partial \gamma_N} + \frac{1}{2} \frac{\partial \gamma_N}{\partial \ln \alpha_S} \left(\frac{\partial}{\partial \gamma_N} \right)^2 \right] + \mathcal{O}(\varepsilon^2)$$

$$(B.13)$$

acting on the integral $I(\gamma_N; \varepsilon)$ in Eq. (B.6). Equating the $\mathcal{O}(\varepsilon^0)$ and $\mathcal{O}(\varepsilon)$ terms in Eq. (B.12) we thus find an implicit equation for $\gamma_N(\alpha_S)$ and a differential equation for $R_N(\alpha_S)$.

To order ε^0, we obtain

$$1 = \frac{\bar{\alpha}_S}{N} \chi(\gamma_N(\alpha_S)) \, , \quad (B.14)$$

i.e. the result (3.14) for the BFKL anomalous dimension, $\chi(\gamma)$ being the characteristic function in Eq. (3.15). To order ε, we have

$$\left(2 \frac{\partial \ln \gamma_N}{\partial \ln \alpha_S} + \frac{\partial \ln R_N}{\partial \ln \alpha_S} \right) \chi'(\gamma_N) + \frac{1}{2} \frac{\partial \gamma_N}{\partial \ln \alpha_S} \chi''(\gamma_N)$$
$$+ \frac{1}{2} \left[2 \psi'(1) - \psi'(\gamma_N) - \psi'(1 - \gamma_N) + \chi^2(\gamma_N) \right] = -\frac{N}{\bar{\alpha}_S \gamma_N} \, , \quad (B.15)$$

where $\chi'(\gamma)$, $\chi''(\gamma)$ are the first and second derivatives of $\chi(\gamma)$ with respect to γ. In order to solve Eq. (B.15), it is convenient to consider γ_N as the independent variable

$$\frac{\partial \ln R_N}{\partial \ln \alpha_S} = \frac{\partial \ln \gamma_N}{\partial \ln \alpha_S} \frac{\partial \ln R_N}{\partial \ln \gamma_N} ,$$ (B.16)

and use Eq. (B.14):

$$\frac{N}{\bar{\alpha}_S} = \chi(\gamma_N) , \qquad \frac{\partial \ln \gamma_N}{\partial \ln \alpha_S} = -\frac{\chi(\gamma_N)}{\gamma_N \, \chi'(\gamma_N)} .$$ (B.17)

Inserting Eqs. (B.16), (B.17) into Eq. (B.15) we get the differential equation

$$\frac{\partial \ln R_N}{\partial \ln \gamma_N} = \frac{1}{2} \gamma_N \left[-\frac{2}{\gamma_N} - \frac{\chi''(\gamma_N)}{\chi'(\gamma_N)} + \chi(\gamma_N) \right.$$
$$\left. + \frac{2\,\psi'(1) - \psi'(\gamma_N) - \psi'(1-\gamma_N)}{\chi(\gamma_N)} \right] ,$$ (B.18)

whose solution is given by Eq. (3.17).

Appendix C

The evaluation of the quark anomalous dimensions starting from the $k_\perp$-factorization formula (4.4) requires the explicit computation of the off-shell kernel $\hat{K}_{qg}$. Inserting Eq. (4.2) into Eq. (4.3) and comparing the latter with Eq. (4.4) we find

$$\hat{K}_{qg,N}(\mathbf{k}^2/Q^2, \alpha_S(Q^2/\mu^2)^\varepsilon; \varepsilon) = \int_0^1 dz \, z^{N-1} \, \hat{K}_{qg}(z, \mathbf{k}^2/Q^2, \alpha_S(Q^2/\mu^2)^\varepsilon; \varepsilon) ,$$ (C.1)

where $\hat{K}_{qg}(z)$ is obtained from $\hat{K}^{(0)}(q,k)$ (see Fig. 7b), after integration over q, as follows

$$\hat{K}_{qg}\left(z, \frac{\mathbf{k}^2}{Q^2}, \alpha_S\left(\frac{Q^2}{\mu^2}\right)^\varepsilon; \varepsilon\right)$$
$$= \int \frac{dq^2 \, d^{2+2\varepsilon}\mathbf{q}}{2(2\pi)^{4+2\varepsilon}} \Theta\left(Q^2 - |q^2|\right) \left(\frac{\bar{p}}{2\bar{p}\cdot q}\right)_{\alpha\beta} \hat{K}^{(0)\,\alpha\beta}_{\mu\nu}(q,k) \frac{k_\perp^\mu \, k_\perp^\nu}{\mathbf{k}^2} .$$ (C.2)

In Eq. (C.2) the following Sudakov parametrization for the momenta k and q is understood:

$$k^\mu = y\,p^\mu + k_\perp^\mu , \quad q^\mu = x\,p^\mu + q_\perp^\mu + \frac{q^2 + \mathbf{q}^2}{2\,x\,p\cdot\bar{p}}\,\bar{p}^\mu , \quad z \equiv \frac{x}{y} .$$ (C.3)

It is also convenient to introduce the boost-invariant (along the k-direction) transverse momentum $\tilde{\mathbf{q}}$

$$\tilde{\mathbf{q}} = \mathbf{q} - z\,\mathbf{k} .$$ (C.4)

Performing the Dirac and colour algebra we get

$$
\left(\frac{\not{\bar{p}}}{2\bar{p}\cdot q}\right)_{\alpha\beta} \hat{K}^{(0)\,\alpha\beta}_{\mu\nu}(q,k) \frac{k^{\mu}_{\perp} k^{\nu}_{\perp}}{\mathbf{k}^2} = g^2_s \,(\mu^2)^{-\varepsilon}\, \frac{\delta^{ab}}{N^2_c - 1} \mathrm{Tr}\left(t^a t^b\right)
$$

$$
\cdot 2\pi\delta_+\left((k-q)^2\right) \frac{z}{2\bar{p}\cdot q}\frac{1}{(q^2)^2}\frac{1}{\mathbf{k}^2}\mathrm{Tr}\left[\not{q}\,\not{\bar{p}}\,\not{q}\,\not{k}_{\perp}\,(\not{q}-\not{k})\,\not{k}_{\perp}\right]
$$

$$
= 16\pi^2\alpha_S\,(\mu^2)^{-\varepsilon}\,T_R\,\frac{z^2}{1-z}\,\Theta(1-z)\frac{1}{(q^2)^2}\,\delta\left(q^2 + \frac{\tilde{\mathbf{q}}^2}{1-z} + z\,\mathbf{k}^2\right)
$$

$$
\cdot\left[\frac{\tilde{\mathbf{q}}^2}{z(1-z)} - \frac{4}{\mathbf{k}^2}(\mathbf{k}\cdot\tilde{\mathbf{q}})^2 + 4(1-2z)\,\mathbf{k}\cdot\tilde{\mathbf{q}} + 4z(1-z)\,\mathbf{k}^2\right], \qquad (\text{C.5})
$$

and, inserting Eq. (C.5) into Eq. (C.2), the azimuthal average over $\tilde{\mathbf{q}}$ and the integration over q^2 can easily be performed, thus leading to

$$
\hat{K}_{qg}\left(z,\frac{\mathbf{k}^2}{Q^2},\alpha_S\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon};\varepsilon\right) = \frac{\alpha_S}{2\pi}\,S_{\varepsilon}\,\frac{e^{\varepsilon\psi(1)}}{\Gamma(1+\varepsilon)}\,T_R\,z\,\Theta(1-z)\int\limits_0^{\infty} d\tilde{\mathbf{q}}^2\left(\frac{\tilde{\mathbf{q}}^2}{\mu^2}\right)^{\varepsilon}
$$

$$
\cdot\,\Theta\left(Q^2 - \frac{\tilde{\mathbf{q}}^2}{1-z} - z\mathbf{k}^2\right)\frac{\tilde{\mathbf{q}}^2}{\left[\tilde{\mathbf{q}}^2 + z(1-z)\,\mathbf{k}^2\right]^2}
$$

$$
\cdot\left[1 - \frac{2z(1-z)}{1+\varepsilon} + 4z^2(1-z)^2\frac{\mathbf{k}^2}{\tilde{\mathbf{q}}^2}\right]. \qquad (\text{C.6})
$$

Eq. (C.6) is precisely the result in Eq. (4.8), expressed in terms of the off-shell splitting function (4.9).

The $\tilde{\mathbf{q}}^2$-integration in Eq. (C.6) is not elementary and provides a representation of the off-shell kernel $\hat{K}_{qg}$ in terms of hypergeometric functions. However, in order to obtain the power series expansion (4.10) for the Green function $G^{(0)}_{qg}$, it is more convenient to carry out first the $\mathbf{k}$-integration in Eq. (4.4). Inserting the expansion (3.3) into Eq. (4.4), we get

$$
G^{(0)}_{qg,N}(\alpha_S(Q^2/\mu^2)^{\varepsilon},\varepsilon) = \frac{\alpha_S}{2\pi}\,T_R\,S_{\varepsilon}\,\frac{e^{\varepsilon\psi(1)}}{\Gamma(1+\varepsilon)}\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon}\,h_{qg,N}(\gamma=0;\varepsilon)
$$

$$
\cdot\left\{1 + \sum_{k=1}^{\infty}\left[\frac{\bar{\alpha}_S}{N}\,S_{\varepsilon}\,\frac{e^{\varepsilon\psi(1)}}{\Gamma(1+\varepsilon)}\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon}\right]^k\right.
$$

$$
\left.\cdot\frac{1}{k\varepsilon}\,c_k(\varepsilon)\,\frac{h_{qg,N}(\gamma=k\varepsilon;\varepsilon)}{h_{qg,N}(\gamma=0;\varepsilon)}\right\}, \qquad (\text{C.7})
$$

where the function $h_{qg,N}(\gamma;\varepsilon)$ is defined by the following $k_{\perp}$-transform of the off-shell kernel $\hat{K}_{qg}$

$$\frac{\alpha_S}{2\pi} T_R S_\varepsilon \frac{e^{\varepsilon\psi(1)}}{\Gamma(1+\varepsilon)} \left(\frac{Q^2}{\mu^2}\right)^\varepsilon h_{qg,N}(\gamma;\varepsilon)$$

$$\equiv \gamma \int_0^\infty \frac{d\mathbf{k}^2}{\mathbf{k}^2} \left(\frac{\mathbf{k}^2}{Q^2}\right)^\gamma \hat{K}_{qg,N}\left(\frac{\mathbf{k}^2}{Q^2}, \alpha_S \left(\frac{Q^2}{\mu^2}\right)^\varepsilon; \varepsilon\right) . \tag{C.8}$$

The evaluation of h_{qg} is straightforward. We insert Eq. (C.6) into Eq. (C.8) and perform first the trivial integration over $\mathbf{k}^2$ with $\tilde{q}^2/(z(1-z)\mathbf{k}^2) = \tau$ fixed. Then, the integrations over τ and z decouple and can be carried out in terms of Euler gamma functions. Note also that the $N \to 0$ limit of $h_{qg,N}$ is regular (i.e., $\hat{K}_{qg}(z) \sim z$, modulo $\ln z$-terms, for $z \to 0$ and any value of $\mathbf{k}^2/Q^2$). Therefore, to the logarithmic accuracy we are interested in, we can limit ourselves to computing $h_{qg,N=0}$. The final result is:

$$h_{qg,N=0}(\gamma;\varepsilon) = \frac{4 + \varepsilon - 3\gamma}{\gamma + \varepsilon} \frac{\Gamma(1+\gamma)\,\Gamma(1-\gamma)\,\Gamma(1+\varepsilon)\,\Gamma(2+\varepsilon)}{\Gamma(1+\varepsilon+\gamma)\,\Gamma(4+\varepsilon-\gamma)} . \tag{C.9}$$

Using Eq. (C.9), we recover Eqs. (4.10) and (4.11) by the identification

$$\frac{d_k(\varepsilon)}{c_k(\varepsilon)} = \frac{h_{qg,N=0}(\gamma = k\varepsilon;\varepsilon)}{h_{qg,N=0}(\gamma = 0;\varepsilon)} . \tag{C.10}$$

As discussed in Section 4, we have not been able to use the power series expansion (4.10) for explicitly resumming all the next-to-leading logarithmic corrections in the quark anomalous dimensions $\gamma_{qg,N}$. In general, to this accuracy we can write

$$\gamma_{qg,N}(\alpha_S) = \frac{\alpha_S}{2\pi} T_R \frac{2}{3} \left\{ 1 + \sum_{k=1}^\infty a_k \left(\frac{\bar{\alpha}_S}{N}\right)^k \right\} , \tag{C.11}$$

and the values of the coefficients a_k for $k \le 5$ are given in Eq. (4.14). The calculation of the higher-order coefficients is much more cumbersome. Here, we present only the result for the rational part of a_k. In other words, let us split a_k as follows

$$a_k = r_k + b_k , \tag{C.12}$$

where b_k is an irrational number given in terms of powers of Riemann zeta functions $\zeta(n)$ ($n \ge 3$) and r_k is the residual rational contribution to a_k. We find the following expression for r_k in the $\overline{\text{MS}}$ scheme:

$$r_k = \frac{2^{k-2}}{k!} \left[3 + \left(\frac{1}{3}\right)^k \right] , \tag{C.13}$$

or, equivalently,

$$1 + \sum_{k=1}^\infty r_k \left(\frac{\bar{\alpha}_S}{N}\right)^k = \frac{3}{4} \left[\exp\left(2\frac{\bar{\alpha}_S}{N}\right) + \frac{1}{3} \exp\left(\frac{2}{3}\frac{\bar{\alpha}_S}{N}\right) \right] . \tag{C.14}$$

One can easily check that Eq. (C.13) reproduces the values of r_k in Eq. (4.14), i.e. $r_1 = 5/3$, $r_2 = 14/9$, $r_3 = 82/81$, $r_4 = 122/243$, $r_5 = 146/729$. The derivation of the result (C.13) is left as an exercise for the reader.

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