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High Energy Nuclear Events

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Abstract

A statistical method for computing high energy collisions of protons with multiple production of particles is discussed. The method consists in assuming that as a result of fairly strong interactions between nucleons and mesons the probabilities of formation of the various possible numbers of particles are determined essentially by the statistical weights of the various possibilities.

1. Introduction.

The meson theory has been a dominant factor in the development of physics since it was announced fifteen years ago by Yukawa. One of its outstanding achievements has been the prediction that mesons should be produced in high energy nuclear collisions. At relatively low energies only one meson can be emitted. At higher energies multiple emission becomes possible.

In this paper an attempt will be made to develop a crude theoretical approach for calculating the outcome of nuclear collisions with very great energy. In particular, phenomena in which two colliding nucleons may give rise to several π-mesons, briefly called hereafter pions, and perhaps also to some anti-nucleons, will be discussed.

In treating this type of processes the conventional perturbation theory solution of the production and destruction of pions breaks down entirely. Indeed, the large value of the interaction constant leads quite commonly to situations in which higher approximations yield larger results than do lower approximations. For this reason it is proposed to explore the possibilities of a method that makes use of this fact. The general idea is the following:

When two nucleons collide with very great energy in their center of mass system this energy will be suddenly released in a small volume surrounding the two nucleons. We may think pictorially of the event as of a collision in which the nucleons with their surrounding retinue of pions hit against each other so that all the portion of space occupied by the nucleons and by their surrounding pion field will be suddenly loaded with a very great amount of energy. Since the interactions of the pion field are strong we may expect that rapidly this energy will be distributed among the various degrees of freedom present in this volume according to statistical laws. One can then compute statistically the probability that in this tiny volume a certain number of pions will be created with a given energy distribution. It is then assumed that the concentration of energy will rapidly dissolve and that the particles into which the energy has been converted will fly out in all directions.

It is realized that this description of the phenomenon is probably as extreme, although in the opposite direction, as is the perturbation theory approach. On the other hand, it might be helpful to explore a theory that deviates from the unknown truth in the opposite direction from that of the conventional theory. It may then be possible to bracket the correct state of fact in between the two theories. One might also make a case that a theory of the kind here proposed may perhaps be a fairly good approximation to actual events at very high energy, since then the number of possible states of the given energy is large and the probability of establishing a state to its average statistical strength will be increased by the very many ways to arrive at the state in question.

The statement that we expect some sort of statistical equilibrium should be qualified as follows. First of all there are conservation laws of charge and of momentum that evidently must be fulfilled. One might expect further that only those states that are easily reachable from the initial state may actually attain statistical equilibrium. So, for example, radiative phenomena in which photons could be created will certainly not have time to develop. The only type of transitions that are believed to be fast enough are the transitions of the Yukawa theory. A succession of such transitions starting with two colliding nucleons may lead only to the formations of a number of charged or neutral pions and also presumably of nucleon-anti-nucleon pairs. The discussion shall be limited, therefore, to these particles only. Notice the additional conservation law for the difference of the numbers of the nucleons and the anti-nucleons.

The proposed theory has some resemblance to a point of view that has been adopted by Heisenberg[1] who describes a very high energy collision of two nucleons by assuming that the pion "fluid" surrounding the nucleons is set in some sort of turbulent motion by the impact energy. He uses qualitative ideas of turbulence in order to estimate the distribution of energy of this turbulent motion among eddies of different sizes. Turbulence represents the beginning of an approach to thermal equilibrium of a fluid. It describes the spreading of the energy of motion to the many states of larger and larger wave number. One might say, therefore, in a qualitative way that the present proposal consists in pushing the Heisenberg point of view to its extreme consequences of actually reaching statistical equilibrium.

The multiple meson production has also been investigated in an interesting paper by Lewis, Oppenheimer and Wouthuysen[2]. These authors stress the importance of the strong coupling expected in the pseudoscalar meson theory for the production of processes of high multiplicity.

In the theory here proposed there is only one adjustable parameter, the volume Ω, into which the energy of the two colliding nucleons is dumped. Since the pion field surrounding the nucleons extends to a distance of the order $\hbar/\mu c$ where μ is the pion mass, Ω is expected to have linear dimensions of this order of magnitude. As long as the Lorentz contraction is neglected one could take for example a sphere of radius $\hbar/\mu c$. However, when the two nucleons approach each other with very high energy in the center of gravity system, their surrounding pion clouds will be Lorentz contracted and the volume will be correspondingly reduced.

For this reason the volume Ω will be taken energy dependent according to the relationship:

$$\Omega = \Omega_0 \frac{2Mc^2}{W}, \tag{1}$$

where Ω_0 is the volume without Lorentz contraction. W is the total energy of the two colliding nucleons in the center of gravity system and M is the nucleon mass. The factor $2Mc^2/W$ is the Lorentz contraction. The uncontracted volume Ω_0 may be taken as a sphere of radius R:

$$\Omega_0 = 4\pi R^3/3. \tag{2}$$

It is found in the applications that one seems to get an acceptable agreement with known facts by assuming:

$$R = \hbar/\mu c = 1.4 \times 10^{-13}\text{cm}. \tag{3}$$

This choice of the volume, although plausible as order of magnitude, is clearly arbitrary and could be changed in order to improve the agreement with experiment. One finds that an increase of Ω_0 would tend to favor processes in which a large number of particles is created.

According to this point of view the total collision cross-section of the two nucleons will be always of the order of magnitude of the geometrical cross-section of the pion cloud. In the numerical calculations actually, the total cross-section has been taken equal to area of a circle of radius R, namely,

$$\sigma_{tot} = \pi R^2. \tag{4}$$

Assuming (3) one finds $\sigma_{tot} = 6 \times 10^{-26}\text{cm}^2$. In order to compute the partial cross-section for a phenomenon in which for example three pions are produced in the collision, one will multiply the total cross-section (4) by the relative probability that three pions instead of any other possible number and kind of particles are produced.

The probability of transition into a state of a given type is proportional to the square of the corresponding effective matrix element and to the density of states per unit energy interval. Our assumption of a statistical equilibrium consists in postulating that the square of the effective matrix element is merely proportional to the probability that, for the state in question, all particles are contained at the same time inside Ω. For example in the case of a state that describes n completely independent particles with momenta $p_1, p_2, \cdots p_n$ this probability is $(\Omega/V)^n$ where V is the large normalization volume. The number of states per unit energy interval is

$$\left(\frac{V}{8\pi^3\hbar^3}\right)^n \frac{d}{dW}Q(W),$$

where $Q(W)$ is the volume of momentum space corresponding to the total energy W. The probability for the formation of the state in question is therefore assumed to be proportional to the product:

$$S(n)=\left(\frac{\Omega}{8\pi^3\hbar^3}\right)^n \frac{dQ(W)}{dW}. \tag{5}$$

There are some complications arising from the fact that the particles are not independent.

a) In the center of mass system the positions and momenta of only $n-1$ of the n particles are independent variables. For this reason the exponent of Ω will be $n-1$ instead of n. Also the momentum space $Q(W)$ will be $3(n-1)$-dimensional instead of $3n$-dimensional.

b) Some of the particles may be identical and this fact should be taken into account in computing $Q(W)$.

c) Some of the particles may carry a spin and one should then allow for the corresponding multiplicity of the states.

d) The conservation of angular momentum restricts the statistical equilibrium to states with angular momentum equal to that of the two colliding nucleons. In all cases considered $\lambda\!\!\!^-$ for the nucleons is smaller than the radius $\hbar/\mu c$ of the sphere of action. It is then meaningful to discuss separately collisions with various values of the impact parameter b (b is the distance of the two straight lines along which the nucleons move before the collision). In units of $\hbar$ the angular momentum is $l=b/\lambda\!\!\!^-$. The cross-section for collisions with impact parameter between b and $b+db$ is $2\pi b db=2\pi\lambda\!\!\!^{-2} l dl$. One should treat separately collisions with different values of the impact parameter and compute for each of them the probability of the various possible events. The cross-section for a special event is then obtained by adding the contribution of the various l-values.

It is found in most cases that the results so obtained differ only by small numerical factors from those obtained by neglecting the conservation of angular momentum.

This has been done as a rule in order to simplify the mathematics. The corrections arising from the conservation of angular momentum have been, however, indicated in typical cases.

2. Example. Pion Production in Low Energy Nucleon Collisions.

As a first example the production of pions in a collision of two nucleons with relative energy barely above the threshold needed for emission of a pion will be discussed. This example is chosen because it is the simplest possible. It is, however, a case in which the statistical approach may be misleading, since only few states of rather low energy are involved. We will first simplify this example by disregarding the spin of the nucleons as well as the possible existence of a spin of the pions and by disregarding also the various possible electric charges of the particles in question. In the center of gravity system we will have therefore two nucleons colliding against each other. $T/2$ is the kinetic energy of each of the two nucleons. A pion can be emitted when $T > \mu c^2$. We shall assume that this inequality is fulfilled; that the kinetic energy, however, exceeds the threshold by only a small amount, so that both the two nucleons and the pion that may be formed will have non-relativistic energies.

Conservation of energy in this case allows only two types of states; Type (a) in which the two nucleons are scattered elastically without formation of pions; and type (b) in which a pion is formed and three particles, two nucleons and a pion, emerge after the collision.

The statistical weight of the states of type (a) is obtained as follows: Since the momenta of the two nucleons are equal and opposite the momentum space will be three-dimensional. We can compute the statistical weight with (5). s will be taken $=1$ because the momentum of one particle determines that of the other. The reduced mass is $M/2$, the momentum $p=\sqrt{MT}$ and the phase space volume $Q(T)=4\pi p^3/3$. According to (5) the statistical weight of this state has therefore the familiar expression:

$$S_2=\frac{\Omega M^{3/2}}{4\pi^2 \hbar^3}\sqrt{T}\,. \tag{6}$$

S_2 should be compared with the statistical weight S_3 for case (b) in which three particles, two nucleons and one pion, emerge. Since the total momentum is zero, only the momenta of two of the three particles are independent and therefore in (5) $s=2$. The calculation of the momentum volume involves some slight complication on account of the conservation of momentum. Let p be the momentum of the pion and let the momenta of the two nucleons be $-\frac{1}{2}p \pm q$. The kinetic energy will then be:

$$T_1=\left(\frac{1}{2\mu}+\frac{1}{4M}\right)p^2+\frac{1}{M}q^2 \tag{7}$$

where $T_1=T-\mu c^2$ is the kinetic energy left over after a pion has been formed. Formula (7) represents an ellipsoid in the six dimensional momentum space of the two vectors p and q. Its volume is:

$$Q_3=\frac{\pi^3}{3!}\left(\frac{4M^2\mu}{2M+\mu}\right)^{3/2}T_1^3. \tag{8}$$

The factor $\pi^3/3!$ is, for a six-dimensional sphere, the analog of the factor $4\pi/3$ in the volume of an ordinary sphere. Substituting in (5) one finds:

$$S_3=\frac{\Omega^2}{16\pi^3\hbar^6}\left(\frac{M^2\mu}{2M+\mu}\right)^{3/2}T_1^2\approx\frac{\Omega^2M^{3/2}\mu^{3/2}T_1^2}{32\sqrt{2}\,\pi^3\hbar^6}. \tag{9}$$

The last expression is simplified by assuming $\mu\ll M$. The probabilities of the two events (a) and (b) are proportional to S_2 and S_3. Since S_3 is very small, we may take the ratio S_3/S_2 to be the probability that the collision leads to pion formation. This is given by:

$$\frac{S_3}{S_2}=\frac{\Omega\mu^{3/2}}{8\sqrt{2}\,\pi\hbar^3}\,\frac{(T-\mu c^2)^2}{\sqrt{T}}\approx\frac{\Omega\mu(T-\mu c^2)^2}{8\sqrt{2}\,\pi\hbar^3c}.$$

Since T is barely larger than the threshold energy μc^2, this value has been substituted for T in the denominator. In this case also the Lorentz contraction of the two colliding nucleons is negligible and we can therefore substitute for Ω the value Ω_0 given by (2) and (3). One finds:

$$\frac{S_3}{S_2}=\frac{1}{6\sqrt{2}}\left(\frac{T}{\mu c^2}-1\right)^2. \tag{11}$$

The cross-section for pion formation is given by the product of the total cross-section (4) and the probability (11). For example, in a bombardment of nucleons at rest with 345 MeV nucleons, (the proton energy available at Berkeley) one finds that the energy available in the center of gravity system is $T=165$MeV. On the other hand $\mu c^2=140$MeV and the previous formula gives therefore S_3/S_2 $=.0038$. This means that at this bombarding energy a pion will be formed in about 0.4 per cent of the nucleon collisions.

If one examines the process more in detail one will recognize that in a collision of two protons the probability of emission of a positive pion is twice (11) namely, .0076. Because if a positive pion is formed a proton and a neutron, instead of two protons, will also emerge. Their statistical weight is twice that for two protons because they are not identical particles. Similarly in the collision of a proton and a neutron the probability of emission of a positive pion is one-half of (10) namely, .0019. The probability of emission of a negative pion is the same.

For example, when a carbon target is bombarded by 345 MeV protons the probabilities that the collision takes place between the proton and another proton or a neutron are the same. Hence, the probability of emission of a positive pion will be .0076/2+.0019/2; that is, .0048; and the probability of emission of a negative pion will be .0019/2=.001. Since the nuclear cross-section of carbon is about 3×10^{-25}cm^2, one will obtain the expected values of the cross-sections for emission of a positive and a negative pion by multiplying the nuclear cross-section by the above probabilities. The results are $1.4\times10^{-27}\text{cm}^2$ for the positive and $3\times10^{-28}\text{cm}^2$ for the negative pions. Considering the extremely crude calculation these values are in surprisingly good agreement with the experimental results.

In the above discussion the conservation of angular momentum has been disregarded. When a pion is produced the kinetic energy of the three emerging particles is small and they will therefore escape in an s-state. Consequently, only the initial states of zero angular momentum can be contribute to this type of final state. Their maximum cross-section has the well known expression $\pi\lambda\!\!\!^{-2}$ which is appreciably smaller than (4). However, also the competition of elastic scattering versus pion production is less since only the scattering states of zero angular momentum will contribute.

By carrying out the calculation one finds that the two effects almost cancel each other and that the conservation of angular momentum changes the previous results for the cross-section for pion production by only a factor 2/3. As long as the conservation of angular momentum is neglected one expects the scattering of the two nucleons to be spherically symmetrical in the center of mass system. This is no longer the case when the angular momentum is conserved. One finds then that the elastic scattering cross-section per steradian in the center of mass system instead of being constant is approximately proportional to $1/\sin\theta$, where θ is the scattering angle in the same system.

3. Formulas for the Statistical Weights.

Some standard formulas expressing the statistical weights, S, for a number of simple cases will be collected here.

First, the case will be considered that after a collision n particles emerge with masses $m_1, m_2\cdots\cdots m_n$. Neglecting spin properties and assuming that the particles are statistically independent, and disregarding also the momentum conservation, one finds for S the following two formulas corresponding to the classical and to the extreme relativistic case:

$$S_n=\frac{(m_1m_2\cdots m_n)^{3/2}\Omega^n}{2^{3n/2}\pi^{3n/2}\hbar^{3n}}\frac{T^{3n/2-1}}{(3n/2-1)!}, \tag{12}$$

(classical case)

$$S_n=\frac{\Omega^n}{\pi^{2n}\hbar^{3n}c^{3n}}\frac{W^{3n-1}}{(3n-1)!}. \tag{13}$$

(extr. relativistic case)

In (12) T is the total classical kinetic energy of the n particles and in (13) W is the total energy including rest energy of the n particles. One can also compute a formula for the case that s of the particles, usually the nucleons, are classical and n, usually the pions, are extreme relativistic. Neglecting again spin statistics and momentum conservation and assuming further that all the classical particles have the nucleon mass, M, one finds:

$$S(s, n)=\frac{M^{3s/2}\Omega^{n+s}}{2^{3s/2}\pi^{2n+3s/2}\hbar^{3s+3n}c^{3n}}\frac{(W-sMc^2)^{3n+3s/2-1}}{(3n+3s/2-1)!}. \tag{14}$$

It is sometimes convenient to re-write (14) in the following form:

$$S(s,n)=\frac{M^{3s/2}\Omega^{s/2+1/3}}{2^{3s/2}\pi^{s/2+2/3}\hbar^{3s/2+1}c^{-3s/2+1}}\frac{\left(\frac{\Omega^{1/3}(W-sMc^2)}{\pi^{2/3}\hbar c}\right)^{3n+3s/2-1}}{(3n+3s/2-1)!}, \tag{15}$$

since it is thus easy to obtain an approximate expression for the sum of the statistical weights $S(s, n)$ over all values of n. The approximation applies to the cases when the average value of n is $\gg 1$. One finds then:

$$\sum_{n=0}^{\infty} S(s, n)\approx\frac{M^{3s/2}\Omega^{s/2+1/3}}{3\times 2^{3s/2}\pi^{s/2+2/3}\hbar^{3s/2+1}c^{-3s/2+1}}\exp\left(\frac{\Omega^{1/3}(W-sMc^2)}{\pi^{2/3}\hbar c}\right). \tag{16}$$

The numerical values of (15) and (16) adopting for Ω (1), (2) and (3) are:

$$Mc^2S(s, n)=\frac{6.31}{w^{1/3}}\left(\frac{98.8}{w}\right)^{s/2}\frac{\left(6.31\,\frac{w-s}{w^{1/3}}\right)^{3n+3s/2-1}}{(3n+3s/2-1)!} \tag{17}$$

and:

$$Mc^2\sum_{n=0}^{\infty} S(s, n)\approx\frac{2.10}{w^{1/3}}\left(\frac{98.8}{w}\right)^{s/2}\exp\left(6.31\,\frac{w-s}{w^{1/3}}\right) \tag{18}$$

where one has put

$$w=W/Mc^2 \tag{19}$$

and it has further been assumed $\mu/M=.15$.

In the previous formulas the momentum conservation has been disregarded. The formulas, however, can be generalized without difficulty so as to introduce at least approximately the requirement that the total momentum be zero. The approximation consists in assuming that the mass of the pion is very small compared to the nucleon mass. The momentum of the nucleons will then be much greater than the momentum of the pions since the kinetic energy is approximately equipartitioned among the various particles. Therefore, one can approximately apply the condition that the sum of the momenta vanishes to the nucleons only. One recognizes then that formula (15) must be changed as follows. a) Instead of s one will write $s-1$ at all places except in the term $W-sMc^2$ since we have now $s-1$ independent momenta of the heavy particles. b) The factor $M^{3s/2}$ must be changed because instead of the mass one should substitute an expression which is the analog of a reduced mass. It is found that the factor in question must be substituted by $M^{3(s-1)/2}/s^{3/2}$. When the conservation of momentum is approximately taken into account formulas (17) and (18) should then be changed as follows:

$$Mc^2 S(s,n) = \frac{6.31}{s^{3/2} w^{1/3}} \left(\frac{98.8}{w}\right)^{(s-1)/2} \frac{(6.31(w-s)/w^{1/3})^{3n+3s/2-1}}{(3n+3s/2-1)!}, \tag{20}$$

$$Mc^2 \sum_{n=0}^{\infty} S(s,n) \approx \frac{2.10}{s^{3/2} w^{1/3}} \left(\frac{98.8}{w}\right)^{(s-1)/2} \exp(6.31(w-s)/w^{1/3}). \tag{21}$$

In all the preceding formulas the particles have been assumed to be statistically independent. As long as few nucleons and pions are involved, the error is not great. Larger errors are expected for high multiplicity. The formulas, however, become quite involved since there are at least three kinds of pions and four kinds of nucleons and anti-nucleons. No attempt has been made to introduce these complications for phenomena of relatively low energy. They have been calculated as if there were only one type of pions, one of nucleons and one of anti-nucleons statistically independent. This procedure is certainly inadequate and will give a too high multiplicity at high energy. For phenomena of extremely high energy it becomes simple to introduce the statistical correlations by substituting the statistical by a thermo-dynamical model. This case will be treated in Section 6.

In the previous expressions also the conservation of angular momentum has been neglected. The error introduced with this omission will be discussed in Section 6, where an appropriate correction factor for it will be given.

4. Transition from Single to Multiple Production of Pions.

In Section 2 the emission of a single low energy pion has been discussed. Collisions of higher energy in which besides the two original nucleons also several pions may be produced will be considered now. A rough indication of the features of this process may be obtained by computing the relative probabilities for the emission of 0, 1, 2,$\cdots\cdots n\cdots$ pions with (20). In that formula one will put $s=2$. Statistical correlations and conservation of angular momentum will be negelcted. Omitting a common factor, the probabilities of the various values of n are proportional to:

$$\left\{\frac{251}{w}(w-2)^3\right\}^n \Big/ \left(\frac{3}{2}\times\frac{5}{2}\times\cdots\times\frac{6n+1}{2}\right). \tag{22}$$

Table I gives the probabilities of pion production of different multiplicities calculated according to this formula. The first column of the table gives the energy, w, of the two nucleons in the center of gravity system in units of Mc^2. The second column gives the energy, w', of the primary particle in the laboratory frame of reference. The next eight columns are labeled by the number, n, of pions produced and give the probabilities of various events in per cents. The last column gives the average number of pions produced.

TABLE I

w	w'	$n=0$	1	2	3	4	5	6	7	$\bar{n}$
2.5	2.1	49	47	4						.6
3	3.5	9	59	30	2					1.2
3.5	5.1	2	31	46	18	3				1.9
4	7.0		13	40	33	11	2			2.5
5	11.5		2	15	34	31	14	3	1	3.5

Notice that already for a bombarding energy of about 1 BeV corresponding to the first line of the table the probability of elastic collision of the two nucleons is 50%. This probability decreases rapidly and drops below one per cent for bombarding energies of about 5 BeV. As the bombarding energy increases the probability of multiple phenomena increases as indicated in the table. The most probable value of n according to (22) should be given approximately by $2.1(w-2)/w^{1/3}$.

It will be seen in Section 6 that at high energy very appreciable errors are are introduced by neglecting the angular momentum conservation and the statistical correlations. Table I gives only a qualitative indication of the transition from elastic scattering to single and then multiple pion production. The quantitative features of the multiple production, however, should be more reliably represented by formula (32).

5. Production of Anti-Nucleons.

When the two colliding pions have a total energy $>4Mc^2$ in the center of gravity system, competition with processes in which a nucleon-anti-nucleon pair is formed becomes possible. When the energy is barely above the $4Mc^2$-threshold, no pions can be formed accompanying the pair. As the energy increases, however, the pair will be as a rule accompanied by a number of pions. For moderate energy $w<10$ one will use formula (20). Substituting in it $s=4$ we obtain the statistical weight for nucleon pair formation associated with the emission of n pions. Substituting $s=2$ we obtain an expression proportional to the probability that no pair is formed and the two original nucleons plus n pions emerge.

Omitting the common factor Mc^2 one obtains from (20) :

$$S(4,n)=\frac{775}{w^{11/6}}\frac{(6.31(w-4)/w^{1/3})^{3n+7/2}}{(3n+7/2)!}. \tag{23}$$

In normalizing these probabilities to total probabilitiy=1, one can make use of the fact that the probability of pair formation in the range of energies here discussed is always less than one per cent. One can therefore disregard the pair formation in the normalization factor which reduces to $\sum_n S(2,n)$. In calculating this sum one can use (21). One obtains in the end the following expression for the probability of pair formation accompanied by n pions :

$$P(4, n)=\frac{105}{w}\left(6.31\ \frac{w-4}{w^{1/3}}\right)^{3n+7/2}\frac{\exp(-6.31(w-2)/w^{1/3})}{(3n+7/2)\,!}.$$

TABLE II

w	w'	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	Total
$4+\varepsilon$	7.0	$1000\ \varepsilon^{7/2}$					$.1\varepsilon^{7/2}$
4.5	9.1	14	.6				15×10^{-4}
5	11.5	27	8	.7			36×10^{-4}
5.5	14.3	21	21	5	.5		47×10^{-4}
6	16.9	12	25	14	3	.3	54×10^{-4}

Columns 3 to 7 inclusive $\times\ 10^{-4}$

Table II is calculated with this formula. Again the first and second columns represent in units of Mc^2 the total energy in the center of gravity system and the total energy of the bombarding particle in the laboratory system. The next five columns give the probabilities of pair formation accompanied by n pions. These probabilities have been multiplied by a factor 10^4. The eighth column is the total probability of pair formation.

Again, in computing this table the statistical correlations mentioned in Section 3 and the conservation of angular momentum have been disregarded. For this reason the data of the table are merely indicative of the results that would be given by a more correct computation.

At the highest energy here considered the probability of anti-nucleon formation is 0.005. Since in a collision of this energy probably two or three pions are formed in the average one concludes that at these energies the ratio of anti-nucleons to pions formed is about 0.002. Therefore, anti-nucleons will be hard to find even in fairly high energy collisions.

6. Collisions of Extremely High Energy.

In discussing the collision of two nucleons with extremely high energy one can simplify the calculations by assuming that all the various particles produced are extreme relativistic and that thermo-dynamics may be applied instead of a detailed statistical computation of the probabilities of the various events.

In this discussion the conservation of angular momentum will first be neglected. Its effect will be given at the end of this section.

The extremely high energy density that is suddenly formed in the volume Ω will give rise to multiple production of pions and of pairs of nucleons and anti-nucleons. Since both kinds of particles are extreme relativistic, the energy density will be proportional to the fourth power of the temperature, T, as in Stefan's law.

The pions, like the photons, obey the Bose-Einstein statistics. Since we further assume that the temperature is so high that the rest mass is negligible, their energy momentum relationship will be the same as for the photons. Consequently the Stefan's law for the pions will be quite similar to the ordinary Stefan's law of the black body radiation. The difference is only in a statistical weight factor. For the photons the statistical weight is the factor, 2, because of the two polarization directions. If we assume that the pions have spin zero and differ only by their charge $\pm e$ or 0, their statistical weight will be 3. Consequently, the energy density of the pions will be obtained by multiplying the energy density of the ordinary Stefan's law by the factor 3/2. This energy density is therefore:

$$\frac{3\times 6.494(kT)^4}{2\pi^2\hbar^3c^3}. \tag{25}$$

The numerical factor $6.494=\pi^4/15$ is six times the sum of the inverse fourth powers of the integral numbers.

The contribution of the nucleons and anti-nucleons to the energy density is given by a similar formula. The differences are that the statistical weight of the nucleons is eight since we have four different types of nucleons and anti-nucleons and for each, two spin orientations. A further difference is due to the fact that these particles obey the Pauli principle. In the extreme relativistic case their energy density is:

$$\frac{4\times 5.682(kT)^4}{\pi^2\hbar^3c^3}. \tag{26}$$

Here the numerical factor 5.682 is $6\sum_1^\infty(-1)^{n+1}/n^4$.

The temperature is obtained by equating the total energy to the product of the volume Ω times the sum of the two energy densities (25) and (26). Making use of (1) one obtains the temperature from the following equation:

$$(kT)^4=.152\frac{\hbar^3c^3W^2}{Mc^2\Omega_0}. \tag{27}$$

In order to compute the number of pions, nucleons and anti-nucleons produced we need formulas for the density of the various particles. These are computed according to standard procedures of statistical mechanics. In the extreme relativistic case the density of the particles turns out to be proportional to the third power of the temperature. The total densities of the pions and of the nucleons are given by the following two expressions:

$$n_\pi=.367\frac{(kT)^3}{\hbar^3c^3}; \qquad n_N=.855\frac{(kT)^3}{\hbar^3c^3}. \tag{28}$$

The total numbers of pions and nucleons are obtained by multiplying the expressions (28) by the volume Ω and by substituting in them the temperature calculated from (27). The result must be finally corrected in order to take into account the conservation of angular momentum. Only the result of this correction will be given. It is found that conservation of angular momentum has the effect of reducing the numbers of pions and nucleons by a factor that has been calculated numerically to be about .51. The conservation of angular momentum has the further effect that the angular distribution of particles produced is no longer isotropical but tends to favor somewhat, particles moving parallel to the original direction of the two colliding nucleons. Introducing these corrections one finds that the number of pions is:

$$\text{No. of pions} = .091\left(\frac{\Omega_0 M W^2}{c\hbar^3}\right)^{1/4} = .54\sqrt{W/Mc^2} \tag{29}$$

and the number of nucleons plus anti-nucleons is:

$$\text{No. of nucleons and anti-nucleons} = .21\left(\frac{\Omega_0 M W^2}{c\hbar^3}\right)^{1/4} = 1.3\sqrt{WMc^2}\,. \tag{30}$$

From this follows that the number of charged particles that emerge out of an extremely high energy collision is given by:

$$1.2(W'/Mc^2)^{1/4}. \tag{30a}$$

(W'=energy in the laboratory system)

In these formulas Ω_0 has been substituted by its value, (2), (3).

These formulas apply only to extreme high energies. Substituting the value, (2), (3), for Ω_0 one finds from (27) that the relationship between temperature and energy can be written in the form:

$$kT/Mc^2 = .105\sqrt{W/Mc^2}\,. \tag{31}$$

Relativistic conditions for the nucleons will be achieved therefore only when $W > 100Mc^2$. This corresponds in the laboratory system to an energy of the bombarding particles of more than 5×10^{12} eV. At somewhat lower energies the number of anti-nucleon pairs formed will decrease very rapidly, especially since an energy $2Mc^2$ is needed in order to form a pair. In this energy range the formation of paris is probably better represented by the computation of Section 5.

A comparison of (29) and (30) indicates that in such collisions of extremely high energy the number of nucleons and anti-nucleons produced exceeds that of the pions. Naturally, the anti-protons which are the particles in which we are most interested from the experimental point of view are only one-fourth of the particles (30). Therefore, a somewhat larger number of pions than of anti-protons is formed, even at these high energies. The reason why so many nucleons of all kinds are formed compared to the pions is their statistical weight (8 for the nucleons, 3 for the pions).

In an intermediate energy range where the multiple production of pions is the relevant phenomenon one can still apply the thermo-dynamic method restricting, however, the thermo-dynamic equilibrium to the pion gas only, and assuming that the activation energy of the pairs is too high for producing a sizeable number of these particles at the given temperature. The energy density in this case will be given by (25). The numerical coefficient in formula (27) will be reduced for this reason from .152 to .046. Also in the same formula one will substitute W by $W-2Mc^2$ since the energy of the two nucleons does not contribute to the energy of the pion gas. Introducing also the factor .51 for the conservation of angular momentum one finds that the number of pions in this approximation is given by:

$$\text{No. of pions} = .323\frac{M^{1/4}R^{3/4}(W-2Mc^2)^{3/2}}{\hbar^{3/4}c^{1/4}W} = 1.34\frac{(w-2)^{3/2}}{w}, \tag{32}$$

where $w = W/Mc^2$.

In the intermediate energy range of bombarding particles from 10 to 100 BeV this formula probably gives a better estimate of the multiplicities than do the computations of Table I. In particular it would appear that especially the multiplicities given in the last two lines of Table I are too large. According to (32) one would expect for these two energies multiplicities of about 2 instead of the considerably higher values given in Table I. The difference is due to two effects which have been disregarded in computing Table I; namely, the statistical correlation between various types of pions and the angular momentum conservation. Both factors are approximately taken into account in formula (32).

Since no observation of multiple production of an isolated nucleon is available at present, the comparison of these findings with experimental results is only tentative. The present theory seems to give rather low multiplicities except at extremely high energies of the order of 10^{12} to 10^{13} eV. As more experimental results become available it may be possible to improve the agreement of the theory with experiment by changing the choice (3) of R. If experimentally the multiplicities should turn out to be larger than according to the theory, one would increase R or decrease it in the opposite case.

In the present theory we have considered only one type of mesons, the pions. If mesons of larger mass strongly bound to nucleons should exist, as seems to be indicated by the recent experiments of Anderson[3], these particles also could reach statistical equilibrium. Since their rest energy is large, however, they would compete unfavorably with the production of pions except at very high energies. One would expect therefore in most collisions that the number of pions produced should be appreciably larger than that of the heavier mesons.

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