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HIGH-FREQUENCY SCATTERING FROM A METAL-LIKE DIELECTRIC LENS*)

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Abstract
The scattering of a plane electromagnetic wave by a dielectric lens which behaves like a metal reflector is considered. At short wavelengths, the leading term of the backscattered field cannot be determined entirely through simple geometrical optics considerations; instead, it is obtained by means of a modified Watson transformation of the exact solution. The difficulties that arise in applying this technique to other lenses are discussed.

§ 1. Introduction

The spherically symmetric dielectric lens considered in this paper has a refractive index N given by the relation

$$N(x) = 2x^{-\frac{1}{2}}(1 + x)^{-1}, \tag{1}$$

where $x = r/a$ is the radial distance from the center $r = 0$ of the lens, normalized to the radius $r = a$ of the rim of the lens. The optical and electromagnetic behaviours of a class of dielectrics to which this lens belongs have recently been studied [1]. In particular, it has been shown that the lens with refractive index given by (1) behaves like a metal sphere with the same radius $r = a$, in the optical limit. While this is certainly true for bistatic scattering, in the case of backscattering the geometrical optics prediction is not very reliable, because the optical ray which is responsible for the leading term contribution to the high-frequency backscattered field follows a path that has an infinite curvature at the center of the lens. In this paper, the high-frequency backscattered field is ob-

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tained by means of an asymptotic analysis of the exact solution of the electromagnetic boundary value problem, and is then compared with the optical answer. It is found that the high-frequency monostatic cross section of the lens is equal to that of a metal sphere of radius $r = a/2$.

The geometrical optics and the exact electromagnetic solutions for the far backscattered field produced by an incident plane wave are presented in sections 2 and 3, respectively. The radial eigenfunctions for the TE and TM modes are asymptotically evaluated in sections 4 and 5, and the high-frequency backscattered field is determined in section 6 by means of a modified Watson transformation. The appendix is devoted to a brief discussion of the difficulties encountered in applying the method used here to other dielectric lenses.

The rationalized MKSA system of units is adopted, and the time-dependence factor $\exp(-i\omega t)$ is omitted throughout.

§ 2. The geometrical optics solution

The geometry of the problem is illustrated in Fig. 1. The primary field is a beam of optical rays parallel to the z-axis. The ray that enters the lens at P at an angle of incidence α is smoothly deviated along a closed path surrounding the center C of the lens, and leaves the lens at P. Thus, a point source at the rim is imaged on to itself. The path is symmetric with respect to the diameter PCQ, and its minimum distance CQ from the center is given by

$$CQ = ax_{\min} = a\left(\frac{1 - \cos \alpha}{\sin \alpha}\right)^2. \tag{2}$$

The optical length R of the loop PQP is independent of α and given by

$$R = 2\pi a. \tag{3}$$

Let us now consider an incident plane electromagnetic wave propagating in the positive z direction with wavenumber k, whose electric vector

$$\boldsymbol{E}^i = \hat{e}\, e^{ikz} \tag{4}$$

has unit amplitude and is polarized in the direction of the constant unit vector $\hat{e}$. On the basis of simple geometric considerations and

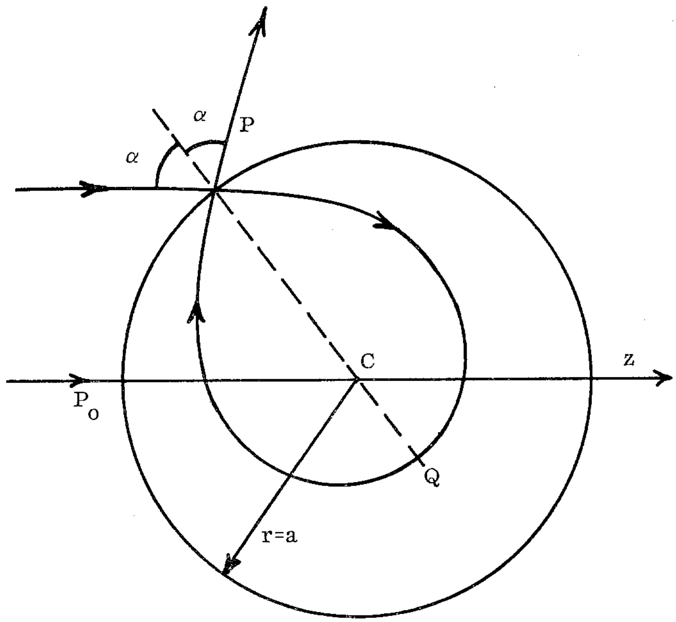

Fig. 1. Geometry for the optical rays.

of formula (3), we would expect a far backscattered field whose leading term at short wavelengths ($ka \gg 1$) is given by:

$$\boldsymbol{E}_{g.o.}^{b.s.} \sim \hat{e}\,\frac{a}{2r}\exp\{ik[r + 2a(\pi - 1)]\}, \tag{5}$$

where $r = -z$ is the distance of the observation point from the center C of the lens. It is instructive to compare (5) with the geometrical optics backscattered field from a perfectly conducting sphere of radius $r = a$:

$$[\boldsymbol{E}_{g.o.}^{b.s.}]_{metal} \sim -\hat{e}\,\frac{a}{2r}\exp\{ik(r - 2a)\}. \tag{6}$$

Result (5) differs from (6) by a factor ($-\exp i2\pi ka$); the minus sign is due to the lack of a 180 deg phase jump at P_0 (see Fig. 1), whereas the phase increase of $2\pi ka$ is due to the optical length of the path P_0CP_0.

Considerations similar to those leading to (5) are certainly justified in deriving the dominant term of the bistatic scattered field, for which α is bounded away from zero and therefore $x_{\min}$ in (2) is not too small compared with unity. However, if α is nearly zero, the optical ray path has a nearly infinite curvature where it bends around C, and the geometric optics result is no longer reliable. In particular, the leading term of the high-frequency backscattered field is obtained by multiplying the right-hand side of (5) by a correction factor, which we write in the form:

$$D\,\mathrm{e}^{-\mathrm{i}(\pi/2)+\mathrm{i}\delta}, \tag{7}$$

where D is real positive and δ is real.

It is intuitively felt that not all the energy propagating within a narrow beam of rays at nearly normal incidence will follow the sharp bend round the center C, and this implies that

$$0 < D < 1. \tag{8}$$

Furthermore, it follows from (2) that all the rays within a narrow cylindrical tube of diameter $2a\alpha$, ($\alpha \ll 1$), and axis z will cross the z-axis at points in the interval

$$0 \leq z \lesssim \frac{\alpha^2}{4}\,a. \tag{9}$$

If the rays of this tube were crossing a caustic, the phase would decrease by $\pi/2$ and therefore we would have $\delta = 0$ in (7). However, the region round (9) is simply a "pseudo-caustic" area in which an extremely high concentration of optical rays occurs, so that the phase will decrease by less than $\pi/2$, and therefore

$$0 < \delta < \frac{\pi}{2}. \tag{10}$$

In conclusion, the high-frequency backscattered field produced by the incident field (4) is:

$$\boldsymbol{E}^{b.s.} \sim \hat{e}D\,\frac{a}{2r}\exp\left\{\mathrm{i}\left[kr + 2ka(\pi-1) - \frac{\pi}{2} + \delta\right]\right\}, \tag{11}$$

where D and δ are real positive quantities subjected to the limitations (8) and (10). This result is based on physical intuition, and it is confirmed by the rigorous analysis of the following sections; in particular, the precise values of D and δ are obtained in section 6.

§ 3. The exact solution

The incident plane wave (4) produces the far backscattered field:

$$\boldsymbol{E}^{b.s.} = \hat{e}\,\frac{e^{ikr}}{ikr}\sum_{n=1}^{\infty}(-1)^n\,(n+\tfrac{1}{2})(a_n - b_n), \tag{12}$$

where

$$a_n = -\frac{\psi_n'(ka) - M_n\,\psi_n(ka)}{\zeta_n'(ka) - M_n\,\zeta_n(ka)}, \tag{13}$$

$$b_n = -\frac{\psi_n'(ka) - \tilde{M}_n\,\psi_n(ka)}{\zeta_n'(ka) - \tilde{M}_n\,\zeta_n(ka)}, \tag{14}$$

$$\psi_n(ka) = \left\{\frac{\pi ka}{2}\right\}^{\frac{1}{2}} J_{n+\frac{1}{2}}(ka), \quad \zeta_n(ka) = \left\{\frac{\pi ka}{2}\right\}^{\frac{1}{2}} H^{(1)}_{n+\frac{1}{2}}(ka), \tag{15}$$

$$M_n = \frac{1}{ka}\left[\frac{S_n^{(1)\prime}(x)}{S_n^{(1)}(x)}\right]_{x=1}, \quad \tilde{M}_{\mathrm{n}} = \frac{1}{ka}\left[\frac{T_n^{(1)\prime}(x)}{T_n^{(1)}(x)}\right]_{x=1}, \tag{16}$$

and the prime indicates the derivative with respect to the argument. The radial eigenfunctions $S_n^{(1)}(x)$ and $T_n^{(1)}(x)$ are those particular solutions of the radial differential equations

$$S_n''(x) + \left\{[kaN(x)]^2 - \frac{n(n+1)}{x^2}\right\} S_n(x) = 0 \tag{17}$$

and

$$T_n''(x) - 2\,\frac{N'(x)}{N(x)}\,T_n'(x) + \left\{[kaN(x)]^2 - \frac{n(n+1)}{x^2}\right\} T_n(x) = 0 \tag{18}$$

with $N(x)$ given by (1), which are finite over the interval $0 \le x \le 1$. They are given by [1]:

$$S^{(1)}_{\nu-\frac{1}{2}}(x) = x^{\nu+\frac{1}{2}}(1+x)^{\beta}\,{}_2F_1(\beta+2\nu, \beta;\, 1+2\nu;\, -x), \tag{19}$$

$$T^{(1)}_{\nu-\frac{1}{2}}(x) = x^{\gamma_1}(1+x)^{\beta-1}\,{}_2F_1(\beta+\gamma_1+\gamma_2, \beta+\gamma_1-\gamma_2;\, 1+2\gamma_1;\, -x), \tag{20}$$

where

$$\nu = n + \tfrac{1}{2}, \tag{21}$$

$$\beta = \tfrac{1}{2}(1 - \sqrt{1+(4ka)^2}), \tag{22}$$

$$\gamma_1 = \sqrt{\nu^2 - \tfrac{1}{4}}, \quad \gamma_2 = \sqrt{\nu^2 + \tfrac{3}{4}}, \tag{23}$$

and ${}_2F_1$ is the hypergeometric function.

If we consider ν as a complex number, then γ_1 has branch-points at $\nu = \pm 1/2$ and γ_2 at $\nu = \pm i\sqrt{3}/2$. We choose the branch-cuts in the complex ν plane along the real axis between $-\frac{1}{2}$ and $+\frac{1}{2}$, and along the imaginary axis between $-i\sqrt{3}/2$ and $+i\sqrt{3}/2$ (see Fig. 2).

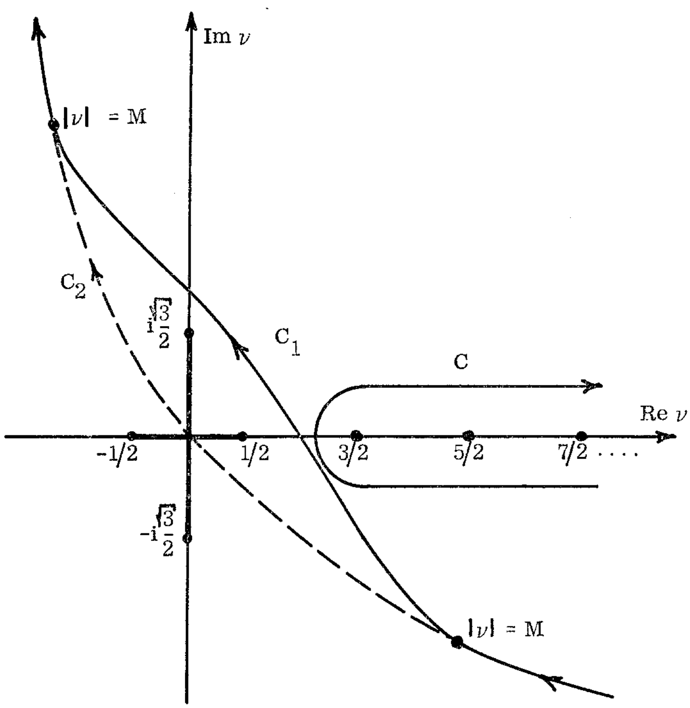

Fig. 2. Contours of integration in the complex ν plane.

Now we replace the summation in (12) with a line integral taken along the clockwise contour C of Fig. 2, which encloses those poles of the integrand that are located at $\nu = \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \ldots$. By following a transformation of the type of Watson's, the line integral along C is replaced by the sum of (i) a line integral whose contour consists of a path C_1 extending from the fourth through the first to the second quadrant, plus the arc of a circle of large radius with center at $\nu = 0$ extending from the second through the first to the fourth quadrant, and (ii) a residue series due to the poles of the integrand which lie in the first quadrant. It can be shown that the line integral along the arc of circle vanishes as the radius tends to infinity. The contour C_1 crosses the real ν axis between $\frac{1}{2}$ and $\frac{3}{2}$ and the imaginary ν axis above $+\mathrm{i}\sqrt{3}/2$, avoiding the branch-cuts (see Fig. 2). The result obtained thus far is still exact.

We may neglect the contributions to the backscattered field due to the poles in the first quadrant because we only want the dominant term of the high-frequency backscattered field, and this arises from an asymptotic estimate of the line integral along the contour C_1; thus we have:

$$\boldsymbol{E}^{b.s.} \sim -\hat{e}\,\frac{\mathrm{e}^{\mathrm{i}kr}}{2kr}\int\limits_{C_1}\frac{\nu}{\cos\pi\nu}\,(a_{\nu-\frac{1}{2}}-b_{\nu-\frac{1}{2}})\,\mathrm{d}\nu, \qquad (ka \gg 1). \tag{24}$$

The quantity $(a_{\nu-\frac{1}{2}}-b_{\nu-\frac{1}{2}})$ in (24) must now be evaluated for $|\nu| = \mathrm{O}[(ka)^{\frac{1}{3}+\varepsilon}]$ where ε is an arbitrarily small positive number.

§ 4. Asymptotic expansions for *TE* modes

By substituting (19) in the first of (16) and by using the differentiation formula for ${}_2F_1$, we find that

$$M_{\nu-\frac{1}{2}} = \frac{1+\beta+2\nu}{2ka} - \frac{\beta(\beta+2\nu)}{ka(1+2\nu)} \times$$
$$\times\,\frac{{}_2F_1(\beta+2\nu+1,\beta+1;2+2\nu;-1)}{{}_2F_1(\beta+2\nu,\beta;1+2\nu;-1)} \tag{25}$$

The functions ${}_2F_1$ in (25) can be expressed in terms of gamma functions [2]:

$${}_2F_1(\beta+2\nu,\beta;1+2\nu;-1) =$$
$$= \pi^{\frac{1}{2}}2^{-\beta-2\nu}\,\frac{\Gamma(1+2\nu)}{\Gamma\left(1+\nu-\dfrac{\beta}{2}\right)\Gamma\left(\dfrac{1+2\nu+\beta}{2}\right)}, \tag{26}$$

valid if $2\nu \neq -1, -2, -3, \ldots;$

$${}_2F_1(\beta+2\nu+1,\beta+1;2+2\nu;-1) = \pi^{\frac{1}{2}}\beta^{-1}\,\frac{\Gamma(2+2\nu)}{2^{1+\beta+2\nu}} \times$$

$$\times \left\{ \frac{1}{\Gamma\left(\frac{1+2\nu+\beta}{2}\right)\Gamma\left(1+\nu-\frac{\beta}{2}\right)} - \right.$$
$$\left. - \frac{1}{\Gamma\left(1+\nu+\frac{\beta}{2}\right)\Gamma\left(\frac{1+2\nu-\beta}{2}\right)} \right\}, \quad (27)$$

valid if $2\nu \neq -2, -3, -4, \ldots$. Therefore we have:

$$M_{\nu-\frac{1}{2}} = \frac{1}{2ka} + \frac{\left(\nu-\frac{\beta}{2}\right)\Gamma\left(\nu-\frac{\beta}{2}\right)\Gamma\left(\nu+\frac{1+\beta}{2}\right)}{ka\Gamma\left(\nu+\frac{\beta}{2}\right)\Gamma\left(\nu+\frac{1-\beta}{2}\right)}. \quad (28)$$

The exact result (28) may be further simplified with the aid of Stirling's formula for the gamma function, yielding:

$$M_{\nu-\frac{1}{2}} \sim \frac{1}{2ka}\left\{1+\sqrt{AB}\left[1+\frac{1}{4A}+\frac{1}{32A^2}+\mathrm{O}(A^{-3})\right]\times\right.$$
$$\left.\times\left[1-\frac{1}{4B}+\frac{1}{32B^2}+\mathrm{O}(B^{-3})\right]\right\}, \quad (29)$$

where

$$A = 2\nu - \beta, \quad B = 2\nu + \beta; \quad (30)$$

formula (29) is valid provided that

$$|A| \gg 1, \quad |B| \gg 1. \quad (31)$$

Let us now suppose that $ka \gg 1$, and let us restrict ν to the domain $|\nu| = \mathrm{O}[(ka)^{\frac{1}{2}+\varepsilon}]$; then result (29) becomes:

$$M_{\nu-\frac{1}{2}} \sim -i + \frac{1}{2ka} + \frac{i}{2}\left(\frac{\nu}{ka}\right)^2 - \frac{i}{16}(ka)^{-2} + \frac{i}{8}\left(\frac{\nu}{ka}\right)^4 +$$
$$+ \mathrm{O}\left[\frac{\nu}{(ka)^3}\right] + \mathrm{O}\left[\frac{\nu^3}{(ka)^4}\right] + \mathrm{O}[(ka)^{-3}]. \quad (32)$$

§ 5. Asymptotic expansions for *TM* modes

By substituting (20) in the second of (16) and by using the differentiation formula for ${}_2F_1$, we find that

$$\tilde{M}_{\nu-\frac{1}{2}} = \frac{2\gamma_1 + \beta - 1}{2ka} - \frac{(\beta+\gamma_1)^2 - \gamma_2^2}{ka(1+2\gamma_1)} R, \quad (33)$$

where

$$R = \frac{{}_2F_1(\beta + \gamma_1 + \gamma_2 + 1, \beta + \gamma_1 - \gamma_2 + 1; 2 + 2\gamma_1; -1)}{{}_2F_1(\beta + \gamma_1 + \gamma_2, \beta + \gamma_1 - \gamma_2; 1 + 2\gamma_1; -1)} =$$

$$= \frac{{}_2F_1(\beta + \gamma_1 + \gamma_2 + 1, 1 - \beta + \gamma_1 + \gamma_2; 2 + 2\gamma_1; \frac{1}{2})}{2\,{}_2F_1(\beta + \gamma_1 + \gamma_2, 1 - \beta + \gamma_1 + \gamma_2; 1 + 2\gamma_1; \frac{1}{2})}. \tag{34}$$

For the hypergeometric functions in the second line of (34), we use the integral representation [2]:

$${}_2F_1\left(b, \lambda; C; \frac{h}{z}\right) = \frac{z^\lambda}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1}\, \mathrm{e}^{-zt}\, {}_1F_1(b; C; ht)\, \mathrm{d}t, \tag{35}$$

which is valid for

$$\operatorname{Re} z > \operatorname{Re} h > 0, \quad \operatorname{Re} \lambda > 0. \tag{36}$$

These conditions are satisfied for $|\nu| = \mathrm{O}[(ka)^{\frac{1}{2}+\varepsilon}]$, and therefore we may write:

$$R = \frac{\int_0^\infty t^{\gamma_1+\gamma_2-\beta}\, \mathrm{e}^{-2t}\, {}_1F_1(\beta + \gamma_1 + \gamma_2 + 1; 2 + 2\gamma_1; t)\, \mathrm{d}t}{2 \int_0^\infty t^{\gamma_1+\gamma_2-\beta}\, \mathrm{e}^{-2t}\, {}_1F_1(\beta + \gamma_1 + \gamma_2; 1 + 2\gamma_1; t)\, \mathrm{d}t}. \tag{37}$$

Now we observe that [2]:

$${}_1F_1(\beta + \gamma_1 + \gamma_2; 1 + 2\gamma_1; t) = \mathrm{e}^{t/2}\, t^{-\frac{1}{2}-\gamma_1} M_{\frac{1}{2}-\beta-\gamma_2, \gamma_1}(t) =$$

$$= \frac{\Gamma(1 + 2\gamma_1)}{\Gamma(1 - \beta + \gamma_1 - \gamma_2)}\, \mathrm{e}^t\, t^{-\gamma_1} \int_0^\infty \mathrm{e}^{-\tau}\, \tau^{-\beta-\gamma_2} J_{2\gamma_1}(2\sqrt{t\tau})\, \mathrm{d}\tau, \tag{38}$$

with a similar result for the ${}_1F_1$ in the numerator of (37); these integral representations are certainly valid if

$$ka - |\nu| \gg 1. \tag{39}$$

With the change of variables

$$t = u^2, \quad \tau = w^2 \tag{40}$$

relation (37) thus becomes

$$R = (\tfrac{1}{2} + \gamma_1) \frac{\int\limits_0^\infty \mathrm{d}u\, \mathrm{e}^{-u^2} u^{2\gamma_2 - 2\beta} \int\limits_0^\infty \mathrm{d}w\, \mathrm{e}^{-w^2} w^{-2\gamma_2 - 2\beta} J_{1+2\gamma_1}(2uw)}{\int\limits_0^\infty \mathrm{d}u\, \mathrm{e}^{-u^2} u^{2\gamma_2 - 2\beta + 1} \int\limits_0^\infty \mathrm{d}w\, \mathrm{e}^{-w^2} w^{-2\gamma_2 - 2\beta + 1} J_{2\gamma_1}(2uw)}. \tag{41}$$

By using Sommerfeld's integral representation

$$J_\mu(2uw) = \frac{1}{2\pi} \int\limits_\Sigma \mathrm{d}\tau\, \mathrm{e}^{\mathrm{i}\mu\tau - \mathrm{i}2uw\sin\tau} \tag{42}$$

where the contour Σ begins at $\tau = -\pi + \mathrm{i}\infty$ and ends at $\tau = +\pi + \mathrm{i}\infty$, making the change of variables

$$u = (ka)^{\frac{1}{2}}\, \xi, \quad w = (ka)^{\frac{1}{2}}\, \eta, \tag{43}$$

and interchanging orders of integration, we finally obtain:

$$R = \frac{\frac{1}{2} + \gamma_1}{ka} \cdot \frac{\int\limits_\Sigma \mathrm{d}\tau\, \mathrm{e}^{\mathrm{i}(1+2\gamma_1)\tau} \int\limits_0^\infty \mathrm{d}\eta\, \mathrm{e}^{-ka\eta^2 - (2\gamma_2 + 2\beta)\ln\eta} \times \int\limits_0^\infty \mathrm{d}\xi\, \mathrm{e}^{-ka\xi^2 - \mathrm{i}2ka\xi\eta\sin\tau + (2\gamma_2 - 2\beta)\ln\xi}}{\int\limits_\Sigma \mathrm{d}\tau\, \mathrm{e}^{\mathrm{i}2\gamma_1\tau} \int\limits_0^\infty \mathrm{d}\eta\, \mathrm{e}^{-ka\eta^2 - (2\gamma_2 + 2\beta - 1)\ln\eta} \times \int\limits_0^\infty \mathrm{d}\xi\, \mathrm{e}^{-ka\xi^2 - \mathrm{i}2ka\xi\eta\sin\tau + (2\gamma_2 - 2\beta + 1)\ln\xi}} \tag{44}$$

This result is exact, and certainly valid for

$$|\nu| = \mathrm{O}[(ka)^{\frac{1}{2}+\varepsilon}].$$

All the integrals appearing in (44) can be asymptotically evaluated by steepest descents. The first two integrals in both numerator and denominator have saddle points at $\xi = \xi_0$ and $\eta = \eta_0$ in the complex ξ and η planes, where:

$$\xi_0 \sim \left\{\frac{s}{2(1 + \mathrm{i}\sin\tau)}\right\}^{\frac{1}{2}} \times \left\{1 + \mathrm{i}\frac{\gamma_2}{ska}\sin\tau + \frac{\mathrm{i}\sin\tau}{2s^2}(2 + \mathrm{i}\sin\tau)\left(\frac{\gamma_2}{ka}\right)^2 + \mathrm{O}\left[\left(\frac{\gamma_2}{ka}\right)^3\right]\right\}, \tag{45}$$

$$\eta_0 \sim \left\{\frac{s}{2(1+\mathrm{i}\sin\tau)}\right\}^{\frac{1}{2}} \times$$
$$\times \left\{1 - (2+\mathrm{i}\sin\tau)\,\frac{\gamma_2}{ska} - \right.$$
$$\left. - \frac{4+2\mathrm{i}\sin\tau+\sin^2\tau}{2s^2}\left(\frac{\gamma_2}{ka}\right)^2 + \mathrm{O}\left[\left(\frac{\gamma_2}{ka}\right)^3\right]\right\}; \tag{46}$$

here and in the following,

$$s = \frac{2\gamma_2 - 2\beta}{ka}, \qquad \text{for the numerator of } R,$$
$$= \frac{2\gamma_2 - 2\beta + 1}{ka}, \qquad \text{for the denominator of } R. \tag{47}$$

Once the first two integrals are asymptotically evaluated, it is found that the integral along Σ has two saddle points at $\tau = \tau_{0\pm}$, where

$$\tau_{0+} \sim \frac{\pi}{2} - \frac{1+\mathrm{i}}{s}\,\Lambda\,\frac{\gamma_2}{ka} - \frac{2(1+\mathrm{i})}{s^2}\,\Lambda\left(\frac{\gamma_2}{ka}\right)^2 + \mathrm{O}\left[\left(\frac{\gamma_2}{ka}\right)^3\right], \tag{48}$$

$$\tau_{0-} \sim -\frac{\pi}{2} + \frac{1-\mathrm{i}}{s}\,\Lambda\,\frac{\gamma_2}{ka} + \frac{2(1-\mathrm{i})}{s^2}\,\Lambda\left(\frac{\gamma_2}{ka}\right)^2 + \mathrm{O}\left[\left(\frac{\gamma_2}{ka}\right)^3\right], \tag{49}$$

with

$$\Lambda = \frac{1+2\gamma_1}{\gamma_2}, \qquad \text{for the numerator of } R,$$
$$= 2\,\frac{\gamma_1}{\gamma_2}, \qquad \text{for the denominator of } R. \tag{50}$$

When all the integrals in (44) are evaluated, it is found that

$$R \sim -\frac{\frac{1}{2}+\gamma_1}{2ka}\left[1 + \tan\left(\frac{\pi}{4} - \pi ka + \pi\gamma_1 - \tfrac{1}{2}\arctan\tfrac{1}{2}\right)\right] \times$$
$$\times \left\{1 + \mathrm{O}\left(\frac{\nu}{ka}\right) + \mathrm{O}\left[\frac{\nu^3}{(ka)^2}\right] + \mathrm{O}\left(\frac{1}{ka}\right)\right\} \tag{51}$$

and therefore from (33):

$$\tilde{M}_{\nu-\frac{1}{2}} \sim \tan\left(\frac{\pi}{4} - \pi ka + \pi\gamma_1 - \tfrac{1}{2}\arctan\tfrac{1}{2}\right) \times$$

$$\times\left\{1 + \mathrm{O}\left(\frac{\nu}{ka}\right) + \mathrm{O}\left[\frac{\nu^3}{(ka)^2}\right] + \mathrm{O}\left(\frac{1}{ka}\right)\right\}. \quad (52)$$

§ 6. The high-frequency backscattered field

The Debye asymptotic expansions are used for the Bessel functions appearing in

$$(a_{\nu-\frac{1}{2}} - b_{\nu-\frac{1}{2}}).$$

In particular, for $|\nu| = [\mathrm{O}(ka)^{\frac{1}{3}+\varepsilon}]$,

$$H_\nu^{(1)}(ka) \sim \sqrt{\frac{2}{\pi\rho ka}}\left\{1 + \mathrm{O}\left(\frac{1}{ka}\right) + \mathrm{O}\left[\frac{\nu^2}{(ka)^3}\right]\right\}, \quad (53)$$

$$H_\nu^{(1)\prime}(ka) \sim \sqrt{\frac{2}{\pi\rho ka}}\left\{\mathrm{i} + \mathrm{O}\left[\left(\frac{\nu}{ka}\right)^2\right] + \mathrm{O}\left(\frac{1}{ka}\right)\right\}, \quad (54)$$

where

$$\rho = \exp\left[\mathrm{i}\left(\pi\nu + \frac{\pi}{2} - 2ka - \frac{\nu^2}{ka}\right)\right] \times$$

$$\times\left\{1 + \mathrm{O}\left[\left(\frac{\nu}{ka}\right)^2\right] + \mathrm{O}\left[\frac{\nu^4}{(ka)^3}\right]\right\}. \quad (55)$$

Substitution of (32), (52), (53) and (54) in the integrand of (24) yields:

$$\boldsymbol{E}^{b.s.} \sim \hat{e}\,\frac{\mathrm{e}^{\mathrm{i}kr}}{2kr}\exp\{\mathrm{i}[2ka(\pi - 1) + \arctan\tfrac{1}{2}]\} \times$$

$$\times \int_{C_1} \mathrm{d}\nu\,\frac{\nu}{1 + \mathrm{e}^{-\mathrm{i}2\pi\nu}}\,\mathrm{e}^{-\mathrm{i}(\nu^2/ka) - \mathrm{i}2\pi\gamma_1}\left[1 + \mathrm{O}\left(\frac{\nu}{ka}\right) + \mathrm{O}\left(\frac{1}{ka}\right)\right]. \quad (56)$$

Let us indicate by M a positive number, large compared with unity but independent of ka, that is:

$$M \gg 1, \quad \lim_{ka\to\infty}\frac{M}{ka} = 0. \quad (57)$$

Let us split the contour C_1 into three parts, by singling out the portion near $\nu = 0$ along which $|\nu| < M$ (see Fig. 2). Along this central portion

$$\exp(-\mathrm{i}\nu^2/ka) \sim 1, \qquad (|\nu| < M) \tag{58}$$

so that the corresponding integral is $O(1)$, whereas the integral along the entire contour C_1 is $O(ka)$. Since we only want the leading term in the asymptotic estimate, we may neglect the central portion of C_1. Along the remaining part of the contour

$$\gamma_1 \sim \nu, \qquad (|\nu| > M), \tag{59}$$

and therefore

$$\int_{C_1} \mathrm{d}\nu \, \frac{\nu}{1 + \mathrm{e}^{-\mathrm{i}2\pi\nu}} \, \mathrm{e}^{-\mathrm{i}(\nu^2/ka) - \mathrm{i}2\pi\gamma_1} \sim - \int_{\substack{C_1 \\ |\nu| > M}} \mathrm{d}\nu \, \frac{\nu \, \mathrm{e}^{-\mathrm{i}(\nu^2/ka)}}{1 + \mathrm{e}^{-\mathrm{i}2\pi\nu}} + \mathrm{O}(1). \tag{60}$$

The integrand in the right-hand side of (60) has no branch-cuts, and we may therefore connect the two portions of the contour with a line through the origin $\nu = 0$; the added term is again $\mathrm{O}(1)$. Thus,

$$\boldsymbol{E}^{b.s.} \sim -\hat{e} \, \frac{\mathrm{e}^{\mathrm{i}kr}}{2kr} \exp\{\mathrm{i}[2ka(\pi - 1) + \arctan \tfrac{1}{2}]\} \times$$

$$\times \left\{ \int_{C_2} \mathrm{d}\nu \, \frac{\nu \, \mathrm{e}^{-\mathrm{i}(\nu^2/ka)}}{1 + \mathrm{e}^{-\mathrm{i}2\pi\nu}} \left[1 + \mathrm{O}\left(\frac{\nu}{ka}\right) + \mathrm{O}\left(\frac{1}{ka}\right) \right] + \mathrm{O}(1) \right\}, \tag{61}$$

where the contour C_2 in the uncut ν plane consists of that portion of C_1 along which $|\nu| > M$, plus the broken line of Fig. 2. Now the leading term of the integral in (61) can be evaluated by the method of Scott [3], and the term containing $\mathrm{O}(\nu/ka)$ can be estimated by reducing it to a Fresnel-type integral. The final result is

$$\boldsymbol{E}^{b.s.} \sim \hat{e} \, \frac{a}{4r} \exp\left\{ \mathrm{i} \left[kr + 2ka(\pi - 1) - \frac{\pi}{2} + \arctan \tfrac{1}{2} \right] \right\} \times$$

$$\times \{1 + \mathrm{O}[(ka)^{-\frac{1}{2}}]\}. \tag{62}$$

A comparison between (62) and (11) yields:

$$D = \tfrac{1}{2}, \quad \delta = \arctan \tfrac{1}{2}, \tag{63}$$

in agreement with the intuitive limitations (8) and (10).

§ 7. Conclusion

It has been shown that the dielectric lens of radius $r = a$ made of a material with a variable refractive index given by (1) has a high-frequency monostatic cross section equal to that of a metal sphere of radius $r = a/2$, whereas its bistatic cross section is equal to that of a metal sphere of radius $r = a$ when the separation angle between transmitter and receiver is not small.

The backscattered field at short wavelengths has been obtained by means of a lengthy asymptotic analysis of the exact solution. However, there seems to be no easy way to derive the values of the constants D and δ; any attempt to refine the limitations (8) and (10) on grounds of physical reasoning would lead to a complicated analysis; among other things, the internal reflections should then be taken into account.

Finally, we observe that the problem treated here has points in common with the non-relativistic quantum-mechanical scattering of a particle by a spherically symmetric potential V of finite range $r = a$, where

$$V(x) = \frac{\hbar^2 k^2}{2m}\left[1 - \frac{4}{x(1+x)^2}\right], \qquad (0 \leq x = r/a \leq 1), \quad (64)$$

m being the mass of the particle and $2\pi\hbar$ Planck's constant. The separation of Schrödinger's equation in spherical coordinates leads to equation (17) for the n-th radial wavefunction. Observe that $V(1) = 0$ and $V(0) = -\infty$ for all k, but the value of the attractive potential $V(x)$ at any point x in $0 < x < 1$ depends on the energy $\hbar^2 k^2/(2m)$ of the incident particle. That is to say, the scattering center at $r = 0$ reacts differently to particles of different energy.

APPENDIX

The lens considered in this paper belongs to the family

$$N(x) = \frac{2x^{(1/c)-1}}{1 + x^{2/c}}, \qquad (A.1)$$

where c is any positive real constant. In fact, the refractive index (1) is obtained by putting $c = 2$ in (A.1). The solutions of the radial differential equations (17) and (18) for this more general case have been given in [1].

When the asymptotic analysis of sections 4 and 5 is applied to the class of lenses with refractive index (A.1), it is found that for

$$|\nu| = \mathrm{O}[(ka)^{\frac{1}{2}+\varepsilon}] \quad \text{and} \quad cka \gg 1:$$

$$M_{\nu-\frac{1}{2}} \sim \left(1 - \frac{c}{2} - \mathrm{i}\right) + \left(\frac{3}{4} - \frac{1}{2c}\right)(ka)^{-1} + \\ + \frac{\mathrm{i}}{2}\left(\frac{\nu}{ka}\right)^2 + \frac{2 - c - 4\mathrm{i}}{16c^2}(ka)^{-2} + \frac{\mathrm{i}}{8}\left(\frac{\nu}{ka}\right)^4 + \\ + \mathrm{O}\left[\frac{\nu}{(ka)^3}\right] + \mathrm{O}\left[\frac{\nu^3}{(ka)^4}\right] + \mathrm{O}[(ka)^{-3}], \tag{A.2}$$

$$\tilde{M}_{\nu-\frac{1}{2}} \sim \left[1 - \frac{c}{2} + \tan f(\nu)\right] \cdot \\ \cdot \left\{1 + \mathrm{O}\left(\frac{\nu}{ka}\right) + \mathrm{O}\left[\frac{\nu^3}{(ka)^2}\right] + \mathrm{O}\left(\frac{1}{ka}\right)\right\}, \tag{A.3}$$

where

$$f(\nu) = \frac{\pi}{4} - \frac{\pi}{2} cka + \pi\gamma - \tfrac{1}{2}\arctan \tfrac{1}{2}, \tag{A.4}$$

$$\gamma = \tfrac{1}{2}\sqrt{c^2\nu^2 - c + 1}. \tag{A.5}$$

We limit our considerations to the case in which i) the optical rays do not make more than one turn about the center of the lens (first-order lenses), and ii) at least one ray emerges in the back-scattering direction; then

$$1 \leq c \leq 2. \tag{A.6}$$

Substitution of expressions (A.2) and (A.3) in the integrand of (24) yields:

$$\int_{C_1} \frac{\nu}{\cos \pi\nu}(a_{\nu-\frac{1}{2}} - b_{\nu-\frac{1}{2}})\,\mathrm{d}\nu \sim \frac{\mathrm{e}^{-\mathrm{i}2ka}}{\dfrac{c}{4} - \dfrac{1}{2} + \mathrm{i}} \times \\ \times \int_{C_1} \mathrm{d}\nu\, \frac{\nu\, \mathrm{e}^{-\mathrm{i}(\nu^2/ka)}}{1 + \mathrm{e}^{-\mathrm{i}2\pi\nu}} \frac{1 - \mathrm{i}\tan f(\nu)}{1 + \mathrm{i}\left(1 - \dfrac{c}{2}\right) + \mathrm{i}\tan f(\nu)} \times \\ \times \left[1 + \mathrm{O}\left(\frac{\nu}{ka}\right) + \mathrm{O}\left(\frac{1}{ka}\right)\right], \tag{A.7}$$

where now C_1 crosses the real ν axis between $\frac{1}{2}$ and $\frac{3}{2}$ and the imaginary ν axis above $+(i/c)\sqrt{c+1}$.

If $c = 2$, expression (A.7) simplifies and yields the result (56). If $c < 2$, the integrand in (A.7) has poles at those points of the complex ν plane for which

$$\operatorname{Re}\gamma = m + \frac{c}{2}ka - \frac{3}{4} + \frac{1}{2\pi}\left(\arctan\frac{1}{2} + \arctan\frac{4}{2-c}\right), \tag{A.8}$$

$$\operatorname{Im}\gamma = \frac{1}{4\pi}\ln\left\{1 + \frac{16}{(2-c)^2}\right\}, \tag{A.9}$$

where m is any integer. The contributions to the backscattered field that are due to the poles enclosed by C_1 and by the semicircle at infinity cannot be neglected when compared with the contour integral contribution. Physically, this means that the dominant term in the high-frequency backscattered field does not arise from specular reflection as in the case of a metal sphere (or of the lens $c = 2$), and therefore is not obtainable by a saddle point evaluation of a contour integral. If $1 \leq c < 2$, the leading term of the backscattered field is not due to a single ray, but to all rays which impinge on the lens with an angle of incidence α given by:

$$\alpha = \pi\left(1 - \frac{c}{2}\right). \tag{A.10}$$

These rays are the generators of a circular cylinder with radius $a \sin\alpha$. The situation is somewhat similar to the phenomenon of glory rays that occurs in the scattering by a uniform dielectric lens with a refractive index between $\sqrt{2}$ and 2.

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Note added in proof.

In a private communication to the first author (Letter of 3 November 1969) Dr. S. M. Sherman of R. C. A. Missile and Surface Radar Division pointed out that if polarization is taken into account, the geometric-optics backscattering cross section of a metal-like lens should be zero. Dr. Sherman's discussion is identical to that commonly applied to the isotropic lens of Eaton; in our case, it fails to give the correct results for reasons similar to those presented in section 2.